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GAUGE INTEGRATION

by

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It is generally accepted that the Riemann integral is more useful as a pedagogical device for introductory analysis than for advanced mathematics. This is simply because there are many meaningful functions that are not Riemann integrable, and the theory of Riemann integration does not contain sufficiently strong convergence theorems. Lebesgue developed his theory of measure and integration to address these shortcomings. His integral is more powerful in the sense that it integrates more functions and possesses more general convergence theorems. However, his techniques are significantly more complicated and require a considerable foundation in measure theory. There is now an impetus to accept the gauge integral as a possible new standard in mathematics. This relatively recent integral possesses the intuitive description of the Riemann integral, with the power of the Lebesgue integral. The purpose of this thesis is to explore the basis of gauge integration theory through its associated preliminary convergence theorems, and to contrast it with other integration techniques through explicit examples.

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GAUGE INTEGRATION

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ABSTRACT

It is generally accepted that the Riemann integral is more useful as a pedagogical device for introductory analysis than for advanced mathematics. This is simply because there are many meaningful functions that are not Riemann integrable, and the theory of Riemann integration does not contain sufficiently strong convergence theorems. Lebesgue developed his theory of measure and integration to address these shortcomings. His integral is more powerful in the sense that it integrates more functions and possesses more general convergence theorems. However, his techniques are significantly more complicated and require a considerable foundation in measure theory. There is now an impetus to accept the gauge integral as a possible new standard in mathematics. This relatively recent integral possesses the intuitive description of the Riemann integral, with the power of the Lebesgue integral. The purpose of this thesis is to explore the basis of gauge integration theory through its associated preliminary convergence theorems, and to contrast it with other integration techniques through explicit examples.

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TABLE OF CONTENTS

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I. INTRODUCTION

Although the mathematical field of real analysis has evolved tremendously over the last century, developing the theory of the integral has remained a problematic and curious issue. As such, the majority of educational institutions normally introduce the Riemann integral for their undergraduate analysis courses, even to future mathematicians. The rationale is that the Riemann integral has an intuitive appeal and its basic theorems are relatively simple to prove. However, this is essentially the end of this integral's usefulness. There is no doubt that Bernhard Riemann's approach to integration advanced mathematics greatly, but that was nearly 150 years ago and his theory is not powerful enough for most modern applications. [Ref.1] Even when pressed to address more complex results (that are still attainable), Riemann's techniques prove no easier than corresponding solutions using more contemporary approaches.

Recall the familiar example of integrating the function 1_Q which represents the characteristic function of the rationals over some given interval. After some exposure to number theory and the idea of cardinality, intuition would indicate that the integral of 1_Q should have a value of zero. However, 1_Q is not Riemann integrable over any interval $[a, b]$ as it is discontinuous everywhere, and is the most obvious example of a non-integrable bounded function. Similar problems exist in the areas of physics and applied mathematics where there are many useful but much more involved functions that exhibit this "bad behavior" for integrability.

The theory of integration now used by professional mathematicians was created by Henri Lebesgue at the beginning of the twentieth century. For many years, his theory was difficult to criticize as it greatly empowered mathematics, especially in the fields of real analysis and probability theory. Unfortunately, although this theory is still relevant, there is a considerable amount of measure theory that needs to be developed before the Lebesgue integral can even be defined. Experience shows that, perhaps because of this and the theory's abstract character, it is generally deemed to be difficult and unpopular with physicists and engineers. Notably, the Lebesgue theory does not cover non-absolutely convergent integrals, and there is a need to consider such improper integrals.

Consider the improper integral introduced by Peter Dirichlet:

$$\lim_{s\to\infty}\int_0^s \frac{\sin(t)}{t}dt$$

This important integral does not even exist as a Lebesgue integral since the absolute value of $t^{-1}\sin(t)$ is not Lebesgue integrable; note that this is not Riemann integrable either.

On or about 1956, Jaroslav Kurzweil gave a new definition of the integral that in many respects is more general than Lebesgue's. Ralph Henstock further developed the theory and started to advocate its use at the elementary level. The Kurzweil-Henstock approach, generally called *gauge theory*, preserves the intuitive appeal of Riemann's definition of the integral but has the power of Lebesgue's approach. The basic premise is to use the standard δ, ϵ definition of the Riemann integral with only one modification, replacing the constant δ with a function. This function, denoted γ, is called a **gauge** and it represents an open interval that varies in length. This small change in the definition has enormous repercussions in applications. As one might suspect, generalizing the constant δ to a function γ yields a wider class of integrands, but it is surprising just how much wider. It turns out that:

$$\left\{\begin{array}{l}\text{Riemann}\\ \text{integrable}\\ \text{functions}\end{array}\right\} \subset \left\{\begin{array}{l}\text{Lebesgue}\\ \text{integrable}\\ \text{functions}\end{array}\right\} \subset \left\{\begin{array}{l}\text{Gauge}\\ \text{integrable}\\ \text{functions}\end{array}\right\}$$

[Ref.2] The classes of gauge integrable functions and Lebesgue integrable functions are closely related. Indeed, it can be shown that:

A function f is Lebesgue integrable if and only if both f and $|f|$ are gauge integrable.

While there are a number of ways to express this new theory, this idea of a gauge function is consistent. Since there are no uniformly accepted titles for this theory, this new integral goes by several names such as: Henstock, Kurzweil, Henstock-Kurzweil (HK), gauge, Denjoy, Denjoy-Perron, or simply the generalized Riemann integral.

The aim of this thesis is to explore the basis of gauge integration theory through its associated preliminary convergence theorems, and to produce comparative examples with other integration techniques. Presumably, the reader is familiar with the basic properties of the Riemann and Lebesgue integrals, along with some knowledge of functional analysis.

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II. INTEGRATION WITH TAGGED PARTITIONS

A. OPENING REMARKS

As gauge integration is, in fact, a generalization of the Riemann integral, it is important to define and review the Riemann integral and its basic properties. As will be seen in the following section, Riemann's integral of 1867 can be generally stated as a limit of summation of a partition over an interval:

$$\int_I f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(\tau_i)(x_i - x_{i-1}) \qquad \tau_i \in [x_{i-1}, x_i], \quad i = 1, \ldots, n$$

Unfortunately, this notation obscures some nuances that will later prove complicated. It does, however, display what needs to be emphasized: that the integral is formed by combining the values $f(\tau_i)$ in a very direct fashion. This method is very intuitive and makes for relatively simple calculation.

Gauge integration can now be expressed relatively easily as it differs from the classical Riemann integral only in that *uniformly fine* partitions of the integration domain are replaced by *locally fine* partitions. Roughly speaking, a *gauge function*, $\gamma(\tau_i)$, defines this locally fine partition which varies from point to point. This idea is critical to understanding gauge integration and will be the focus of this chapter.

We shall consider this and other integrals only over compact intervals $[a, b]$ where $-\infty < a < b < +\infty$. Unless otherwise stated, we will consider integrals of functions $f : I \to \mathcal{R}$, where I represents a closed interval $[a, b]$. While the most important cases to consider are when the functional ranges are $\mathcal{R}$ and $\mathcal{C}$, other spaces of interest (e.g. $\mathcal{R}^n$) can be investigated without significantly increasing the complexity of the theory as long as the range of f remains a normed vector space. Eric Schecter's book contains extensive coverage of this material. [Ref.3]

B. DEFINITION OF THE RIEMANN INTEGRAL

Suppose a function f is to be integrated over the interval $[a,b]$. Form a partition of $[a,b]$ with subintervals $[x_{i-1},x_i]$ by selecting numbers x_i such that $a = x_0 < x_1 < x_2 < \cdots < x_n = b$. A *tagged partition*, $\mathcal{D}$, is created by choosing a number τ_i, called a *tag*, from each interval such that:

$$\mathcal{D}: \qquad a = x_0 < x_1 < x_2 < \cdots < x_n = b, \qquad \tau_i \in [x_{i-1},x_i]$$

The corresponding Riemann sum for f on the interval $[a,b]$ is then:

$$f(\mathcal{D}) = \sum_{i=1}^{n} f(\tau_i)(x_i - x_{i-1})$$

Armed with the above, we are ready to define the number $A \in \mathcal{R}$ as the Riemann integral of $f : [a,b] \to \mathcal{R}$ if $\forall\ \epsilon > 0, \quad \exists\ \delta > 0$ such that if $\mathcal{D}$ is any tagged partition of $[a,b]$ where $0 < x_i - x_{i-1} < \delta$, for $i = 1,\ldots,n$ then:

$$\Big| A - \sum_{i=1}^{n} f(\tau_i)(x_i - x_{i-1}) \Big| < \epsilon$$

It is only after some pursuit of this definition that it becomes apparent that the limitation induced by the *constant* $\delta > 0$ on the integral is a significant drawback.

Similar to the methodology of introducing integration in undergraduate calculus, consider $f(x) > 0$ for $a \le x \le b$ where S is the area under the graph of f. Then each term $f(\tau_i)(x_i - x_{i-1})$ is the area of a rectangle, and the Riemann sum of these rectangles will approximate S. Clearly, the approximation will not be greatly affected if $(x_i - x_{i-1})$ is relatively large over intervals where f changes little. Conversely, the rectangles must be small where f is steep or behaves erratically. Since the partitions need not be uniform and the selection of partitions depends on the behavior of f, this indicates a strategy for selecting Riemann sums.

C. γ-FINE PARTITIONS

Using the behavior of f at τ, assign to τ a neighborhood $\gamma(\tau)$. This results in a interval-valued function γ defined on $[a, b]$. Consider the sums formed from tagged partitions where $[x_{i-1}, x_i] \subseteq \gamma(\tau_i)$. As previously stated, γ is called a *gauge*.

A tagged partition is said to be γ-*fine* when:

$$[x_{i-1}, x_i] \subseteq \gamma(\tau_i), \qquad \forall\ i = 1, 2, \ldots, n$$

The right hand side will often be denoted $\gamma(\tau)$. This definition shows that the function γ determines the size of the interval associated with a given tag.

An alternate manner of thinking about the gauge function, γ, is to define $\gamma(\tau_i) = \big(\tau_i - \delta(\tau_i), \tau_i + \delta(\tau_i)\big)$, where $\delta(\tau)$ is a strictly positive function. Then $\delta(\tau)$ will depend on the behavior of f at τ and will produce *variable* length intervals. Notice that with this definition, the Riemann integral will have constant length intervals $\gamma(\tau_i) = \big(\tau_i - \delta, \tau_i + \delta\big)$, with each length now being less than 2δ. This is equivalent to the previous definition as δ can be made arbitrarily small.

In choosing a Riemann sum, the traditional way of thinking is to choose the partition first, then the associated tags afterward. A critical difference for gauge integration is to think of the tags $\tau_1, \ldots, \tau_n$ as fixed, then deciding if x_{i-1} and x_i are close enough to τ_i to make $f(\tau_i)(x_i, x_{i-1})$ a good approximation. Consider the following illustration:

Let $\gamma : [0, 1] \to \mathcal{R}$ be $\gamma(0) = .01$ and $\gamma(x) = x/2$ for $0 < x \leq 1$. Note that the interval $\gamma(x) = \big(x - \delta(x), x + \delta(x)\big)$ does not contain 0 unless $x = 0$. As a result, any γ-fine partition of $[0, 1]$ must have 0 as a tag. Using similar ideas, it is possible to force any finite number of points to be tags.

The strategy of forming a Riemann sum and the use of γ-fine partitions is best seen in the following example; note that the properties of the function dictate the choice of γ.

1. Unbounded Function Example

Let $f(0) = 0$ and $f(x) = 1/\sqrt{x}$ for $0 < x \leq 1$. Find a gauge γ on $[0, 1]$ which correlates to a Riemann sum differing from the actual area by less than ϵ. As in ordinary calculus, the improper integral technique will be used for this unbounded example.

Since $2\sqrt{x}$ is an antiderivative of $1/\sqrt{x}$ when $x > 0$, the area of the region is:

$$\lim_{s \to 0^+} \int_s^1 \frac{1}{\sqrt{x}} dx = \lim_{s \to 0^+} (2 - 2\sqrt{s}) = 2$$

Note that the area of the strip bounded by $x = u$ and $x = v$ is $2\sqrt{v} - 2\sqrt{u}$, even when $u = 0$. The reader should recall that this computation is not a formal proof using the improper Riemann integral technique. The value of 2 was attained following an easy calculation based on the results of a proof of the general case. The Kenneth Ross text contains a very readable explanation of the development of this technique. [Ref.4]

Consider choosing $\gamma(\tau)$ such that $\gamma(\tau) \subseteq (0, \infty)$ when $0 < \tau \leq 1$. Similar to the previous illustration, the first interval $[0, x_1]$ must have the tag $\tau_1 = 0$ to control the error. Since $f(0) = 0$, the error for the strip between $x = 0$ and $x = x_1$ will be $2\sqrt{x_1}$. By forcing x_1 to approach zero by the choice of $\gamma(0)$, that action suffices to make the error arbitrarily small. In other words, $\forall x_1 < \epsilon^2/2, \quad 2\sqrt{x_1} < \epsilon$.

For the error $2\sqrt{v} - 2\sqrt{u} - (1/\sqrt{\tau})(v - u)$ when $0 < u \leq \tau \leq v$, the number of strips to be used is not known and the error in each strip must be estimated such that the sum of all errors can be controlled.

Walking through the steps, consider $\sqrt{v} - \sqrt{u} = (v - u)/(\sqrt{v} + \sqrt{u})$. This implies: $|2\sqrt{v} - 2\sqrt{u} - 1/\sqrt{\tau}(v - u)| = (v - u) \cdot |2/(\sqrt{v} + \sqrt{u}) - 1/\sqrt{\tau}|$. After getting a common denominator, replace $\sqrt{v} + \sqrt{u}$ by $\sqrt{\tau}$. This leads to:

$$\left|2\sqrt{v} - 2\sqrt{u} - \frac{1}{\sqrt{\tau}}(v - u)\right| \qquad \leq \qquad \frac{u - v}{\tau}\left|2\sqrt{\tau} - \sqrt{v} - \sqrt{u}\right|$$

With $|\sqrt{\tau} - \sqrt{v}| \leq (v - \tau)/\sqrt{\tau}$ and $|\sqrt{\tau} - \sqrt{v}| \leq (\tau - u)/\sqrt{\tau}$, use these and apply the triangle inequality which results in:

$$\frac{u-v}{\tau}\Big|\sqrt{\tau}-\sqrt{v}+\sqrt{\tau}-\sqrt{u}\Big| \le \frac{u-v}{\tau}\cdot\big(\frac{v-\tau}{\sqrt{\tau}}+\frac{\tau-u}{\sqrt{\tau}}\big)=\frac{(u-v)^2}{\tau\sqrt{\tau}}$$

The factor $v-u$ in $(v-u)^2/(\tau\sqrt{\tau})$ will be used to cancel $\tau\sqrt{\tau}$ through the choice of $\gamma(\tau)$. The remaining factor $v-u$ will control the increase in error through summation.

To make this expressly clear, define the γ functions as:

$$\gamma(0)=(-\epsilon^2/16,\epsilon^2/16) \qquad \textit{and} \qquad \gamma(\tau)=(\tau-\delta_\tau,\tau+\delta_\tau)$$

with $\delta_\tau=\epsilon\tau\sqrt{\tau}/4$ when $0<\tau\le 1$. Note that 0 is not in $\gamma(\tau)$ when $\tau>0$.

The choices of $\gamma(\tau)$ imply that $|2\sqrt{(v)}-2\sqrt{(u)}-f(\tau)(v-u)|$ is less than $\epsilon/2$ when $\tau=0$, and less than $(\epsilon/2)(v-u)$ when $0<u\le\tau\le v$, provided that $[u,v]\subseteq\gamma(\tau)$.

Now consider a γ-fine partition of $[0,1]$. The first strip error is at most $\epsilon/2$. The summation of errors for the remaining strips is less than $\sum_{i=2}^{n}(\epsilon/2)(x_i-x_{i-1})$ or $(\epsilon/2)(1-x_1)$. Thus,

$$\left|2-\sum_{i=1}^{n}f(\tau_i)(x_i-x_{i-1})\right|<\epsilon,$$

as required.

The only remaining question is whether γ-fine partitions exist over $[0,1]$. Since γ has the values $\gamma(0)=(-\epsilon^2/16,\epsilon^2/16)$ and $\gamma(\tau)=(\tau-\delta_\tau,\tau+\delta_\tau)$ with $\delta_\tau=\epsilon\tau\sqrt{(\tau)}/4$ when $0<\tau\le 1$. Assign x_1 so that $0<x_1<\epsilon^2/16$ and h so that $0<h<\epsilon x_1\sqrt{x_1}/4$. Choose the least integer n such that $x_1+(n-1)h\ge 1$. Let $x_2=x_1+h, \quad x_3=x_1+2h,\ldots,x_{n-1}=x_1+(n-2)h$, and $x_n=1$. Also set $x_0=0$ and let $\tau_i=x_{i-1}$ for $k=1,2\ldots,n$. The selection of x_1 implies that $[x_0,x_1]\subseteq\gamma(0)$. The choice of h implies that $[x_1,x_2]\subseteq\gamma(x_1)$. Since the length of $\gamma(\tau)$ is an increasing function of τ for $0<\tau\le 1$, it is also true that $[x_{i-1},x_i]\subseteq\gamma(x_{i-1})$ for $i=3,4,\ldots,n$ and thus the partition is γ-fine.

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III. THE GAUGE INTEGRAL

Recall that when applying the techniques associated with the Riemann integral, the intervals are thought of as being chosen first with each width being less than some constant δ. As the tag is considered next, there are no questions as to whether a tagged partition exists, or what conditions may or may not be satisfied. However, as alluded to in the unbounded example, gauge integration techniques brings forth these questions as the sequence of choosing partitions and tags is reversed. Since gauge functions are arbitrary, and the partitions must be made the "correct" width, the existence of γ-fine partitions becomes abstract and is no longer obvious. If the infimum of a set of arbitrary positive functions is itself positive, i.e. $inf\{\gamma(x) : x \in [a, b]\} > 0$, then it is fairly clear that γ-fine partitions exist as this is essentially the constant δ case. By contrast, if the infimum is 0 which is often the case for interesting functions, then a proof of the existence of γ-fine tagged partitions is required. The following theorem and subsequent proof will address this issue for the general case.

A. γ-FINE EXISTENCE THEOREM

If γ is some gauge function defined over the interval $[a, b]$, then there exists a γ-fine tagged partition of $[a, b]$.

Proof: Let S be the set of points $x \in (a, b]$ such that there exists a γ-fine tagged partition of $[a, x]$. Note that S is non-empty since it contains the interval $(a, a+\delta(a))$. In effect, $(a, a + \delta(a)) \subset (a - \delta(a), a + \delta(a)) = \gamma(a)$. So, the set $\{[a, x]\}$ with the tag a, is itself a γ-fine tagged partition of $[a, x] \quad \forall x \in (a, a + \delta(a))$. Let $y = sup\ S$ and note that $y \in [a, b]$. We now need only to show that y belongs to to S and that $y = b$; this will then cover the whole interval.

Since $y = sup\ S$, this means that either $y \in S$, or there is a point $p_1 \in S$ such that $y - \delta(y) < p_1 < y$. To address the latter, let $\mathcal{D}$ be a γ-fine tagged partition of

$[a, p_1]$ and let $\mathcal{E} = \mathcal{D} \cup \{[p_1, y]\}$, with y as the tag for the last interval. Now $\mathcal{E}$ is a γ-fine tagged partition of $[a, y]$ and this shows that $y \in S$. Now suppose $y < b$ and let $p_2 \in [a, b]$ be such that $y < p_2 < y + \delta(y)$ and let $\mathcal{E} = \mathcal{F} \cup \{[y, p_2]\}$ with y as the last interval tag. Then $\mathcal{E}$ is a γ-fine tagged partition of $[a, p_2]$ and it follows that $p_2 \in S$, a contradiction to the fact that y is an upper bound of the set S. Hence, $y \not< b$, $\Rightarrow \quad y = b$ as required. Notice that this proof relies directly on the Completeness Axiom and, in fact, can be shown to be equivalent to it.

The language and methods have now been established in order to pursue a definition of the integral that differs only slightly from the previous Riemann definition, yet is significantly superior in integrating power. It is very desirable to develop a precise, yet flexible notation for tagged partitions and Riemann sums. The capital script lettering shall normally be $\mathcal{D}, \mathcal{E}, \mathcal{F}$ for tagged partitions; the definition of which can now be rephrased. A *tagged partition* $\mathcal{D}$ of $[a, b]$ is a set of ordered pairs, $[(\tau_1, I_1), \ldots, (\tau_n, I_n)]$ where $I_1, \ldots, I_n$ are non-overlapping closed intervals whose union is $[a, b]$ and $\tau_i \in I_i$ for $i = 1, 2, \ldots, n$. Further, suppose $L(I)$ is the *length* of I in the Euclidean sense. The $f(\tau)L(I)$ is the term in the Riemann sum representing the tagged interval(τ, I). Thus, we let $fL(\mathcal{D})$ denote the Riemann sum given by the tagged partition $\mathcal{D}$. That is,

$$fL(\mathcal{D}) = \sum_{I \in \mathcal{D}} f(\tau)L(I)$$

Furthermore, the terms "integral" and "integrate," along with the associated symbols, will refer to the gauge definitions unless otherwise stated.

B. DEFINITION OF THE GAUGE INTEGRAL

Let $f : [a, b] \to \mathcal{R}$ be given. A number $A \in \mathcal{R}$ is called the integral of f on $[a, b]$ provided $\forall \epsilon > 0, \quad \exists$ a function γ such that $|A - fL(\mathcal{D})| < \epsilon$ whenever $\mathcal{D}$ is a γ-fine tagged partition of $[a, b]$.

A key goal in pursuing a modification of the classical Riemann integral is to be able to integrate more functions. The previous "unbounded example" uses a function that is not Riemann integrable as it is not bounded, and an improper integral technique had to be used. The following example will show that even within the class of bounded functions, the gauge integral is more applicable.

1. Bounded Function Example

Evaluate the integral of a function that is constant over the complement of a countably infinite subset of $[a, b]$, and prove its integrability. Note that there is no value assigned to the countable set itself.

To address this problem, recall that a set S is countably infinite when $S = \{s_1, s_2, \ldots\}$. Hence $g(x) = C$ (constant) for all x not in S. So, the only realistic possibility for $\int_a^b g$ is $C(b-a)$ since this is $\int_a^b g$ when g is constant on $[a, b]$. The aim is to develop a gauge so that $|C(b-a) - gL(\mathcal{D})| < \epsilon$ for all γ-fine partitions I of $\mathcal{D}$. Since

$$C(b-a) - gL(\mathcal{D}) = \sum_{I \in \mathcal{D}} \Big[C - g(\tau)\Big] L(I)$$

and $g(\tau) = C$ when τ is not in S with the only remaining difficulty is the construction of $\gamma(\tau)$ when $\tau \in S$. Note, for τ not in S, there is no restriction on $L(I)$ and $\gamma(\tau)$ can be chosen as $(\tau - 1, \tau + 1)$.

Every s_n in S is the tag for at most two intervals in a partition of $[a, b]$. The simple choice for δ_n is such that $|C - g(s_n)|\delta_n \leq \epsilon/2^{n+2}$ when $\tau = s_n$ and $I \subseteq \gamma(\tau)$. Also, the sum of all terms tagged with s_n is at most $\epsilon/2^n$.

Grouping and ordering all nonzero terms $[C - g(\tau)]L(I)$ by subscript n such that $\tau = s_n$. Hence,

$$\Big|C(b-a) - gL(\mathcal{D})\Big| < \sum_{n=1}^{\infty} \frac{\epsilon}{2^n}$$

when $\mathcal{D}$ is γ-fine. The solution is now complete as the last sum equals ϵ.

Notice that there are no limitations on the last example other than it needs to be countable. Thus, we can let S be the rational numbers over some interval $[a, b]$, say $[0, 1]$. Choosing the function to be one over the irrationals and zero over the rationals, we now have a special case of the example where the integral exists and is equal to one. In similar fashion, if we choose the function to be $1_{\mathcal{Q}}$ we have answered the initial problem in the introduction.

C. COMMENTS ON UNBOUNDED INTERVALS

The process of explaining integration over unbounded intervals, such as $\int_0^\infty f(x)dx$, is detailed in undergraduate calculus courses as limits of integrals over bounded intervals. Here, we will form a definition in terms of Riemann sums. Thus, the definition will be equivalent to the one given previously for bounded intervals.

Using the previous analogy of introducing Riemann sums, we can again consider the area under a positive curve; say f over $[a, \infty)$. Making the assumption that the area under f is finite, we can again consider how to approximate the area with rectangles. As before, it is beneficial to have narrow rectangles over the steep portions of f. Clearly, a finite number of partitions cannot cover $[a, \infty)$, so the rectangles should cover a "large" interval $[a, s]$. Using a carefully selected tagged partition of $[a, s]$ and large enough s, the resulting Riemann sum will closely approximate the area.

The strategy for using the familiar ideas above is to extend $\mathcal{R}$ with the points at positive and negative infinity. Henceforth, $\bar{\mathcal{R}} = \mathcal{R} \cup \{-\infty, \infty\}$. The extended real numbers will be ordered as expected, $-\infty < x < \infty$ for all $x \in \mathcal{R}$. An interval in $\bar{\mathcal{R}}$ is said to be unbounded if at least one endpoint is $-\infty$ or ∞, and bounded otherwise.

For $[a, b] \in \bar{\mathcal{R}}$, the previous definitions for a gauge and tagged partition will remain unchanged. The corresponding Riemann sum, however, cannot transfer directly as there is no length that can cover an infinite interval. We know that the sum must have a value of zero for all unbounded intervals in the partition, and the most direct method is to extend the definition of length. So, $L(I_k) = 0$ for all unbounded intervals, then $fL(\mathcal{D})$ is as before.

Clearly, the only intervals that can have $-\infty$ or ∞ as a tag are unbounded ones, and the values of f at these tags will not affect the sum. Integrals in the form of:

$$\int_a^\infty f(x)dx \qquad \int_{-\infty}^a g(x)dx \qquad \int_{-\infty}^\infty h(x)dx$$

can now be expressed exactly as detailed in the previous chapter. In order to apply the definition of the integral on unbounded I_K, it is necessary to use similar methodology in addressing $\int_a^b f(x)dx$ when a, b are finite but f is unbounded approaching a or b.

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IV. FUNDAMENTAL THEOREM OF CALCULUS

The Fundamental Theorem of Calculus (FTC) generally asserts that:

$$\int_a^b f(x)dx = F(b) - F(a)$$

where F is the antiderivative, and it is assumed that the necessity that f be integrable over $[a, b]$ is satisfied; notice that our previous examples fail these conditions. However, the antiderivative does play a significant role, and we will review the major points of the example solutions to ease into the FTC for gauge integration.

In our first example, it was demonstrated that $\int_0^1 1/\sqrt{x}dx = 2$. Setting $F(x) = 2\sqrt{x}$ for $x \in (0,1)$, then $F'(x) = f(x)$ when $x \in (0,1]$ but $F'(0)$ does not exist. Regardless, it was shown that $\int_0^1 f(x)dx = F(1) - F(0)$ by manipulating $2\sqrt{v} - 2\sqrt{u} - (1/\sqrt{\tau})(v-u)$. This last statement is $F(v) - F(u) - f(\tau)(v-u)$. The solution was attained by observing that $(v-u)^2/(\tau\sqrt{\tau})$ was a dominate term over $|F(v) - F(u) - f(\tau)(v-u)|$ when $\tau \in [u, v]$. By applying a limit calculation, it can be observed that this dominate term is only possible if $F'(x) = f(x)$.

In the second example, the function f can also be considered an application of the FTC where $F(x) = Cx$. The continuity of F is critical at the points where there is no standing assumption that $F'(x) = f(x)$. The fact that S was countable ensured that it did not have "too many" points, and thus the properties of F could be used for continuity.

These examples show that the integrability of f is proved rather than assumed. We have also seen that the interval need not be bounded. They also show that $f(x)$ need not equal $F'(x)$ for all x. Recall a countable set can be used with the requirement that F is continuous over $[a, b]$. These will all be notable features of the FTC that will now be stated and proved.

A. FUNDAMENTAL THEOREM

If $F : [a,b] \to \mathcal{R}$ is differentiable at each point of $I = [a,b]$ then $f = F'$ is gauge integrable and:

$$\int_a^b f = F(b) - F(a)$$

1. FTC Proof

Given $\epsilon > 0$, $\exists\, \delta(\tau) > 0$ such that when $0 < |z - \tau| \leq \delta(\tau)$ for $z \in I$, then since $f(\tau) = F'(\tau)$ exists we have:

$$\left| \frac{F(z) - F(\tau)}{z - \tau} - f(\tau) \right| \leq \epsilon.$$

Furthermore, if $|z - \tau| \leq \delta(\tau)$ for $z \in I$, it follows that:

$$|F(z) - F(\tau) - (z - \tau)f(\tau)| \leq \epsilon|z - \tau|$$

Thus, if $a \leq u \leq \tau \leq v \leq b$ and $0 < v - u \leq \delta(\tau)$, the triangle inequality implies:

$$|F(v) - F(u) - (v-u)f(\tau)| \leq |F(v) - F(\tau) - (v-\tau)f(\tau)| + |F(\tau) - F(u) - (\tau - u)f(\tau)|$$

$$\leq \epsilon(v - \tau) + \epsilon(\tau - u) = \epsilon(v - u).$$

If $\mathcal{D}$ is a γ-fine partition with Riemann sum of $f(\mathcal{D})$ over I, then the sum $\sum_{i=1}^{n} (F(x_i) - F(x_{i-1})) = \Big[- F(x_0) + (F(x_1) - F(x_1)) + \cdots$
$\cdots + (F(x_{n-1} - F(x_{n-1}) + F(x_n)\Big] = F(b) - F(a)$ will satisfy:

$$|F(b) - F(a) - f(\mathcal{D})| = \left| \sum_{i=1}^{n} \Big(F(x_i) - F(x_{i-1}) - f(\tau_i)(x_i - x_{i-1}) \Big) \right|$$

$$\leq \sum_{i=1}^{n} \Big| F(x_i) - F(x_{i-1}) - f(\tau_i)(x_i - x_{i-1}) \Big| \leq \sum_{i=1}^{n} \epsilon(x_i - x_{i-1}) = \epsilon(b - a).$$

Since $\epsilon > 0$ is arbitrary, this shows that f is gauge integrable and that $\int_a^b f = F(b) - F(a)$ as required. [Ref.5]

2. Alternate Statement of FTC

Another approach in stating the FTC is to fully develop the definition of the primitive. Let I be an interval in $\bar{\mathcal{R}}$ where $F : I \to \mathcal{R}$ is given. A function $F : I \to \mathcal{R}$ is a *primitive* of f on I given that F is continuous on I and $F'(x) = f(x)$ for all $x \in I$ save a countable set of x values.

Notice that when F is a primitive over an unbounded interval I, each infinite endpoint of I (which is in I) will be in the countable set for which $F'(x) \neq f(x)$. This follows the notion of the derivative being meaningful only at points in $\mathcal{R}$. We can now restate the FTC as follows:

If $f : [a, b] \to \mathcal{R}$ has a primitive F on $[a, b]$, f is integrable and $\int_a^b f(x)dx = F(b) - F(a)$. The proof is similar to the previous one, but will not be stated here. See [Ref.6] for details.

B. IMPROPER INTEGRALS

In general terms, an *improper integral* is one that does not exist by the definition, but can be interpreted by use of a limit. In effect:

$$\lim_{s \to a} \int_s^b f(x)dx \qquad or \qquad \lim_{s \to b} \int_a^s g(x)dx$$

Notice the function must still be integrable over both $[s, b]$ $\forall$ $s < a$, and $[a, s]$ $\forall$ $b < s$ respectively. Our first example in Chapter 2 demonstrated this idea.

The FTC permits us to address many types of problems that have this property. Namely, when f has a primitive F over $(a, b]$ and $\lim_{s \to a} F(s)$ exists $\in \mathcal{R}$. It can then be said that $F(a) = \lim_{s \to a} F(s)$, and the result is a primitive of f on $[a, b]$.

Hence, applying the FTC on $[a, b]$ we have:

$$\int_a^b f(x)dx = F(b) - F(a) = F(b) - \lim_{s \to a} F(s) = \lim_{s \to a} \Big(F(b) - F(s) \Big) = \lim_{s \to a} \int_s^b f(x)dx$$

As a result, the integral exists as a gauge integral and has the same value if evaluated as an improper Riemann integral.

1. Example

Suppose that $\int_a^b g$ and $\int_s^b g$ exist $\forall\ s$ where $a < s < b$. Demonstrate that:

$$\int_a^b g = \lim_{s \to a^+} \int_s^b g$$

In order to pursue the solution, we must consider the particular gauge γ over $[a, b]$ such that $\left|\int_a^b g - gL(\mathcal{D})\right| < \epsilon$ when $\mathcal{D}$ is a γ-fine partition of $[a, b]$. Choose some $s \in (a, b)$. Because g is integrable on $[s, b]$, $\exists\ \gamma_s$ where $|\int_s^b g - gL(\mathcal{F})| < \epsilon$ when $\mathcal{F}$ is a γ_s-fine partition of $[s, b]$.

Now, select γ_s so that $\gamma_s(\tau) \subseteq \gamma(\tau)$, and choose $k \in \gamma(a)$ and notice $|g(a)| \cdot L([a, k]) < \epsilon$. Let $s \in (a, k)$ with $\mathcal{F}$ being a γ_s-fine partition of $[s, b]$. Choose $\mathcal{D} = [a, s] \cup \mathcal{F}$ with the first partition having endpoint a as the tag. This results in $\mathcal{D}$ being a γ-fine partition of $[a, b]$ with:

$$\left|\int_a^b g - \int_s^b g\right| \leq \left|\int_a^b g - gL(\mathcal{D})\right| + \left|gL(\mathcal{F}) - \int_s^b g\right| + \left|g(a)L\big([a, s]\big)\right|$$

As each term on the right is less than ϵ, we have $\lim_{s \to a^+} \int_s^b g = \int_a^b g$ as ϵ can be made arbitrarily small. This example is important as it demonstrates that if a function fails to have an integral according to the improper integral definition, then it will fail to have a meaningful gauge integral.

For instance, consider $\int_a^b 1/y^2 dy$. This function has no useful improper integral when $\lim_{s \to 0^+} \int_s^1 1/y^2 dy = \lim_{s \to 0^+}(-1 + s^{-1}) = \infty$, and hence no gauge integral either. Notice that the example could have been restated as $\lim_{t \to b^-} \int_a^t g$ with similar results.

This leaves one unresolved issue for addressing improper integrals. Namely, that $\int_a^b f$ will still exist even when $\lim_{s \to b^-} f$ exists without a primitive of f on $(a, b]$.

2. Improper Integral Theorem

Let $f : [a, b] \to \mathcal{R}$ have an integral on $[s, b]\ \ \forall\ s$ such that $a < s < b$. Then $\int_a^b f$ exists iff $\lim_{s \to a^+} \int_s^b f$ exists; furthermore, those values will be equal.

The proof of this theorem is a bit lengthy and is full of technical details that add little in understanding gauge integration; hence, it is omitted. The McLeod book [Ref.7] contains the most readable version of this proof.

This theorem gives insight on the nature of improper integrals within gauge integration theory. First, the gauge integral exists precisely when the improper Riemann integral exists and has the same value. Secondly, that the gauge integral over intervals in $\bar{\mathcal{R}}$ has no improper extensions; it is either integrable or it is not.

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V. THE DEVELOPMENT OF GAUGE INTEGRATION THEORY

The majority of theorems and topics discussed in this chapter should be quite familiar from the usual study of the Riemann integral. The purpose of this chapter is three-fold; first, to build a mathematical bridge to the more advanced topics presented later. The next purpose is to reassure the reader that basic properties of the Riemann and Lebesgue integrals also hold for the gauge integral, and finally, to demonstrate that the use of gauges and tagged partitions often simplify long-established proofs. In most cases where the proposition and proof nearly parallel the traditional ones, the proofs shall be omitted. In some instances, such as property 5, the proof is highlighted as it gives additional insight to the use of gauge theory within an argument. The proof of the last theorem, which concerns absolutely continuous singular functions being constant, is an elegant example of how gauge theory can be a versatile tool in greatly simplifying difficult concepts.

A. BASIC PROPERTIES

Among the properties of the gauge integral, there are basically two that determine its value. First is the function f, called the integrand, secondly is the interval $I = [a, b]$ over which integration is defined. We shall first deal with f, and consider the integral as a function of the integrand.

1. Linearity

Let $f : I \to \mathcal{R}$ and $g : I \to \mathcal{R}$ be integrable. For $c \in \mathcal{R}$, then cf and $f + g$ are integrable. Also, $\int_I cf = c \int_I f$ and $\int_I (f + g) = \int_I f + \int_I g$. Notice that this extends to all finite linear combinations $\sum_{k=1}^{n} c_k f_k$ as well.

2. Positivity

If $f : I \to \mathcal{R}$ is integrable over I and $f(x) \geq 0$ for $x \in I$, then $\int_I f \geq 0$.

Corollary: If $f,\ g : I \to \mathcal{R}$ are both integrable over I and $f(x) \leq g(x)$ for all $x \in I$, then $\int_I f \leq \int_I g$.

3. Absolute Integral Inequality

A function is said to be *absolutely integrable* over I if both f and $|f|$ are integrable over I. Note that this concept is one of the key issues explored in the following chapter.

If $f : I \to \mathcal{R}$ is absolutely integrable over I, then:

$$\left|\int_I f\right| \leq \int_I |f|$$

4. Integration by Parts

Let F and G be primitives of f and g on I. Then fG is integrable $\Leftrightarrow$ Fg is integrable. Furthermore:

$$\int_a^b fG = F(b)G(b) - F(a)G(a) - \int_a^b Fg$$

5. Bounded Continuity Theorem

If $f : I \to \mathcal{R}$ is continuous on I, then it is bounded on I.

Proof: Since f is continuous for every $x \in I$, then $\exists\ \gamma(x) > 0$ such that $|f(t) - f(x)| < 1, \quad \forall\ t \in I$ that satisfy $|t - x| < \gamma(x)$. Let this define a positive function γ on I. With $\mathcal{D}$ as a γ-fine tagged partition of I, let M be the maximum value of all tags within $\mathcal{D}$; *i.e.* $M = max\{|f(\tau_i)|\}$. Given some $x \in I$, there is an index j such that $x \in [x_{j-1}, x_j]$ and hence:

$$|f(x)| \leq |f(x) - f(c_j)| + |f(c_j)| < 1 + M$$

Then f is bounded by $1 + M$ as required.

B. CAUCHY CRITERION

Recall that a sequence of real numbers $(a_i)_{i=1}^{\infty}$ has a limit $A \in \mathcal{R} \Leftrightarrow \ \forall\ \epsilon > 0, \quad \exists\ N \in \mathcal{N}$ such that $|a_i - a_j| < \epsilon$ when $i, j > N$. This is the Cauchy criterion for convergence and is quite useful in that the value of the limit need not be known. Similarly, this idea extends integrals and is used in the proofs of many subsequent results.

Theorem

The function $f : I \to \mathcal{R}$ is integrable over $I \ \Leftrightarrow\ \forall\ \epsilon > 0, \quad \exists$ a gauge γ such that $|fL(\mathcal{D}) - fL(\mathcal{E})| < \epsilon$ for all γ-fine partitions $\mathcal{D}$ and $\mathcal{E}$ of I.

Proof:

"$\Rightarrow$" Let A represent the integral of f over I. Given some $\epsilon > 0$, there exists a γ function for γ-fine partitions $\mathcal{D}, \ \mathcal{E}$ of I, such that $|A - fL(\mathcal{D})| < \epsilon/2$ and $|A - fL(\mathcal{E})| < \epsilon/2$. It immediately follows from the triangle inequality that $|fL(\mathcal{D}) - fL(\mathcal{E})| \leq |A - fL(\mathcal{D})| + |A - fL(\mathcal{E})| < \epsilon$

"$\Leftarrow$" We are given that for all n, $\exists$ a gauge γ_n such that $|fL(\mathcal{D}) - fL(\mathcal{E})| < 1/n$ when $\mathcal{D}, \ \mathcal{E}$ are γ-fine partitions of I. Notice that we can replace $\gamma_n(\tau)$ by $\gamma_1(\tau) \cap \gamma_2(\tau) \cap \cdots \cap \gamma_n(\tau)$ so that $\gamma_j(\tau) \subseteq \gamma_i(\tau)$ when $i < j$. Now for each n, fix a γ_n-fine partition $\mathcal{D}_n$. Consider the sequence of elements $fL(\mathcal{D}_n) \in \mathcal{R}$ and suppose that $i < j$. Then $\mathcal{D}_j$ is not only γ_j-fine, it is also γ_i-fine since γ_j is a subset of γ_i. Hence:

$$|fL(\mathcal{D}_j) - fL(\mathcal{D}_i)| < 1/i$$

Then $\Big(fL(\mathcal{D}_n)\Big)_{n=1}^{\infty}$ is a Cauchy sequence converging to a limit $A \in \mathcal{R}$.

As mentioned previously, the most significant advantage of the gauge theory approach compared to the Lebesgue approach is its relative simplicity. An excellent example of this is the theorem proof that an absolutely continuous singular function is constant. This result is very important for the development of Lebesgue integration.

The traditional proof is quite complex and involves, among other results, the Vitali Covering Lemma. A typical presentation of this argument is detailed in Royden's *Real Analysis*, [Ref.8]. The concepts in just this lemma alone are very involved and often difficult for students to grasp. The following proof bypasses all of that machinery and uses only elementary facts from measure theory in conjunction with gauge theory.

C. ABSOLUTELY CONTINUOUS SINGULAR FUNCTION THEOREM

Given that F is absolutely continuous on I, if $F' = 0$ almost everywhere on I, then F is constant on I.

Proof: Define $G = \{x \in I : \ F'(x) = 0, \text{ or doesn't exist}\}$. Then $m(G) = 0$ ($m \equiv$ Lebesgue measure). For $\epsilon > 0$, choose $\rho > 0$ such that $\sum_{i=1}^{n} |F(t_i) - f(s_i)| < \epsilon$ when $\{[s_i, t_i]\}$ is a finite collection of non-overlapping intervals in $[a, b]$ that satisfy $\sum_{i=1}^{n} |t_i - s_i| < \rho$. Since $m(G) = 0$, there exists a sequence of open intervals $\{O_k\}$ such that $G \subseteq \bigcup_{k=1}^{\infty} O_k$ and $\sum_{k=1}^{\infty} L(O_k) < \rho$. Define a positive function δ on I as follows:

If $x \notin G$, fix $\delta(x) > 0$ such that $|F(t) - F(x)| \leq \epsilon \cdot |t - x|, \ \ \forall\, t \in I$ where $|t - x| < \delta(x)$.

If $x \in G$, fix $\delta(x) > 0$ such that $\Big(x - \delta(x), x + \delta(x)\Big) \subseteq O_k$ for some k.

Let $\mathcal{D}$ be a γ-fine tagged partition of I and define:

$$S_{\widetilde{E}} = \{i : \ \tau_i \notin E\} \qquad \textit{and} \qquad S_E = \{i : \ \tau_i \in E\}$$

Notice $|F(\tau_i) - F(\tau_{i-1})| \leq \epsilon \cdot (x_i - x_{i-1}) \quad \forall\, i \in S_{\widetilde{E}}$, and that:

$$\sum_{i \in S_E} (x_i - x_{i-1}) \leq \sum_{k=1}^{\infty} L(I_k) < \rho$$

Now we have:

$$|F(b){-}F(a)| = \left|\sum_{i=1}^{n}\Big(F(x_i) - F(x_{i-1})\Big)\right| \leq \sum_{i\in S_{\widetilde{E}}} |(F(x_i){-}F(x_{i-1})| + \sum_{i\in S_E} |(F(x_i){-}F(x_{i-1})|$$

$$\leq \sum_{i\in S_{\widetilde{E}}} \epsilon \cdot (x_i - x_{i-1}) + \epsilon \ \leq \ \epsilon \cdot (b - a + 1)$$

Since $\epsilon > 0$ was arbitrary, $\Rightarrow \quad F(b) = F(a)$. Then it follows that the exact same argument will show $F(x) = F(a) \quad \forall\, x \in (a, b)$, and thus F is constant on I. [Ref. 9]

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VI. ABSOLUTE INTEGRABILITY

An important and useful characteristic when analyzing a function is whether or not $|f|$ is integrable when f is known to be integrable. Knowledge of this attribute is important for investigating convergence behavior, and in determining what functional space the function exists in. This is also an important element for the calculation of the length of curves. While examining the integrability of $|f|$, we need to consider the summation:

$$\sum_{i=1}^{n} \left| \int_{x_{i-1}}^{x_i} f \right|$$

that is taken over the intervals of a partition of I. The connection between this sum and a Riemann sum for $|f|$ is given by a fundamentally important result in gauge theory called "Henstock's Lemma." The concepts in this lemma will be evident in most of the significant results about the gauge integral.

A. HENSTOCK'S LEMMA

For $f : I \to \mathcal{R}$ integrable over I, there is a gauge γ on I such that $|fL(\mathcal{D}) - \int_I f| < \epsilon$ when $\mathcal{D}$ is any γ-fine partition of I. Let $\mathcal{J}$ be a subset of $\mathcal{D}$ where $\mathcal{J} = \{J_1, \ldots, J_n\}$ is a collection of tagged, closed intervals such that $\tau \in J_i \subset \gamma(\tau_i)$, then:

$$\left| \sum_{i=1}^{n} \left[fL(J_i) - \int_{J_i} f \right] \right| \leq \epsilon \quad \textit{and} \quad \sum_{i=1}^{n} \left| fL(J_i) - \int_{J_i} f \right| \leq 2\epsilon$$

[Ref.10] In general, the above lemma states that gauge γ defines Riemann sums equally well on subintervals of I as it does on the whole. Thus, the partition $\mathcal{D}$ can be broken down into subsets without losing the close approximation of the sum of integrals. This assertion of the lemma is a central idea in proving the monotone and dominated convergence theorems as we shall see in the next chapter.

The second portion of the lemma claims that $\sum |f(\tau)| \cdot L(J)$ closely approximates $\sum | \int_J f|$. This is a consequence of the inequality $\Big||A| - |B|\Big| \leq \Big|A - B\Big|$. This aspect of the lemma is critical for proving the integrability of $|f|$.

1. Proof of Lemma

The set $I \setminus \bigcup_{i=1}^{n} J_i$ consists of a finite number of disjoint intervals.
(Note that $\mathcal{J}$ is not said to be a γ-fine partition as it is possible that $\bigcup_{i=1}^{n} J_i \neq I$)
Let $E_1, \ldots, E_m$ be these disjoint intervals with their endpoints adjoined. For an arbitrary $\epsilon' > 0$, the integrability of f over each E_j implies that $\exists$ γ-fine tagged partition $\mathcal{E}_j$ for each E_j such that:

$$\left| fL(\mathcal{E}_j) - \int_{E_j} f \right| < \frac{\epsilon'}{m}$$

Then $\mathcal{D}' = \mathcal{J} \cup \mathcal{E}_1 \cup \cdots \cup \mathcal{E}_m$ is a γ-fine tagged partition of I. For ease of notation we'll say $fL(\mathcal{J}) = \sum_{i=1}^{n} fL(J_i)$, then:

$$\left| fL(\mathcal{D}') - \int_I f \right| = \left| fL(\mathcal{J}) - \sum_{i-1}^{n} \int_{J_i} f + \sum_{i=1}^{m} \left[fL(\mathcal{E}_i) - \int_{E_i} f \right] \right| < \epsilon$$

This implies:

$$\left| fL(\mathcal{J}) - \sum_{i-1}^{n} \int_{J_i} f \right| - \left| \sum_{i=1}^{m} \left[\int_{E_i} f - fL(\mathcal{E}_i) \right] \right| < \epsilon,$$

and after applying the triangle inequality m times we have:

$$\left| fL(\mathcal{J}) - \sum_{i-1}^{n} \int_{J_i} f \right| < \epsilon + \sum_{i=1}^{m} \left| \int_{E_i} f - fL(\mathcal{E}_i) \right| < \epsilon + m \cdot \frac{\epsilon'}{m} = \epsilon + \epsilon' \qquad (\forall \epsilon' > 0)$$

Thus, it follows that $|fL(\mathcal{J}) - \sum_{i=1}^{n} \int_{J_i} f| \leq \epsilon$.
To address the second portion of the lemma, let $\mathcal{J}^+$ be the collection $\{J_i\}$ such that $fL(J_i) - \int_{J_i} f \geq 0$, $\quad$ ($\mathcal{J}^-$ being ≤ 0). Hence, we have:

$$0 \leq \sum_{\mathcal{J}^+} \left[fL(J_i) - \int_{J_i} f \right] = \sum_{\mathcal{J}^+} \left| fL(J_i) - \int_{J_i} f \right| \leq \epsilon$$

and

$$-\sum_{\mathcal{J}^-}\left[fL(J_i)-\int_{J_i} f\right]=\sum_{\mathcal{J}^-}\left|fL(J_i)-\int_{J_i} f\right|\leq \epsilon$$

Therefore,

$$\sum_{i=1}^{n}\left|fL(J_i)-\int_{J_i} f\right|\leq 2\epsilon$$

2. Lemma Example

Suppose $\int_a^c f = 0 \quad \forall\ c \in (a,b]$ and we want to demonstrate that $\int_a^b |f| = 0$. Choose γ so that $|fL(\mathcal{D}) - \int_a^b f| < \epsilon$ when $\mathcal{D}$ is γ-fine. Suppose $a < c_1 < c_2 \leq b$, then:

$$\int_{c_1}^{c_2} f = \int_a^{c_2} f - \int_a^{c_1} f = 0$$

Now, since the integral of f is zero for all subintervals J_i, then

$$\left|f(\tau)L(J_i)-\int_{J_i} f\right| = \left|f(\tau)\right|\cdot L(J_i) \qquad (\forall\ J_i \in \mathcal{D})$$

As a result from Henstock's lemma we have, $|f|\cdot L(\mathcal{D}) \leq 2\epsilon$ which can be made arbitrarily small. This shows $\int_a^b |f| = 0$.

B. FUNCTION VARIATION

Recall that a function f is *absolutely integrable* if f and $|f|$ are both integrable. In the following sections, we shall give conditions for the absolute integrability of a function, and demonstrate that integrable functions need not be absolutely integrable. For such a function, one can imagine that there is some sort of cancellation taking place within the functional range of f so that the Riemann sums converge to a limit (the integral). Conversely, when the absolute value is taken, the cancellation fails to take place for $|f|$ and thus the Riemann sums diverge. The crititical point in understanding this concept is deciding whether these oscillations can be controlled.

The *variation* of $\Phi : [a,b] \to \mathcal{R}$ denoted: $Var(\Phi : [a,b])$ is defined by:

$$Var\Big(\Phi : [a,b]\Big) = \sup_P\Big\{\sum_{i=1}^{n}|\Phi(x_i)-\Phi(x_{i-1})| : \quad P = \{a = x_0 < x_1 < \ldots < x_n = b\}\ \Big\}$$

Φ is of *bounded variation* if $Var(\Phi : [a,b]) < \infty$. The set of all functions that are of bounded variation on $[a,b]$ is denoted by $BV[a,b]$. Geometrically speaking, the variation of a function is a measure of how much the function oscillates over an interval. The next example demonstrates that even a continuous function can have an infinite variation.

1. Infinite Variation Example

Let

$$f(x) = \begin{cases} 0 & \text{if } t = 0 \\ t \cdot \sin(1/t) & \text{if } 0 < t \leq 1 \end{cases}$$

Set $x_n = \frac{1}{(n-1/2)\pi}$ Then,

$$f(x_n) = \begin{cases} \frac{1}{(n+1/2)\pi} & n \text{ even} \\ \frac{-1}{(n+1/2)\pi} & n \text{ odd} \end{cases}$$

If P_n is the partition $\{0 = x_n < x_{n-1} < \ldots < x_1 = 1\}$, then

$$\sum_{i=1}^{n-1} \Big| f(x_i) - f(x_{i+1}) \Big| \geq \frac{2}{\pi} \cdot \sum_{i=1}^{n-1} \frac{1}{i+1}$$

However, $\sum_{i=1}^{\infty} \frac{1}{i+1}$ diverges. Hence, $Var(f : [0,1]) = \infty$.

Understanding the properties of bounded variation will give an insight to the necessary and sufficient condition for the absolute integrability of an integrable function.

C. INTEGRATION OF ABSOLUTE VALUES THEOREM

Let $f : I \to \mathcal{R}$ be integrable over $I = [a,b]$. Then $|f|$ is integrable over I if and only if, the indefinite integral $F(x) = \int_a^x f$ is of bounded variation over I.

In effect:

$$Var\Big(F : [a,b]\Big) = \int_a^b |f|$$

The proof here will be omitted as it contains rather lengthy and tedious arguments. [Ref.10] contains a detailed justification.

We shall next investigate a specific example of an integrable function that is not absolutely integrable. Such functions are said to be *conditionally integrable.*

1. Conditionally Integrable Example

$$f(x) = \begin{cases} x^2\cos(\pi/x^2) & 0 < x \leq 1 \\ 0 & x = 0 \end{cases}$$

with derivative:

$$f'(x) = \begin{cases} 2x\cos(\pi/x^2) + \frac{2\pi}{x}\sin(\pi/x^2) & 0 < x \leq 1 \\ 0 & x = 0 \end{cases}$$

Henceforth, f' is integrable by the FTC. However, much like the argument presented in the previous example, $|f'|$ is not integrable. To see this, let $\beta_n = 1/\sqrt{2n}$ and $\alpha_n = \sqrt{2/(4n+1)}$. then:

$$\int_{\beta_n}^{\alpha_n} f' = \frac{1}{2n}$$

The intervals $\{\ [\alpha_n, \beta_n] : n \in \mathcal{N}\}$ are pairwise disjoint. Thus, it follows that:

$$Var\Big(f : [0,1]\Big) \geq \sum_{n=1}^{N} \left| \int_{\beta_n}^{\alpha_n} f' \right| = \sum_{n=1}^{N} \frac{1}{2n}$$

for all N, and $f \notin BV[0,1]$.

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VII. CONVERGENCE THEOREMS

Some of the most use useful tools in general integration theory will be developed in this chapter. Now that absolute integrability and Henstock's Lemma have been explored, we have the foundations to start examining the convergence theorems.

Recall that one of the principal reasons that Lebesgue integration is preferred over the Riemann approach concerns the associated convergence theorems. Namely, those of the form $\lim \int_I f_k = \int_I (\lim f_k)$ hold for the Lebesgue integral under quite general conditions. The most significant theorems of this type are the Monotone Convergence Theorem (MCT) and the Dominated Convergence Theorem (DCT). We shall demonstrate that these convergence theorems hold for the gauge integral as well, thus showing that gauge integration possesses the same advantages over Riemann integration as does the Lebesgue integral.

We must first introduce the concept of *uniform integrability* which is central to understanding the following convergence theorems. We say that $\{f_k\}$ is uniformly integrable over I if each f_k is integrable over I and $\forall\ \epsilon > 0$, there is a gauge γ and γ-fine partition $\mathcal{D}$ on I such that:

$$|fL(\mathcal{D}) - \int_I f_k| < \epsilon, \qquad \forall\ k \in \mathcal{N}$$

The point of this definition is that that same gauge works uniformly for all k. For uniformly integrable sequences of integrable functions we have the following convergence theorem.

A. UNIFORMLY INTEGRABLE SEQUENCE CONVERGENCE THEOREM

Let $\{f_k\}$ be uniformly integrable over I and assume that $f_k \to f$ pointwise. Then f is integrable over I with:

$$\lim \int_I f_k = \int_I (\lim f_k) = \int_I f$$

Proof: Let $\epsilon > 0$. Let γ be a gauge on I such that $|fL(\mathcal{D}) - \int_I f_k| < \epsilon/3$ for every k when $\mathcal{D}$ is γ-fine. Choose a γ-fine tagged partition $\mathcal{E}$ of I. Pick N such that $|f_iL(\mathcal{E}) - f_jL(\mathcal{E})| < \epsilon/3$ when $i, j \geq N$ (note that this is possible by the pointwise convergence of $\{f_k\}$). We now have:

$$\left|\int_I f_i - \int_I f_j\right| \leq \left|\int_I f_i - f_iL(\mathcal{E})\right| + \left|f_iL(\mathcal{E}) - f_jL(\mathcal{E})\right| + \left|f_jL(\mathcal{E}) - \int_I f_j\right| < \epsilon$$

and thus $\lim \int_I f_k = L$ exists.

Now suppose that $\mathcal{D}$ is γ-fine tagged partition of I. Similar to the above, choose N such that $|f_NL(\mathcal{D}) - fL(\mathcal{D})| \leq \epsilon/3$ and also that $|L - \int_I f_N| < \epsilon/3$. Then:

$$\left|fL(\mathcal{D}) - L\right| \leq \left|fL(\mathcal{D}) - f_NL(\mathcal{D})\right| + \left|f_NL(\mathcal{D}) - \int_I f_N\right| + \left|\int_I f_N - L\right| < \epsilon$$

Hence, f is integrable over I with $\int_I f = L = \lim \int_I f_k$ as required.

B. UNIFORMLY INTEGRABLE SERIES CONVERGENCE THEOREM

Let $f_k, f : I \to \mathcal{R}$ be non-negative with each f_k integrable over I and suppose $f = \sum_{k=1}^{\infty} f_k$ pointwise on I, then:

(i) $\sum_{k=1}^{\infty} f_k$ is uniformly integrable over I

(ii) f is integrable over I and $\sum_{k=1}^{\infty} \int_I f_k = \int_i f = \int_i \sum_{k=1}^{\infty} f_k$

Discussion: The proof of (i) is a fairly straightforward application of Henstock's lemma that creates a dominant term over the series that, as one would expect, can be made arbitrarily small. Unfortunately, the argument introduces a significant amount of new definitions and notation that only distorts the central ideas. The interested reader should look to [Ref.11] for details. However, from (i) and the proof of the uniformly integrable sequence convergence theorem, (ii) immediately follows.

We now have the necessary machinery to state and prove the first significant convergence theorem.

C. MONOTONE CONVERGENCE THEOREM

Recall that a sequence of functions $\{f_k\}$ is *monotone* when it is increasing or decreasing. It is increasing if $f_n(x) \leq f_{n+1}(x)$ for all n and x, and this will be denoted $f_k \uparrow f$. Reversal of the inequality produces a decreasing sequence, $f_k \downarrow f$. A monotone sequence has, for every x, two options for its behavior. Either $L = \lim_{n\to\infty} f_n(x)$ is an element of $\mathcal{R}$ or is infinite. The version of the monotone convergence theorem that is forthcoming shall assume a finite limit. Further study will reveal that this assumption can be removed.

1. MCT

Let $f_k : I \to \mathcal{R}$ be integrable over $\mathcal{R}$, and suppose that $f_k(x) \uparrow f(x) \in \mathcal{R}$, $\forall x \in \mathcal{R}$. If $sup_k \int_I f_k < \infty$, then:

(i) $\{f_k\}$ is uniformly integrable over I

(ii) f is integrable over I and $\lim \int_I f_k = \int_I f = \int_I \lim f_k$

Proof: Choose $f_0 = 0$, $\;h_k = f_k - f_{k-1}$ for $k \geq 1$. Then $h_k \geq 0$, $\sum_{k=1}^{n} h_k = f_n \to f$ pointwise, and:

$$\sum_{k=1}^{\infty} \int_I h_k = \lim_{n\to\infty} \sum_{k=1}^{n} \int_I (f_k - f_{k-1}) = \lim_{n\to\infty} \int_I f_n = sup_n \int_I f_n < \infty$$

Thus, the uniformly integrable sequence convergence theorem can now be directly applied to give the desired result.

Of course, a similar result holds for decreasing sequences. The MCT gives a very useful and powerful sufficient condition by granting the interchange of integration with the sequential limit. The following example shows a typical application of monotone convergence.

2. MCT Example

Suppose f, h are nonnegative functions over I. Suppose f is integrable on I, and h is integrable over each subinterval J_n. Now, we need to demonstrate that $f \vee h$ is integrable on I. [Recall that the function $f \vee h = max\{f(x), h(x)\}$]

Begin by fixing the expanding bounded intervals J_n such that:

$$\bigcup_{n=1}^{\infty} J_n = I \cap \mathcal{R}$$

Let h_n equal h on J_n and zero elsewhere. Choose $v_n = f \vee h_n$, and now v_n is integrable on I by design, and also increases to $f \vee g$ on I. Since $\int_I v_n \leq \int_I f$, the MCT can be applied to show the integrability of $f \vee h$ on I.

Notice that the MCT allows for the interchange of integration with the sequence limit without limiting the interval I in any way. The Monotone Convergence Theorem gives further evidence of the difference between the Riemann and gauge integral. An earlier example can be used to show that the Riemann integral does not have an equivalent theorem.

3. Riemann MCT Counterexample

Let r_n be a sequential arrangement of the rationals ($\mathcal{Q}$) in $[0, 1]$ with $r_m \neq r_n$ when $m \neq n$. Let $f_n(x) = 1$ when $x = r_n$ and zero otherwise. Then $\int_0^1 f_n$ exists in the Riemann sense and has value zero.

Thus, $\sum_{n=1}^{\infty} \int_0^1 f_n$ is convergent, but $\sum_{n=1}^{\infty} f_n$ is not Riemann integrable since it is one on $\mathcal{Q}$ and zero otherwise over $[0, 1]$.

D. DOMINATED CONVERGENCE THEOREM

It should be observed that not even the gauge integral permits the interchange of integral and sequential limit.

1. DCT Example

Let $g_n(x) = n$ for $x \in (0, 1/n)$, and $g(x) = 0$ otherwise. Then $\lim_{n\to\infty} g_n(x) = 0$ for all $x \in [0,1]$. However, $\int_0^1 g_n = 1, \ \forall \ n$. As a result, $\lim_{n\to\infty} \int_0^1 g_n$ and $\int_0^1 (\lim_{n\to\infty} g_n)$ both exist, but fail to be equal.

The DCT rules out this type of behavior displayed in this example by a condition which is simple to state and easy to check for specific instances. Also note that in many applications, the condition that f_k be monotone is not satisfied. The dominated convergence theorem gives conditions sufficient for $\int_I f_k = \int_I (\lim f_k)$ to hold, but is generally easier to work with. [Ref.12]

2. DCT

Let $f_k : \ I \to \mathcal{R}$ be integrable over $I, \ \forall \ k$. Let $f, g : \ I \to \mathcal{R}$ be such that $\{f_n\}$ converges pointwise to f on I, with g being integrable over I. If $|f_k(t)| \leq g(t)$ for $k \in \mathcal{N}$ and $t \in I$, then f integrable such that:

$$\lim_{k\to\infty} \int_I f_k = \int_I (\lim_{k\to\infty} f_k) = \int_I f$$

Proof: Since $|f_k(t)| \leq g(t)$, then each f_k is absolutely integrable. Define $U_1 = sup\{f_k : \ k \in \mathcal{N}\}$. If $u_k = f_1 \vee \cdots \vee f_k$, then each u_k integrable such that $u_k \uparrow U_1$ and $\int_I u_k \leq \int_I g$. The MCT $\Rightarrow$ U_1 integrable over I. Similarly, $U_k = sup\{f_j : \ j \geq k\}$ is integrable over I. Now we have $U_k \downarrow f$ pointwise and $\int_I U_k \geq -\int_I g$, and thus the MCT $\Rightarrow$ f is integrable with $\lim_k \int_I U_k = \int_I f$.

Analogously, the same argument is made for the lower dominate term with $L_1 = inf\{f_k : \ k \in \mathcal{N}\}$ and $L_k = inf\{f_j : \ j \geq k\}$. Again, the MCT implies that f is integrable with $\lim_k \int_I L_k = \int_I f$. Thus, with $L_k \leq f_k \leq U_k$, we have:

$$\lim_{k\to\infty} \int_I L_k = \lim_{k\to\infty} \int_I U_k \qquad \Rightarrow \qquad \lim_{k\to\infty} \int_I f = \int_I f$$

as required.

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VIII. FINAL COMMENTS

While this paper concludes with the monotone and dominated convergence theorems, it is clear that there is much at the elementary level that has not been addressed. The basic convergence theorems that the gauge integral enjoys over the Riemann integral are indeed very powerful, and make for a natural stopping point for this exposition. However, it is apparent that the majority of available material after the basic convergence theorems diverges significantly in both style and content. In other words, there is little similarity in either material, or in what format the material is approached, between various authors. This is in stark contrast to the typical elementary analysis books based on the Riemann integral. For general acceptance of this theory, the mathematical community should come to a consensus on how to present this material in both approach and notation. Perhaps after this point it would be realistic to consider replacing the Riemann integral with the gauge integral for elementary analysis courses.

Additionally, this paper only discussed integrals of functions from $[a, b]$ to $\mathcal{R}$. The associated functional space created by the collection of gauge integrable functions is called Denjoy space, after Arnaud Denjoy who first explored this aspect. It is analogous, but not equal to, the Banach space $L^1[a, b]$ for Lebesgue integrable functions in $\mathcal{R}$. In fact, $L^1[a, b]$ is a subspace of Denjoy space. Recall that all Riemann integrable functions are Lebesgue integrable, and clearly all Riemann integrable functions will be in the space of improper Riemann integrable functions. However, there are functions that are integrable with the improper Riemann integral technique, but not Lebesgue integrable, and vice-versa.

All of these functional spaces are subsets of Denjoy space. This is both an advantage and disadvantage. The advantage is clear as we have discussed the integrating power of the gauge integral. With that power, however, comes a significantly larger functional space that implies some very troublesome functions can live in this space. The result is that the metric becomes more complicated to work with, as does some of the more advanced convergence theorems. In fact, Denjoy space has a natural semi-norm called the Alexiewicz norm. In contrast to the case of Lebesgue integrable functions with the L^1-norm, Denjoy space is not complete under the Alexiewicz norm. Thus, the real advantage of the gauge integral isn't really its expanded functional space, but rather the insight it gives through its simplicity.

In his analysis book, Eric Schechter draws the analogy between these integrals and series. He compares gauge integrals to convergent series, and the Lebesgue integral to absolutely convergent series. The absolutely convergent series are much easier to work with, and provide a clean, consistent theory. Series that are just convergent are more general and thus more complex. However, it is a rare event to have to work with a series that is conditionally convergent. [Ref.3]

As mentioned in the opening remarks of Chapter 2, the theory of gauge integration can be easily extended to complex functions, or functions in $\mathcal{R}^n$. The theory can also be extended to non-compact intervals without great difficulty or loss of simplicity. However, gauge theory has also been extended to infinite dimensional spaces and even more abstract spaces, but at the cost of losing its intuitive advantage. The theory quickly becomes quite abstract and difficult to follow. Ralph Henstock has written a number of documents in this area, but it doesn't appear that there is much current research in this direction.

As a note on a type of function that is not gauge integrable, it would be nice to have a bounded, non-integrable function on a finite domain to explore. This would allow the reader to really grasp the meaning of "non-integrable" outside the obvious examples concerning unbounded functions or unbounded domains. In fact, these types of functions exist, but are very difficult to demonstrate and are somewhat less than satisfying. Actually, for bounded functions on a finite interval, the Lebesgue and gauge integrals are equivalent. Hence, the same kind of non-integrable functions will exist for both. Recall the example provided by Vitali that proves the existence of a non-Lebesgue measurable set. [Ref.8] The proof is centered about the Axiom of Choice, and is non-constructive in nature. Thus, the existence is proved without actually finding the function, or even giving specific criterion. So for most students just being introduced to real analysis, the example of such a function will have to wait until a later course.

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APPENDIX. A BRIEF HISTORY OF THE GAUGE INTEGRAL

The history of integration theory is both quite extensive and remarkable. A sizeable treatise could easily be devoted just to the non-mathematical portions of it. Here only a few remarks are made in order to add chronological perspective to the topic, and to add emphasis to important achievements.

The roots of integration can clearly be traced to Archimedes' "The Method." This work, while stunning in its achievement, is less important towards the development of the integral as it was lost for so many centuries. The real story of integration starts with Newton and Leibniz. Even today, if $F : [a, b] \to \mathcal{R}$ and $F'(x) = f(x)$ for every $x \in [a, b]$ we say that $F(b) - F(a)$ is the definite integral of f from a to b, in symbols:

$$F(b) - F(a) = \int_a^b f$$

We also refer to the function F as the Newton indefinite integral of f.

By today's standards, the Newtonian definition looks much more concrete than the Leibniz definition consisting of an integral as a sum of infinitely many infinitesimal quantities. This is because the concept of derivative is firmly fixed in our thinking as a well defined mathematical concept. During Newton's era, however, the concepts of derivative and limit were somewhat vague. Despite the logical inconsistencies associated with the development of integral calculus, the founding masters of these new calculus techniques were able to make wonderful discoveries with the newly formed tools. The Bernoulli brothers and Euler made the most significant discoveries during this period.

Out of all the many definitions that survived modern analytical scrutiny, by far the simplest and most intuitive is that which was given at the beginning of the modern era by Cauchy (1789-1857) and completed and fully investigated by Riemann

(1826-1866). As mentioned in the introduction, it is still the Riemann theory that is taught today at universities to mathematicians, physicists and engineers alike.

Among non-mathematicians, there is an almost universal identification of the integral as the *Riemann integral* with little or no concept of differing methods of integration. This can generally be identified with no advancement in mathematical analysis beyond the introductory courses. Regardless, this identification is somewhat surprising for two reasons. Firstly the Riemann integral, despite its wide use and its intuitive appeal, has serious shortcomings that have been identified for well over a century. Secondly, over eighty years ago Henri Lebesgue (1875-1941) gave another definition of what is now known as the Lebesgue integral. As discussed in this paper, his integral turns out to be the correct one for almost all uses and is the one currently used (almost exclusively) by professional mathematicians.

Aside from the complexities in developing the Lebesgue method of integration mentioned in the introduction, both the Lebesgue and Riemann definitions require the assumption that the derivative F' be integrable to obtain the basic formula $\int_a^b F' = F(b) - F(a)$. This encouraged mathematicians in the early 1900s to seek a more general fundamental theorem (such as the gauge FTC). In 1914 Oskar Perron proposed yet another definition that had certain advantages over the Lebesgue definition. Namely that it had the more general fundamental theorem, and included the Newton integral and all improper integrals as well. Note that the Perron integral has since been proven to be equivalent to the gauge integral. Perron was joined by Arnaud Denjoy, and the two further developed this more general theory. However, while their theory solved some lingering problems, it further exasperated the complexity of understanding it.

In 1957, the Czech mathematician Jaroslav Kurzweil, in connection with research in differential equations, gave an elementary definition of the integral equivalent to the one given by Perron. In 1961, Ralph Henstock independently rediscovered Kurzweil's approach and advanced it further. Henstock quickly recognized that the most significant repercussion of gauge theory is that it preserves the intuitive geometrical background of the Riemann theory, and yet it has the integrating advantages of the Lebesgue theory.

E. J. McShane made a further essential contribution when he noticed that a simple alteration in the definition of the gauge integral produces exactly the Lebesgue integral. Thus, he recaptured Lebesgue integration in the Kurzweil-Henstock framework and by doing so made it accessible to non-specialists. In our definition of a tagged partition, $\mathcal{D}: \quad a = x_0 < x_1 < x_2 < \cdots < x_n = b; \quad \tau_i \in [x_{i-1}, x_i]$. McShane simply drops the requirement that the tag τ_i must belong to the subinterval $[x_{i-1}, x_i]$. McShane also was the first to make a serious attempt in having the gauge integral be the primary integral for undergraduate courses in real analysis. He claimed the theory is so simple that it can be presented in introductory courses, and wrote a text suitable for such purposes.

The field of gauge integration now has many mathematicians working in many directions. This paper addresses mostly the pedagogical advances for developing the fundamental ideas of real analysis, and the majority of recent works suitable for this type of study are in the list of references. However, there are many working on the cutting edge of analysis research in this field. The interested reader should explore the periodical, *Real Analysis Exchange*, for the most up-to-date research level articles.

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