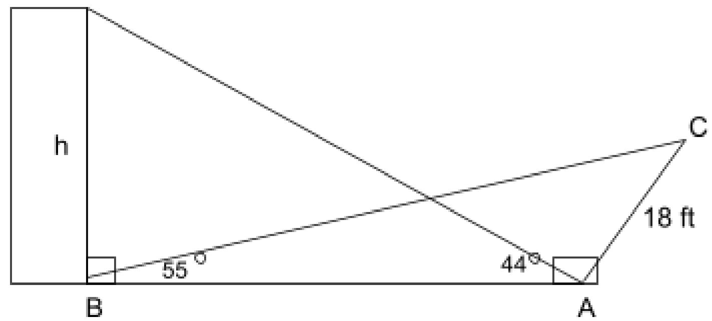
Kaitlyn Eck

Subarea 3: Trigonometry and Calculus 0011 Applying the Principles and Techniques of Trigonometry to Model and Solve Problems

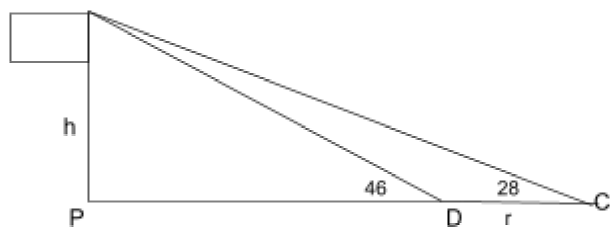
Exemplar 1: Applying trigonometric functions to solve problems involving length, area, volume, or angle measure (e.g arcs, angles, and sectors associated with a circle, unknown sides and angles of polygons, vectors)

- Using the Trig Functions to find length/distance.
    - Knowing the formulas for sine, cosine, and tangent, we can apply these formulas in multiple steps to examine more 3D examples.
    - Example: At a street corner, point A, you are perpendicular from a building at point B, with an angle of elevation of 44 degrees. You turn the corner at a 90 degree angle and walk a distance of 18 feet to point C, and are now at an 55 degree angle from point B, as shown in the figure. Find the height, h, of the building, to the nearest hundreth.



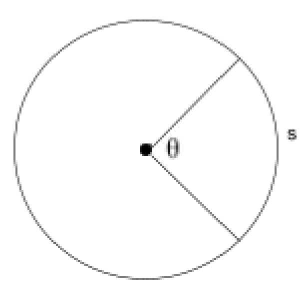
Solution: First, we must find the length from point A to B. Consider tan(55), we can use this to find the length from A to B, since tan(55)=$\frac{18}{AB}$. This gives us the length of A to B to be about 12.6037 feet. Then, we can use the tangent of the angle of elevation, 44 degrees, in order to find the height of the building. So, tan(44)=$\frac{h}{12.6037}$ and our height, h=12.1713 feet.

- Example: There is a river that you are unable to cross, since its width is r feet. You are standing at point C and directly on the other side of the river is point D. There is a flagpole in line with your positioning, at point P. We know that the angle of elevation from the flagpole to where you are standing is 28 degrees and the angle of elevation from the base of the other side of the river to the flagpole is 46 degrees. Using this, find the height of the flagpole in terms of r.



Solution: We can express the height in terms of two different tangent functions, the first using the angle of elevation of 46 degrees. We have from this tan(46)=$\frac{h}{PD}$, and we do not know the length of PD yet. We can also use the angle of elevation of 28 degrees to get tan(28)=$\frac{h}{PD+r}$. We isolate h in both of these functions and set the equations equal to each other to produce the equation tan(46)PD=tan(28)(PD+r), or tan(46)PD=tan(28)PD+tan(28)r. Subtracting tan(28)PD from both sides of the equation and factoring out PD, we now have PD(tan(46)-tan(28))=tan(28)r. Since we want to express height in terms of r, we can solve for PD in terms of r and then use that equation and substitute it in for one of the height formulas. We will divide both sides of our equation by (tan(46)-tan(28)) to get PD= $\frac{tan(28)r}{(tan(46)-tan(28))}$. Evaluating the tan values, we are left with PD=1.05535r. We will substitute that into our height equation of h=tan(46)PD to get h=tan(46)(1.05535r), or h=1.09285r.

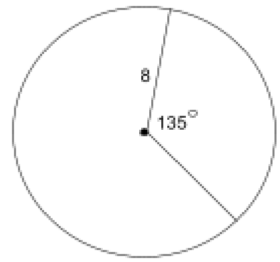
- Using the Trig Functions to find information associated with a circle.
    - <u>Arc length formula</u>: s=θr, where θ is the central angle in radians and r is the radius.
    - **Derivation of the arc length formula:**
    - Consider a circle having radius r. We want to find the length of arc s, made by central angle theta, where theta is in radians. We will do this by using proportions. The length of the arc around the entire circle is just the circumference, and there are 2Π radians in a circle. So, between the whole



circle, and the part of a circle with arc length s, we have the following proportion:

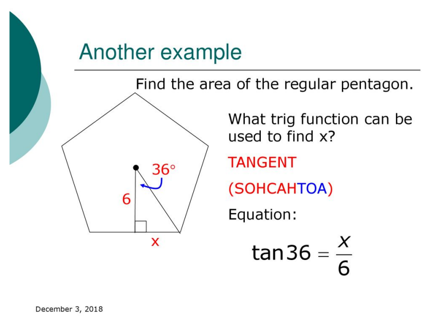
$$\frac{2\Pi r}{2\Pi} = \frac{s}{\theta}$$

Where $2\Pi r$ is equal to the circumference of the whole circle. So we have the proportion set up with the arc lengths in the numerator and the angle measures of the central angle in the denominators, comparing the whole to the part. Through simplification, the 2Π cancels out from the first fraction, leaving us with: $r = \frac{s}{\theta}$, and we can multiply both sides of the equation by theta to get $s = r\theta$, as desired.



- **Ex**: Given the circle to the right with radius of 8 and a central angle of 135 degrees, find the length of the intercepted arc.
  - Solution: First we must convert the central angle from degrees to radian. So, $135(\frac{\pi}{180}) = \frac{3\pi}{4}$ radians. Now, we will apply the formula, so $s = \frac{3\pi}{4}(8)$ to get an arc length of 6π.

- Using Trig Functions to find the unknown sides, angles, and areas of regular polygons.
  - A <u>polygon</u> is defined to be a closed shape made up of line segments. Specifically, a regular polygon is a closed shape made up of equivalent line segments.
  - After locating the center of a regular polygon, one can draw a line connecting the center to the perpendicular bisector of a side of the polygon, and then draw another line connecting the center to the vertex of that line segment, to form a right triangle, as shown in the picture.
  - <u>Apothem</u>: a line drawn from the center of any regular polygon to the midpoint of one of the sides.
  - The formula for an apothem is: $a = R\cos(\frac{\Pi}{n})$, where R is the circumradius and n is the number of sides.
  - Using trigonometric functions (such as $\tan\theta = \frac{o}{a}$) to find the length of half of the polygon side, one can multiply that by 2 to solve for unknown side length of a polygon.



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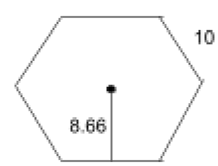
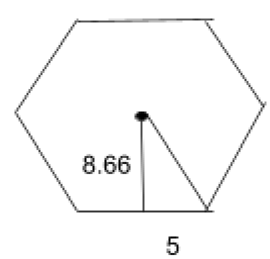
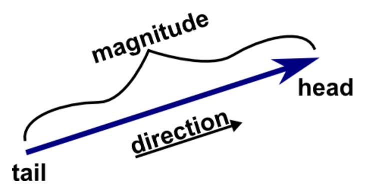
  - To find the unknown angles of regular polygons, one can use trigonometric functions with the same triangle, in order to solve for the angle that is not the central nor the right angle in the triangle. The hypotenuse of the triangle bisects the unknown angle, so after using $\tan\theta = \frac{o}{a}$, multiply that degree value by 2 to find the angle measure of each polygon angle. One can also deduce the central angle by using the formula $\frac{(n-2)180}{n}$ if little information is given.
  - To find the area of the polygon, we can use the formula $A = \frac{1}{2}aP$, where a is the apothem and P is the perimeter of the regular polygon.
  - **Ex**: Find the angle measure of the central angle and area of a regular hexagon with the distance of the apothem equaling 8.66 units and each side having a length of 10 units.
  - Solution: A hexagon has 6 sides, so each interior angle would be $\frac{(4)180}{6}$, or 60 degrees. We can draw a line segment to one of the vertices to form a right triangle with the apex, with a base of 5 (half of the side length). Using $\tan\theta = \frac{o}{a}$, we have $\tan\theta = \frac{5}{8.66}$, and θ=30 degrees. We can now use the area formula utilizing the apothem, A=





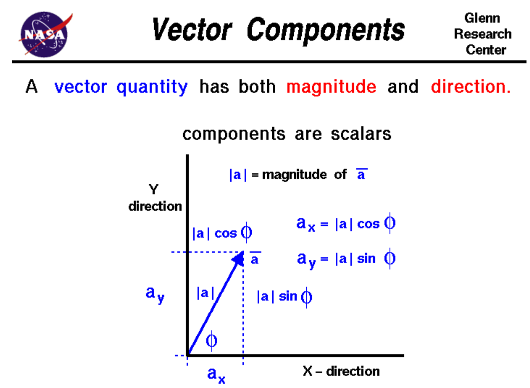
$\frac{1}{2}$aP. First, we must find the perimeter, which is just (10)(6)=60. So A=$\frac{1}{2}$(8.66)(60) and the area is equal to 259.8 units squared.

- Usin Trig Functions with vectors.
  - A vector is a value that has both direction and magnitude. A vector can be modeled graphically on a coordinate plane whose length is the magnitude and direction is indicated by the direction of the arrow.
  - Magnitude here can be further examined referring to the hypotenuse of the triangle formed from the distance traveled on the x axis and the distance traveled on the y axis



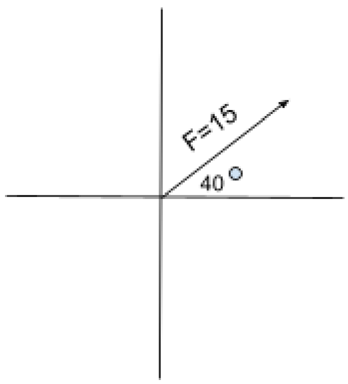
https://mathinsight.org/vector_introduction

    - To find the horizontal vector quantity, we use: $a_x = |a|cos\theta$
    - To find the vertical vector quantity, we use: $a_y = |a|sin\theta$



https://www.grc.nasa.gov/www/BGH/vectpart.html

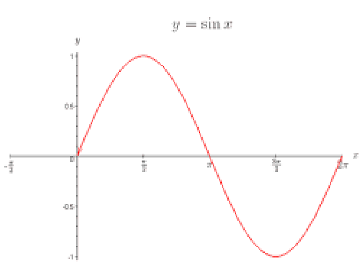
  - **Application Example with the Force Vector**: A force of 15 pounds is acting at a 40 degree angle to the horizon. Find the horizontal force (F_x) and the vertical force (F_y).



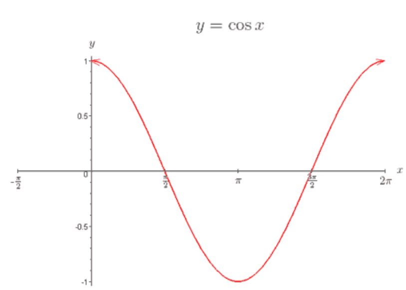
    - **Solution**: To find the horizontal force, we use the formula $a_x = |a|cos\theta$, so F_x=|15|cos40=11.49. To find the vertical force, we use the formula $a_y = |a|sin\theta$, so F_y=|15|sin40=9.64
    - For more vector practice, click here

Exemplar 2: Using circular functions to model periodical phenomena

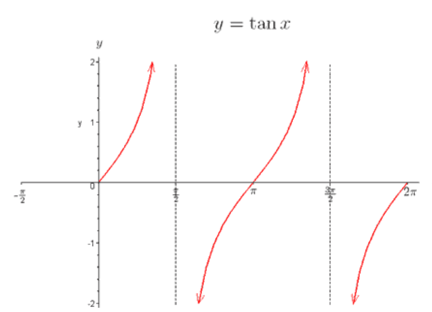
Graph of sin(x):



Graph of cos(x):

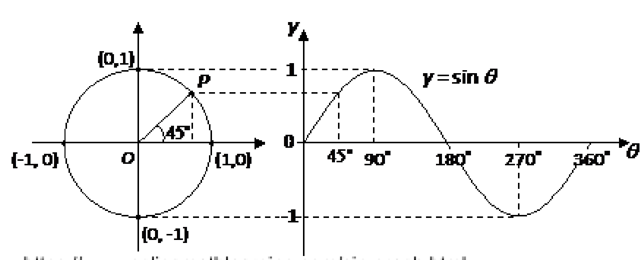


Graph of tan(x):



Graphs from: http://www.biology.arizona.edu/biomath/tutorials/trigonometric/graphtrigfunctions.html

- Explanation of why the sin function graphs the way it does:
  - In terms of the unit circle, the sinθ as θ goes from 0-90 degrees, increases from 0 to 1, then as θ goes from 90-180 degrees in accordance to the unit circle, sinθ

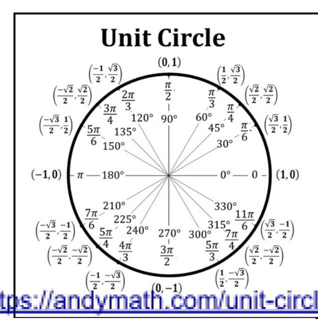


https://www.onlinemathlearning.com/sin-graph.html decreases from 1 to 0. When θ goes from 180 to 270 degrees in the unit circle, sinθ goes from 0 to -1, and lastly as θ completes the unit circle rotation from 270-360 degrees, sinθ increases from -1 to 0. This pattern of behavior from the unit circle is exhibited by the graph y=sin(x). The same type of explanation goes for cos(x).

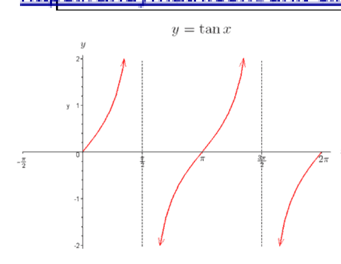
- One can model periodic concepts by manipulating the sine function to fit each scenario. The sine graph can be shifted using this formula y=Asin(B(x-C))+D where A represents the amplitude, B is the value used to help determine the period (the period is $\frac{2\Pi}{B}$), C is a phase/horizontal shift, and D is a vertical shift. The same goes with the cosine function, using the formula y=Acos(B(x-C))+D. The graph used to change tangent is similar to that of sine and cosine, however, in the formula y=Atan(B(x-C))+D, the period is found by doing $\frac{\Pi}{B}$. These formulas can be used to create graphs that fit specific curves and data entries.
- For further clarification and more examples, watch the video here
- **Ex**: State the amplitude, period, horizontal, and vertical shift of the following equation: y=5sin($\frac{\pi}{3}$x-2Π)+3.
  - **Solution**: Using the formula y=Asin(B(x-C))+D, A=5 which tells us that the amplitude is 5. B=$\frac{\pi}{3}$, since we can factor out $\frac{\pi}{3}$ from ($\frac{\pi}{3}$x-2Π) to get $\frac{\pi}{3}$(x-6), telling us our period is 6. Also from the factorization, there is a horizontal phase shift 6 units to the right. D=3, so there is also a vertical shift 3 units up.
- For more information: see this website https://www.onlinemathlearning.com/sin-graph.html

Exemplar 3: Solving trigonometric equations using analytic or graphing techniques

- When first solving trigonometric equations using graphing techniques, the first thing one must do is graph the trigonometric function in question, whether that be sin(x), cos(x), or tan(x). Then, find the value of y that is equal to the function in order to find the corresponding x as a solution.
- One can use the unit circle or graphing the trigonometry function in order to solve these equations. When using the unit circle, one looks for the solution by examining the trigonometry function itself. In accordance with the unit circle, the cosine function cos(θ)=x represents the distance from the origin in the x axis, and the sine function, sin(θ)=y represents the distance from the origin in the y direction, in accordance to the angle measure theta.
- The difference between using the unit circle and solving trigonometric functions graphically are the variables at play. When graphing sine and cosine, the input values are x, and the output values are y. When examining the unit circles, theta is considered, and the coordinates (x,y) are the coordinates that lie on the unit circle when the vector from the origin of angle theta is formed.
- **Example using both methods**: Find all solutions between 0 and 2pi for $4sec^2(x) - 3tan^2x - 5 = 0$
  - First, we must simplify the equation at hand $4sec^2(x) - 3tan^2x - 5 = 4(tan^2x + 1) - 3tan^2x - 5 = 4tan^2x + 4 - 3tan^2x - 5$ Now, combining like terms, we have $tan^2x - 1 = 0 \text{ and so } tan^2x = 1 \text{ and } tanx = +1, -1.$ We will evaluate this using two methods.
  - Unit Circle: Using the unit circle method, we are looking for where the x coordinates on the unit circle are -1 and 1. This occurs when theta is equal to $\frac{\Pi}{4}, \frac{3\Pi}{4}, \frac{6\Pi}{4}, \text{ and } \frac{7\Pi}{4}$. More practice with this method can be found here.
  - Graph: First, graph the function y=tanx from 0 to 2 pi. The graph should look like the picture below. Solving for the solutions, we are looking for the x values where the output is our wanted y value, in this problem being -1 or 1. So, we are looking for where y=-1, which gives us the solution of $\frac{3\Pi}{4}$ and



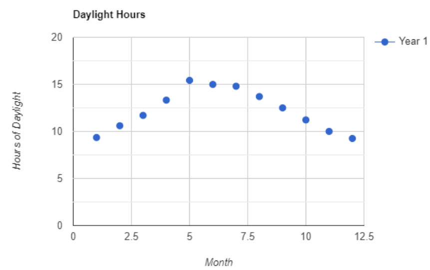
https://andymath.com/unit-circle/



$\frac{7\Pi}{4}$. We are also looking for where y=1, which gives us a solution of $\frac{\Pi}{4}$ and $\frac{6\Pi}{4}$. This gives us the same for solutions as the other strategy.

Exemplar 4: Modeling and solving problems involving trigonometric functions

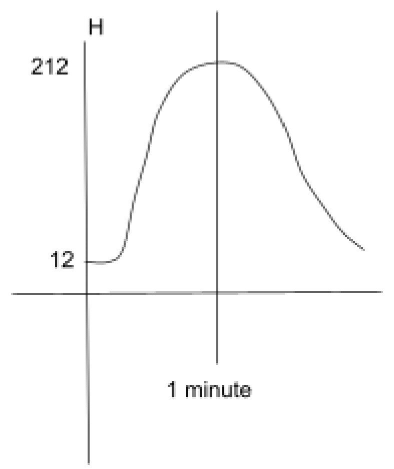
- The types of problems being modeled by trigonometric functions are those representing cyclic motions.
- Below are two types of examples of problems displaying a cyclic pattern, where they all can go through various cycles of y as a change in x occurs.
  - The number of hours and minutes in a city located at a latitude of 50 degrees on the 15th of the month is shown by the table here:
  - Using the data, model this with a trig function.
  - Solution: First, make a scatter plot of the data to examine the relationship. From the graph, it looks like the data can be modeled using a sin function, with a year being around half of a sin curve period (11 months). Using the TI-nSpire, we will input this data into a scatterplot and use the graph to fit a sin equation. First, input the data into a table, then go to Menu→Statistics→Stats Calculation→Sinusoidal Regression. From this, we get the equation y=1.6777sin(2.89789x+2.81774)+11.4813, or in the form that we use y=1.6777sin(2.89789(x+.97234))+11.4813. So, the amplitude is 1.6777 and the period is $\frac{2\Pi}{2.89789}$=2.16819. There is a phase shift to the left of .97234, which makes sense in context, since it is close to 1 and the data starts at month 1 and not 0. Lastly, there is a vertical shift of 11.4813 up.



https://www.rapidtables.com/tools/scatter-plot.html

January	9.35
February	10.6
March	11.7
April	13.322
May	15.421
June	14.99
July	14.798
August	13.7
September	12.499
October	11.221
November	10
December	9.25

- Geneseo is hosting a carnival, including a Ferris wheel. It takes 2 minutes for the wheel to make a full rotation. The start of the Ferris wheel, the lowest point, point S, is 12 feet from the ground. How high will point S be after t seconds, if the radius of the Ferris wheel is 100 feet and turns counterclockwise?
- Solution: We can sketch the graph pictured, knowing that the minimum height is 12 feet and the maximum height is 12 plus the diameter of the Ferris wheel (200 feet) equalling 212 feet. The point S will reach maximum height halfway through the revolution, so at 1minute. We can use a cosine function to model this behavior. The amplitude would be 100, since it is halfway between the maximum and minimum height. The A in our equation, however, will be -100, since our graph here is a negative reflection of the regular cosine curve. The period is 2 minutes since it takes 2 minutes, or 120 seconds, for the Ferris wheel to make a full rotation, meaning the B in our cosine equation is $\frac{2\Pi}{120} = \frac{\Pi}{60}$, since t is in seconds. There is a vertical shift up 112 feet and no phase shift. As a result, our equation is H=-100cos($\frac{\Pi}{60}$t)+112.



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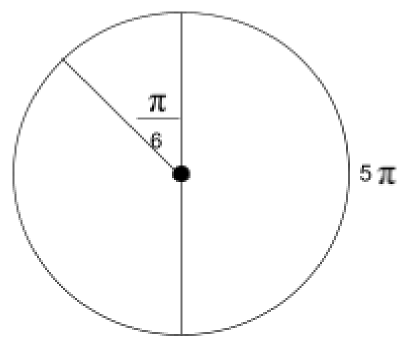
Problems:

1. Solve the equation tan(3θ)=1 for all solutions between 0 and 2π using:
    a. The unit circle

    b. The graph of tanθ

2. Suppose a circle had a semicircle such that the arc length of the semi circle (in radians) were 5Π. In that same circle, what is the arc length of an arc where theta equaled $\frac{\Pi}{6}$.



3. Eric is parasailing in a seated harness with a rope attaching the harness to a speedboat. The rope makes a 63 degree angle with the horizon and has a tension of 420 Newtons. Find the horizontal and vertican components of the tension force.

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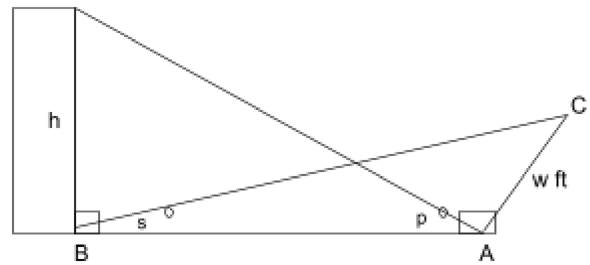
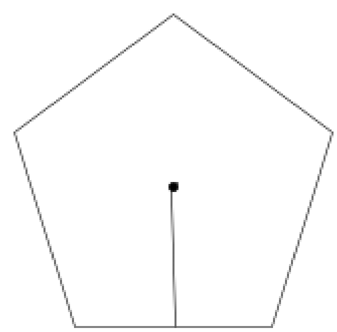
4. You were given the following example in the summary above. “At a street corner, point A, you are perpendicular from a building at point B, with an angle of elevation of 44 degrees. You turn the corner at a 90 degree angle and walk a distance of 18 feet to point C, and are now at an 55 degree angle from point B, as shown in the figure. Find the height, h, of the building, to the nearest hundreth.” Generalize this problem for any angle of elevation at point A, of p degrees, a walking distance of w feet from point B to C, and any angle of elevation of s degrees from point B.



5. Explain what B means in the formula y=Asin(B(x-C))+D and how that value relates to the period and frequency of the sine curve.

6. Find the apothem, side length, and perimeter of a regular pentagon with an area of 247.75.



7. The diameter of a cars tire is 3 feet. While the car is being drive, the front driver’s side tire picks up a nail in it. How high above the ground is the nail after the car has been driving for 52 seconds after being hit with the nail, if it takes 40 seconds to make 1 full rotation of a tire?

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8. The average monthly temperature for a year was recorded below in a small town in Washington. Find a sine function that models this data and sketch the graph.

January	43
February	45
March	49
April	51.75
May	58.2
June	62.9
July	68.95
August	69.5
September	65
October	56.2
November	46.7
December	44.1

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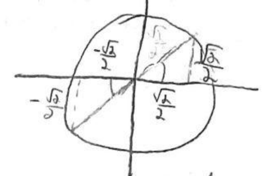
Problems:

1. Solve the equation tan(3θ)=1 for all solutions between 0 and 2π using:
   a. The unit circle

Let $3\theta = \phi$. Solve for $\tan\phi = 1$

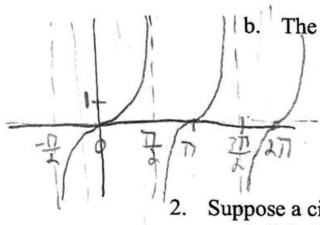
We are looking for where on the unit circle, $\frac{y}{x} = 1$. This occurs when $\phi = \frac{\pi}{4}, \frac{5\pi}{4}$ since both the x and y coordinates are $\frac{\sqrt{2}}{2}$ or $-\frac{\sqrt{2}}{2}$, so, solving for θ, $3\theta = \frac{\pi}{4} \to \boxed{\theta = \frac{\pi}{12}}$

$3\theta = \frac{5\pi}{4} \to \boxed{\theta = \frac{5\pi}{12}}$



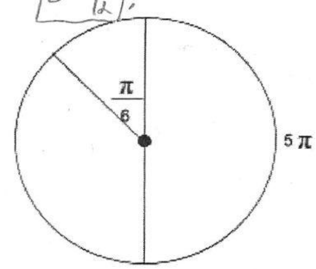
$\tan\phi = y$

   b. The graph of tanθ



This is a rough sketch of the tangent graph, one can also make a more accurate one in their calculator. We are looking for where $y=1$, and then we will solve for θ. Between 0 and 2π, $x = \frac{\pi}{4}$ and $\frac{5\pi}{4}$ when $y=1$. So, $3\theta = \frac{\pi}{4}$ and $\boxed{\theta = \frac{\pi}{12}}$, and $3\theta = \frac{5\pi}{4} \to \boxed{\theta = \frac{5\pi}{12}}$.

2. Suppose a circle had a semicircle such that the arc length of the semi circle (in radians) were 5Π. In that same circle, what is the arc length of an arc where theta equaled $\frac{\Pi}{6}$. $s = \theta r$

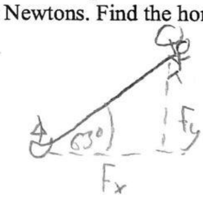


Find radius; $5\pi = 5 \cdot r \to r = 5$

Find arc length when $\theta = \frac{\pi}{6}$; $s = \frac{\pi}{6} \cdot 5 = \frac{5\pi}{6}$

The arc length is $\frac{5\pi}{6}$ radians.

3. Eric is parasailing in a seated harness with a rope attaching the harness to a speedboat. The rope makes a 63 degree angle with the horizon and has a tension of 420 Newtons. Find the horizontal and vertican components of the tension force.



$F_x = |F|\cos\theta$

$= |420|\cos 63°$

$= 190.676\,N$

$F_y = |F| \cdot \sin\theta$

$F_y = |420|\sin 63°$

$= 374.223\,N$

$\therefore$ $\boxed{F_x = 190.676\,N \quad F_y = 374.223\,N}$.

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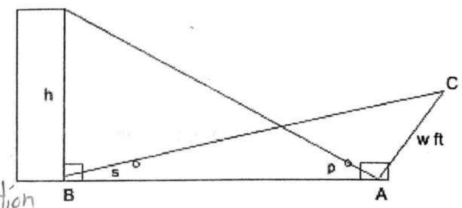
4. You were given the following example in the summary above. "At a street corner, point A, you are perpendicular from a building at point B, with an angle of elevation of 44 degrees. You turn the corner at a 90 degree angle and walk a distance of 18 feet to point C, and are now at an 55 degree angle from point B, as shown in the figure. Find the height, h, of the building, to the nearest hundreth." Generalize this problem for any angle of elevation at point A, of p degrees, a walking distance of w feet from point B to C, and any angle of elevation of s degrees from point B.



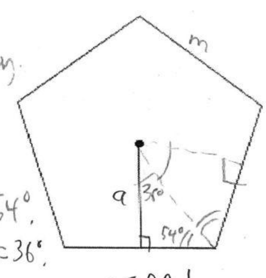
First, we must find length AB by using the tangent function of angles. $\tan(s)=\frac{w}{AB} \to AB=\frac{w}{\tan(s)}$

To find the height, h, we use the tangent function of angle p. $\tan(p)=\frac{h}{AB}$. Since $AB=\frac{w}{\tan(s)}$, by substitution, $\tan(p)=\frac{h}{\frac{w}{\tan(s)}}=\frac{h\tan(s)}{w}$. Solving for h, $\frac{w\tan(p)}{\tan(s)}=\frac{h\tan(s)}{\tan(s)}$ and $h=\frac{w\tan(p)}{\tan(s)}$

5. Explain what B means in the formula y=Asin(B(x-C))+D and how that value relates to the period and frequency of the sine curve.

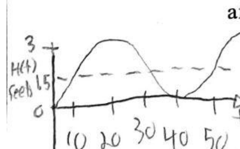
B refers to the coefficient in front of (x-c) applied in the sine function. The period can be found by dividing 2 pi by B. If B is less than 0, as B decreases, the the period of the graph increases. If B is greater than 0, then as B increases, the period of the graph decreases. The frequency of the graph is $\frac{1}{period}$ of the graph.

6. Find the apothem, side length, and perimeter of a regular pentagon with an area of 247.75.



$A=\frac{1}{2}aP$, but since it is a regular pentagon, $P=5m$, where m is the side length.
The interior angle of a pentagon is equal to $\theta=\frac{3\cdot180}{5}=108°$, so for the right triangle here, a $\frac{1}{2}m$, the base angle is $\frac{108}{2}=54°$. The central angle in the triangle would be $180-54-90=36°$.
Now solving for the apothem, $a=\frac{2A}{5m}$, and area is 247.75, so $a=\frac{99.1}{m}$, $\tan 36°=\frac{m}{2a}$, or $\tan 36°=\frac{m^2}{198.2}$ and $m^2=144 \to \boxed{m=12}$. Therefore, $\boxed{P=5(12)=60}$ and $\boxed{a=8.25833}$

7. The diameter of a cars tire is 3 feet. While the car is being drive, the front driver's side tire picks up a nail in it. How high above the ground is the nail after the car has been driving for 52 seconds after being hit with the nail, if it takes 40 seconds to make 1 full rotation of a tire?



$A=-1.5$ (starting at $H=0$, means it is negative cos graph)
Period is 40 secs so $B=\frac{2\pi}{40}=\frac{\pi}{20}$
There is no phase shift, but a vertical shift up 1.5.
$H(t)=-1.5\cos(\frac{\pi}{20}x)+1.5$
$H(52)=-1.5\cos(\frac{\pi}{20}(52))+1.5$ and $\boxed{H(52)=1.96353 \text{ feet}}$

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8. The average monthly temperature for a year was recorded below in a small town in Washington. Find a sine function that models this data and sketch the graph.

January	43
February	45
March	49
April	51.75
May	58.2
June	62.9
July	68.95
August	69.5
September	65
October	56.2
November	46.7
December	44.1

I will make a sinusoidal regression in order to find a sine function that models this data, by using the calculator.

First, I will input the data into my calculator's table.

Then, using a TI-nspire CX II I go:

'Menu → statistics → Stats Calculation → Sinusoidal Regression,

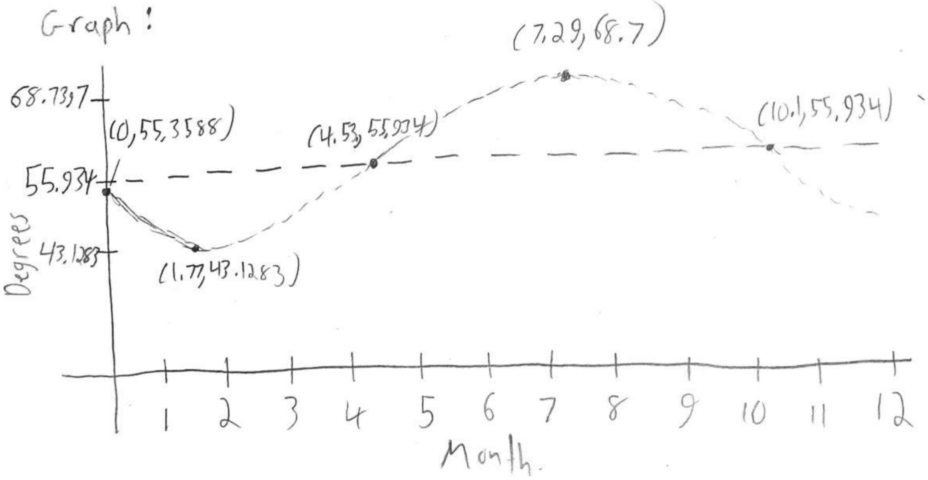
resulting in the following equation:

$$y = 12.8057\sin(.568362x - 2.5746) + 55.934$$

We can rewrite this in the more familiar format by factoring out .568362 to get $y = 12.8057\sin(.568362(x - 4.52986)) + 55.934$

so our amplitude, A, is 12.8057, our period is $\frac{1}{.568362}$, or 1.75944, there is a phase shift right 4.52986 months and a vertical shift up 55.934 degrees.

Graph:



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Resources:

https://slideplayer.com/slide/14976890/

https://mathinsight.org/vector_introduction

https://www.grc.nasa.gov/www/BGH/vectpart.html

http://www.biology.arizona.edu/biomath/tutorials/trigonometric/graphtrigfunctions.html

https://www.youtube.com/watch?v=pNEcCYuLr94&feature=youtu.be

https://www.onlinemathlearning.com/sin-graph.html

https://andymath.com/unit-circle/

https://www.rapidtables.com/tools/scatter-plot.html

Sultan, A., & Artzt, A. F. (2018). *The mathematics that every secondary school math teacher needs to know*. Routledge, Taylor & Francis Group.

https://www.youtube.com/watch?v=qlD76kOU2wY

Usiskin, Z. (2010). *Mathematics for high school teachers: An advanced perspective*. Cram101.

https://www.google.com/url?q=https://opentextbc.ca/algebratrigonometryopenstax/chapter/solving-trigonometric-equations/&sa=D&source=docs&ust=1652206578272278&usg=AOvVaw1LklrH8NCvPrG5vhLlsjZu

https://www.cuemath.com/geometry/apothem/

https://opentextbc.ca/algebratrigonometryopenstax/chapter/solving-trigonometric-equations/

https://www.physicsclassroom.com/calcpad/vecforce/problems

https://courses.lumenlearning.com/precalctwo/chapter/modeling-with-trigonometric-equations/

http://jwilson.coe.uga.edu/EMAT6680Fa09/Gonterman/Gonterman1/Gonterman1.html

https://www.cbsd.org/cms/lib/PA01916442/Centricity/Domain/2746/PC%20-%20Section%207.5%20-%20Worksheet%20-%20KEY.pdf