THE CERN ACCELERATOR SCHOOL

Theory of Electromagnetic Fields

Part I: Maxwell's Equations

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In these lectures, we shall discuss the theory of electromagnetic fields, with an emphasis on aspects relevant to RF systems in accelerators:

1. Maxwell's equations
   - Maxwell's equations and their physical significance
   - Electromagnetic potentials
   - Electromagnetic waves and their generation
   - Electromagnetic energy
2. Standing Waves
   - Boundary conditions on electromagnetic fields
   - Modes in rectangular and cylindrical cavities
   - Energy stored in a cavity
3. Travelling Waves
   - Rectangular waveguides
   - Transmission lines

I shall assume some familiarity with the following topics:

- vector calculus in Cartesian and polar coordinate systems;
- Stokes' and Gauss' theorems;
- Maxwell's equations and their physical significance;
- types of cavities and waveguides commonly used in accelerators.

The fundamental physics and mathematics is presented in many textbooks; for example:

I.S. Grant and W.R. Phillips, "Electromagnetism,"
2nd Edition (1990), Wiley.

In cartesian coordinates:

$$\text{grad}\, f \equiv \nabla f \equiv \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) \tag{1}$$

$$\text{div}\, \vec{A} \equiv \nabla \cdot \vec{A} \equiv \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \tag{2}$$

$$\text{curl}\, \vec{A} \equiv \nabla \times \vec{A} \equiv \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} \tag{3}$$

$$\nabla^2 f \equiv \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \tag{4}$$

Note that $\hat{x}$, $\hat{y}$ and $\hat{z}$ are unit vectors parallel to the x, y and z axes, respectively.

Gauss' theorem:

$$\int_V \nabla \cdot \vec{A}\, dV = \oint_S \vec{A} \cdot d\vec{S}, \tag{5}$$

for any smooth vector field $\vec{A}$, where the closed surface S bounds the volume V.

Stokes' theorem:

$$\int_S \nabla \times \vec{A} \cdot d\vec{S} = \oint_C \vec{A} \cdot d\vec{\ell}, \tag{6}$$

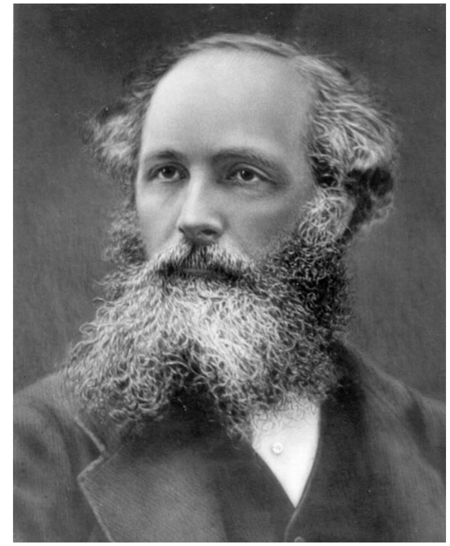
for any smooth vector field $\vec{A}$, where the closed loop C bounds the surface S.

A useful identity:

$$\nabla \times \nabla \times \vec{A} \equiv \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}. \tag{7}$$

$$\nabla \cdot \vec{D} = \rho \qquad \nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$



James Clerk Maxwell

1831 – 1879

Note that ρ is the electric charge density; and $\vec{J}$ is the current density.

The *constitutive relations* are:

$$\vec{D} = \varepsilon \vec{E}, \qquad \vec{B} = \mu \vec{H}, \tag{8}$$

where ε is the permittivity, and μ is the permeability of the material in which the fields exist.

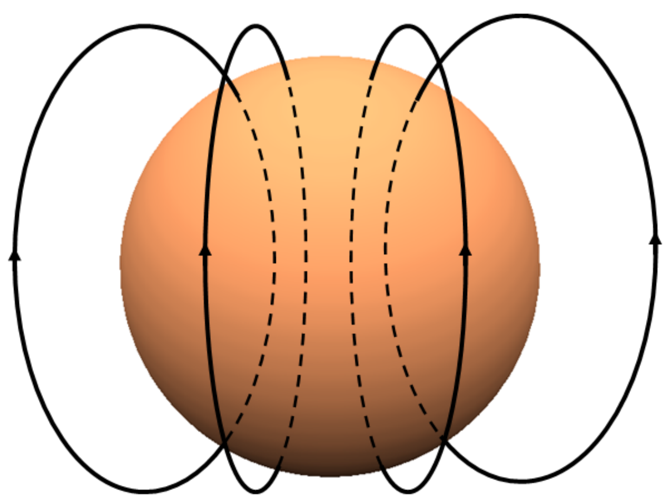
Physical interpretation of $\nabla \cdot \vec{B} = 0$

Gauss' theorem tells us that for *any* smooth vector field $\vec{B}$:

$$\int_V \nabla \cdot \vec{B} \, dV = \oint_S \vec{B} \cdot d\vec{S}, \tag{9}$$

where the closed surface S bounds the region V.

Applied to Maxwell's equation $\nabla \cdot \vec{B} = 0$, Gauss' theorem tells us that the total flux entering a bounded region equals the total flux leaving the same region.



Applying Gauss' theorem to Maxwell's equation $\nabla \cdot \vec{D} = \rho$, we find that:

$$\int_V \nabla \cdot \vec{D} \, dV = \oint_S \vec{D} \cdot d\vec{S} = Q, \tag{10}$$

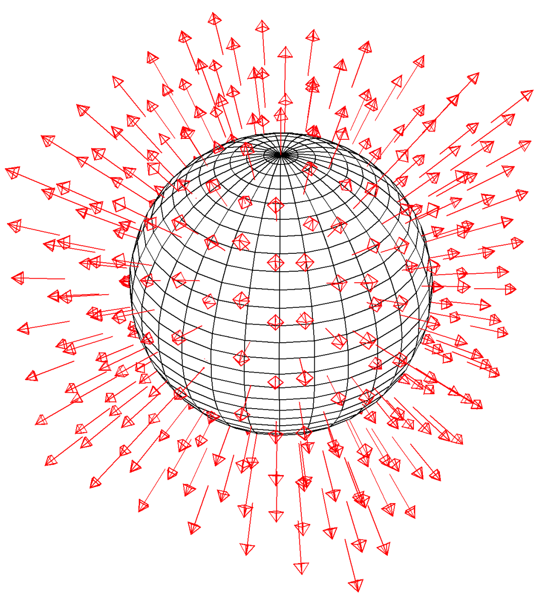
where $Q = \int_V \rho \, dV$ is the total charge within the region V, bounded by the closed surface S.

The total flux of electric displacement crossing a closed surface equals the total electric charge enclosed by that surface.

In particular, at a distance r from the centre of any spherically symmetric charge distribution, the electric displacement is:

$$\vec{D} = \frac{Q}{4\pi r^2} \hat{r}, \tag{11}$$

where Q is the total charge within radius r, and $\hat{r}$ is a unit vector in the radial direction.



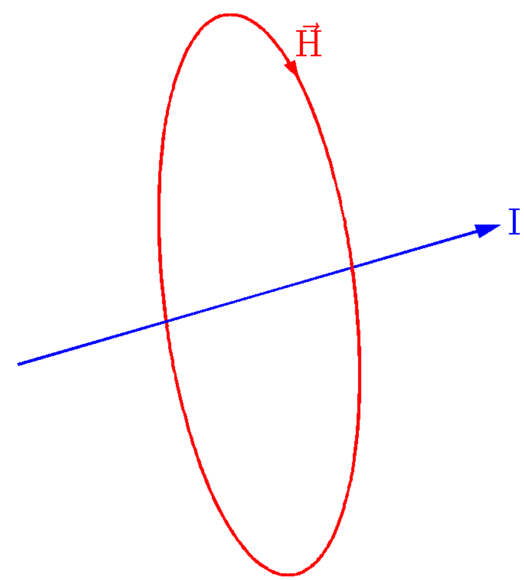
Stokes' theorem tells us that for *any* smooth vector field $\vec{H}$:

$$\int_S \nabla \times \vec{H} \cdot d\vec{S} = \oint_C \vec{H} \cdot d\vec{\ell}, \qquad (12)$$

where the closed loop C bounds the surface S.

Applied to Maxwell's equation $\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$, Stokes' theorem tells us that the magnetic field $\vec{H}$ integrated around a closed loop equals the total current passing through that loop. For the static case (constant currents and fields):

$$\oint_C \vec{H} \cdot d\vec{\ell} = \int_S \vec{J} \cdot d\vec{S} = I. \qquad (13)$$



The term $\frac{\partial \vec{D}}{\partial t}$ in Maxwell's equation $\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$ is known as the *displacement current density*, and has an important physical consequence.

Since, for any smooth vector field $\vec{H}$:

$$\nabla \cdot \nabla \times \vec{H} \equiv 0, \tag{14}$$

it follows that:

$$\nabla \cdot \vec{J} + \nabla \cdot \frac{\partial \vec{D}}{\partial t} = \nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0. \tag{15}$$

This is the continuity equation, that expresses the local conservation of electric charge. The significance is perhaps clearer if we use Gauss' theorem to express the equation in integral form:

$$\oint_S \vec{J} \cdot d\vec{S} = -\frac{dQ}{dt}, \tag{16}$$

where Q is the total charge enclosed by the surface S.

Applied to Maxwell's equation $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$, Stokes' theorem tells us that a time-dependent magnetic field generates an electric field.

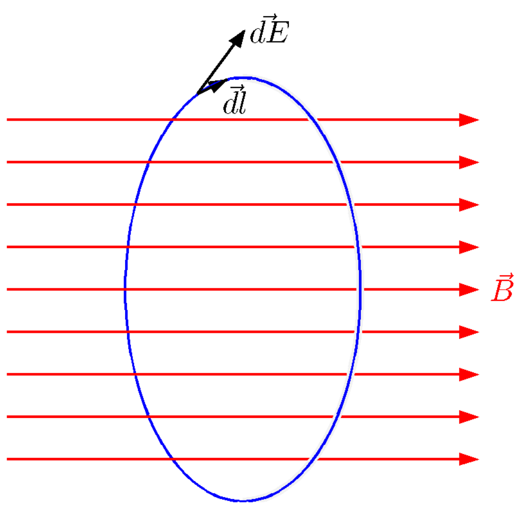
In particular, the total electric field around a closed loop equals the rate of change of the total magnetic flux through that loop:

$$\int_S \nabla \times \vec{E} \cdot d\vec{S} = \oint_C \vec{E} \cdot d\vec{\ell} = -\frac{\partial}{\partial t} \int_S \vec{B} \cdot d\vec{S}. \tag{17}$$

This is Faraday's law of electromagnetic induction:

$$\mathcal{E} = -\frac{\partial \Phi}{\partial t}, \tag{18}$$

where $\mathcal{E}$ is the electromotive force (the integral of the electric field) around a closed loop, and Φ is the total magnetic flux through that loop.



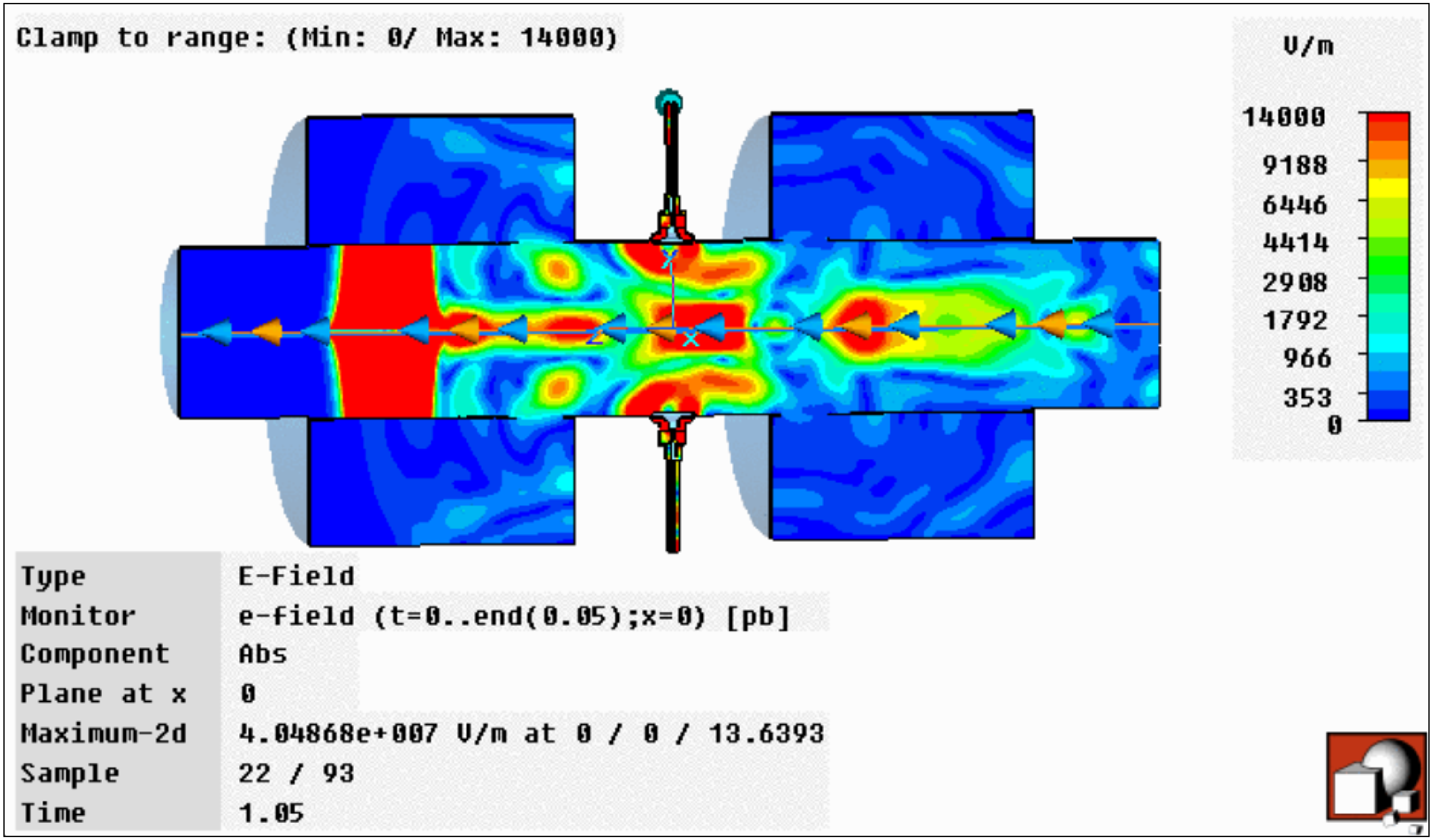
Maxwell's equations are of fundamental importance in electromagnetism, because they tell us the fields that exist in the presence of various charges and materials.

In accelerator physics (and many other branches of applied physics), there are two basic problems:

- Find the electric and magnetic fields in a system of charges and materials of specified size, shape and electromagnetic characteristics.

- Find a system of charges and materials to generate electric and magnetic fields with specified properties.

Example: fields induced by a bunch in an accelerator



Neither problem is particularly easy to solve in general; but fortunately, there are ways to decompose complex problems into simpler ones...

Maxwell's equations are *linear*:

$$\nabla \cdot \left(\vec{B}_1 + \vec{B}_2 \right) = \nabla \cdot \vec{B}_1 + \nabla \cdot \vec{B}_2, \tag{19}$$

and:

$$\nabla \times \left(\vec{H}_1 + \vec{H}_2 \right) = \nabla \times \vec{H}_1 + \nabla \times \vec{H}_2. \tag{20}$$

This means that if two fields $\vec{B}_1$ and $\vec{B}_2$ satisfy Maxwell's equations, so does their sum $\vec{B}_1 + \vec{B}_2$.

As a result, we can apply the *principle of superposition* to construct complicated electric and magnetic fields just by adding together sets of simpler fields.

Perhaps the simplest system is one in which there are no charges or materials at all: a perfect, unbounded vacuum. Then, the constitutive relations are:

$$\vec{D} = \varepsilon_0 \vec{E}, \quad \text{and} \quad \vec{B} = \mu_0 \vec{H}, \tag{21}$$

and Maxwell's equations take the form:

$$\nabla \cdot \vec{E} = 0 \qquad\qquad \nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad\qquad \nabla \times \vec{B} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$$

where $1/c^2 = \mu_0 \varepsilon_0$.

There is a trivial solution, in which all the fields are zero. But there are also interesting non-trivial solutions, where the fields are not zero. To find such a solution, we first "separate" the electric and magnetic fields.

If we take the curl of the equation for $\nabla \times \vec{E}$ we obtain:

$$\nabla \times \nabla \times \vec{E} \equiv \nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E} = -\frac{\partial}{\partial t} \nabla \times \vec{B}. \tag{22}$$

Then, using $\nabla \cdot \vec{E} = 0$, and $\nabla \times \vec{B} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$, we find:

$$\nabla^2 \vec{E} - \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} = 0. \tag{23}$$

This is the equation for a plane wave, which is solved by:

$$\vec{E} = \vec{E}_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)}, \tag{24}$$

where $\vec{E}_0$ is a constant vector, and the phase velocity c of the wave is given by the *dispersion relation*:

$$c = \frac{\omega}{|k|} = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}. \tag{25}$$

Similarly, if we take the curl of the equation for $\nabla \times \vec{B}$ we obtain:

$$\nabla^2 \vec{B} - \frac{1}{c^2} \frac{\partial^2 \vec{B}}{\partial t^2} = 0. \tag{26}$$

This is again the equation for a plane wave, which is solved by:

$$\vec{B} = \vec{B}_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)}, \tag{27}$$

where $\vec{B}_0$ is a constant vector, and the phase velocity c of the wave is again given by the dispersion relation (25):

$$c = \frac{\omega}{|k|} = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}.$$

Although it appears that we obtained independent equations for the electric and magnetic fields, we did so by taking derivatives. Therefore, the original Maxwell's equations impose constraints on the solutions.

For example, substituting the solutions (24) and (27) into Maxwell's equation:

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \tag{28}$$

we find:

$$\vec{k} \times \vec{E}_0 = \omega \vec{B}_0. \tag{29}$$

This imposes a constraint on both the directions and the relative magnitudes of the electric and magnetic fields.

Similarly we find:

$$\nabla \times \vec{B} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} \qquad \Rightarrow \qquad \vec{k} \times \vec{B}_0 = -\frac{\omega}{c^2}\vec{E}_0 \tag{30}$$

$$\nabla \cdot \vec{E} = 0 \qquad \Rightarrow \qquad \vec{k} \cdot \vec{E}_0 = 0 \tag{31}$$

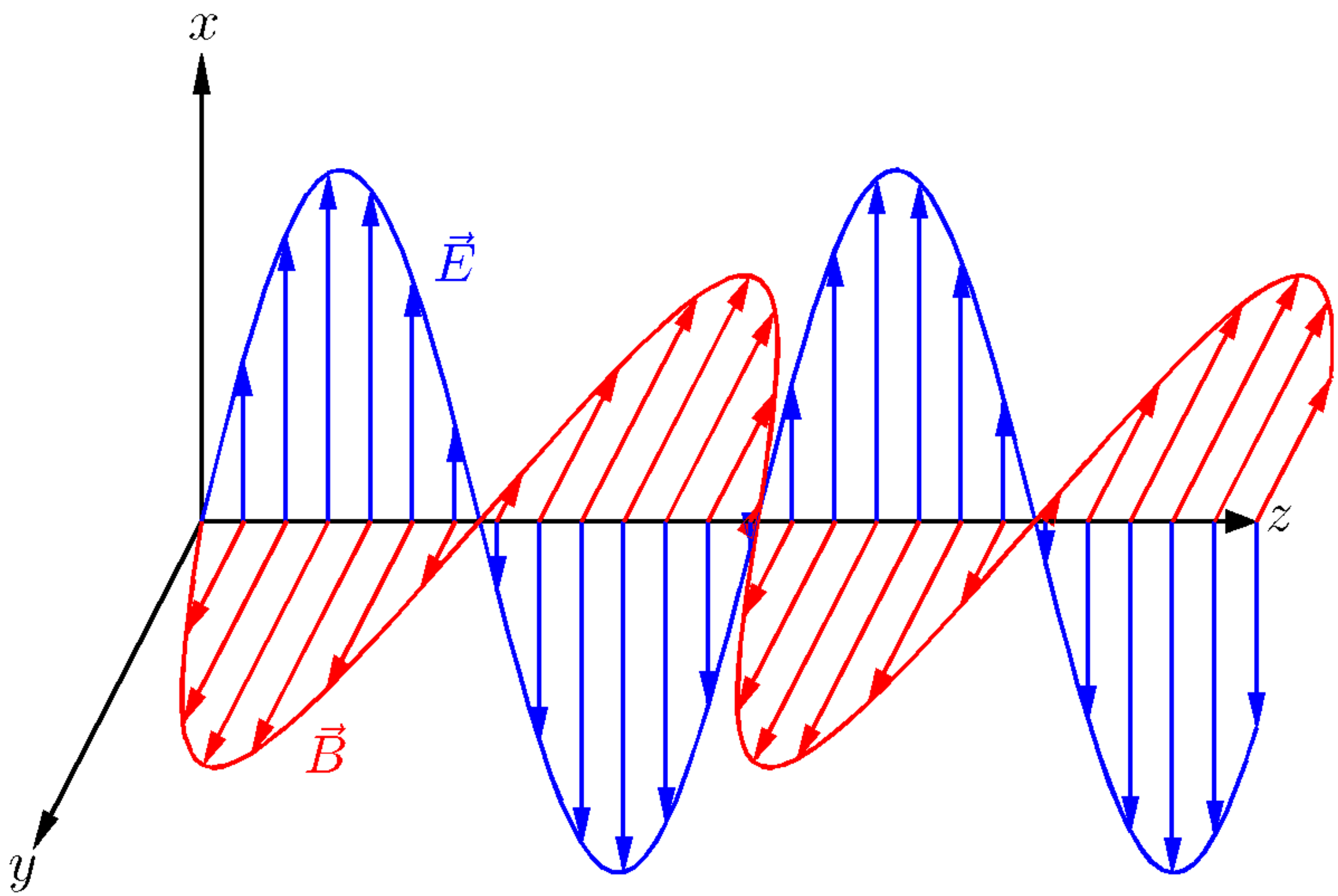
$$\nabla \cdot \vec{B} = 0 \qquad \Rightarrow \qquad \vec{k} \cdot \vec{B}_0 = 0 \tag{32}$$

These equations impose constraints on the relative amplitudes and directions of the electric and magnetic fields in the waves:

- $\vec{E}_0$, $\vec{B}_0$, and $\vec{k}$ are mutually perpendicular;

- The field amplitudes are related by $\frac{E_0}{B_0} = c$.

Example: plane electromagnetic waves in free space



Note that the wave vector $\vec{k}$ can be chosen freely. We can refer to a wave specified by a particular value of $\vec{k}$ as a "mode" of the electromagnetic fields in free space.

The frequency of each mode is determined by the dispersion relation (25):

$$c = \frac{\omega}{|k|} = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}.$$

A single mode represents a plane wave of a single frequency, with infinite extent in space and time. More realistic waves can be obtained by summing together (superposing) different modes.

Sometimes, problems can be simplified by working with the electromagnetic potentials, rather than the fields.

The potentials ϕ and $\vec{A}$ are defined as functions of space and time, whose derivatives give the fields:

$$\vec{B} = \nabla \times \vec{A}, \tag{33}$$

$$\vec{E} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t}. \tag{34}$$

Note that because the fields are obtained by taking derivatives of the potentials, there is more than one set of potential functions that will produce the same fields. This feature is known as *gauge invariance*.

To define the potentials uniquely, we need to specify not just the fields, but also an additional condition – known as a *gauge condition* – on the potentials.

For time-dependent fields (and potentials), the conventional choice of gauge is the Lorenz gauge:

$$\nabla \cdot \vec{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} = 0, \tag{35}$$

where c is the speed of light.

The Lorenz gauge is convenient because it allows us to write wave equations for the potentials in the presence of sources, in a convenient form.

If we take Maxwell's equation $\nabla \cdot \vec{D} = \rho$, and substitute for the electric field in terms of the potentials (34), we find:

$$\nabla \cdot \vec{E} = -\nabla^2 \phi - \frac{\partial}{\partial t} \nabla \cdot \vec{A} = \frac{\rho}{\varepsilon_0}. \tag{36}$$

Then, using the Lorenz gauge (35), we find:

$$\nabla^2 \phi - \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} = -\frac{\rho}{\varepsilon_0}. \tag{37}$$

The Lorenz gauge allows us to write a wave equation for the scalar potential ϕ, with a source term given by the charge density ρ; and without the appearance of either the vector potential $\vec{A}$ or the current density $\vec{J}$.

We can find a similar wave equation for the vector potential, $\vec{A}$.

Substituting $\vec{B} = \nabla \times \vec{A}$ into Maxwell's equations, we obtain:

$$\nabla \times \vec{B} = \nabla \times \nabla \times \vec{A} \equiv \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A} = \mu_0 \vec{J} + \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}. \qquad (38)$$

Then, using the Lorenz gauge (35), and substituting for the electric field $\vec{E} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t}$, we find:

$$\nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J}. \qquad (39)$$

Using the Lorenz gauge allows us to write a wave equation for the vector potential, with a source term given by the current density $\vec{J}$, and without the appearance of either the charge density ρ, or the scalar potential ϕ.

The wave equations for the potentials are useful, because they allow us to calculate the fields around time-dependent charge and current distributions.

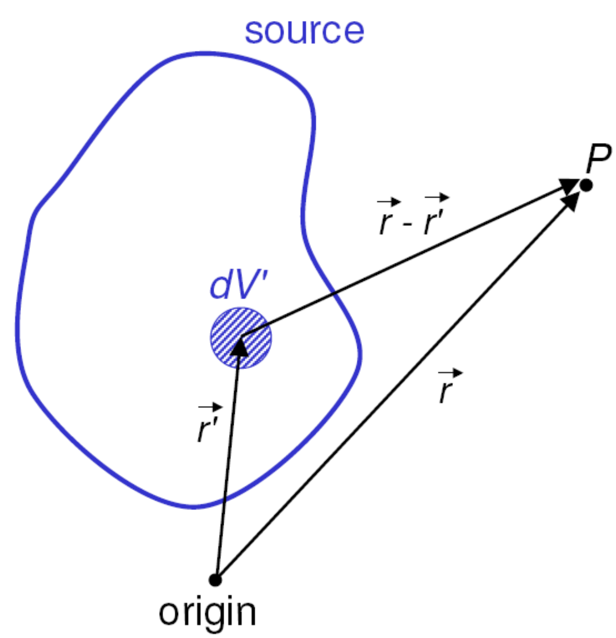
The general solutions to the wave equations can be written:

$$\phi(\vec{r},t) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}',t')}{|\vec{r}-\vec{r}'|}\,dV', \tag{40}$$

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}',t')}{|\vec{r}-\vec{r}'|}\,dV', \tag{41}$$

where:

$$t' = t - \frac{|\vec{r}-\vec{r}'|}{c}. \tag{42}$$



It is easy to show that, in the static case, the expression for the scalar potential gives the result expected from Coulomb's law.

A more interesting exercise is to calculate the fields around an infinitesimal oscillating dipole (a Hertzian dipole). We can model the current associated with a Hertzian dipole oriented parallel to the z axis as:

$$\vec{I} = I_0 e^{-i\omega t}\,\hat{z}. \tag{43}$$

We can think of the current as being associated with a charge oscillating between two points either side of the origin, along the z axis.

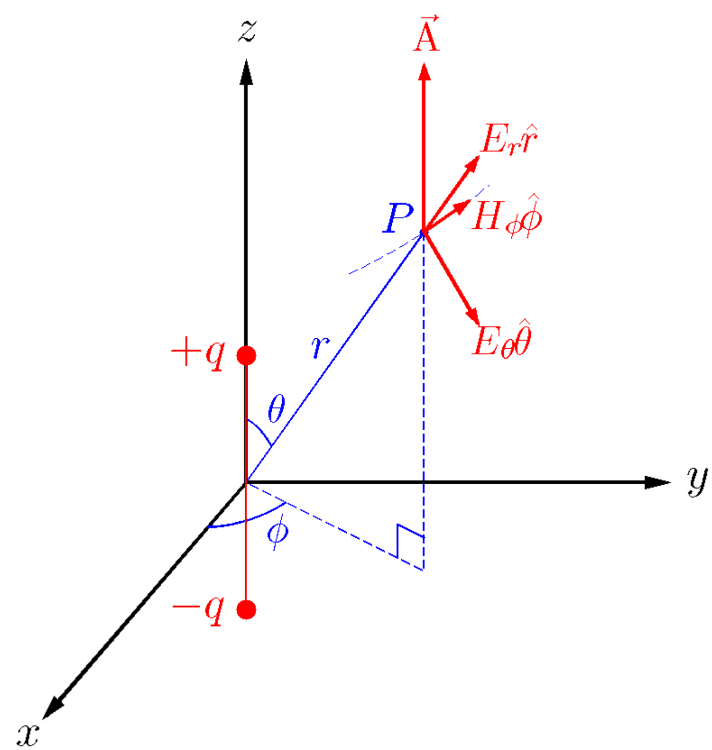
Since the current is located only at the origin, it is straightforward to perform the integral (41) to find the vector potential:

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}(I_0\ell)\frac{e^{i(kr-\omega t)}}{r}\,\hat{z}, \quad (44)$$

where:

$$k = \frac{\omega}{c}. \quad (45)$$

Note that ℓ is the length of the dipole: strictly speaking, we take the limit $\ell \to 0$, but with the amplitude $I_0\ell$ remaining constant.



Having obtained the vector potential, we can find the magnetic field from $\vec{B} = \nabla \times \vec{A}$. For the curl in spherical polar coordinates, see Appendix A. The result is:

$$B_r = 0, \tag{46}$$

$$B_\theta = 0, \tag{47}$$

$$B_\phi = \frac{\mu_0}{4\pi}(I_0\ell)k \sin\theta \left(\frac{1}{kr} - i\right) \frac{e^{i(kr-\omega t)}}{r}. \tag{48}$$

The electric field can be obtained from $\nabla \times \vec{B} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$. The result is:

$$E_r = \frac{1}{4\pi\varepsilon_0}\frac{2}{c}(I_0\ell)\left(1 + \frac{i}{kr}\right)\frac{e^{i(kr-\omega t)}}{r^2}, \tag{49}$$

$$E_\theta = \frac{1}{4\pi\varepsilon_0}(I_0\ell)\frac{k}{c}\sin\theta\left(\frac{i}{k^2r^2} + \frac{1}{kr} - i\right)\frac{e^{i(kr-\omega t)}}{r}, \tag{50}$$

$$E_\phi = 0. \tag{51}$$

At distances from the dipole large compared with the wavelength, $kr \gg 1$, and we can find approximate expressions for the dominant field components:

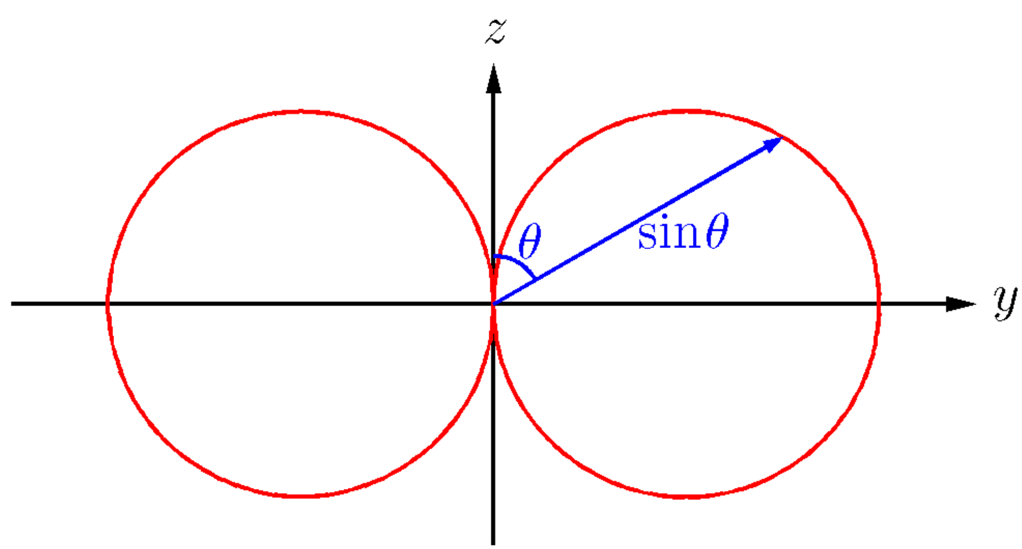
$$E_\theta \approx -i\frac{1}{4\pi\varepsilon_0}(I_0\ell)\frac{k}{c}\sin\theta\,\frac{e^{i(kr-\omega t)}}{r}, \tag{52}$$

$$B_\phi \approx -i\frac{\mu_0}{4\pi}(I_0\ell)k\sin\theta\,\frac{e^{i(kr-\omega t)}}{r}. \tag{53}$$

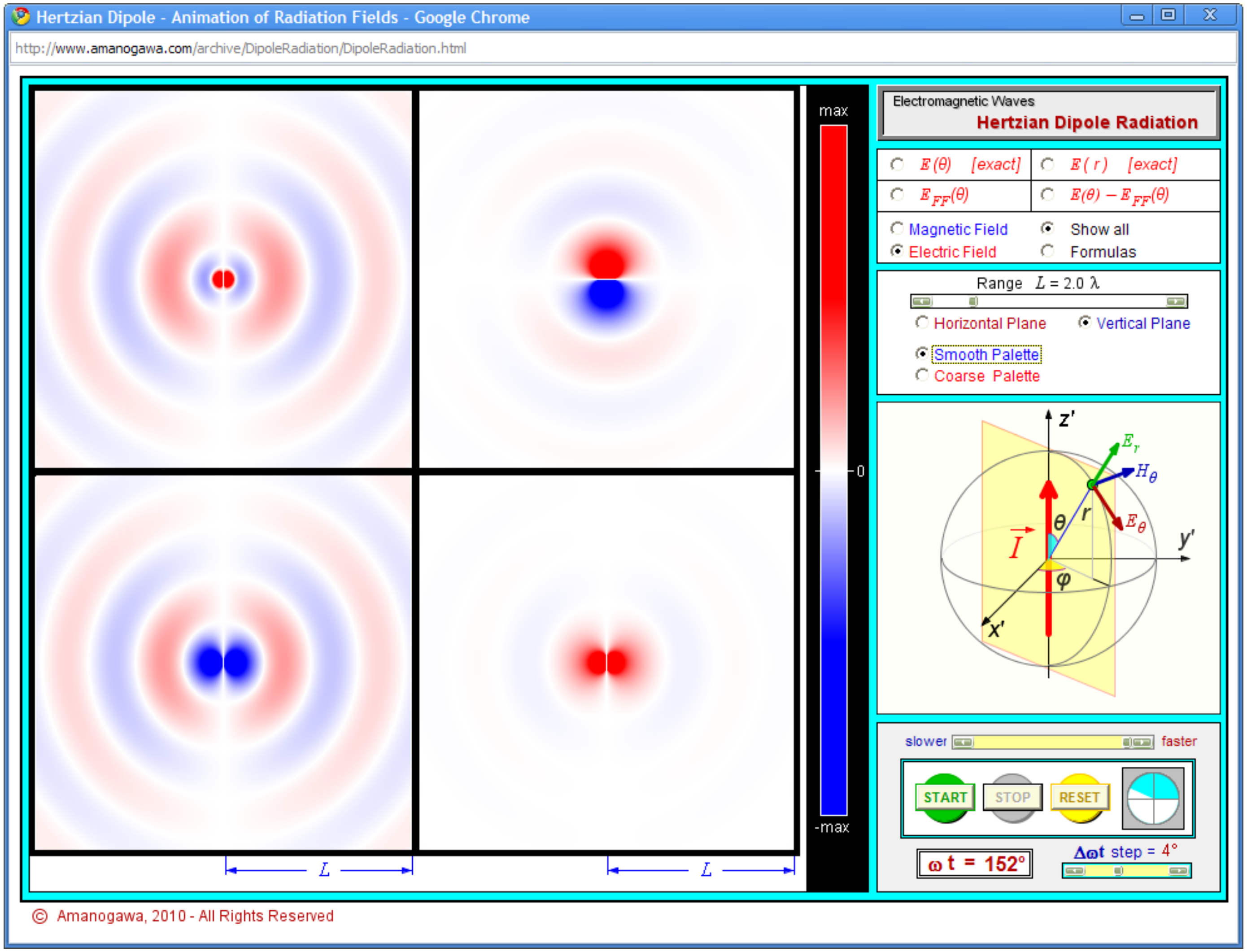
This is known as the "far field regime". Note that the fields take the form of a wave propagating in the radial direction: the electric and magnetic fields are perpendicular to each other, and to the direction of the wave (as we found for the case of the plane wave).

The relative amplitudes of the electric and magnetic fields are also as we found for a plane wave.

Note that the field amplitudes fall off as $1/r$; and that there is a directional dependence on $\sin\theta$, so that the amplitudes are zero in the direction of the current ($\theta = 0^\circ$, and $\theta = 180^\circ$), and are maximum in the plane perpendicular to the current ($\theta = 90^\circ$).

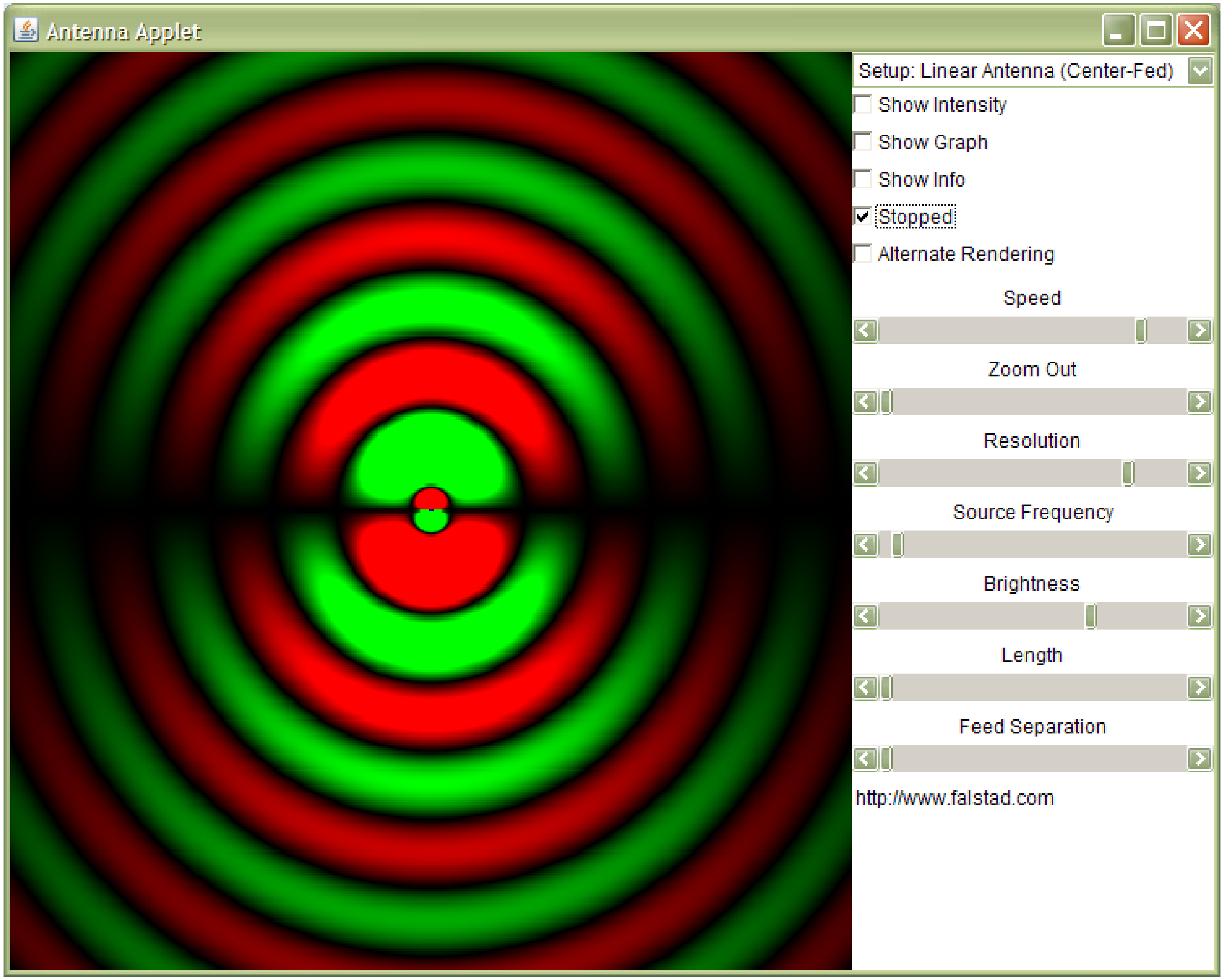


Fields around a Hertzian dipole



http://www.amanogawa.com

Fields around a Hertzian dipole



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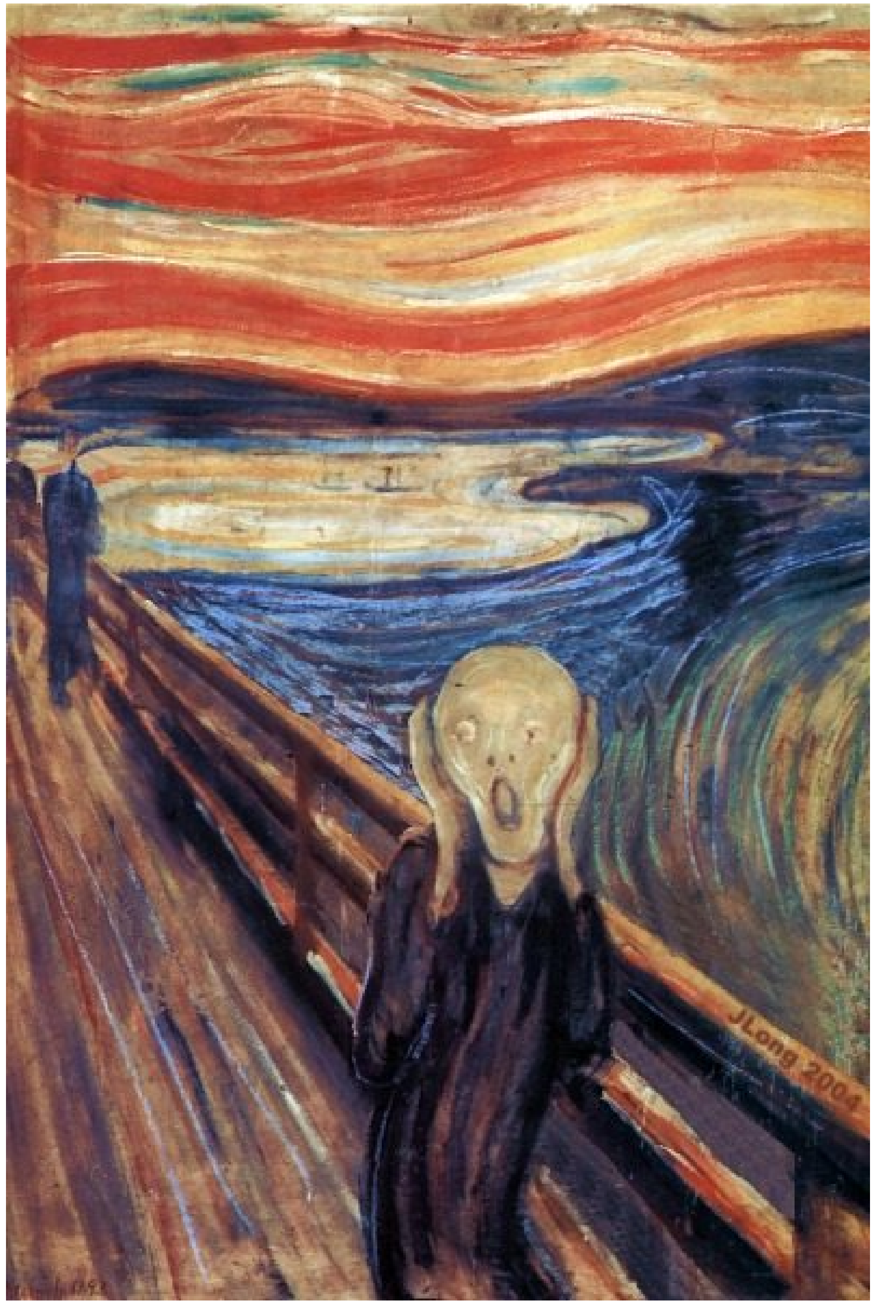
The Scream (Edvard Munch, 1893)



There are some important differences between the behaviour of electromagnetic waves in free space, and the behaviour of electromagnetic waves in conductors.

One significant difference is that the electric field in the wave drives a flow of electric current in the conductor: this leads to ohmic energy losses, and results in attentuation of the wave.

A key concept is the skin depth: this is the distance over which the amplitude of the wave falls by a factor $1/e$.

We shall derive an expression for the skin depth in terms of the properties of the conductor.

To start with, we define an *ohmic conductor* as a material in which the current density is proportional to the electric field:

$$\vec{J} = \sigma \vec{E}. \tag{54}$$

The constant σ is the conductivity of the material.

In practice, σ depends on many factors, including (in the case of an oscillating electric field) on the frequency of oscillation of the field. However, we shall regard σ as a constant.

In an ohmic conductor with absolute permittivity ε, absolute permeability μ, and conductivity σ, Maxwell's equations take the form:

$$\nabla \cdot \vec{E} = 0 \qquad \nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \nabla \times \vec{B} = \mu\sigma\vec{E} + \frac{1}{v^2}\frac{\partial \vec{E}}{\partial t}$$

where $1/v^2 = \mu\varepsilon$.

We can derive wave equations for the electric and magnetic fields as before; but with the additional term in σ, the wave equation for the electric field takes the form:

$$\nabla^2\vec{E} - \mu\sigma\frac{\partial \vec{E}}{\partial t} - \frac{1}{v^2}\frac{\partial^2 \vec{E}}{\partial t^2} = 0. \tag{55}$$

The first order derivative with respect to time in equation (55) describes the attenuation of the wave.

We can write a solution to the wave equation (55) in the usual form:

$$\vec{E}(\vec{r},t) = \vec{E}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}. \tag{56}$$

But now, if we substitute this into equation (55), we find that the dispersion relation is:

$$-\vec{k}^2 + i\omega\sigma\mu + \frac{\omega^2}{v^2} = 0. \tag{57}$$

In general, the wave vector $\vec{k}$ will be complex. We can write:

$$\vec{k} = \vec{\alpha} + i\vec{\beta}, \tag{58}$$

where $\vec{\alpha}$ and $\vec{\beta}$ are real vectors, that we shall assume are parallel.

In terms of the real vectors α and β, the electric field (56) can be written:

$$\vec{E}(\vec{r},t) = \vec{E}_0\, e^{-\vec{\beta}\cdot\vec{r}}\, e^{i(\vec{\alpha}\cdot\vec{r}-\omega t)}. \tag{59}$$

The amplitude of the wave falls by a factor $1/e$ in a distance $\delta = 1/\beta$. δ is known as the *skin depth*.

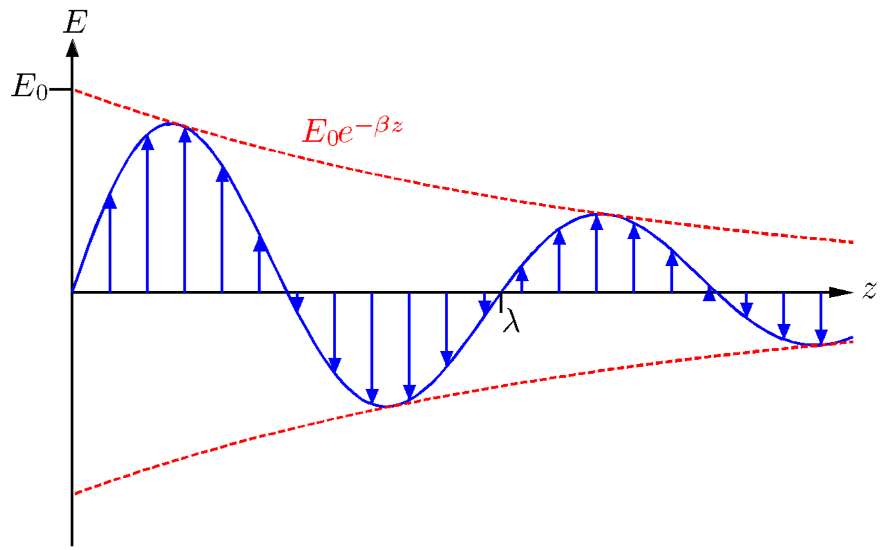
The dispersion relation (57) can be solved to find the magnitudes of the vectors $\vec{\alpha}$ and $\vec{\beta}$. The algebra is left as an exercise! The result is:

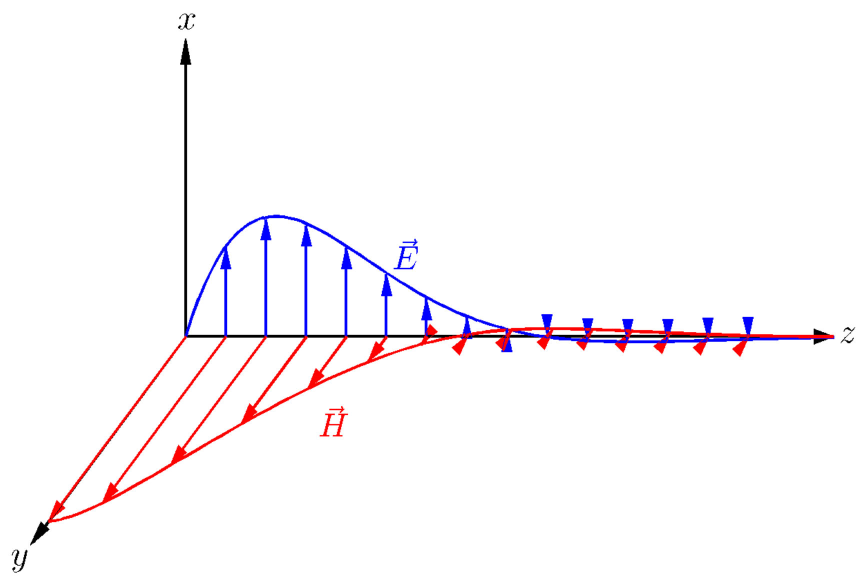
$$\alpha = \frac{\omega}{v}\left(\frac{1}{2}+\frac{1}{2}\sqrt{1+\frac{\sigma^2}{\omega^2\varepsilon^2}}\right)^{\frac{1}{2}}, \tag{60}$$

and:

$$\beta = \frac{\omega\mu\sigma}{2\alpha}. \tag{61}$$

Waves in conducting media





Equations (60) and (61) are exact expressions for the real and imaginary parts of the wave vector for an electromagnetic wave in an ohmic conductor.

In the case that $\sigma \gg \omega\varepsilon$ (a good conductor), we can make the approximations:

$$\alpha \approx \frac{\omega}{v}\sqrt{\frac{\sigma}{2\omega\varepsilon}}, \tag{62}$$

and (using $v = 1/\sqrt{\mu\varepsilon}$):

$$\delta = \frac{1}{\beta} \approx \sqrt{\frac{2}{\omega\sigma\mu}}. \tag{63}$$

Note that the skin depth is *smaller* for *larger* conductivity: the better the conductivity of a material, the less well an electromagnetic wave can penetrate the material. This has important consequences for RF components in accelerators, as we shall see in the next lecture.

Energy in Electromagnetic Fields: Poynting's Theorem

Electromagnetic waves carry energy.

The energy density in an electric field is given by:

$$U_E = \frac{1}{2}\varepsilon\vec{E}^2 \tag{64}$$

The energy density in a magnetic field is given by:

$$U_H = \frac{1}{2}\mu\vec{H}^2 \tag{65}$$

The energy flux (energy crossing unit area per unit time) is given by the Poynting vector:

$$\vec{S} = \vec{E} \times \vec{H} \tag{66}$$

These results follow from Poynting's theorem...

To derive Poynting's theorem, we start with Maxwell's equations. First, we use:

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{67}$$

Take the scalar product on both sides with the magnetic intensity $\vec{H}$:

$$\vec{H} \cdot \nabla \times \vec{E} = -\vec{H} \cdot \frac{\partial \vec{B}}{\partial t} \tag{68}$$

Next, we use:

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \tag{69}$$

Take the scalar product on both sides with the electric field $\vec{E}$:

$$\vec{E} \cdot \nabla \times \vec{H} = \vec{E} \cdot \vec{J} + \vec{E} \cdot \frac{\partial \vec{D}}{\partial t} \tag{70}$$

Now we take equation (68) minus equation (70):

$$\vec{H} \cdot \nabla \times \vec{E} - \vec{E} \cdot \nabla \times \vec{H} = -\vec{E} \cdot \vec{J} - \vec{E} \cdot \frac{\partial \vec{D}}{\partial t} - \vec{H} \cdot \frac{\partial \vec{B}}{\partial t} \tag{71}$$

which can be written as:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\varepsilon \vec{E}^2 + \frac{1}{2}\mu \vec{H}^2\right) = -\nabla \cdot \left(\vec{E} \times \vec{H}\right) - \vec{E} \cdot \vec{J} \tag{72}$$

Equation (72) is Poynting's theorem. Using Gauss' theorem, it may be written in integral form:

$$\frac{\partial}{\partial t}\int_V (U_E + U_H)\, dV = -\oint_A \vec{S} \cdot d\vec{A} - \int_V \vec{E} \cdot \vec{J}\, dV \tag{73}$$

where the closed surface A bounds the volume V,

$$U_E = \frac{1}{2}\varepsilon \vec{E}^2 \qquad U_H = \frac{1}{2}\mu \vec{H}^2 \tag{74}$$

and:

$$\vec{S} = \vec{E} \times \vec{H} \tag{75}$$

Poynting's theorem in integral form is equation (73):

$$\frac{\partial}{\partial t}\int_V (U_E + U_H)\, dV = -\oint_A \vec{S}\cdot d\vec{A} - \int_V \vec{E}\cdot\vec{J}\, dV$$

We note that the last term on the right hand side represents the rate at which the electric field does work on electric charges within the bounded volume V. It is then natural to interpret the first term on the right hand side as the flow of energy in the electromagnetic field across the boundary of the volume V, and the left hand side as the rate of change of the total energy in the electromagnetic field.

With this interpretation, Poynting's theorem expresses the local conservation of energy.

As an example, let us calculate the average energy density and the energy flux in a plane electromagnetic wave in free space.

From equations (24) and (27), the (real) electric and magnetic fields are given by:

$$\vec{E} = \vec{E}_0 \cos(\vec{k} \cdot \vec{r} - \omega t), \tag{76}$$

$$\vec{H} = \vec{H}_0 \cos(\vec{k} \cdot \vec{r} - \omega t), \tag{77}$$

where:

$$\frac{E_0}{B_0} = c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}, \qquad \therefore \quad \frac{E_0}{H_0} = Z_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}}. \tag{78}$$

Note (in passing) that Z_0 is the *impedance* of free space. Impedance will play an important role when we come to consider waves on boundaries.

The energy densities in the electric and magnetic fields are:

$$U_E = \frac{1}{2}\varepsilon_0 \vec{E}^2, \qquad U_H = \frac{1}{2}\mu_0 \vec{H}^2. \tag{79}$$

But since $H_0 = E_0/Z_0$, we find:

$$U_H = \frac{1}{4}\varepsilon_0 \vec{E}^2 = U_H. \tag{80}$$

In other words, the energy density in the magnetic field is equal to the energy density in the electric field (for a plane electromagnetic wave in free space).

The Poynting vector (which gives the energy flow per unit area per unit time) $\vec{S}$ is defined by:

$$\vec{S} = \vec{E} \times \vec{H} \tag{81}$$

Since $\vec{E}_0$ and $\vec{H}_0$ are perpendicular to each other and to $\vec{k}$ (the direction in which the wave is travelling), and the amplitudes of the fields are related by the impedance Z_0, we find that:

$$\vec{S} = \frac{E_0^2}{Z_0} \hat{k} \cos^2(\vec{k} \cdot \vec{x} - \omega t). \tag{82}$$

We see that (as expected) the energy flow is in the direction of the wave vector.

The amount of energy carried by the wave depends on the square of the electric field amplitude, divided by the impedance.

- Maxwell's equations describe the constraints on physical electric and magnetic fields.

- In free space, electromagnetic waves can propogate as transverse plane waves. For such waves, the wave vector $\vec{k}$ defines the "mode" of the electromagnetic fields.

- In conductors, electromagnetic waves are attenuated because the energy is dissipated by currents driven by the electric field in the waves.

- Electromagnetic waves can be generated by oscillating electric charges.

- Poynting's theorem provides expressions for the energy density and energy flux in an electromagnetic field.

In spherical polar coordinates, the curl of a vector field is given by:

$$\nabla \times \vec{A} \equiv \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\,\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_r & rA_\theta & r\sin\theta A_\phi \end{vmatrix} \tag{83}$$

1. Derive the wave equation for the electric field in a conductor. Show that the real and imaginary parts of the wave vector have magnitudes given by equations (60) and (61).

2. Estimate the skin depth for microwaves in copper.

3. Find an expression for the total power radiated by a Hertzian dipole.