Theorems with Balls

Carleton Algorithms Seminar

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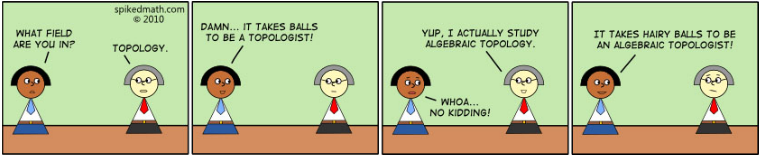
Ottawa – May 9, 2014



Combinatorial proofs for topological theorems

- **Brouwer's fixed point theorem**
  - **Proof:** Sperner's lemma
- **Hairy ball theorem**
  - **Proof:** generalized Sperner's lemma
  - **Corollary:** fixed points on spheres
- **Borsuk–Ulam theorem**
  - **Proof:** Tucker's lemma
  - **Corollary:** Lusternik–Schnirelmann theorem
  - **Corollary:** ham sandwich theorem

Brouwer's fixed point theorem

Theorem (Brouwer, 1910)

Every continuous mapping from an n-dimensional ball into itself has a fixed point.

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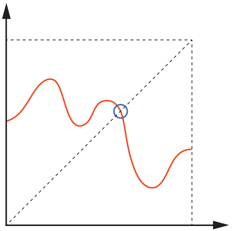


For $n = 1$, it easily follows from the intermediate value theorem.

Brouwer's fixed point theorem

Theorem (Brouwer, 1910)

Every continuous mapping from an n-dimensional ball into itself has a fixed point.

$n = 2$: if we crumple up the tablecloth and put it back on the table, one point ends up in its original position.

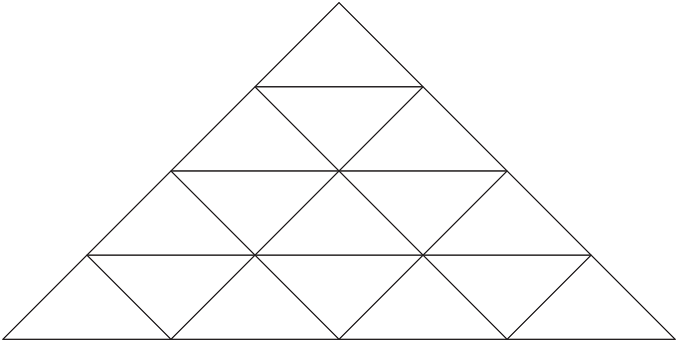
Brouwer's fixed point theorem

Theorem (Brouwer, 1910)

Every continuous mapping from an n-dimensional ball into itself has a fixed point.

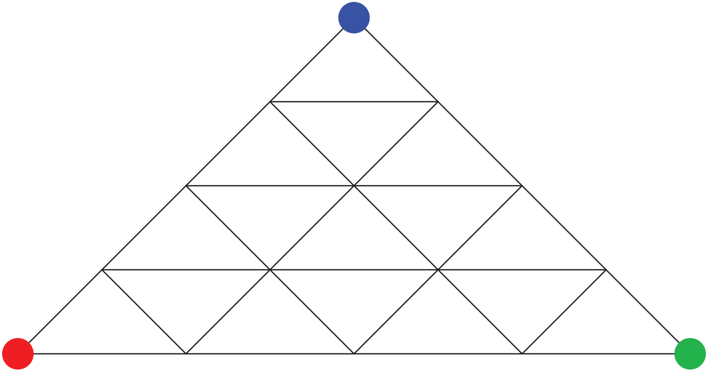
$n = 3$: if we stir a cocktail and let it rest, one point in the liquid ends up in its initial position.

Sperner's lemma



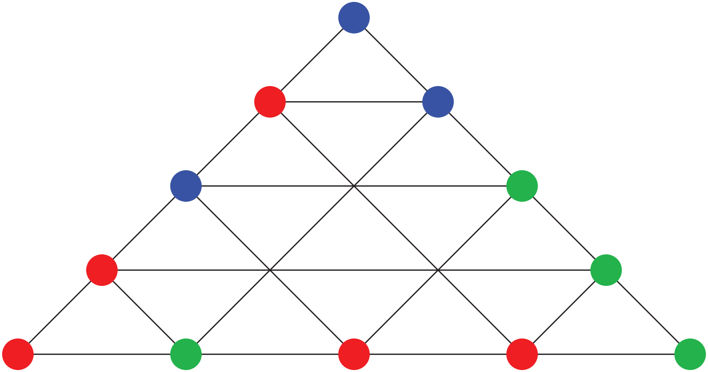
Start from a triangulated triangle.

Sperner's lemma



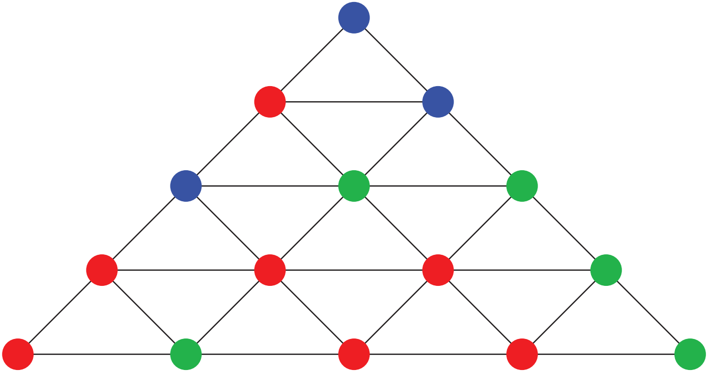
Color the vertices red, green and blue.

Sperner's lemma



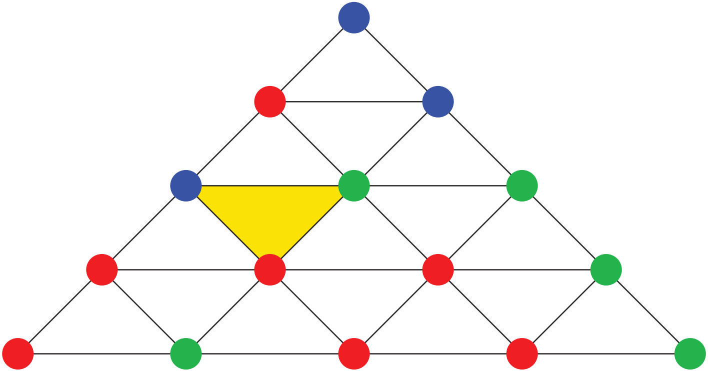
Color each vertex on an edge with one of the two colors of the endpoints of that edge.

Sperner's lemma



Color the internal vertices red, green or blue, arbitrarily.

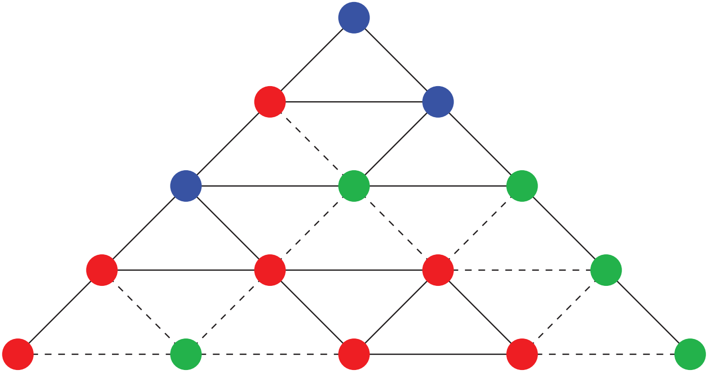
Sperner's lemma



Lemma (Sperner, 1928)

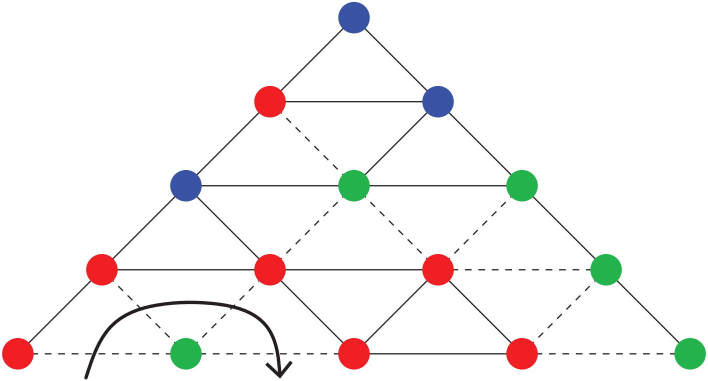
There exists at least a triangle with vertices of all three colors.

Sperner's lemma: proof



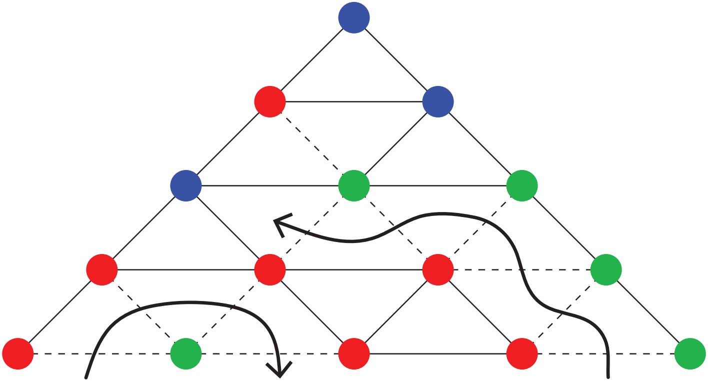
The red-green edges are *permeable*.

Sperner's lemma: proof



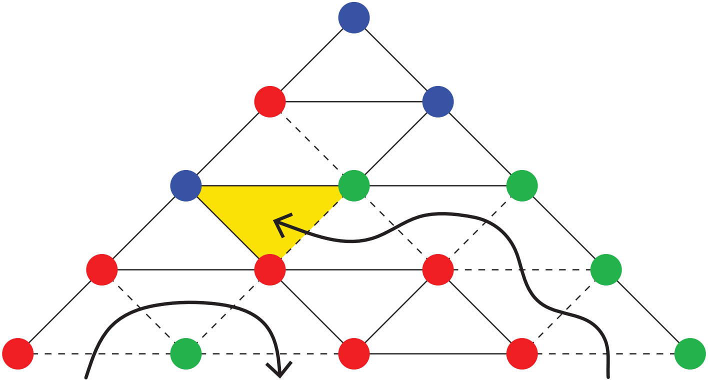
Let us enter the triangulation from a red-green edge. We may exit from another red-green edge...

Sperner's lemma: proof



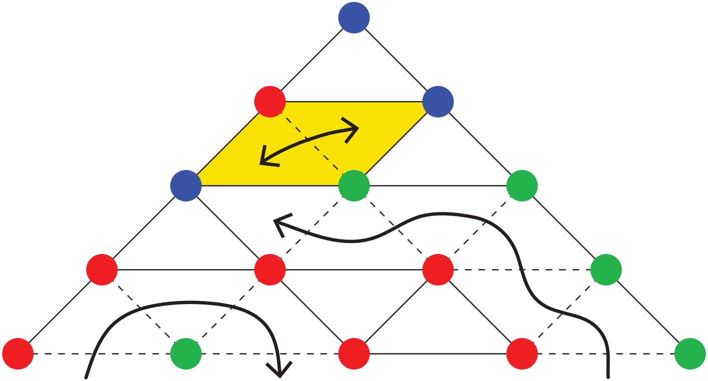
...But, because the external red-green edges are odd, an odd number of paths end inside the triangle.

Sperner's lemma: proof



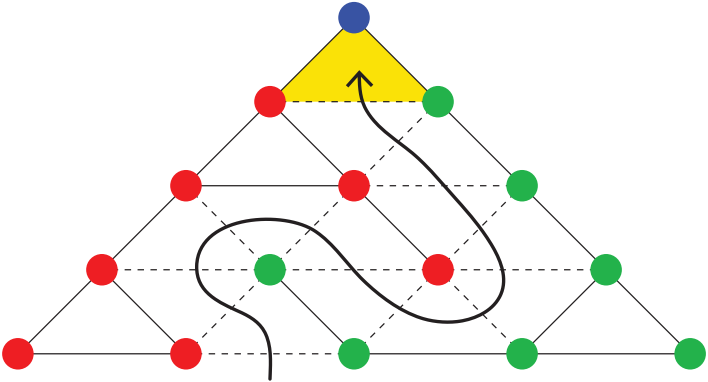
When the path ends, a 3-colored triangle has been found.

Sperner's lemma: proof



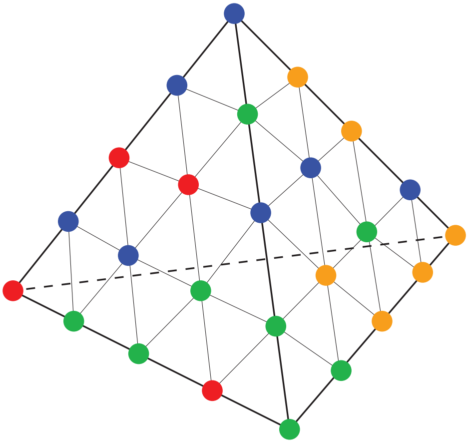
There may be other 3-colored triangles, which are endpoints of internal paths. In total, the 3-colored triangles are odd.

Sperner's lemma: proof



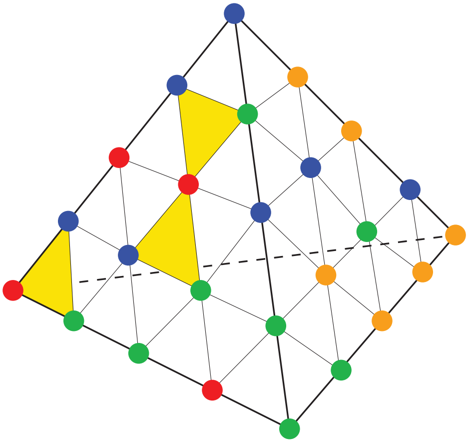
Another example.

Sperner's lemma: proof



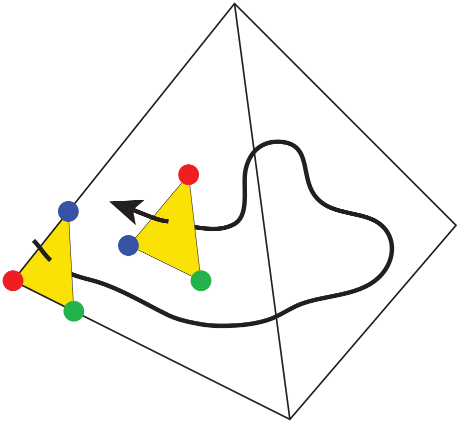
The proof generalizes to n-dimensional simplices and $n + 1$ colors.

Sperner's lemma: proof



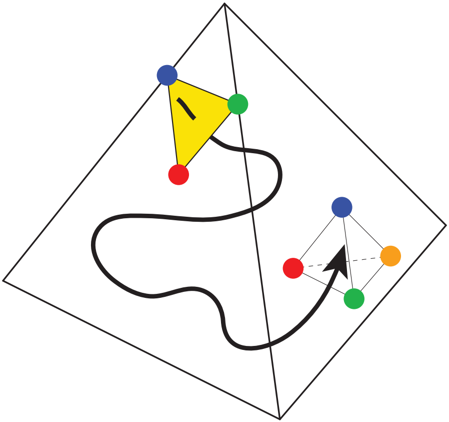
By inductive hypothesis, a face contains an odd number of 3-colored simplices.

Sperner's lemma: proof



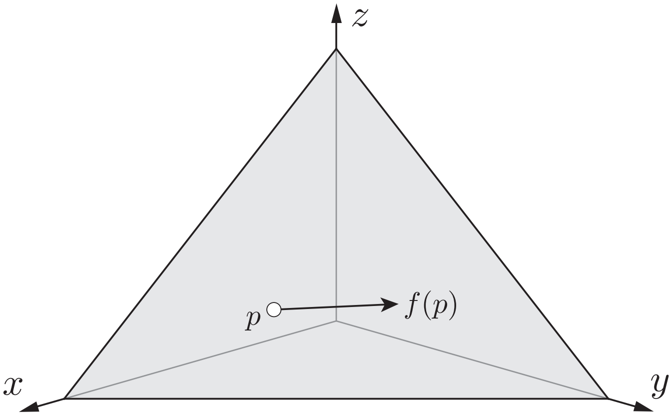
We enter from one of them, and we keep walking through 3-colored triangles. We either exit from another 3-colored triangle...

Sperner's lemma: proof



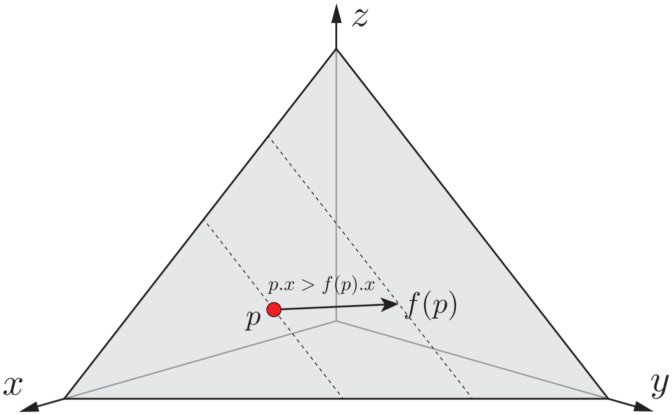
...Or we end up in a 4-colored tetrahedron. The 4-colored tetrahedra are again odd.

Brouwer's fixed point theorem: proof



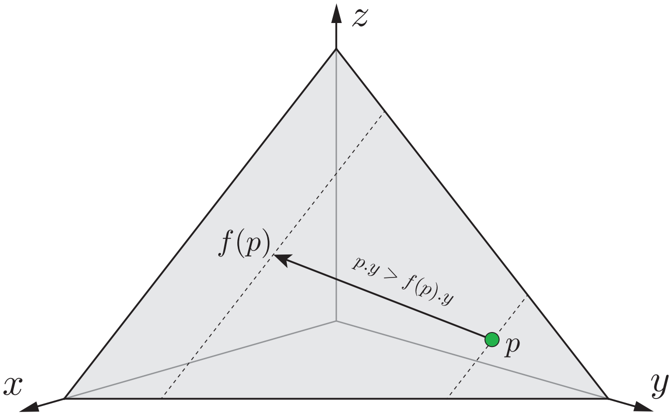
Consider the convex hull of $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, and a continuous function f from this set to itself.

Brouwer's fixed point theorem: proof



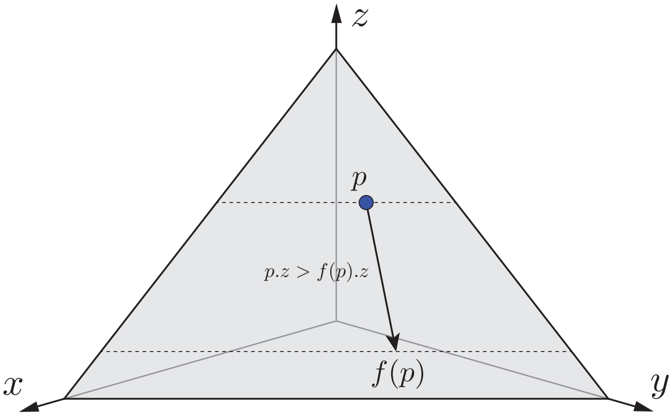
If f strictly decreases the x-coordinate of p, color p red.

Brouwer's fixed point theorem: proof



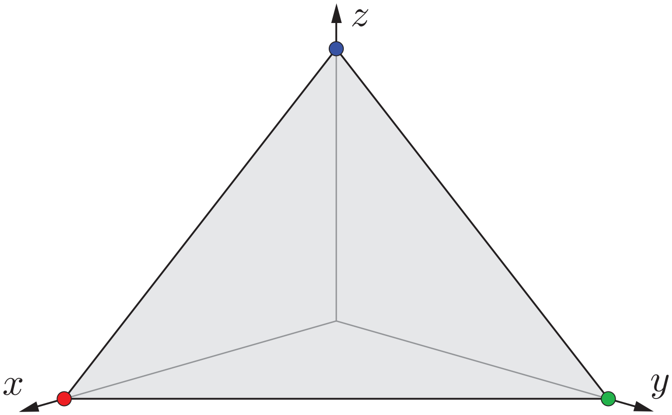
Otherwise, if f strictly decreases the y-coordinate of p, color p green.

Brouwer's fixed point theorem: proof



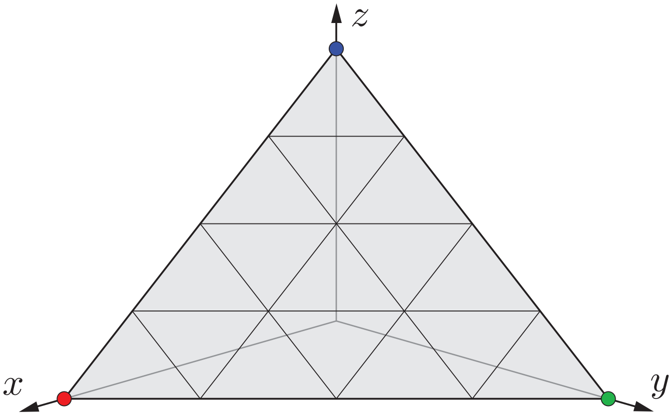
Otherwise, if f strictly decreases the z-coordinate of p, color p blue.

Brouwer's fixed point theorem: proof



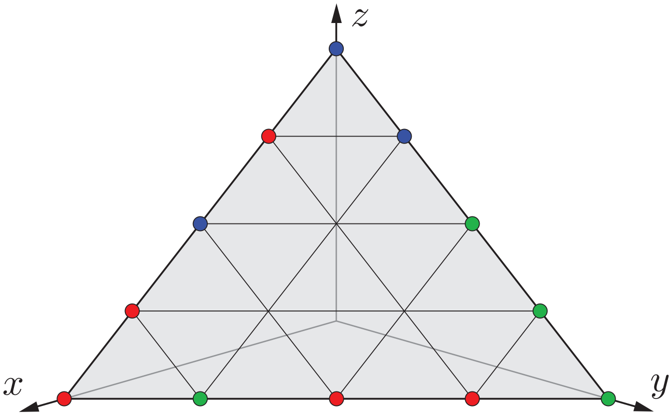
Suppose that f has no fixed points. Then $(1, 0, 0)$ is red, $(0, 1, 0)$ is green, and $(0, 0, 1)$ is blue.

Brouwer's fixed point theorem: proof



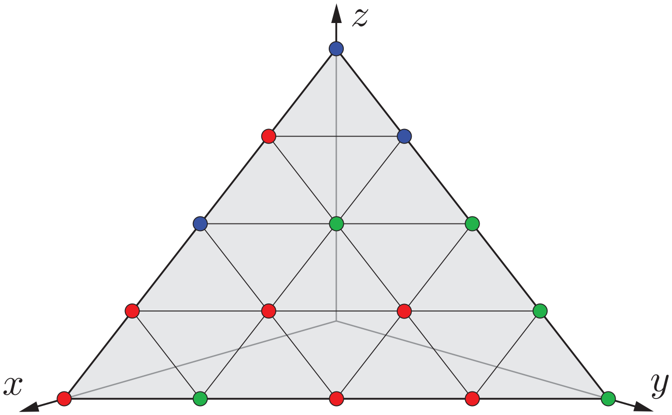
Triangulate the triangle.

Brouwer's fixed point theorem: proof



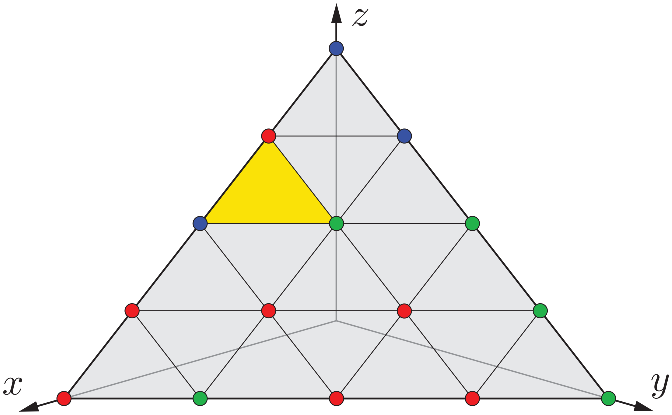
The points with $x = 0$ cannot be colored red, and similarly for y and z.

Brouwer's fixed point theorem: proof



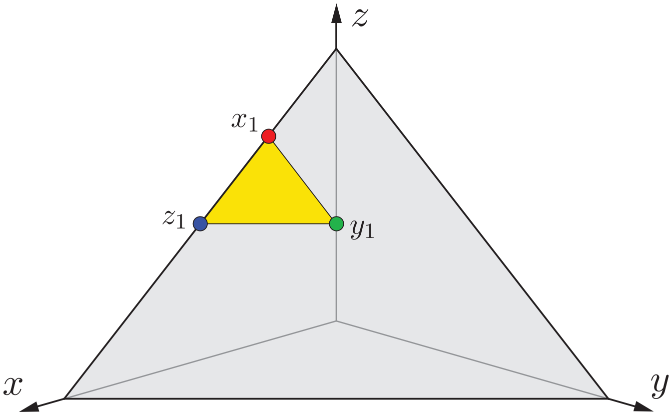
The coloring of the vertices satisfies the hypotheses of Sperner's lemma.

Brouwer's fixed point theorem: proof



Hence there is a 3-colored triangle.

Brouwer's fixed point theorem: proof



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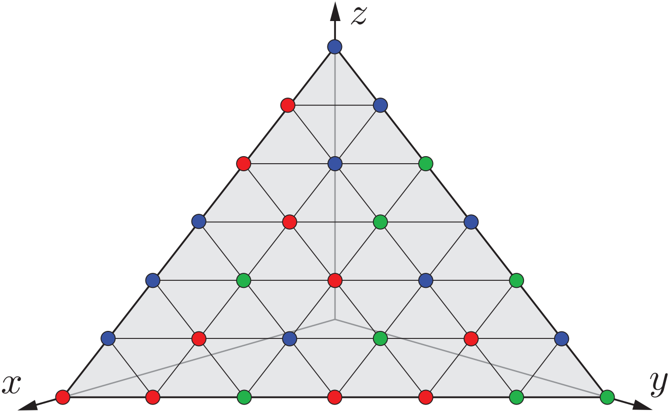
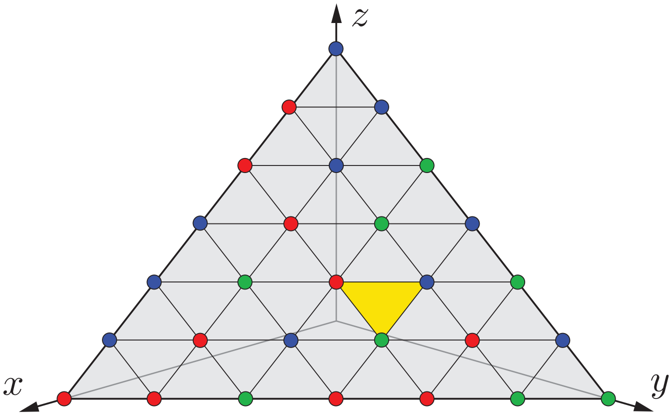
Brouwer's fixed point theorem: proof



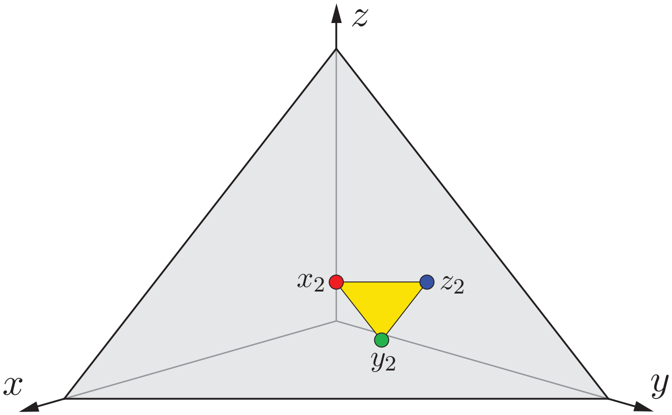
Construct a finer triangulation.

Brouwer's fixed point theorem: proof



Again, Sperner's lemma yields a smaller 3-colored triangle.

Brouwer's fixed point theorem: proof



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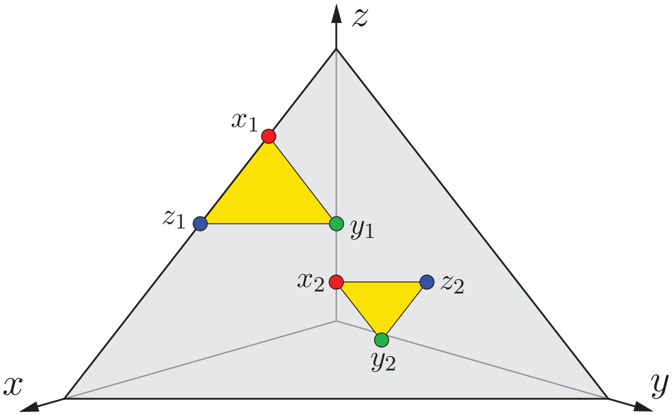

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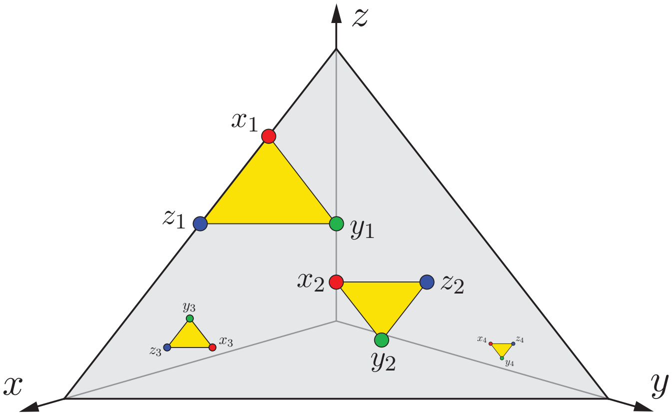
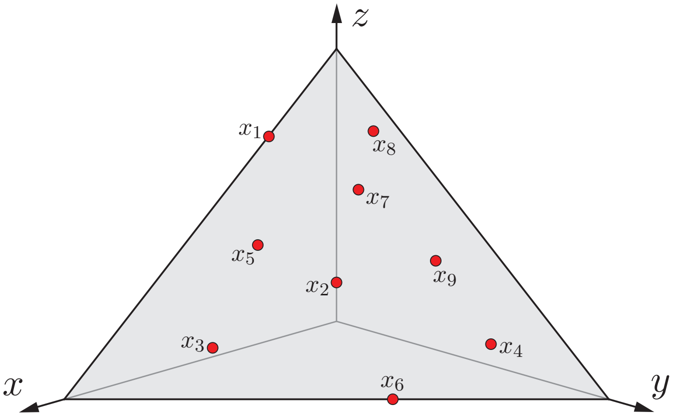
Brouwer's fixed point theorem: proof



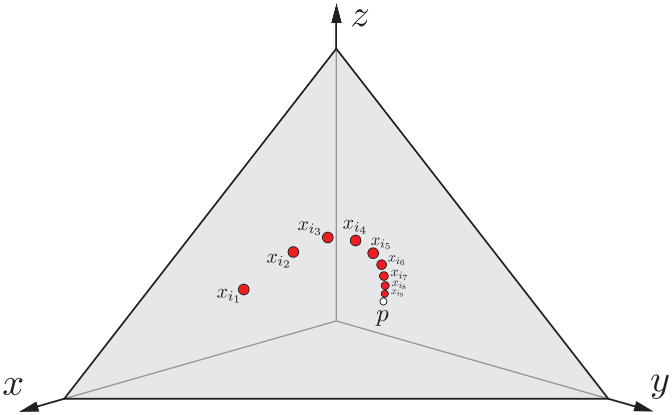
Proceeding in this fashion, we obtain a sequence of 3-colored triangles with vanishing edge lengths.

Brouwer's fixed point theorem: proof



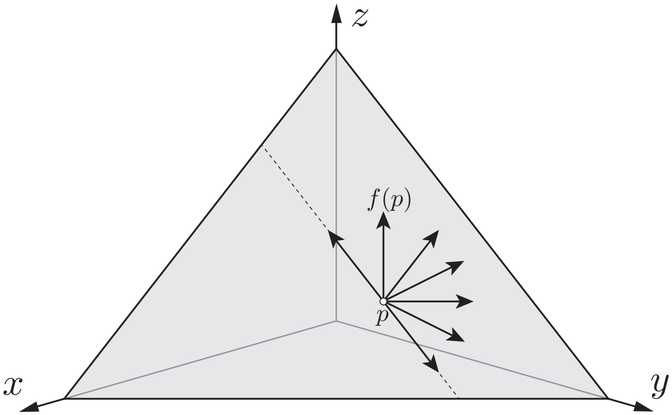
Consider the sequence of the red vertices of such triangles.

Brouwer's fixed point theorem: proof



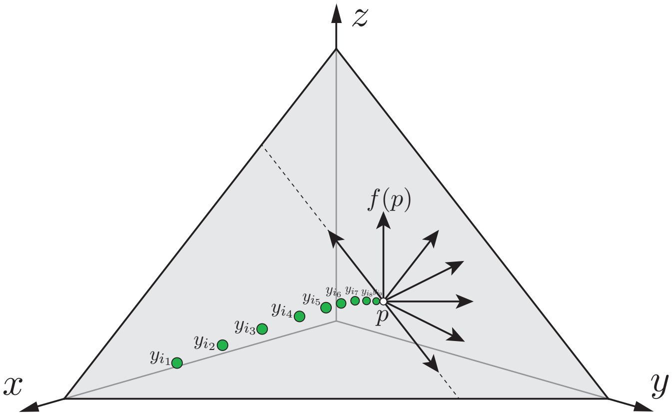
By the Bolzano–Weierstrass theorem, this sequence has a subsequence that converges to a point p in the triangle.

Brouwer's fixed point theorem: proof



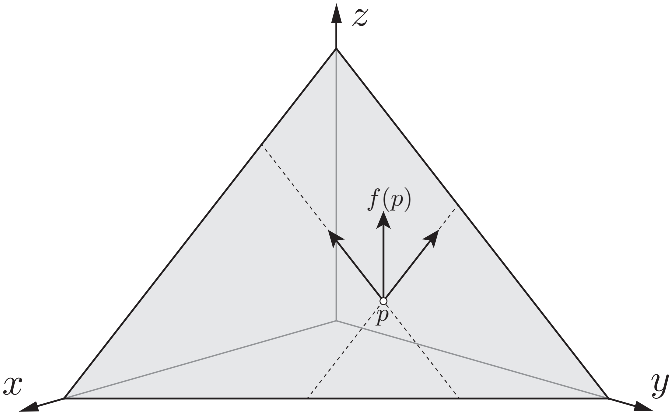
Since p is a limit of red points and f is continuous, $f(p).x \leqslant p.x$.

Brouwer's fixed point theorem: proof



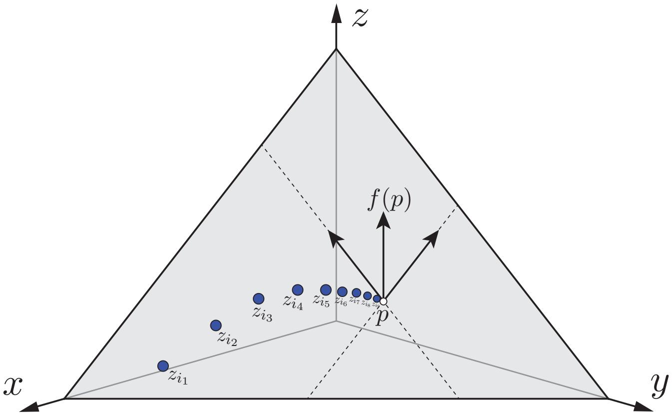
The corresponding subsequence of green vertices must also converge to p, because their distances to the red vertices vanish.

Brouwer's fixed point theorem: proof



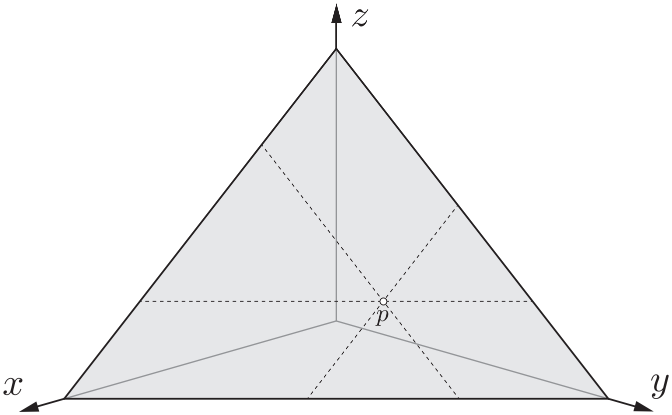
Hence $f(p).y \leqslant p.y$.

Brouwer's fixed point theorem: proof



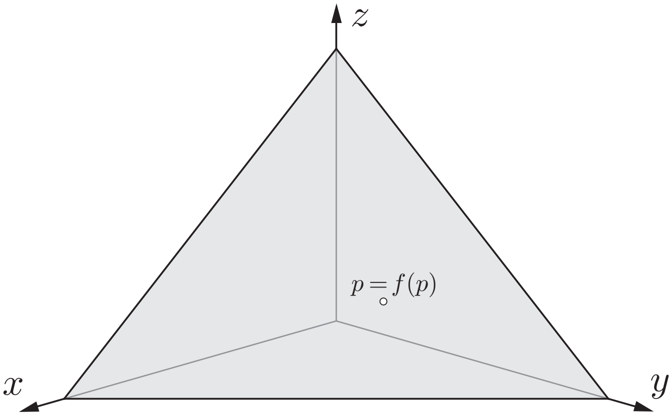
The sub-sequence of blue vertices also converges to *p*.

Brouwer's fixed point theorem: proof



Hence $f(p).z \leqslant p.z$.

Brouwer's fixed point theorem: proof



Because $x + y + z = 1$ for every point in the triangle, it follows that p is a fixed point of f.

Hairy ball theorem

Theorem (Brouwer, 1912)

An even-dimensional sphere does not admit any continuous field of non-zero tangent vectors.

Hairy ball theorem

Theorem (Brouwer, 1912)

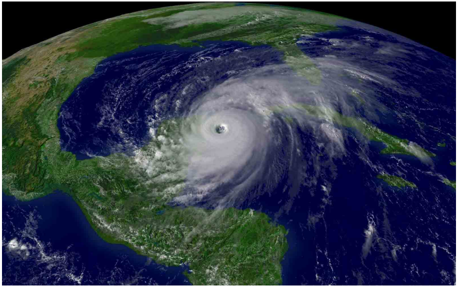
An even-dimensional sphere does not admit any continuous field of non-zero tangent vectors.

It is impossible to comb a hairy ball flat without creating cowlicks.

Hairy ball theorem

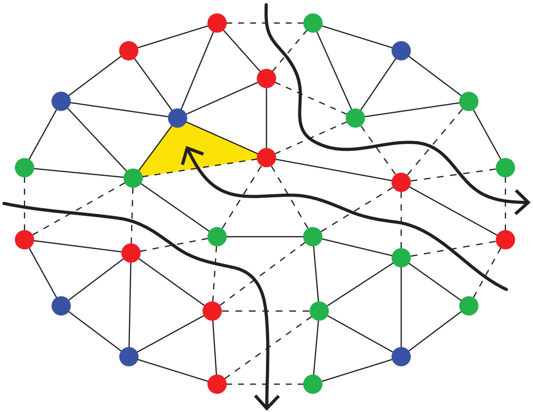
Theorem (Brouwer, 1912)

An even-dimensional sphere does not admit any continuous field of non-zero tangent vectors.



Given at least some wind on Earth, there must at all times be a cyclone somewhere.

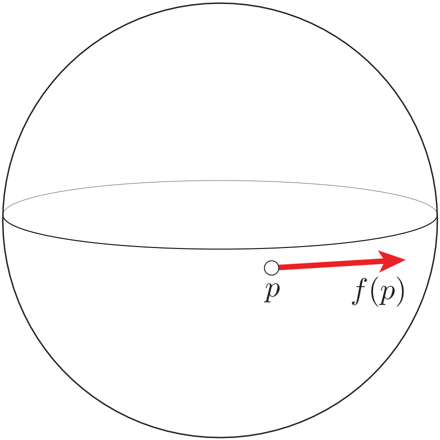
Generalized Sperner's lemma



Lemma

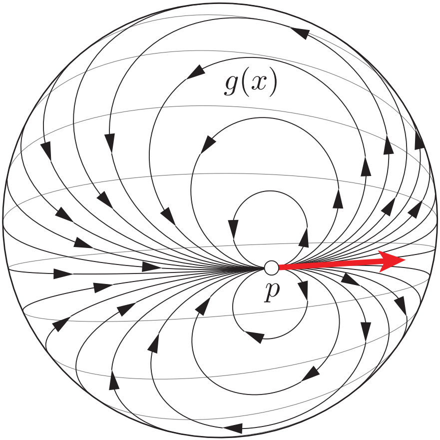
In any 3-colored triangulation with a different number of red-green and green-red outer edges, there is a 3-colored triangle.

Hairy ball theorem: proof



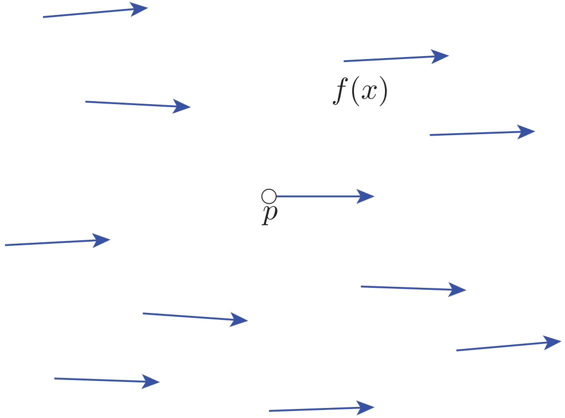
Assume that $f(x)$ is continuous and nowhere zero. Let p be any point on the sphere.

Hairy ball theorem: proof



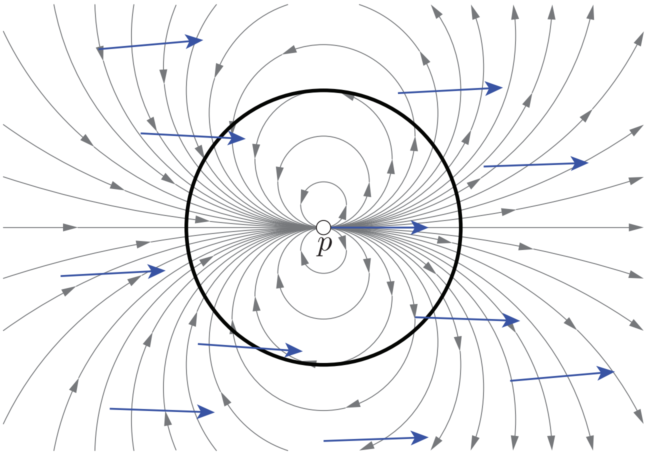
Overlay the vector field $g(x)$, which is continuous everywhere except in p.

Hairy ball theorem: proof



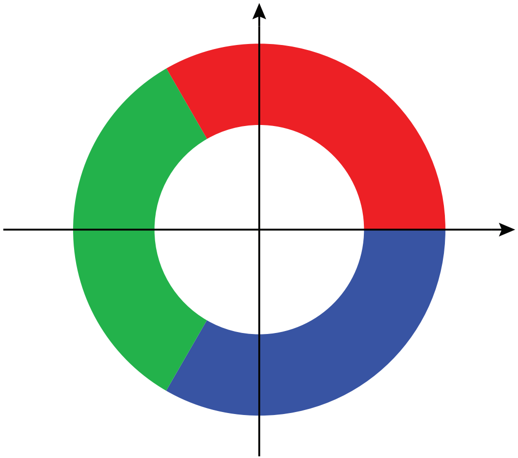
By the continuity of f in p, there is a neighborhood of p in which f varies by at most 1° from $f(p)$.

Hairy ball theorem: proof



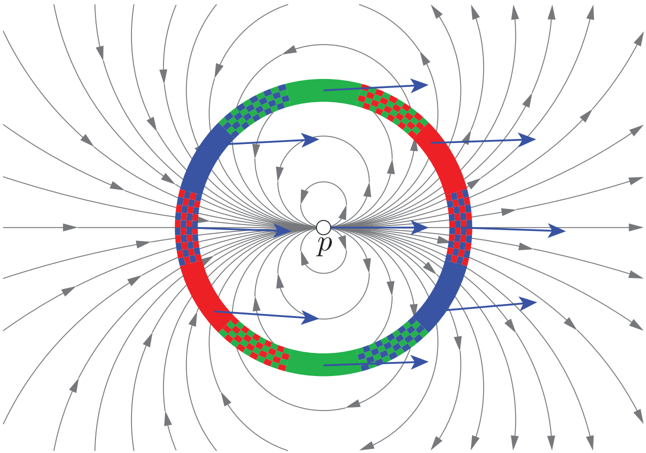
The angle between $f(x)$ and $g(x)$ makes two complete turns as x moves around the circle.

Hairy ball theorem: proof



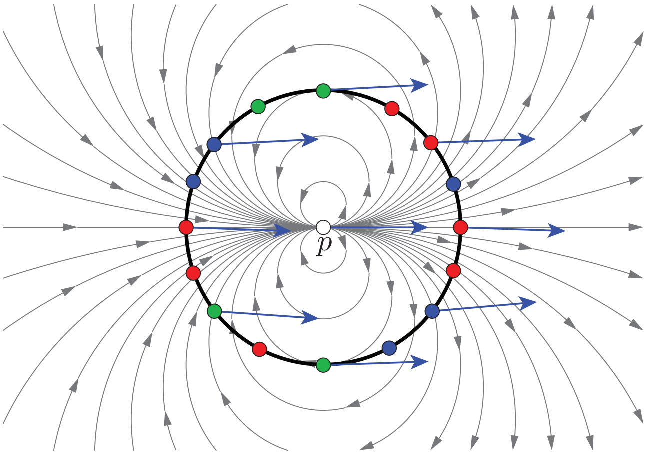
3-color the sphere (minus p) according to the angle between $f(x)$ and $g(x)$.

Hairy ball theorem: proof



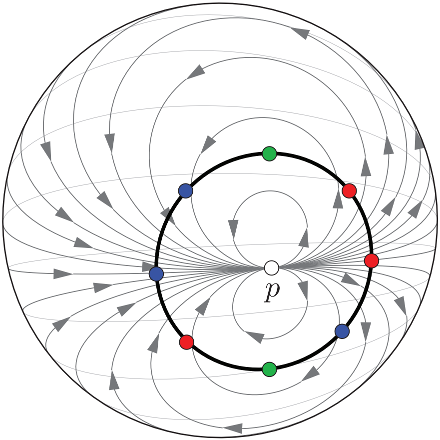
Because $f(x)$ is almost constant, the colors of the points around the circle must follow a precise order.

Hairy ball theorem: proof



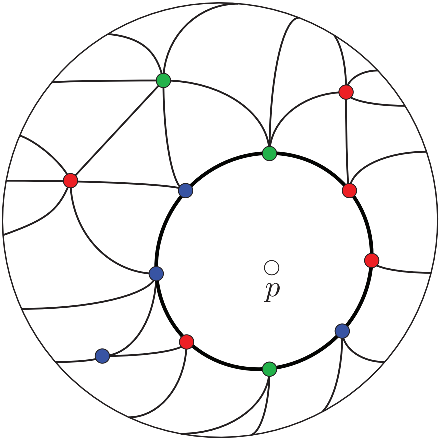
If we pick enough points on the circle and follow them ccw, we have more red-green transitions than green-red transitions.

Hairy ball theorem: proof



Triangulate the part of the sphere not containing p.

Hairy ball theorem: proof



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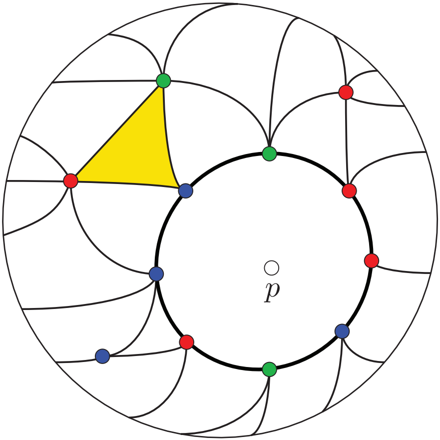
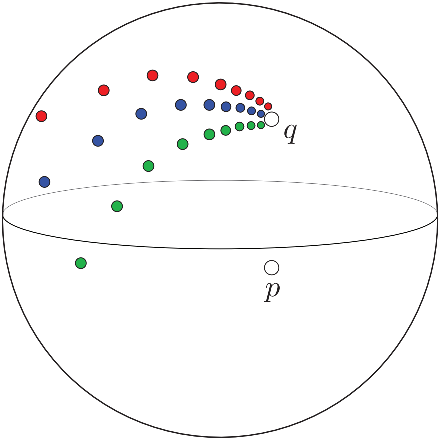


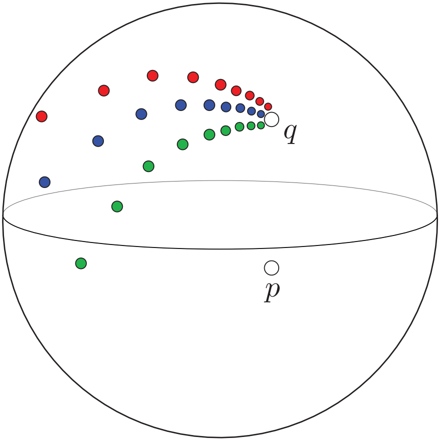
The generalized Sperner's lemma applies, and a 3-colored triangle is found.

Hairy ball theorem: proof



There exists a vanishing sequence of 3-colored triangles. By the Bolzano–Weierstrass theorem, we can extract sequences of all three colors that converge to the same point q.

Hairy ball theorem: proof



The angle between $f(q)$ and $g(q)$ belongs to the intersection of $[0^\circ, 120^\circ]$, $[120^\circ, 240^\circ]$ and $[240^\circ, 360^\circ]$, which is empty.

Hairy ball theorem: corollary

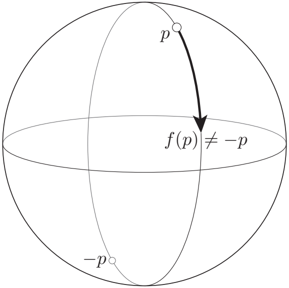
Corollary (Brouwer, 1912)

Any continuous function that maps an even-dimensional sphere into itself has either a fixed point or a point that is mapped onto its own antipodal point.

Hairy ball theorem: corollary

Corollary (Brouwer, 1912)

Any continuous function that maps an even-dimensional sphere into itself has either a fixed point or a point that is mapped onto its own antipodal point.



Suppose that $f(x)$ is continuous and no point is mapped onto its antipodal point.

Hairy ball theorem: corollary

Corollary (Brouwer, 1912)

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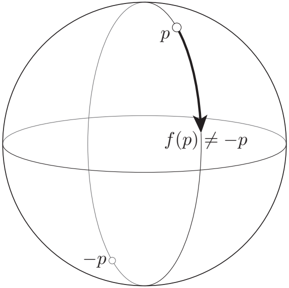
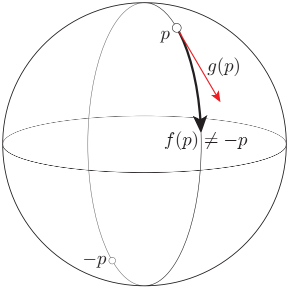


Then there is a unique geodesic between p and $f(p)$.

Hairy ball theorem: corollary

Corollary (Brouwer, 1912)

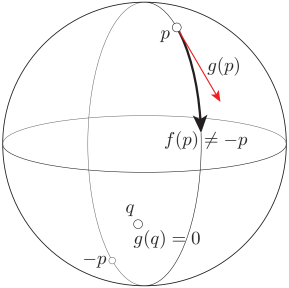
Any continuous function that maps an even-dimensional sphere into itself has either a fixed point or a point that is mapped onto its own antipodal point.



If $f(p) \neq p$, let $g(p)$ be the vector tangent to the geodesic at p. Otherwise, let $g(p) = 0$.

Corollary (Brouwer, 1912)

Any continuous function that maps an even-dimensional sphere into itself has either a fixed point or a point that is mapped onto its own antipodal point.

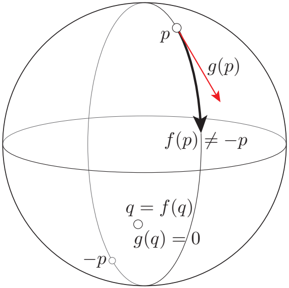


$g(x)$ is a continuous field tangent to the sphere, hence it has a zero in q due to the hairy ball theorem.

Hairy ball theorem: corollary

Corollary (Brouwer, 1912)

Any continuous function that maps an even-dimensional sphere into itself has either a fixed point or a point that is mapped onto its own antipodal point.



Therefore q is a fixed point of f.

Borsuk–Ulam theorem

Theorem (Borsuk–Ulam, 1933)

Every continuous function from an n-dimensional sphere into $\mathbb{R}^n$ *maps some pair of antipodal points into the same point.*

Borsuk–Ulam theorem

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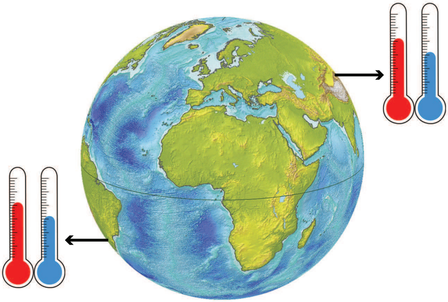
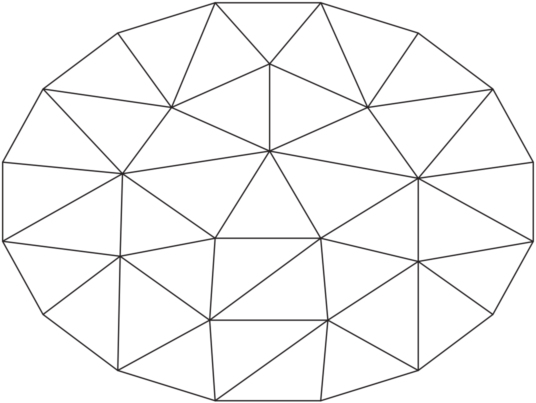


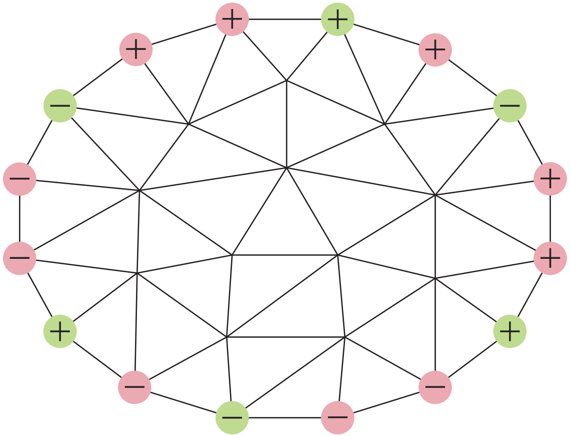
At any moment there is a pair of antipodal points on the Earth's surface with equal temperature and pressure.

Tucker's lemma



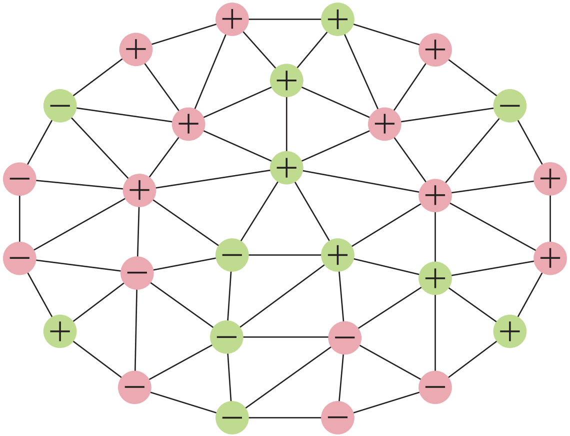
Start from a triangulated polygon with a centrally symmetric boundary.

Tucker's lemma



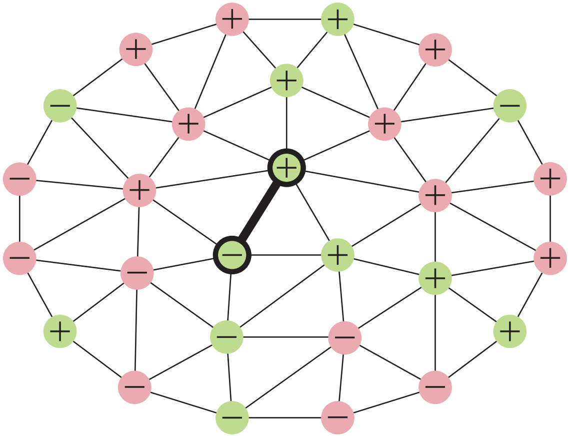
Color the external vertices so that opposite vertices have the same color and opposite sign.

Tucker's lemma



Color the internal vertices arbitrarily.

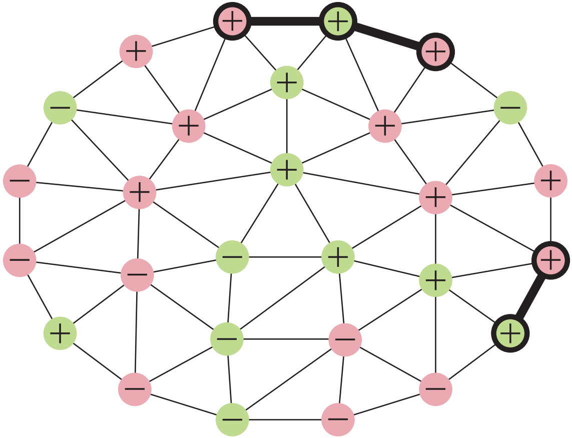
Tucker's lemma



Lemma (Tucker, 1946)

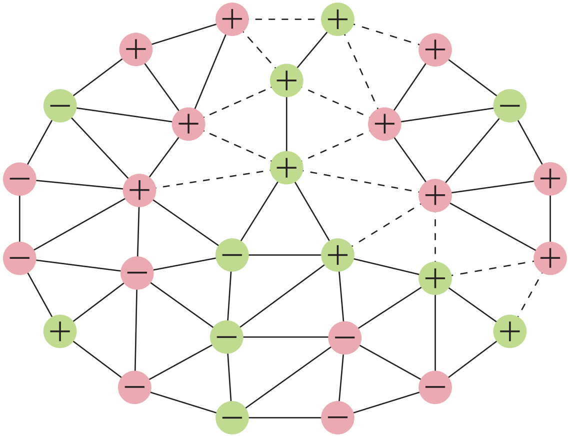
There are adjacent vertices with the same color and opposite sign.

Tucker's lemma: proof



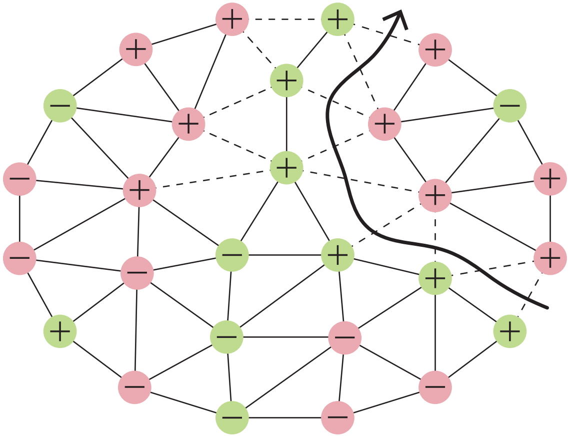
On the boundary, there is either a monochromatic $+-$ edge, or there is an odd number of bi-chromatic $++$ edges.

Tucker's lemma: proof



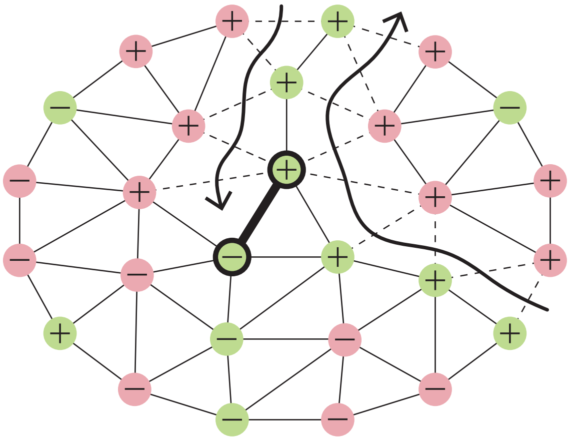
The bi-chromatic $++$ edges are *permeable*.

Tucker's lemma: proof



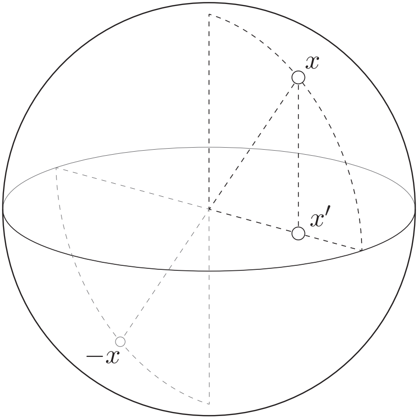
If we enter from a bi-chromatic $++$ edge, we may exit from another bi-chromatic $++$ edge...

Tucker's lemma: proof



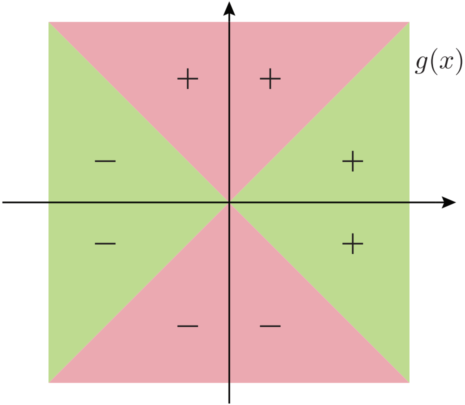
...Or we get stuck in a triangle with a monochromatic $+-$ edge.
This happens at least once, because the entrances/exits are odd.

Borsuk–Ulam theorem: proof



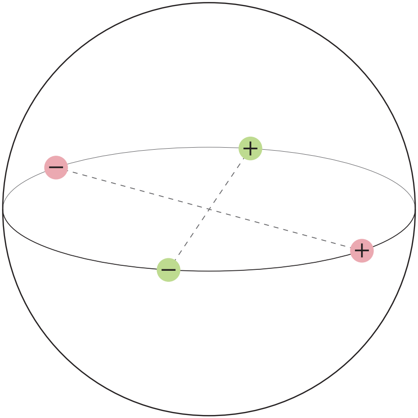
Project x on the horizontal disk, and let $g(x') = f(x) - f(-x)$.

Borsuk–Ulam theorem: proof

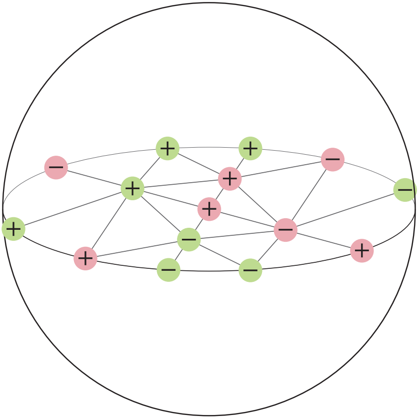


Color x' according to the value of $g(x')$.

Borsuk–Ulam theorem: proof

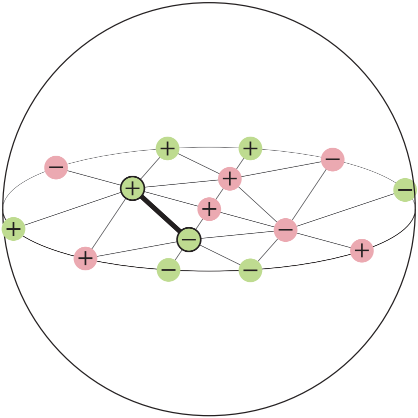


By construction, $g(-x) = -g(x)$.



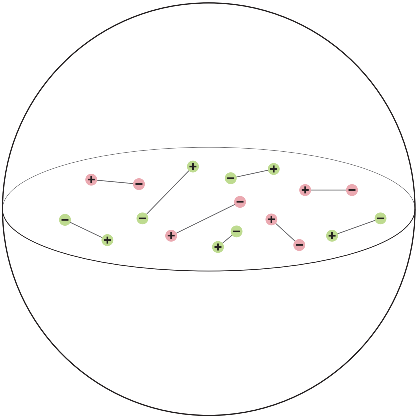
Triangulate the disk. The coloring satisfies the hypotheses of Tucker's lemma.

Borsuk–Ulam theorem: proof



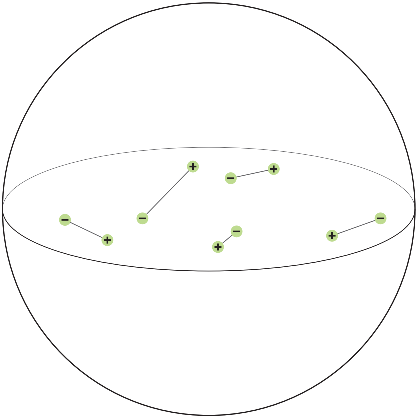
By Tucker's lemma, there are two adjacent vertices with the same color and opposite sign.

Borsuk–Ulam theorem: proof



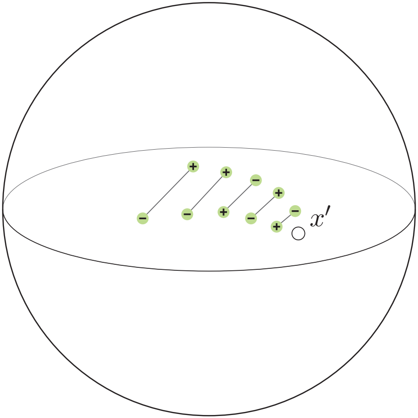
Repeat with finer triangulations to get a vanishing sequence of monochromatic pairs with opposite signs.

Borsuk–Ulam theorem: proof



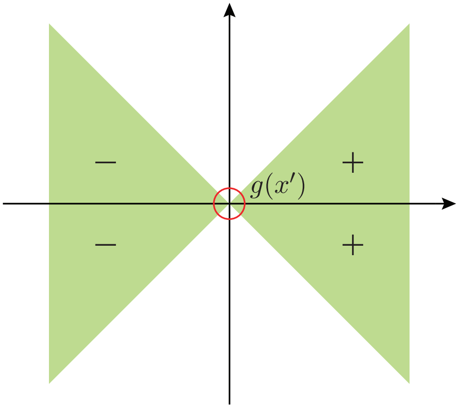
At least one of the colors appears infinitely often in the sequence.

Borsuk–Ulam theorem: proof



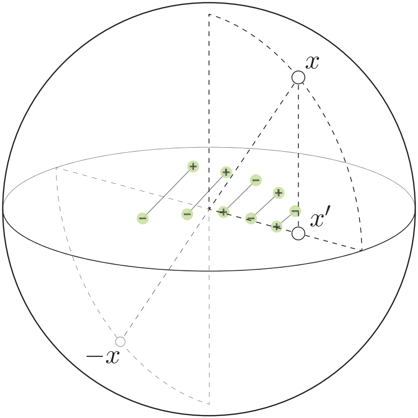
Due to the Bolzano–Weierstrass theorem, a sequence of $+$'s and a sequence of $-$'s of the same color converge to a point x'.

Borsuk–Ulam theorem: proof



By continuity, $g(x')$ belongs to the closure of both areas. Hence $g(x') = 0$.

Borsuk–Ulam theorem: proof

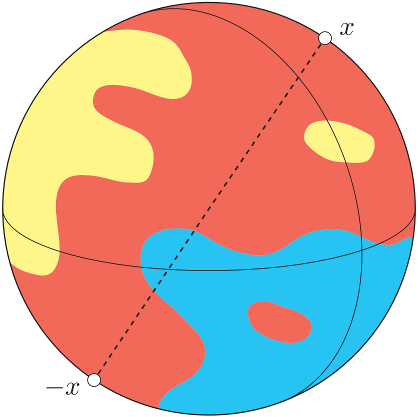


But $g(x') = f(x) - f(-x)$, hence $f(x) = f(-x)$.

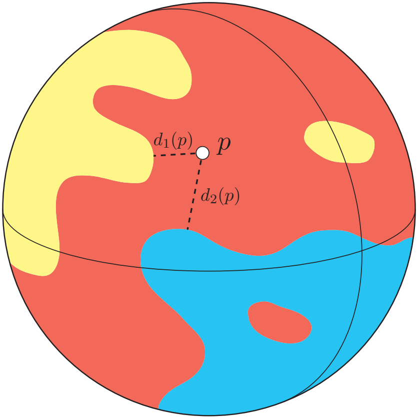
Corollary: Lusternik–Schnirelmann theorem

Theorem (Lusternik–Schnirelmann, 1930)

If the n-dimensional sphere is covered by $n + 1$ closed sets, one of them contains a pair of antipodal points.

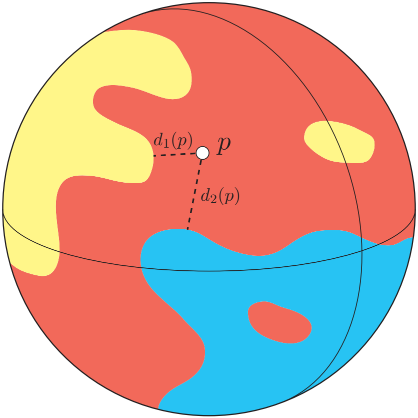


Corollary: Lusternik–Schnirelmann theorem: proof



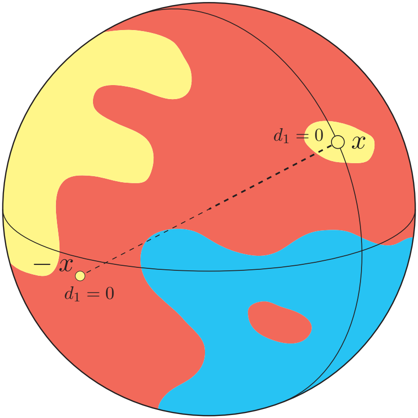
Let $d_1(p)$ be the distance from the first set, and $d_2(p)$ be the distance from the second. d_1 and d_2 are continuous functions.

Corollary: Lusternik–Schnirelmann theorem: proof



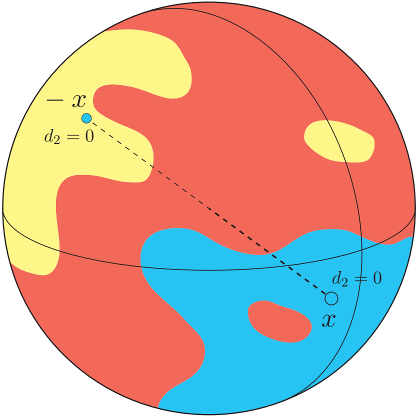
By the Borsuk–Ulam theorem, there is x such that $d_1(x) = d_1(-x)$ and $d_2(x) = d_2(-x)$.

Corollary: Lusternik–Schnirelmann theorem: proof



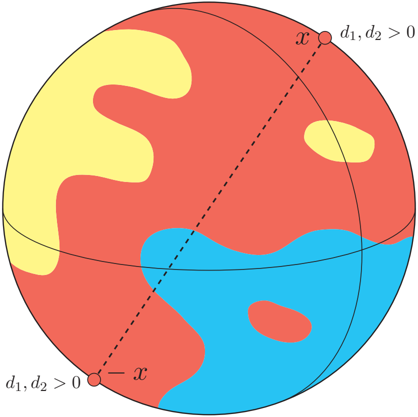
If $d_1(x) = d_1(-x) = 0$, both x and $-x$ belong to the first set (because it is closed).

Corollary: Lusternik–Schnirelmann theorem: proof



If $d_2(x) = d_2(-x) = 0$, both x and $-x$ belong to the second set (because it is closed).

Corollary: Lusternik–Schnirelmann theorem: proof

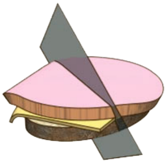


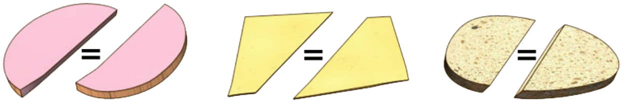
If all distances are positive, both x and $-x$ belong to the third set.

Corollary: ham sandwich theorem

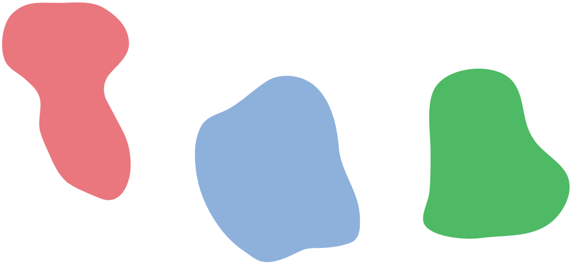
Theorem (Steinhaus–Banach, 1938)

Given n measurable sets in $\mathbb{R}^n$, there exists a hyperplane dividing each of them in two subsets of equal measure.



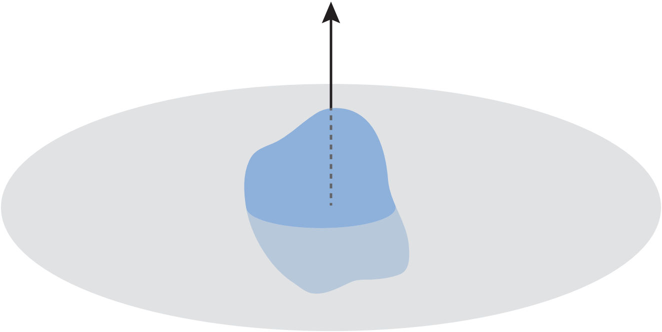


Corollary: ham sandwich theorem: proof



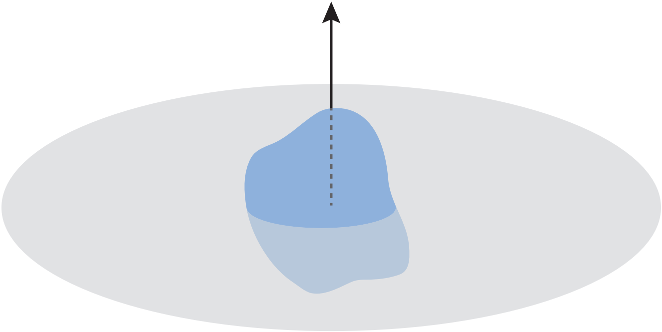
Let three measurable sets be given in $\mathbb{R}^3$.

Corollary: ham sandwich theorem: proof



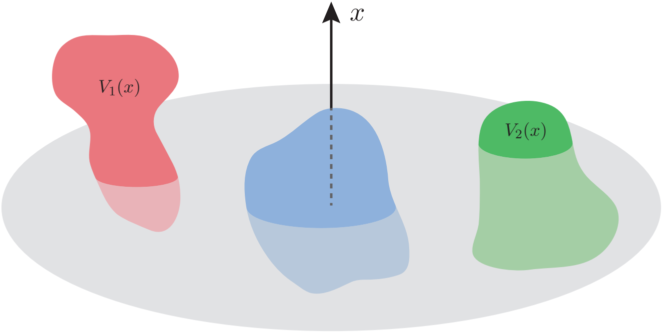
For any given direction, consider the orthogonal plane that divides the third set in two parts of equal measure.

Corollary: ham sandwich theorem: proof



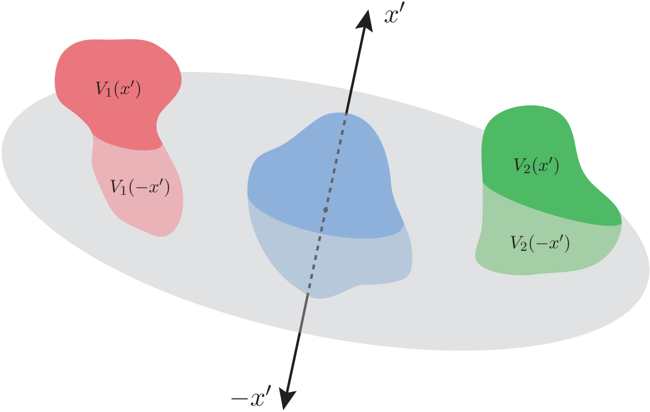
If an interval of parallel planes is eligible, take the middle plane.

Corollary: ham sandwich theorem: proof



Let $V_1(x)$ and $V_2(x)$ be the measures of the parts of the first and second set that lie in the positive half-space determined by x.

Corollary: ham sandwich theorem: proof



$V_1(x)$ and $V_2(x)$ are continuous. By the Borsuk–Ulam theorem, there is a direction x' such that $V_1(x') = V_1(-x')$, and similarly for V_2. The plane determined by x' equipartitions the three sets.