Theory on Solution of Algebraic and Transcendental Equations

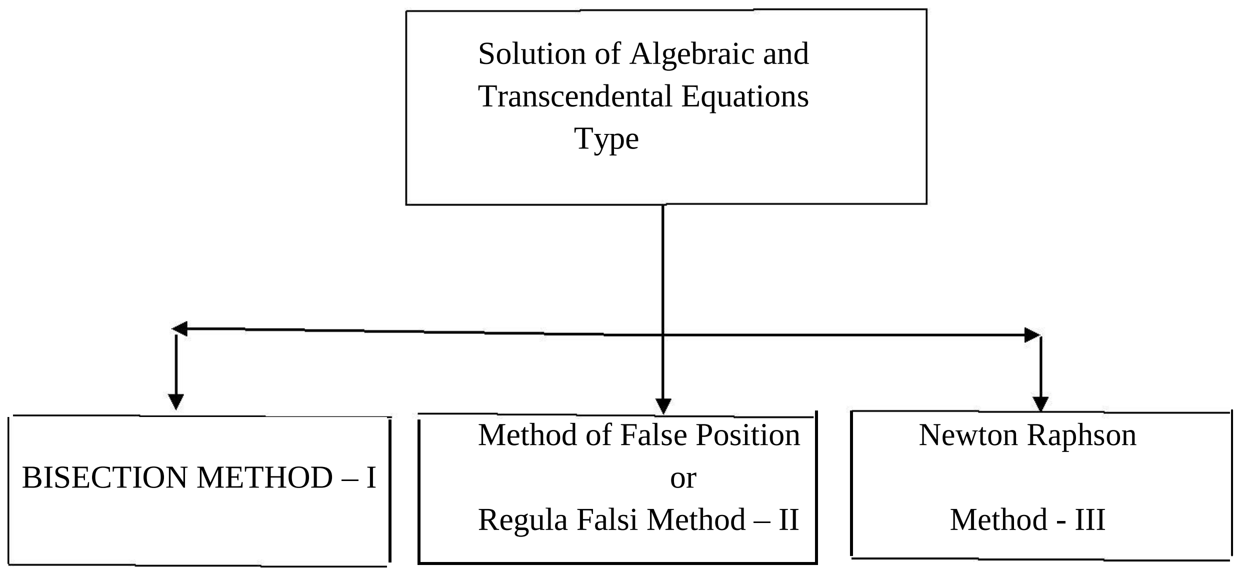
Introduction: In many engineering problems, it is required to find the solution of the equation of the form f(x) = 0 where f(x) = 0 may be algebraic or transcendental equation of higher order.

In this chapter, various numerical approximation methods are used
to solve such algebraic and transcendental equations.

The limitations of analytical methods led the engineers and scientists to evolve graphical and numerical methods.

Numerical methods often a repetitive nature. These consist in repeated execution of the same process where at each step the result of the proceeding step is used. This is known as "Iteration Process" and is repeated till the result obtained to desired degree of accuracy.

In this, we shall discuss some numerical methods for the solution of algebraic and transcendental equations and simultaneous linear and non-linear equations. We can closed the chapter by describing an iterative method for the solution of Eigen value Problem(E V P).



Definition: <u>Polynomial</u>: An expression of the form $f(x) = a_0 x^n + a_1 x^{n-1} + ---- + a_n$ is called a , polynomial in x of degree n, provided $a_0 \neq 0$ where a_0 , a_1, a_3 ,- - - - , a_n are constants, real a_2 or complex.

Definition: <u>Algebraic Equation</u>: A polynomial in x of degree n, when equated to zero i.e., f(x) = 0 is called an algebraic equation of degree n.

Definition: <u>Transcendental Equation</u>: If the polynomial f(x) involves the functions of the form such as trigonometric, logarithmic, exponential etc., then f(x) = 0 is called a transcendental equation.

Properties of Polynomial Equation:

(i) Every polynomial equation of degree n has exactly n roots, real or complex is known as **Fundamental theorem of algebra.**

(ii) Every polynomial equation f(x) = 0 of degree $n \geq 1$ has atleast one root, real or imaginary.

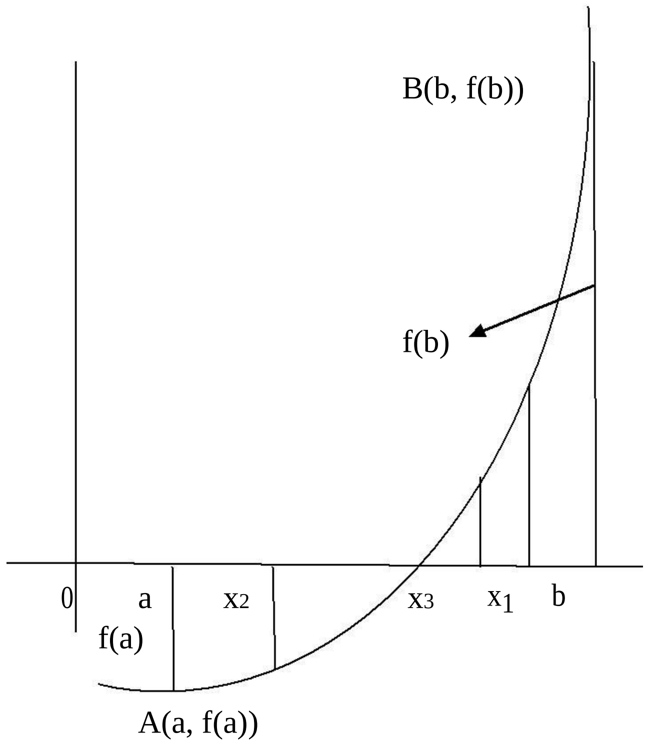
(iii) imaginary roots occur in pairs i.e., if $\alpha + i\beta$ is a root, $\alpha - i\beta$ is also a root of the equation and so every equation of the odd degree has atleast one real root.

(iv) Irrational roots occur in pair i.e., if $a + \sqrt{b}$ is a root of an equation, $a - \sqrt{b}$ must be its root.

(v) If f(x) is continuous in the closed interval [a, b] and f(a) and f(b) are of opposite signs, then the equation f(x) = 0 has atleast one root between x = a and x = b is known as **Intermediate value theorem.**

(vi) The number of positive roots of a polynomial equation f(x) = 0 with real coefficients can not exceed the number of changes of sign of the coefficients in f(x) and the number of negative roots can not exceed the number of changes of sign of the coefficients in f(- x), is known as **Descarte's rule of signs.**

Method – I BISECION METHOD: Solution of Algebraic and Transcendental quations



THE BISECTION METHOD

This method is based on repeated application of the intermediate value theorem.

Let the function f(x) be continuous in a closed interval [a, b].

If f(a) and f(b) are of opposite signs, atleast one real root lies between a and b.

For definiteness, let f(a) be – ve and f(b) be + ve so that f(a) f(b) < 0. Bisection method is used to find the root between a and b.

(i) The approximation of the required root is given by x_1 by $x_1 = \frac{a+b}{2}$.

(ii) If f(x_1) = 0 then x_1 is a root of the equation. If f(x_1) ≠ 0 then the root lies between a and x_1 provided f(x_1) is positive. (or) the root lies between x1 and provided f(x_1) is – ve.

(iii) If f(x_1) is positive the second approximation to the required root is given by

$$x_2 = \frac{a+x_1}{2}.$$

(iv) If f(x_2) is -ve the third approximation to the required root is given by

$$x_3 = \frac{x_1+x_2}{2}.$$

(v) Bisect the interval in which the root lies and continue the process to obtain the root to the desired level of accuracy.

Covergence of Bisection Method:

Let a function f(x) be continuous in a closed interval in [a, b] and there exists a number c between a and b.

Let f(a) and f(b) are of positive signs and x_1, x_2, x_3 ,......... is the sequence of mid points obtained by bi-section method then $|c - x_n| \le \frac{b-a}{2^n}$ *for n is* 1,2,3,......and therefore the sequence { x_n } converges to the root c i.e., Lim x_n = c as n tends to ∞.

The Bi-section Method is simple and the most reliable one. At each iteration, we bisect the interval and so only one binary digit precision can be obtained at each iteration. The convergence is very slow but definite.

Problems on Solution of Algebraic and Transcendental Equations

Problem 01 Solve $x^3 – 9x + 1 = 0$ for the root to three decimals between $x = 2$ and $x = 4$ by the method of interval halving?
[Ans. $2 < 2.943 < 4$]

Problem 02 Find the root between 2 and 3 of the equation $x^4 – x^3 – 2x^2 – 6x - 4 = 0$?
[Ans. $2 < 2.7315 < 3$]

Problem 03 Using Bi-section Method, find the negative root of $x^3 – 4x - 9 = 0$?
[Ans. $-3 < -2.711 < -2$]

Problem 04 Find a real root of equation $x^3 – x - 11 = 0$ by using Bi-section method ?
[Ans. $2.25 < 2.375 < 2.5$]

Problem 05 Find a real root of equation $x^3 – 5x + 3 = 0$ by using Bi-section method?
[Ans. $1 < 1.8125 < 2$]

Problem 06 Find a real root of equation $x^3 – 6x - 4 = 0$ by using Bi-section method?
[Ans. $2 < 2.71875 < 3$]

Solutions on Algebraic and Transcendental Equations

Problem 1. Find a real root of equation $x \log_{10}^{x} = 1.2$ which lies between 2 and 3 by using Bi-section method?
[Ans. 2 < 2.6875 < 3]

Solution: Let $f(x) \cong x \log_{10}^{x} - 1.2 = 0$
Since $f(2) = 2.\log_{10}^{2} - 1.2 = -0.598$ negative and
$f(3) = 3.\log_{10}^{3} - 1.2 = 0.2313$ positive
Therefore, a root lies between 2 and 3.

$\therefore$ Ist approximation to the root is $\boxed{x_1 = \frac{1}{2}(2 + 3) \approx 2.5}$

Then, $f(x_1) = (2.5).\,10^{2.5} - 1.2 = \quad - 0.2053$ is negative sign
$\therefore$ f(3) is positive and The root lies between x_1 and 3.

$\therefore$ The second approximation to the root is $\boxed{x_2 = \frac{1}{2}(x_1 + 3) \approx 2.75}$

Then $f(x_2) = (2.75).\log_{10}^{2.75} - 1.2 = 0.008$ is positive sign
$\therefore$ f(2.5) is positive and The root lies between x_1 and x_2.

$\therefore$ The third approximation to the root is $\boxed{x_3 = \frac{1}{2}(x_1 + x_2) \approx 2.625}$

Then $f(x_3) = (2.625).\log_{10}^{2.625} - 1.2 = -0.10$ is negative sign
$\therefore$ f(2.75) is positive and The root lies between x_2 and x_3.

$\therefore$ The fourth approximation to the root is $\boxed{x_4 = \frac{1}{2}(x_2 + x_3) \approx 2.6875}$

$\boxed{\therefore \text{ The root is 2.6875 approximately.}}$

Hence the required solution.

Problem 2 Find a positive root of equation $x^3 - 4x - 9 = 0$ by using Bi-section method in four stages?
[Ans. 2 < 2.6875 < 3]

Solution: Let $f(x) \cong x^3 - 4x - 9 = 0$

Since $f(2) = 8 - 8 - 9 = -9$ negative and
$f(3) = 27 - 12 - 9 = 6$ positive
Therefore, a root lies between 2 and 3.

$\therefore$ Ist approximation to the root is $\boxed{x_1 = \frac{1}{2}(2 + 3) \approx 2.5}$

Then, $f(x_1) = (2.5)^3 - 4(2.5) - 9 = -3.375$ is negative sign
$\therefore$ The root lies between x_1 and 3.
$\therefore$ The second approximation to the root is $\boxed{x_2 = \frac{1}{2}(x_1 + 3) \approx 2.75}$

Then $f(x_2) = (2.75)^3 - 4(2.75) - 9 = 0.7969$ is positive sign
$\therefore$ The root lies between x_1 and x_2.
$\therefore$ The third approximation to the root is $\boxed{x_3 = \frac{1}{2}(x_1 + x_2) \approx 2.625}$

Then $f(x_3) = (2.625)^3 - 4(2.625) - 9 = -1.4121$ is negative sign
$\therefore$ The root lies between x_2 and x_3.
$\therefore$ The fourth approximation to the root is $\boxed{x_4 = \frac{1}{2}(x_2 + x_3) \approx 2.6875}$

$\boxed{\therefore \text{ The root is 2.6875 approximately.}}$

Continuing the procedure, the successive apprxns. are x5 = 2.71875, x_6 = 2.703125, x_7 = 2.7109375, x_8 = 2.7070313, x_9 = 2.7050782.

Hence the required solution.

Problem 3 Find a positive root of equation $x^3 - x - 1 = 0$ correct to two decimal places by using Bi-section method ?
[Ans. root is 1.32 such that 1 < 1.32 < 2]

Solution: Let $f(x) \cong x^3 - x - 1 = 0$

Since $f(1) = 1 - 1 - 1 = -1$ negative and
$f(2) = 8 - 2 - 1 = 5$ positive
Therefore, a root lies between 1 and 2. Here f(0)= -1 = f(1) and consider f(1)= -1 to get closer approximation between two consecutive values of x. Also f(1.5) = 0.8750
$\therefore$ The root lies between 1 and 1.5 (instead of shorten range 1 and 2).

$\therefore$ Ist approximation to the root is $\boxed{x_1 = \frac{1}{2}(1 + 1.5) \approx 1.2500}$

Then, $f(x_1) = (1.25)^3 - (1.25) - 1 = -0.29688$ is negative sign

$\therefore$ f(1.5) = 0.8750 is positive and The root lies between x_1 and 1.5.

$\therefore$ The second approximation to the root is $\boxed{x_2 = \frac{1}{2}(x_1 + 1.5) \approx 1.375}$

Then $f(x_2) = (1.375)^3 - (1.375) - 1 = 0.22461$ is positive sign
$\therefore$ f(1.25) = - 0.29688 is negative and The root lies between x_1 and x_2.

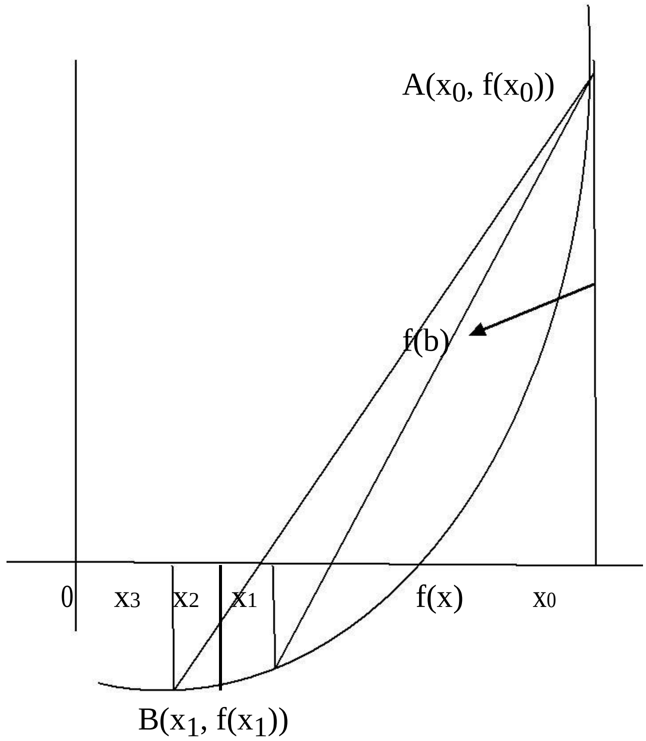
$\therefore$ The third approximation to the root is $\boxed{x_3 = \frac{1}{2}(x_1 + 1.375) \approx 1.3125}$

Continuing the procedure, the successive approximations are $x_4 = 1.3438$, $x_5 = 1.3282$, $x_6 = 1.3204$, $x_7 = 1.3243$, $x_8 = 1.3263$, $x_9 = 1.3253$, $x_{10} = 1.3248$, $x_{11} = 1.32455$, $x_8 = 1.3247$.

$\boxed{\therefore \text{ The root is 1.3247 approximately.}}$

Hence the required solution.

METHOD – II: THE METHOD OF FLASE POSITION(REGULA FALSI METHOD)



False Position or Regula – Falsi Method Figure

This is the oldest method of finding the real root of an equation f(x) = 0 and closely resembles the Bi-section Method.

Let us choose two points x_0 and x_1 such that f(x_0) and f(x_1) are of opposite signs i.e., the graph of y = f(x) crosses the X – axis between these points.

This indicates that root lies between x_0 and x_1 consequently f(x_0) – f(x) < 0.

Equation of the chord joining two points A(x_0, f(x_0)) and B(x_1, f(x_1)) is defined as

$$y - f(x_0) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0) \text{ ......} equation(1)$$

This method consists in replacing the curve AB by means of the chord AB and taking the point of intersection of the chord with the x-axis as an approximation to the root.

So, the abscissa of the point where the chord cuts the x-axis (y = 0) is given by as below

$$x_2 = x_1 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0) \text{............} equation(2)$$

which is the approximation to the root. If now f(x_0) and f(x_2) are of opposite signs, then the root lies between x_0 and x_2. So replacing x_1 by x_2 in equation (2) we obtain approximation x_3. Root lies between x_1 and x_2 and we would obtain x_3 approximately. Accordingly, this procedure is repeated till the root is found to be desired accuracy. The iteration process based on equation (1) is known as the “Method of false position” or “Regula Falsi Method”.

Problems on Method of False Position

Problem 1 Find the smallest real root $x^2 - \log_e x - 12 = 0$ by the method of false position?
[Ans. $3 < 3.6461 < 4$]

Problem 2 Find the smallest real root $x - \cos x = 0$ by the method of false position?
[Ans. $0 < 0.7391 < 1$]

Problem 3 Find a smallest real root of equation $x^3 - x - 1 = 0$ by the method of false position?
[Ans. root is 1.32 such that $1 < 1.3232 < 2$]

Problem 4 Find a smallest real root of equation $x^3 - 2x - 5 = 0$ by the method of false position?
[Ans. root is 1.32 such that $2 < 2.0945 < 3$]

Problems and Solutions on Method False Position (M F P):

(REGULA - FALSI METHOD)

Problem 1 Find a real root of the equation $x^3 - 2x - 5 = 0$ by the method of false position correct to three decimal places?

Solution: Let $f(x) \cong x^3 - 2x - 5 = 0$

$f(2) = 8 - 4 - 5 = -1$ is negative sign

$f(3) = 27 - 6 - 5 \times 1 = 16$ is positive sign

$\therefore$ A root between 2 and 3.

$\therefore$ Taking $x_0 = 2$, $x_1 = 3$ so $f(x_0) = -1$

$f(x_1) = 16$ in this method of false position

we get, $\boxed{x_2 = x_0 - \dfrac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0) = 2 + \dfrac{1}{7} = 2.0588}$

$f(x_2) = f(2.0588) = (2.0588)^3 - 2(2.0588) - 2(2.0588) - 5 = -0.3908$ so, $2 < -0.3908 < 3$

$\therefore$ Taking $x_0 = 2.0588$, $x_1 = 3$ so $f(x_0) = -0.3908$

$f(x_1) = 16$ in this method of false position

we get, $\boxed{x_3 = x_0 - \dfrac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0) = 2.0588 - \dfrac{0.94121}{16.3908} (-0.3908) = 2.0813}$

On repeating this process,

The successive approximations are $x_4 = 2.0862$, $x_5 = 2.0915$, $x_6 = 2.0934$, $x_7 = 2.0941$, $x_8 = 0.0943$ etc.,

$\therefore$ The root is 2.094 correct to three decimals. Hence the solution.

Problem 2 Find the root of the equation $x^3 - 3x + 4 = 0$ correct to three decimals by Method of False Position (M.F.P.).

Solution: Let $f(x) \cong x^3 - 3x + 4 = 0$

So that $f(2) = 8 - 6 + 4 = 6$ is positive sign

$f(3) = 27 - 9 + 4 = 22$ is positive sign

$\therefore$ A root between 2 and 3.

$\therefore$ Taking $x_0 = 2$, $x_1 = 3$ so $f(x_0) = 6$

$f(x_1) = 22$ in this method of false position(M.F.P.)

we get, $\boxed{x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0) = 2 - \frac{3-2}{22-6}.(6) \approx 2 - \frac{6}{16} \approx \frac{13}{16} = 1.9375}$

therefore, $2 < 1.9375 < 3$ and $x_2 = 1.9375$

$f(x_2) = f(1.9375) = (1.9375)^3 - 3(1.9375) + 4 = 5.9185$; so, $2 < 1.9375 < 3$

$\therefore$ Taking $x_0 = 1.9375$, $x_1 = 3$ so $f(x_0) = 6$

$f(x_1) = 22$ in this method of false position

we get, $\boxed{x_3 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0) = 1.9375 - \frac{3 - 1.9375}{5.9185 - 22} (22) = 2.0035}$

Therefore, $2 < 2.0035 < 3$

On repeating this process,

$\therefore$ The root is 2.0035 correct to four decimals. Hence the solution.

Problem 3 Find a real root of the equation $x.\log_{10} x = 1.2$ by Regula - Falsi method correct to 4 decimals?

Solution: Let $f(x) = x.\log_{10} x - 1.2$

So that $f(1) = 1.\text{Log}_{10}1 - 1.2$ is neagative sign

$f(2) = 2.\text{Log}_{10}2 - 1.2$ is neagative sign

$f(3) = 3.\text{Log}_{10}3 - 1.2$ is positive sign

Therefore, a root lies 2 < root < 3

$\therefore$ Taking $x_0 = 2$, $x_1 = 3$ so $f(x_0) = -0.59794$

$f(x_1) = 0.23136$ in this method of false position

we get, $$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0) = 2 - \frac{3-2}{0.23136 + 0.59794}(-0.59794) \approx 2 + \frac{1}{0.82930}(0.59794) \quad \text{.....(1)}$$
$$x_2 \approx 2.72102$$

$\Rightarrow f(x_2) = f(2.72102) = 2.72102 \log_{10} 2.72102 - 1.2 = -0.01709$

And $f(x_1) = 0.23136$ in (1) we get

$$x_3 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0) = 2.72102 + \frac{0.27898}{0.23136 + 0.01709}(0.01709) \quad \text{.....(1)}$$
$$\approx 2.72102 + \frac{0.27898}{0.24845}(0.01709)$$
$$x_3 \approx 2.74021$$

On repeating this process th successive approximations are

$x_4 = 2.74024$; $x_5 = 2.74063$ so on.

$\therefore$ the root is 2.7406 correct to 4 decimals.

Hence the required solution.

Problem 02 Find a real root of the equation $x.e^x = \cos x$ by the Regula - Falsi method correct to 4 decimals?

Solution: Let $f(x) = x.e^x - \cos x = \cos x - x.e^x = 0$

So that $f(0) = \cos 0 - 0.e^0 = 1$ is positive sign

$f(1) = \cos 1 - 1.e^1 = \cos 1 - e = -2.17798$ is - ve sign

Therefore, a root lies 1 < root < 1

$\therefore$ Taking $x_0 = 0, x_1 = 1$ so $f(x_0) = 1$

$f(x_1) = -2.17798$ in this method of Regula falsi-method

we get, $$\boxed{\begin{aligned} x_2 &= x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0) = 0 - \frac{1-0}{-2.17798-1}(1) \quad .....(1) \\ &\approx 0 + \frac{1}{3.17798}(1) \\ x_2 &\approx 0.31467 \end{aligned}}$$

$\Rightarrow f(x_2) = f(0.34167) = \cos(0.34167) - 0.34167.e^{0.34167}$

$= 0.15900 + 0.36087 = 0.51987$

$\therefore f(0.34167) = 0.51987$ Hence $0.34167 <$ root < 1 And $f(x_1) = -2.17798$ in (1) we get

$\therefore$ Taking $x_0 = 0.34167, x_1 = 1$ so $f(x_0) = 0.51987$

$f(x_1) = -2.17798$ in (1) this method of Regula falsi-method we get,

$$\boxed{\begin{aligned} x_3 &= x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0) = 0.34167 + \frac{1 - 0.34167}{-2.17798 - \underset{0.68533}{0.51987}}(0.51978) \quad .....(1) \\ &\approx 0.34167 + \overline{0.69785}(0.51978) \end{aligned}}$$

$\Rightarrow f(x_3) = f(0.44673) = \cos(0.44673) - 0.44673.e^{0.44673} = 0.20356$

$\therefore f(0.44673) = 0.20356$ Hence $0.44673 <$ root < 1.

And $f(x_1) = -2.17798$ in (1) we get

$\therefore$ Taking $x_0 = 0.44673$, $x_1 = 1$ so $f(x_0) = 0.20356$

$f(x_1) = -2.17798$ in (1) this method of Regula falsi-method we get,

$$x_4 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0) = 0.44673 + \frac{1 - 0.44673}{-2.17798 - 0.20356}(0.20356) \quad \ldots\ldots(1)$$

$$\approx 0.44673 + \frac{0.55327}{0.38154}(0.20356)$$

$$x_4 \approx 0.49402$$

$x_4 = 0.49402$; $x_5 = 0.50995$

$X_6 = 0.51520$; $x_7 = 0.51692$

$x_8 = 0.52748$; $x_9 = 0.51767$

$x_{10} = 0.51775$ .... so on.

$\therefore$ the root is 0.5177 correct to 4 decimals.

Hence the required solution.

METHOD–III : NEWTON – RAPHSON METHOD

The newton – Raphson method is a powerful and elegant method to find the root of an equation.

Let the equation be f(x) = 0. Let x0 be an approximate value of the desired root and h be the small correction to it so that $\boxed{x_1 = x_0 + h}$ ……………………(1)

Is the root of the equation f(x) = 0

$f(x_1) = 0 \Rightarrow f(x_0 + h) = 0$

Expanding $f(x_0 + h)$ by Taylor's theorem, we get

$$f(x_0 + h) = f(x_0) + h.f'(x_0) + \frac{h^2}{2!} f''(x_0) + \text{------} = 0$$

Since h is small, neglect h^2 and higher order of h, to get $f(x_0) + h f'(x_0) = 0$

$$\boxed{h = \frac{-f(x_0)}{f'(x_0)}} \qquad (2)$$

On substitution (2) in (1), we get the first approximation to the required root as

$$\boxed{x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}} \qquad (3)$$

Successive approximations can be written as

$$\boxed{x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}} \quad (4)\ ; \quad \boxed{x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}} \text{ .... } (5)$$

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}} \text{ .... } (6)$$

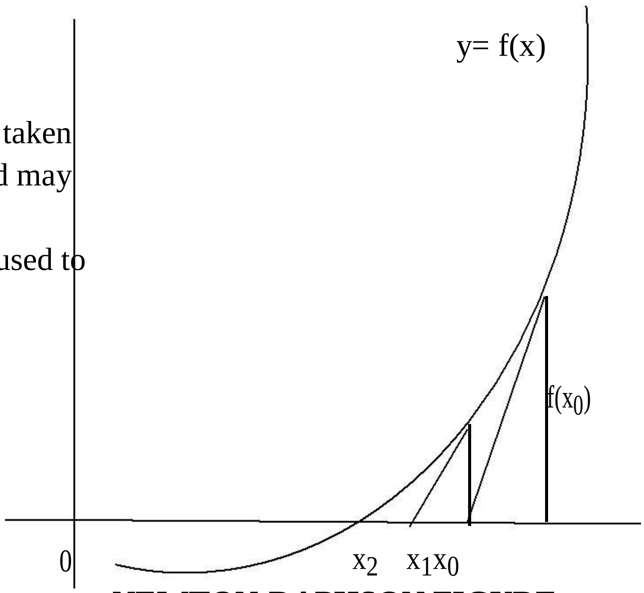
The sequence $\{x_n\}$ if it converges gives the root. The Newton – Raphson method is also known as Newton's method of tangent.

Note 1: This method fails if $f'(x) = 0$

Note 2: the initial approximation should be taken Very close to the root, otherwise the method may Diverges.

Note 3: Newton – Raphson method can be used to find complex root if x0 is complex.

Note 4: Newton‟s formula converges if $|f(x)f''(x)| < |f'(x)|^2$.



NEWTON-RAPHSON FIGURE

Solution of non-linear simultaneous Equations – Newton – Raphson Method:

Consider the equations $f(x, y) = 0$, $g(x, y) = 0$ .... (1)

If an initial approximation (x_0, y_0) to a solution has been found by graphical method or otherwise, then a better approximation (x_1, y_1) can be as below:

Let $x = x_0 + h$, $y = y_0 + k \Rightarrow f(x_0+h, y_0+k) = 0$, $f(x_0+h, y_0+k) = 0$ .... (2)

On expansion each of the function in (2) by Taylor's series theorem to first degree terms

$$\Rightarrow \left.\begin{aligned} f_0 + h.\frac{\partial f}{\partial x_0} + k.\frac{\partial f}{\partial y_0} &= 0, \\ g_0 + h.\frac{\partial g}{\partial x_0} + k.\frac{\partial g}{\partial y_0} &= 0 \end{aligned}\right\} \qquad (3)$$

Where $f_0 (x_0, y_0) = 0$, $\dfrac{\partial f}{\partial x_0} = \left(\dfrac{\partial f}{\partial x}\right)_{x_0, y_0}$ etc.,

On solving equations (3) for h, k we get new approximations to the root as

$\boxed{x_1 = x_0 + h, y_1 = y_0 + k}$

On continuation process till we get the desired degree of accuracy.

Problems and Solutions on Newton Raphson Method :

Problem 01 Find the root of x^4 –x 10 = 0 by using Newton–Raphson Method? [Ans. x_2 = 1.855587]

Problem 02 Find the approximation root of 2x – $\log_{10}^{x}$ – 7 = 0 by using Newton – Raphson Method ?
[Ans. x_2 = 3.789278 ≈ 3.7893]
Problem 03 Solve the system of non-linear equations x_2 + y =11; y_2 + x = 7?
[Ans. x_2 = 3.5844, y_2 = - 1.8482]
Problem 04 By Newton - Raphson method to solve the equations x =x_2 + y_2, y =x_2 – y_2 correct to two decimals starting with approximation (0.8, 0.4)? [Ans. x = 0.7974, y = 0.4006]
Problem 05 By Newton - Raphson method to solve non-linear system of the equations $x^2 - y^2 = 4$, $x^2 + y^2 = 16$ with $x_0 = y_0$ = 2.828 using Newton Raphson Method. Carry out to two iterations? [Ans. x = 3.162, y = 6.450]

Problem 06 By Newton - Raphson method to solve non-linear system of the equations x = 2 (y + 1); y_2 = 3xy - 7 using Newton Raphson Method. Correct to three decimals?
[Ans. x = - 1.853; y = -1.927]

Solutions on Newton Raphson Method (N R M):

Problem 01 Find the root of x_4 – x = 10 by using Newton – Raphson Method?

Solution: Given f(x) ≅ x_4 – x - 10= 0. To find the root of equation by N-R M. By giving values of x is 0, 1 , 2 till we get f(x) opposite signs as below:

x	0	1	2
f(x)= x_4 – x – 10	-10	-10	+4

∴ The root lies between 1 and 2 i.e., 1 < root < 2.

∴ For Further accuracy we observe f(x) by giving values as below:

x	0	1	1.5	1.75	1.8	1.9	2
f(x)	-10	-10	-6.4375	-2.3711	-1.3024	1.1321	+4

On observation f(x) values having opposite signs for the values of x 1.8 and 1.9.

∴ The root lies between 1.8 and 1.9 i.e., 1.8 < root < 1.9.

Now, $f'(x) = 4x^3 - 1$; ∴ $x_{n+1} = x_n - \frac{x_n^4 - x_n - 10}{}$

$4.x_n^3 - 1$

Since, f and f″ have the same sign at x = 1.9 choose x_0 = 1.9 as the starting point.

Now $x_1 = x_0 - \dfrac{x_0^4 - x_0 - 10}{4.x_0^3 - 1}$

$$= 1.9 - \frac{(1.9)^4 - 1.9 - 10}{4.(1.9)^3 - 1} \approx 1.9 - \frac{1.1321}{26.436} \approx 1.9 - 0.042824 \approx 1.8572$$

$f(x_1) = (1.8572)^4 - (1.8572) - 10 = 11.89692435 - 1.8572 - 10 = 0.03972$

$x_2 = 1.8572 - \dfrac{0.03972}{24.623} \approx 1.8572 - 0.0016131 \approx 1.855586895$

$\boxed{\therefore\ x_2 = 1.855587}$ Find $f(x_2) = (1.85587)^4 - (1.855587) - 10 = +0.000058169$

$\therefore\ f(x_2) = +0.000058169$

Hence $\boxed{x_2 = 1.855587 \text{ is required desired degree of accuracy solution.}}$

Problem 02 Find the approximation root of $2x - \log_{10}^{x} - 7 = 0$ by using Newton – Raphson Method ?

[Ans. $x_2 = 3.789278 \approx 3.7893$]

Solution: Given $f(x) \cong 2x - \log_{10}^{x} - 7 = 0$. In f(x) log function involved and to find the root of equation by N-R M.

By giving values of x is 1 , 2, 3 and 4 till we get f(x) opposite signs as below:

X	1	2	3	4
$f(x) = 2x - \log_{10}^{x} - 7$	-5	-3.301	-1.4771	0.3979

∴ The root lies between 3 and 4 i.e., 3 < root < 4.

∴ For Further accuracy we observe f(x) by giving x values as below:

X	3.5	3.7	3.8
$f(x) = 2x - \log_{10}^{x} - 7$	-0.5441	-0.1682	0.0202

On observation f(x) values having opposite signs for the values of x 3.7 and 3.8.

∴ The root lies between 3.7 and 3.8 i.e., 3.7 < root < 3.8.

Now, $f'(x) = 2 - \dfrac{\log_{10}^{e}}{x}$; $f''(x) = \dfrac{\log_{10}^{e}}{x^2} = \dfrac{0.4343}{x^2}$;

Since, f and f″ have the same sign at x = 3.8 choose x_0 = 3.8 as the starting point. Then $\boxed{f(x_0) =}$ 0.0202.

Now $\therefore\ x_{n+1} = x_n - \dfrac{f(x)}{f'(x)} = x_n - \dfrac{2.x_n - \log_{10}^{x_n} - 7}{\left(\qquad \log^{e} \right)}$

$\left(2 - \frac{{}_{10}}{x_n} \right)$

Taking n = 0, $x_1 = 3.8 - \frac{0.0202}{1.88571} \approx 3.8 - 0.010712 \approx 3.7893$

$\boxed{\therefore\ x_1 = 3.7893}$

$f(x_1) = 2(3.7893) - \log_{10}(3.7893) - 7 = 0.000041$

$x_2 = 3.7893 - \frac{0.000041}{1.88538} \approx 3.7893 - 0.00002175 \approx 3.789278$

$\boxed{\therefore\ x_2 = 3.7893}$

Hence $\boxed{x_2 = 3.7893 \text{ is required desired degree of accuracy solution.}}$