THOMISTS AND THOMAS AQUINAS ON THE FOUNDATION OF MATHEMATICS

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I

SOME MODERN THOMISTS, claiming to follow the lead of Thomas Aquinas, hold that the objects of the types of mathematics known in the thirteenth century, such as the arithmetic of whole numbers and Euclidean geometry, are real entities. In scholastic terms they are not beings of reason (*entia rationis*) but real beings (*entia realia*). In his once-popular scholastic manual, *Elementa Philosophiae Aristotelico-Thomisticae*, Joseph Gredt maintains that, according to Aristotle and Thomas Aquinas, the object of mathematics is real quantity, either discrete quantity in arithmetic or continuous quantity in geometry. The mathematician considers the essence of quantity in abstraction from its relation to real existence in bodily substance. "When quantity is considered in this way," he writes, "it is not a being of reason (*ens rationis*) but a real being (*ens reale*). Nevertheless it is so abstractly considered that it leaves out of account both real and conceptual existence." Recent mathematicians, Gredt continues, extend their speculation to fictitious quantity, which has conceptual but not real being; for example, the fourth dimension, which by its essence positively excludes a relation to real existence. According to Gredt this is a special, transcendental mathematics essentially distinct from "real mathematics," and belonging to it only by reduction.[1]

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[1] "Obiectum Matheseos est quantitas realis ita tamen secundum quidditatem suam abstracte et inadaequate considerata, ut non dicat ordinem ad esse reale in substantia corporea seu in ente mobili. . . . Quantitas ita considerata non est quidem ens rationis, sed ens reale, tamen ita abstracte consideratur, ut abstrahat etiam ab esse reali et esse rationis. Recentes mathematici speculationem mathematicam usque ad quantitatem fictam extendunt, quae non est ens reale, sed rationis tantum, ut est quarta dimensio, quae secundum essentiam suam positive excludit ordinem ad esse reale. Ita constituitur Mathesis quaedam specialis, quae vocatur Mathesis transcendentalis et quae a Mathesi reali essentialiter distinguitur neque ad eam pertinet nisi reductive"; Joseph Gredt, *Elementa Philosophiae Aristotelico-Thomisticae*, 5th ed. (Freiburg: Herder, 1929) 1194.

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Jacques Maritain read the works of Gredt, including his *Elementa*, and in his magistral *Degrees of Knowledge* he agrees with Gredt that at least the objects of Euclidean geometry and the arithmetic of whole numbers are *entia realia* in distinction to the objects of modern types of mathematics, which he calls *entia rationis*. The objects of the former types of mathematics, Maritain says, are real in the philosophical sense that they can exist outside the mind in the physical world, whereas the objects of the newer types of mathematics cannot so exist. A point, a line, and a whole number are real beings, but not irrational numbers or the constructions of non-Euclidean geometries.[2]

Delving more deeply than Gredt into the nature of mathematics, Maritain stresses that when the mathematician conceives his objects they acquire an ideal purity in his mind which they lack in their real existence. By abstracting these entities from the sensible world the intellect idealizes them in such a way that not only their mode of being but their very definition is affected. There are no points, lines, or whole numbers in the real world with the conditions proper to mathematical abstraction.[3] Maritain also describes the purification or idealization of quantity in mathematics as a construction or reconstruction. He writes in his *Preface to Metaphysics*, "Quantity [in mathematics] is not now studied as a real accident of corporeal substance, but as the common material of entities reconstructed or constructed by the reason. Nevertheless even when thus idealized it remains something corporeal, continues to bear in itself witness of the matter whence it is derived."[4] He makes the same point in the *Degrees of Knowledge*: "In . . . mathematical knowledge, the mind grasps entities it has drawn from sensible data or which it has built on them. It grasps them through their constitutive elements, and constructs or reconstructs them on the same level. These things in the real [world] (when they are *entia realia*) are accidents or properties of bodies, but the mind treats them *as*

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[2] Jacques Maritain, *Distinguish to Unite; or, The Degrees of Knowledge*, trans. under the supervision of Gerald B. Phelan (London: Geoffrey Bles, 1959), 145–6. Maritain refers to Gredt's *Elementa* on p. 460.

[3] Ibid., 166.

[4] Jacques Maritain, *A Preface to Metaphysics* (London: Sheed and Ward, 1943), 82.

though they were subsistent beings and *as though* the notion it makes of them were free of any experimental origin."[5]

In *The Philosophy of Mathematics* Edward Maziarz sees the essence of the scientific habit of mathematical abstraction as the mind's becoming "conformable and identifiable to the nature of substance solely as quantified."[6] He hastens to assure us that the objects of mathematics are not completely discovered in nature, as the scientist discovers the properties of the elements, but neither are they pure products of the mind. The mind, he says, discovers quantified substance in nature and "inducts" mathematical entities with the help of the imagination. It is only in the imagination and not in extramental reality that the objects of mathematics as such exist. Thus the universe of the mathematician is one of imaginative construction and invention with the materials of experience.[7]

Does this mean that these objects are purely fictitious? Not at all according to Maziarz. Mathematics is not an innate or sheerly self-creative affair, as the idealist would have it. Experience furnishes at least the point of departure for mathematics. For this reason Maziarz does not regard a mathematical object as a logical entity or *ens rationis* but as an *ens naturae.* Referring to Maritain's *The Degrees of Knowledge*, he calls these entities *entia realia.* He goes beyond Maritain, however, in claiming that the objects of modern mathematics are also real entities.[8]

Taking up a suggestion of John of St. Thomas, Yves Simon, a distinguished pupil of Maritain, places the objects of mathematics somewhere between real beings and beings of reason. On the one hand, he asserts that mathematics is not an ontology of real quantity. On the other hand, he is equally insistent that mathematical entities are not beings of reason, for this would imply that none can have physical counterparts. On the contrary the square root of sixteen

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[5] Maritain, *The Degrees of Knowledge*, 32. In *The Range of Reason*, Maritain writes of science using as its instruments "explanatory symbols which are ideal entities (*entia rationis*) *founded on reality*, above all mathematical entities built on the observations and measurements collected by the senses"; Jacques Maritain, *The Range of Reason*, (London: Geoffery Bles, 1953), 32. See also p. 36.

[6] Edward Maziarz, *The Philosophy of Mathematics* (New York: Philosophical Library, 1950), 194. On pp. 208–209, however, the mathematical nature is said to be "treated as if it were a substantial thing."

[7] Ibid., 240.

[8] Ibid., 208. See also p. 227 n. 125.

does have such a counterpart, though the square root of minus one does not. All mathematical objects, however, have an element of unreality, which Simon calls a "condition of reason"; for they abstract from sensible qualities and primary matter, which makes it impossible for them, as mathematical, to exist outside the mind. They are said enigmatically to exist "beside the world of reality."[9]

Simon quotes John of St. Thomas's statement that mathematical quantity admits of real and true being; indeed, it does not exclude the reality of quantity itself. In this respect it differs from purely imaginary quantity, which is a being of reason. "Mathematical quantity," says John of St. Thomas, "is not determinately a being of reason; neither is it determinately a real being, but it is indifferent to either condition and admits of either of them."[10]

Some Thomists interpret Aquinas as teaching that the objects of mathematics not only have counterparts in the real world but are in fact the real quantities of bodies abstracted from them. Thomas Anderson writes that the mathematician studies real quantities, though not precisely as real. Like the philosopher of nature he treats of physical quantities, but unlike the philosopher of nature he does so in abstraction from the sensible qualities and motions of natural things. Furthermore, since quantity is an accident of substance and is only intelligible as such, quantity must enter into the object of mathematics.[11] Though Anderson states that mathematics treats of real quantities, he grants with John of St. Thomas that "mathematical entities, as Thomas describes them, are sort of intermediate beings, i.e. they are neither purely beings of reason nor real beings, but share features of both."[12]

That mathematics—more exactly geometry—is a science of the real world is the theme of Vincent Smith's Aquinas Lecture, *St. Thomas*

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[9] Yves Simon, "Nature and the Process of Mathematical Abstraction," *The Thomist* 29 (1965): 135 n. 23. Commenting on Maritain's philosophy of mathematics, Simon describes Maritain's conception of a mathematical object as "a *preter-real* entity always affected by some *conditio rationis* and which often turns out to be a mere *ens rationis* with a foundation in the real"; Yves Simon, "Maritain's Philosophy of the Sciences," in *The Philosophy of Physics*, ed. Vincent E. Smith (Jamaica: St. John's University Press, 1961), 36.

[10] Simon, "Nature and the Process," 135–6 n. 23. See also John of St. Thomas, *Cursus theologicus*, disp. 6, a. 2 (Paris: Desclée, 1931) 1:534 n. 20.

[11] Thomas C. Anderson, "Intelligible Matter and the Objects of Mathematics in Aquinas," *The New Scholasticism* 43 (1969): 555–76.

[12] Ibid., 559 n. 13.

on the Object of Geometry. How can it be called a science, he asks, if it is not a study of the real world?[13] In fact, geometry studies substance along with its dimensions of quantity, and this is something real. Its objects, he writes, "exist in the sensible world although they are not thought of mathematically as bearing this existence."[14] He concludes his lecture with the statement, "Euclidean geometry is the science of what is real but not physical, imaginable but not sensible, truly essential but not natural and mobile."[15] In his *Science and Philosophy* he presents a similar realistic view of mathematics. Contrary to Bertrand Russell's conception of mathematics as an affair of logic and David Hilbert's view of mathematics as a linguistic or symbolic system, he insists that the subject of mathematics is quantified being—something that exists in matter though it is not thought of as so existing.[16]

From this brief survey of some of the leading Thomistic interpretations of Aquinas's conception of mathematics it is readily apparent that there are significant differences among them. The Thomists generally agree that in Aquinas's view the objects of mathematics are not purely mental beings or constructs but forms of quantity, either continuous or discrete, such as circles, triangles, and numbers. The mind abstracts these forms from sensible matter. There is general consensus that these objects are real, for they exist in the external world. Most qualify this by adding that there is also an element of unreality in the object of mathematics: they are reconstructed in the mind, or they do not exist in reality as mathematical objects, or they are subject to a "condition of reason." It is not clear, however, how the reality and unreality of these objects can be reconciled.

Some Thomists contend that, since the object of mathematics is real quantity, it cannot be abstracted from substance, on which it depends for its being and intelligibility. Others see no place for substance, in the categorical sense, in the object, for this would imply that

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[13] Vincent E. Smith, *St. Thomas on the Object of Geometry* (Milwaukee: Marquette University Press, 1954), 65.

[14] Ibid., 66.

[15] Ibid., 84.

[16] Vincent E. Smith, *Science and Philosophy* (Milwaukee: Bruce, 1965), 219. See also Vincent E. Smith, *Philosophical Physics* (New York: Harper and Row, 1950), 11–12.

the object is a subsistent entity.[17] It will be recalled that Maritain holds that the mathematician treats of quantity *as though* it were a subsistent being.[18]

II

Both the Thomists who insist on the reality of the object of mathematics and those who regard that object as containing an element of unreality, such as a reconstruction by the mind or a "condition of reason," find support for their interpretations in the works of Aquinas. In his commentary on the *De Trinitate* of Boethius (1258-59; his earliest and most extended treatment of the subject), he describes pure mathematics as the science of abstract quantity and its properties, such as the circle and triangle.[19] Geometry considers magnitude and arithmetic number.[20] Mathematical objects, such as lines and numbers, are said to depend on sensible matter for their existence but not for our knowledge of them, for they do not include this sort of matter in their definitions.[21] Following Boethius, Aquinas distinguishes between the theoretical sciences of physics, mathematics, and divine science (namely, metaphysics) by the kind of form which they study. Mathematics is said to concern the form of quantity, and its mode of abstraction is called the abstraction of form (*abstractio formae*).[22] Quantity is understood as the first accidental form inhering in substance. Since quantity depends on substance for its existence and intelligibility, the mathematician cannot abstract it from substance but only from the sensible qualities and movements of bodies.[23]

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[17] For Aquinas's notion of substance see *Scriptum super libros Sententiarum* 2, d. 3, q. 1, a. 6, ed. P. Mandonnet (Paris: Lethielleux, 1929), 2:103. See also Etienne Gilson, "Quasi Definitio Substantiae," in *St. Thomas Aquinas 1274–1974: Commemorative Studies*, ed. Armand Maurer (Toronto: Pontifical Institute of Mediaeval Studies, 1974), 1:111–29.

[18] See note 5 above.

[19] *Expositio super librum Boethii De Trinitate* (hereafter *EBT*), q. 5, a. 3, ed. Bruno Decker (Leiden: E. J. Brill, 1955), 184.20–22; *The Division and Methods of the Sciences*, trans. Armand Maurer, 4th ed. (Toronto: Pontifical Institute of Mediaeval Studies, 1986), 38–9.

[20] *EBT*, q. 5, a. 3, ad 6, 188.25; *Division and Methods*, 44.

[21] *EBT*, q. 5, a. 1, 165.21–24; *Division and Methods*, 14.

[22] *EBT*, q. 5, a. 3, 186.16–18; *Division and Methods*, 41.

[23] *EBT*, q. 5, a. 3, 184.2–186.12; *Division and Methods*, 37–40.

The presence of substance in the object of mathematics is also required as the intelligible matter of quantity. As Aristotle maintains, there must be some matter wherever there is form instantiated in several individuals.[24] Since there are many mathematical circles, triangles, and numbers, there must be a material principle of individuation in mathematics, not perceptible to the senses but known by the intellect. This principle, according to Aquinas, is substance.[25] The notion of substance as the intelligible matter of quantity also appears in Aquinas's *Summa theologiae* and his commentary on Aristotle's *Physics*.[26] It should be noticed, however, that in his commentaries on the *Metaphysics* and *De anima* it is not substance but the continuum, surface, or body that is said to be the intelligible matter of mathematical forms.[27] This conforms to Aristotle's use of the term; for example, he calls the generic notion of plane figure the intelligible matter of the circle.[28]

This description of the object of mathematics in terms of the form of quantity and its underlying substance easily lends itself to a realist interpretation of mathematics. As the physicist treats of material substance with its sensible qualities and actions, so it would seem the mathematician takes for his domain substance with quantity, abstracting from the sensible qualities and active and passive properties of bodies. The object of mathematics would appear to be a real substantial entity characterized only by the form of quantity.

Aquinas's commentaries on Aristotle's *Metaphysics* and *Physics* also lend themselves to a realist point of view in mathematics. In the former commentary the forms of figures, such as circles or triangles, are said to exist in the continuum or body as the form of human nature

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[24] Aristotle, *Metaphysics* 7.10.1036a9–12, 7.11.1037a4–5. For Aristotle's notion of intelligible matter see Hippocrates G. Apostle, *Aristotle's Philosophy of Mathematics* (Chicago: University of Chicago Press, 1952), 50–2, 106; and Joseph Owens, *The Doctrine of Being in the Aristotelian Metaphysics*, 3d ed. (Toronto: Pontifical Institute of Mediaeval Studies, 1978), 342–3.

[25] *EBT*, q. 5, a. 3, 184.16–20; q. 5, a. 3, ad 2, 187.2–13; *Division and Methods*, 38, 42.

[26] *Summa Theologiae* (hereafter *ST*) I, q. 85, a. 1, ad 2; *Sentencia super Physicam* 2, lect. 3, n. 332, ed. Angeli M. Pirotta (Naples: M. d'Auria Pontificus, 1953).

[27] *Sententia super Metaphysicam* 7, lect. 11, n. 1508, ed. M.-R. Cathala and R. M. Spiazzi (Turin: Marietti, 1950). See also lect. 10, n. 1496; and *Sententia super De anima* 3, lect. 8, n. 714, ed. Pirotta (Turin: Marietti, 1938).

[28] *Metaphysics* 8.6.1045a33–35.

exists in the organic body.[29] In the latter commentary physics and mathematics are said to have the same objects of inquiry—for example, points, lines, and surfaces—but they consider them in different ways. Physics, he says, studies them as the termini of natural bodies, whereas mathematics studies them in abstraction from these bodies and their motions.[30] Statements such as these would lead to the conclusion that for Aquinas the objects of mathematics, like those of physics, are found in the real world; in short, that they are *entia realia*.

Along with these statements we find others in the same works of Aquinas implying that the objects of mathematics are abstracted in such a way that, as Maritain says, they are affected not only in their mode of existing but in their very definition.[31] The fact that a mathematical line has only one dimension—length without breadth—gives it properties different from those of a real or natural line. Aquinas recognizes that, although lines and circles exist in the real world, they are not of the same sort as those studied by mathematics.[32] He points out that Aristotle was aware of this, for he realized that in geometry a straight line and a circle have properties that do not belong to real circles and lines. For example, in Euclid's geometry a straight line touches a circle at only one point, but this is not true of a real circle and straight line.[33]

Aquinas clearly differentiates between properties of the real world and those of mathematical objects when he explains how Christ could have entered his disciples' room while the doors were shut. This seems to be contrary to the principles of geometry, for two mathematical straight lines cannot coincide but must differ in place, each having its own starting point and end point. When Christ passed through the closed door, at that moment two bodies occupied the same

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[29] *In VII Metaph.*, lect. 10, n. 1496.

[30] *In II Phys.*, lect. 3, n. 329.

[31] See note 3 above. Anderson lists texts of Aquinas and works on Aquinas that "say in effect that mathematicals exist with their specific characteristics only in the mind of the mathematician."; Anderson "Intelligible Matter and the Objects of Mathematics in Aquinas," 558 n. 10.

[32] *In III Metaph.*, lect. 7, n. 416; 11, lect. l, n. 2161. The statement is in an objection, but Aquinas does not deny it. He only denies Plato's position that there are separately existing objects of mathematics.

[33] *EBT*, q. 6, a. 2, 216.20–26; *Division and Methods*, 78. See Aristotle, *De anima* 1.1.403a12–16; *Metaphysics* 3.2.997b35–998a4. The tangency of a straight line to a circle at only one point is demonstrated by Euclid, *Elements* 3, prop. 15, 16, as Aquinas points out; *In III Metaph.*, lect. 7, n. 416.

place, and two lines of these bodies terminated at only two points, and each line at the same two points. Since this contradicts the notion of a mathematical straight line, God would have done what is mathematically impossible.

In replying, Aquinas distinguishes between mathematical and natural lines. The former must be distinct in place, so that two lines of this sort cannot be thought of as coinciding. Two natural lines, on the contrary, are distinct in the bodies in which they exist. Now, if we assume that two bodies exist in the same place, as in the miracle of Christ's entering the room with the doors shut, it follows that two lines, two points, and two surfaces, coincide. This could only happen by a miracle, but the miracle did not violate the principles of mathematics.[34] This shows that Aquinas clearly conceives the objects of mathematics as having properties quite different from those of nature or reality.

This is true not only in geometry but also in arithmetic. Aquinas distinguishes between a multitude that is numbered or numerable and the number by which we number it. In a sense things numbered or numerable can be called a number, as we speak of ten men or horses. Number itself, however, is that by which they are numbered. This numeration or counting is an act of the human mind. The existence of the multitude is due to the divine mind; its numeration is owing to ours.[35]

Numbers themselves originate through an act of our mind. For Aquinas, neither zero nor one is a number; one is the starting point of what today are called natural numbers.[36] Each natural number is an aggregate of ones, produced by adding one to its immediate predecessor; for example, four is produced by adding one to three. In other words, each number is caused by taking one several times.[37] Number is also said to be caused by the division of the continuum, but this seems to be number in the sense of a numbered or numerable multitude.[38] Number itself, while presupposing this multitude, depends on

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[34] *De potentia*, q. 1, a. 3, *sed contra* 8 and reply, ed. Paul M. Pession (Turin: Marietti, 1953). See also *In IV Sent.*, q. 2, a. 2, sol. 3, ad 2 (Paris: Vivès, 1874), 11:325.

[35] *In IV Phys.*, lect. 23, n. 1209; lect. 17, n. 1113; and *In VII Metaph.*, lect. 3, n. 1722.

[36] *ST* I, q. 11, a. 1, ad 1.

[37] *In V Metaph.*, lect. 17, n. 1020.

[38] *De potentia*, q. 1, a. 16, ad 3.

an act of the mind. Incidentally, Aquinas would not agree with Kronecker's saying that God made the whole numbers but all the others are the work of humans. In fact we make all of them.

From this it seems clear that, in Aquinas's opinion, the objects of mathematics are not simply abstracted from the real world but owe their existence to a constructive or reconstructive activity of the mind.

It is not immediately clear, however, how this view of mathematics can be reconciled with the statements of Aquinas cited above, which seem to imply a realist interpretation of mathematics. If the objects of mathematics have properties different from those of nature or reality, and if they differ from it even in their definitions, how can they be said in any sense to exist in the real world or be abstracted from it?

In a neglected *quaestio disputata* Aquinas approaches the object of mathematics in a new and original way which offers a possible solution to these problems. Composed at Rome between 1265 and 1267, Aquinas regarded the *quaestio* as so important that he inserted it into his commentary on the *Sentences*, which he had written a decade earlier (1252-1256).[39] Thus the *quaestio* dates from his mature years, when he was a master in theology, unlike the rest of the commentary, which he wrote as a bachelor of the *Sentences*.[40] It cannot be dismissed, therefore, as an early expression of Aquinas's teaching on mathematics.

In the *quaestio* Aquinas inquires into the distinction between divine attributes, like goodness and wisdom, and their possible foundation in God. We are not concerned here with his reconciliation of the plurality of these attributes with the absolute oneness of God.

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[39] It is so important, Aquinas says, that practically nothing in book 1 of the *Sentences* can be understood without it. See *In I Sent.*, d. 2, q. 1, a. 3, sol., ed. Mandonnet, 1:66.

[40] Ibid., d. 2, q. 1, a. 3; ed. Mandonnet 1:63–72. On the origin and authenticity of this *quaestio* see Antoine Dondaine, "Saint Thomas et la dispute des attributs divins (I *Sent.* d. 2, a. 3): authenticité et origine," *Archivum Fratrum Praedicatorum* 8 (1938): 253–62. For an analysis of the notion of mathematics in this *quaestio* see my article, "A Neglected Thomistic Text on the Foundation of Mathematics," *Mediaeval Studies* 21 (1959): 185–92; reprinted in Armand Maurer, *Being and Knowing: Studies in Thomas Aquinas and Later Medieval Philosophers* (Toronto: Pontifical Institute of Mediaeval Studies, 1990), 33–41. The *quaestio disputata* has been a part of Aquinas's commentary on the *Sentences* since his own day, but for the sake of convenience I shall refer to it simply as the *quaestio*.

What is of importance for our subject is his threefold distinction between the ways concepts are related to extramental reality. (1) A concept may be a likeness of a reality outside the mind, for example "man." A concept of this sort has an immediate foundation in extramental reality: the reality causes the truth of the concept through the conformity of mind and reality, and the term signifying the concept is properly predicated of the reality. (2) A concept may not be a likeness of an extramental reality, but the mind may devise (*adinvenit*) it as a consequence of our way of knowing extramental reality. A concept of this sort has only a remote foundation in reality; its immediate basis is an activity of the mind itself. An example is the concept of genus. There is nothing outside the mind corresponding to this concept, but from the fact that we know there are many species of animals we attribute to animal the notion of genus. The proximate foundation of a concept of this sort is a constructive act of the mind; but the concept has a remote basis in the extramental world, so the mind is not mistaken in forming it. Another example of this type of concept suggested by Aquinas is the abstraction of the mathematicians, or the abstraction of mathematics (*abstractio mathematicorum*). He is not referring to the mathematician's act of abstracting but to the mathematical concept or *intentio* he forms by means of this act. But more about this later. (3) A concept may have neither a proximate nor a remote foundation in reality, like the concept of a chimera. This is not a likeness of anything in the world, nor do we form it as a consequence of our way of knowing the world. Hence Aquinas calls it a false concept.[41]

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[41] "Unde sciendum, quod ipsa conceptio intellectus tripliciter se habet ad rem quae est extra animam. Aliquando enim hoc quod intellectus concipit, est similitudo rei existentis extra animam, sicut hoc quod concipitur de hoc nomine 'homo'; et talis conceptio intellectus habet fundamentum in re immediate, inquantum res ipsa, ex sua conformitate ad intellectum, facit quod intellectus sit verus, et quod nomen significans illum intellectum proprie de re dicatur. Aliquando autem hoc quod significat nomen non est similitudo rei existentis extra animam, sed est aliquid quod consequitur ex modo intelligendi rem quae est extra animam; et hujusmodi sunt intentiones quas intellectus noster adinvenit; sicut significatum hujus nominis 'genus' non est similitudo alicujus rei extra animam existentis; sed ex hoc quod intellectus intelligit animal ut in pluribus speciebus, attribuit ei intentionem generis[;] et hujusmodi intentionis licet proximum fundamentum non sit in re, sed in intellectu, tamen remotum fundamentum est res ipsa. Unde intellectus non est falsus, qui has intentiones adinvenit. Et simile est de omnibus aliis qui consequuntur ex modo intelligendi, sicut est abstractio mathematicorum et

When Aquinas first commented on the *Sentences* a decade before he wrote the *quaestio* we have been following, he made a similar distinction between three orders of "things signified by names": (1) those having a complete being in extramental reality, like man and stone; (2) those having no being at all in reality outside the mind, like dreams and the image of a chimera; (3) those having a foundation in extramental reality but whose notions are completed by an act of the mind. Aquinas gives as an example of the latter type of concept a universal, such as a species. Humanity, for instance, is something real, but outside the mind it does not have the nature of a universal, for there is no common humanity in the external world. When the mind forms the notion of humanity, however, it acts upon it and adds to it the meaning (*intentio*) of a species. The same is true of time, continues Aquinas. It has a basis in the before and after of motion, but its formal character as time, namely the numbering of the before and after, is completed by an act of the mind. Truth also has a foundation in reality, primarily the being (*esse*) of things, but its notion (*ratio*) is completed by an act of the mind, when it knows things such as they are.[42]

Aquinas here distinguishes between the same three orders of concepts that he described in the *quaestio* on the divine attributes.

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hujusmodi. Aliquando vero id quod significatur per nomen, non habet fundamentum in re, neque proximum, neque remotum, sicut conceptio chimerae: quia neque est similitudo alicujus rei extra animam, neque consequitur ex modo intelligendi rem aliquam vere: et ideo ista conceptio est falsa"; *In I Sent.*, d. 2, q. 1, a. 3; ed. Mandonnet 1:67. For an analysis of this text from the viewpoint of logic see Robert W. Schmidt, *The Domain of Logic according to Saint Thomas Aquinas* (The Hague: Martinus Nijhoff, 1966), 85–9.

[42] "Respondeo dicendum, quod eorum quae significantur nominibus, invenitur triplex diversitas. Quaedam enim sunt quae secundum esse totum completum sunt extra animam; et hujusmodi sunt entia completa, sicut homo et lapis. Quaedam autem sunt quae nihil habent extra animam, sicut somnia et imaginatio chimerae. Quaedam autem sunt quae habent fundamentum in re extra animam, sed complementum rationis eorum quantum ad id quod est formale, est per operationem animae, ut patet in universali. Humanitas enim est aliquid in re, non tamen ibi habet rationem universalis, cum non sit extra animam aliqua humanitas multis communis; sed secundum quod accipitur in intellectu, adjungitur ei per operationem intellectus intentio, secundum quam dicitur species: et similiter est de tempore, quod habet fundamentum in motu, scilicet prius et posterius ipsius motus; sed quantum ad id quod est formale in tempore, scilicet numeratio, completur per operationem intellectus numerantis. Similiter dico de veritate, quod habet fundamentum in re, sed ratio ejus completur per actionem intellectus, quando scilicet apprehenditur eo modo quo est"; *In I Sent.*, d. 19, q. 5, a. 1; ed. Mandonnet, 1:486. See also *De potentia Dei* q. 7, a. 6; Schmidt, *The Domain of Logic*, 82–5.

Unlike the *quaestio*, this passage does not mention mathematical concepts as an example of notions with a foundation in reality but completed by an act of the mind. They readily fit into this category, however, as does the logical notion of species or genus.

The language of the two passages is somewhat different. The notions in this category are here said to be "completed" by an act of the mind in order to achieve their formal character. In the *quaestio* an act of the mind "devises" (*adinvenit*) them on a real foundation. It would seem, however, to be the same mental act that does the devising and completing. This act results in an *ens rationis*, not an *ens reale*. Aquinas regularly describes an *ens rationis* as a being that the mind does not discover in reality but devises (*adinvenit*) as a consequence of knowing reality.[43] He uses this term of logical notions and others, for example, the craftsman's idea of what he intends to make.[44] Among them he places the objects of mathematics, as mental elaborations remotely based on real quantity but proximately on the mind's constructive activity.

If this be true, for Aquinas the mode of abstraction in mathematics is only analogous to the modes employed in the other sciences. Like abstraction in general, it is a way of knowing in which the mind considers one aspect of a thing, leaving out of consideration other aspects of the same thing.[45] Specifically, it considers the quantity of bodies, abstracting from their sensible qualities and motions. But the abstraction is constructive or completive as well as selective. The mind must add to the real foundation of the mathematical notion and complete its formal character, as it does with the notions of species, universal, time, and truth. None of these, Aquinas has told us, enjoys a complete being in extramental reality. They have a foundation there, but their formal character comes from the mind. The same would seem to be true of mathematical notions.

In this new notion of mathematical objects real substance does not play an intrinsic role as the intelligible matter of quantity. Intelligible

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[43] "Ens autem rationis dicitur proprie de illis intentionibus, quas ratio adinvenit in rebus consideratis, sicut intentio generis, speciei et similium, quae quidem non inveniuntur in rerum natura, sed considerationem rationis consequuntur"; *In IV Metaph.*, lect. 4, n. 574. It should be noted that here Aquinas asserts that beings of reason are properly the subject of logic. For the meaning of *ens rationis* see Schmidt, *The Domain of Logic*, 75–93.

[44] *Quodlibet*, q. 8, a. 1, ed. R. Spiazzi (Turin: Marietti, 1949), 162.

[45] *EBT*, q. 5, a. 3, 183.26–184.3; *Division and Methods*, 37.

matter is within the order of quantity itself, where it is placed by Aristotle and sometimes by Aquinas himself when commenting on his works.[46] The mathematical object is still essentially related to substance, however, for sensible substance is the remote reality from which it is abstracted.

There are important consequences of Aquinas's placing the notions of mathematics in the second order of his *quaestio disputata* instead of the first. Unlike concepts on the first level, those on the second do not properly speaking exist outside the mind. Their proper subject of existence is the mind itself. They are not signs of anything in the external world. Hence mathematical terms cannot properly be predicated of anything real: there is no referent in the external world for a mathematical line, circle, or number. Finally, mathematical notions are not false; but neither are they said to be true, in that they conform to anything outside the mind. Aquinas does not suggest that they might be true in some other sense.

The originality of this notion of mathematics should not go unnoticed. It seems to have had few, if any, precedents in the Middle Ages, and to the best of my knowledge none that placed logic and mathematics in the same order in relation to the real world.[47] It has the important advantage of enabling the Thomist to reconcile the real and unreal features of the mathematical object: it is real in its remote foundation in the sensible world, but it is unreal in the construction or completion the minds adds to it through its act of abstraction.

III

Not long after the death of Aquinas his views on mathematics in the *quaestio disputata* were challenged by Giles of Rome (1247–1316),

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[46] See notes 27 and 28 above.

[47] Andrew Molland shows that Albert the Great, Aquinas's teacher, places mathematics in the real order, but that he strongly inclines towards a conceptualist view of mathematics; Andrew G. Molland, "Mathematics in the Thought of Albertus Magnus," *Albertus Magnus and the Sciences*: Commemorative Essays 1980, ed. James A. Weisheipl (Toronto: Pontifical Institute of Mediaeval Studies, 1980): 466–7. Molland quotes Albert, on pp. 469–70: "Many of the geometers' figures are in no way found in natural bodies"; Albert, *Physica* 3, tr. 2, c. 17; ed. A. Borget, *Opera omnia* (Paris: Vivès, 1890). Augustine, in a Platonic vein, differentiates between thin lines, which are like a spider's web, and the lines of pure mathematics, which the eye does not see and which are known within ourselves; also between the numbers by which we count and mathematical numbers (See Augustine, *Confessions* 10.12.19). I owe the reference to Augustine to Edward Synan.

a member of the Hermits of St. Augustine. Giles studied under Aquinas in Paris in 1269–1272, shortly after the latter had left Rome, where he had disputed the *quaestio* on the divine attributes. Giles is best known for teaching that in creatures essence and existence are not, as Aquinas taught, two principles of a being, but two things (*res*).[48]

In his own commentary on the *Sentences* Giles takes notice of Aquinas's *quaestio* and heartily disagrees with it. He quotes almost verbatim the passage in which Aquinas distinguished between three orders of concepts: those with an immediate foundation in reality, those with only a remote foundation, and those with no foundation at all. He notes that Aquinas placed the conceptions of natural things and the divine attributes in the first order, logical and mathematical conceptions in the second. But according to Giles this misconstrues both the divine attributes and the objects of mathematics. Perfections truly exist in God and in a higher way than in creatures, he says, but when we affirm them of God our affirmations fall short of their object. When I understand the divine wisdom, the divine essence furnishes an immediate foundation for my knowledge as regards the reality I understand (*quantum ad rem intellectam)*, but not for my way of knowing it (*quantum ad intelligendi modum*), for wisdom does not exist in God as we understand it. Our conceptions of God are not like those of natural things but rather like conceptions of mathematics. A mathematical line, Giles continues, has an immediate foundation in a natural line as regards the *thing* that is understood, for a mathematical line exists in the natural world. But *my way of understanding* it does not have an immediate foundation in reality, for I know it without the sensible and qualitative matter with which it exists in that world. My way of knowing it has only a remote foundation in reality, like all "second intentions." The remote foundation for my knowing mathematics is the fact that quantity lends itself to be abstracted from quality and the imaginable from the sensible.[49]

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[48] Giles of Rome, *Theoremata de esse et essentia*, ed. E. Hocedez (Louvain: Museum Lessianum, 1930). See Etienne Gilson, *The Christian Philosophy of St. Thomas Aquinas*, trans. L. K. Shook (New York: Random House, 1956), 369.

[49] "Cum intelligo lineam mathematicam, isti conceptioni intellectus quantum ad rem respondet linea naturalis etiam ut fundamentum immediatum, quia linea mathematica esse habet in re naturali. Sed quantum ad modum intelligendi non respondet ei aliquid immediatum fundamentum in re, quia licet linea mathematica sit in re naturali et illam intelligam, tamen non habet esse sine materia quali vel sensibili, quam tamen sine materia quali et sensibili

Consequently, to the three classes of concepts distinguished by Aquinas Giles adds a fourth, in between concepts with an immediate foundation in reality and those with a remote foundation. Concepts in Giles's fourth class, which include divine attributes and mathematical notions, have, as noted above, both an immediate and a remote foundation in reality. They have an immediate foundation in the external world as regards the reality they designate, but they have a remote basis as regards our way of understanding that reality. In the latter respect they are like "second intentions."[50]

With the introduction of Giles's fourth class of concepts he proposes a radically different notion of mathematics from that of Aquinas's *quaestio.* Unlike Giles, Aquinas does not grant that mathematical concepts have a proximate basis in reality; their proximate foundation is a constructive act of the mind. There are natural lines and circles but these are not the lines and circles of mathematics. There are also numbered or numerable multitudes in reality but they are not the numbers of mathematics. The objects of mathematics, though remotely based on the sensible world and abstracted from it, are products of a mental activity which alters their definitions and

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intelligo. Et licet isti modo intelligendi res immediate non respondeat, respondet immediate quia in ipsa re est quod sic intelligi possit, cum quantum possit separari a quali et imaginabile a sensibili"; Giles of Rome, *In libros Sententiarum* 1, d. 2, q. 3, ql. 2 (Venice, 1521), f. 18rb. Giles does not refer to Aquinas by name but, as was the custom of the time, as *quidam.* Antoine Dondaine has shown conclusively that the reference is to Aquinas. See his article, "Saint Thomas et la dispute des attributs divines," 256. For Giles' doctrine of mathematical abstraction see Robert J. McLaughlin, *Abstraction as Constitutive of Science according to Aristotle and Saint Thomas Aquinas* (Ph.D. diss., University of Toronto, 1965), 33–9.

[50] "Cum intelligo sapientiam divinam quantum ad rem intellectam ut immediatum fundamentum respondet ipsa divina essentia in qua verissime talis perfectio existit. Sed quantum ad intelligendi modum non respondet divina essentia ut immediatum fundamentum, quia non est eo modo sapientia in Deo ut nos intelligimus. Unde non assimilatur conceptioni rerum naturalium, sed magis conceptioni mathematicorum, quae conceptio nec penitus immediatum fundamentum habet in re, nec penitus mediatum. Sed quantum ad rem habet immediatum et convenit cum naturalibus, quantum ad modum [intelligendi] habet mediatum, et convenit cum secundis intentionibus."; Giles of Rome, *In I Sent.*, f. 18rb-va.

A "first intention" is immediately the concept of an external thing. A "second intention" is a concept of something consequent upon that first knowledge. It is formed by reflection on the primary act of knowledge and its concept. See St. Thomas, *De potentia*, q. 1, a. 1, ad 10; q. 7, a. 9. For the meaning of these intentions see Schmidt, *The Domain of Logic*, 117–29.

natures. Nothing like them exists in the external world. Aquinas was clearly aware of this from his knowledge of contemporary mathematics. Had he known modern types of mathematics, he would have seen the almost unlimited range of the mind's mathematical inventiveness.

As a witness and defender in Thomas's behalf we have Robert of Orford's polemic against Giles of Rome, written about 1288–1292. A Dominican and Thomist, Orford supported his confrere's threefold division of concepts in his *quaestio disputata* against Giles's fourfold division. Like Aquinas, he places the divine attributes in the first class and logical concepts in the second, but unfortunately he does not mention mathematics.[51]

IV

Aquinas did not bring together in one place his views on mathematics. They are scattered throughout his many writings, chiefly in commentaries on Boethius and Aristotle and often expressed while treating of other topics. Only in the highly original questions of his treatise on the *De Trinitate* of Boethius do we find an approach to a philosophy of mathematics. In his literal commentaries on Aristotle he stays close to the Greek philosopher's thought, echoing his doctrine of mathematical abstraction as a mental act that concentrates exclusively on the extensive and numerical quantities of physical bodies, to the disregard of their sensible qualities and motions.

In these commentaries Aquinas presents arithmetic and geometry (which he knew from Boethius's *Arithmetic* and Euclid's *Elements of Geometry*)[52] as frankly realistic liberal arts and sciences. Their objects are not separately existing forms, as Plato taught, but neither are they free constructions of the imagination and intellect. They exist in sensible bodies, though considered apart from them. Logic alone—

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[51] Robert of Orford, *Reprobationes dictorum a fratre Egidio in primum Sententiarum*, q. 1, d. 2, q. 9, ed. A. P. Vella (Paris: J. Vrin, 1968), 57–9.

[52] Boethius, *De Institutione arithmetica*, ed. G. Friedlein (Leipzig: Minerva, 1867). Euclid, *Elementa Geometriae*, ed. I. L. Heiberg, *Opera omnia*, vols. 1–5 (Leipzig: Teubner, 1883–1888).

which Aquinas regarded not so much as a science as an instrument of science—treats of mental constructs (*entia rationis*).

The realistic view of mathematics seems to have been commonly accepted in the thirteenth century. It was held by Albert the Great,[53] Robert Kilwardby,[54] and in general the masters of arts at Paris.[55] We have seen indications in the writings of Aquinas, however, that he qualified this generally held view of mathematics as a science of reality, suggesting a constructive role for the imagination and intellect in the elaboration of mathematical objects. It has been claimed that Albert the Great preceded him in this regard. "Albert's general strategy," writes Andrew Molland, "was to make more tenuous than had Aristotle the link between the geometer's mind and the outside world."[56] While commenting on Euclid's *Geometry*, Albert could hardly have failed to be struck by the mind's constructive function in that science.[57]

Aquinas explicitly attributes a quasi-factive role to the mind in mathematics in the *quaestio disputata* we have been considering. There he places the concepts of mathematics in the same order as those of logic, as devised by the mind on the proximate basis of its own activity and only remotely on reality. He uses the same word *adinvenit* (which has the double meaning of discovering and inventing) to denote the genesis of both kinds of concepts. Consequently it seems inevitable to conclude that both are *entia rationis* and "second intentions," though not of the same kind. Logical notions are relations consequent on acts of the mind. Mathematical notions include relations but primarily such items as lines, circles, and numbers. These exist in reality as natural and visual quantitative features of bodies. The mathematician visualizes them in the imagination and

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[53] Albert, *Physica* 1, tr. 1, c. 1, ed. A. Borgnet, *Opera omnia*, 3:2a. See also Molland, "Mathematics in the Thought of Albertus Magnus," 466, and note 47 above.

[54] Robert Kilwardby, *De Ortu Scientiarum*, ed. Albert G. Judy (Toronto: Pontifical Institute of Mediaeval Studies, 1976), 13–14, 29–31, 36–41, 53–81. This work is dated c. 1250.

[55] See *Quatre introductions à la philosophie au XIIIe siècle*, ed. Claude Lafleur (Paris: J. Vrin, 1988). This is a compilation of four treatises by masters of arts at Paris before 1250.

[56] Molland, "Mathematics in the Thought of Albertus Magnus," 470.

[57] See Paul M. J. E. Tummers, "The Commentary of Albert on Euclid's Elements of Geometry," in *Albert the Great and the Sciences*, 479–99.

mentally abstracts them from their natural setting, while reconstructing them in an ideal way. The objects of geometry are proximately founded on such a reconstructive act and numbers are directly based on the acts of adding and counting.

Giles of Rome criticized his former teacher for deviating from the commonly held Aristotelian doctrine of mathematics and even from his usual teaching on the subject. Giles granted that mathematical notions are second intentions, in the sense that our way of knowing mathematical entities in abstraction from sensible and qualitative matter depends on the mind and not on reality. But he rejected Aquinas's contention that in mathematics *what* is known (the *res intellecta*) is devised by the mind, with only a remote foundation in the sensible world, and that as such it is an *ens rationis*.

If this be indeed the case, the ambiguities in Thomistic literature on the subject can be removed. Are all—or at least some—of the objects of mathematics real beings, or are they *entia rationis*, or do they lie somewhere between real beings and *entia rationis*? The present study leads to the conclusion that they are all constructs of the mind, but they have a real remote foundation in the sensible world. Is substance needed as the intelligible matter of mathematicals, or is substance only thought of as fulfilling this function? If real quantity is only the remote foundation of mathematical objects, substance itself does not enter into a mathematical notion. A mathematical object is not a substantial entity; as an *ens rationis* it does not fall within the ten categories of Aristotle. The intelligible matter of the mathematicals is within the conceptual order, like an *ens rationis* itself; for example, the generic concept of line as divisible into straight and curved, and the generic concept of number as divisible into whole and fraction. These difficulties and ambiguities in Thomistic accounts of mathematics can be resolved in the light of Thomas's neglected text on the subject.[58]

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[58] See note 40 above.