Topic 5

Energy & Power Spectra, and Correlation

In Lecture 1 we reviewed the notion of average signal power in a periodic signal and related it to the A_n and B_n coefficients of a Fourier series, giving a method of calculating power in the domain of discrete frequencies. In this lecture we want to revisit power for the continuous time domain, with a view to expressing it in terms of the frequency spectrum.

First though we should review the derivation of average power using the complex Fourier series.

5.1 Review of Discrete Parseval for the Complex Fourier Series

You did this as a part of 1st tute sheet

Recall that the average power in a periodic signal with period $T = 2\pi/\omega$ is

$$\textbf{Ave sig pwr} \;=\; \frac{1}{T}\int_{-T/2}^{+T/2} |f(t)|^2 \,\mathrm{d}t \;=\; \frac{1}{T}\int_{-T/2}^{+T/2} f^*(t)f(t)\,\mathrm{d}t\ .$$

Now replace $f(t)$ with its complex Fourier series

$$f(t) = \sum_{n=-\infty}^{\infty} C_n \mathrm{e}^{\mathrm{i}n\omega t}\ .$$

It follows that

$$\begin{aligned}
\textbf{Ave sig pwr} &= \frac{1}{T}\int_{-T/2}^{+T/2} \sum_{n=-\infty}^{\infty} C_n \mathrm{e}^{\mathrm{i}n\omega t} \sum_{m=-\infty}^{\infty} (C_m)^* \mathrm{e}^{-\mathrm{i}m\omega t}\mathrm{d}t \\
&= \sum_{n=-\infty}^{\infty} C_n (C_n)^* \quad \text{(because of orthogonality)} \\
&= \sum_{n=-\infty}^{\infty} |C_n|^2 \;=\; |C_0|^2 + 2\sum_{n=1}^{\infty} |C_n|^2\ , \quad \text{using } \ C_n = (C_{-n})^*.
\end{aligned}$$

5.1.1 A quick check

It is worth checking this using the relationships found in Lecture 1:

$$C_m = \begin{cases} \frac{1}{2}(A_m - \mathrm{i}B_m) & \text{for} \quad m > 0 \\ A_0/2 & \text{for} \quad m = 0 \\ \frac{1}{2}(A_{|m|} + \mathrm{i}B_{|m|}) & \text{for} \quad m < 0 \end{cases}$$

For $n \geq 0$ the quantities are

$$|C_0|^2 = \left(\frac{1}{2}A_0\right)^2 \qquad 2|C_n|^2 = 2\frac{1}{2}(A_m - \mathrm{i}B_m)\frac{1}{2}(A_m + \mathrm{i}B_m) = \frac{1}{2}\left(A_n^2 + B_n^2\right)$$

in agreement with the expression in Lecture 1.

5.2 Energy signals vs Power signals

When considering signals in the continuous time domain, it is necessary to distinguish between "finite energy signals", or "energy signals" for short, and "finite power signals".

First let us be absolutely clear that
All signals $f(t)$ are such that $|f(t)|^2$ is a power.

An energy signal is one where the total energy is finite:

$$E_{\mathrm{Tot}} = \int_{-\infty}^{\infty} |f(t)|^2 \mathrm{d}t \qquad 0 < E_{\mathrm{Tot}} < \infty \ .$$

It is said that $f(t)$ is "square integrable". As E_{Tot} is finite, dividing by the infinite duration indicates that energy signals have zero average power.

To summarize before knowing what all these terms mean: An Energy signal $f(t)$

- *always* has a Fourier transform $F(\omega)$
- *always* has an energy spectral density (ESD) given by $\mathcal{E}_{ff}(\omega) = |F(\omega)|^2$
- *always* has an autocorrelation $R_{ff}(\tau) = \int_{-\infty}^{\infty} f(t)f(t+\tau)\mathrm{d}t$
- *always* has an ESD which is the FT of the autocorrelation $R_{ff}(\tau) \Leftrightarrow \mathcal{E}_{ff}(\omega)$
- *always* has total energy $E_{\mathrm{Tot}} = R_{ff}(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{E}_{ff}(\omega)\mathrm{d}\omega$
- *always* has an ESD which transfers as $\mathcal{E}_{gg}(\omega) = |H(\omega)|^2\mathcal{E}_{ff}(\omega)$

A power signal is one where the total energy is infinite, and we consider average power

$$P_{\text{Ave}} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |f(t)|^2 \mathrm{d}t \qquad 0 < P_{\text{Ave}} < \infty \ .$$

A Power signal $f(t)$

- *may* have a Fourier transform $F(\omega)$
- *may* have an power spectral density (PSD) given $S_{ff}(\omega) = |F(\omega)|^2$
- *always* has an autocorrelation $R_{ff}(\tau) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} f(t)f(t+\tau)\mathrm{d}t$
- *always* has a PSD which is the FT of the autocorrelation $R_{ff}(\tau) \Leftrightarrow \mathcal{S}_{ff}(\omega)$
- *always* has integrated average power $P_{\text{Ave}} = R_{ff}(0)$
- *always* has a PSD which transfers through a system as $S_{gg}(\omega) = |H(\omega)|^2 S_{ff}(\omega)$

The distinction is all to do with avoiding infinities, but it results in the autocorrelation having different dimensions. Instinct tells you this is going to be a bit messy. We discuss finite energy signals first.

5.3 Parseval's theorem revisited

Let us assume an energy signal, and recall a general result from Lecture 3:

$$f(t)g(t) \Leftrightarrow \frac{1}{2\pi} F(\omega) * G(\omega) \ ,$$

where $F(\omega)$ and $G(\omega)$ are the Fourier transforms of $f(t)$ and $g(t)$.

Writing the Fourier transform and the convolution integral out fully gives

$$\int_{-\infty}^{\infty} f(t)g(t)\mathrm{e}^{-\mathrm{i}\omega t}\mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(p)G(\omega - p)\ \mathrm{d}p \ ,$$

where p is a dummy variable used for integration.

Note that ω is not involved in the integrations above — it just a free variable on both the left and right of the above equation — and we can give it any value we wish to. Choosing $\omega = 0$, it must be the case that

$$\int_{-\infty}^{\infty} f(t)g(t)\mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(p)G(-p)\ \mathrm{d}p \ .$$

Now suppose $g(t) = f^*(t)$. We know that

$$\begin{aligned} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt &= F(\omega) \\ \Rightarrow \int_{-\infty}^{\infty} f^*(t) e^{+i\omega t} dt &= F^*(\omega) \qquad \Rightarrow \int_{-\infty}^{\infty} f^*(t) e^{-i\omega t} dt = F^*(-\omega) \end{aligned}$$

This is, of course, a quite general result which could have been stuck in Lecture 2, and which is worth highlighting:

The Fourier Transform of a complex conjugate is

$$\int_{-\infty}^{\infty} f^*(t) e^{-i\omega t} dt = F^*(-\omega)$$

Take care with the $-\omega$.

Back to the argument. In the earlier expression we had

$$\begin{aligned} \int_{-\infty}^{\infty} f(t) g(t) dt &= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(p) G(-p) dp \\ \Rightarrow \int_{-\infty}^{\infty} f(t) f^*(t) dt &= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(p) F^*(p) dp \end{aligned}$$

Now p is just any parameter, so it is possible to tidy the expression by replacing it with ω. Then we arrive at the following important result

Parseval's Theorem: The total energy in a signal is

$$E_{\mathrm{Tot}} = \int_{-\infty}^{\infty} |f(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 \, d\omega \;=\; \int_{-\infty}^{\infty} |F(\omega)|^2 \, df$$

NB! The $df = d\omega/2\pi$, and is nothing to do with the signal being called $f(t)$.

5.4 The Energy Spectral Density

If the integral gives the total energy, it must be that $|F(\omega)|^2$ is the energy per Hz. That is:

The ENERGY Spectral Density of a signal $f(t) \Leftrightarrow F(\omega)$ is defined as

$$\mathcal{E}_{ff}(\omega) = |F(\omega)|^2$$

5.5 ♣ Example

[Q] Determine the energy in the signal $f(t) = u(t)\mathrm{e}^{-t}$ (i) in the time domain, and (ii) by determining the energy spectral density and integrating over frequency.

[A] **Part (i):** To find the total energy in the time domain

$$\begin{aligned} f(t) &= u(t)\exp(-t) \\ \Rightarrow \int_{-\infty}^{\infty} |f(t)|^2 \mathrm{d}t &= \int_0^{\infty} \exp(-2t)\mathrm{d}t \\ &= \left[\frac{\exp(-2t)}{-2}\right|_0^{\infty} \mathrm{d}t \\ &= 0 - \frac{1}{-2} \qquad = \quad \frac{1}{2} \end{aligned}$$

Part (ii): In the frequency domain

$$\begin{aligned} F(\omega) &= \int_{-\infty}^{\infty} u(t)\exp(-t)\exp(-\mathrm{i}\omega t)\mathrm{d}t \\ &= \int_0^{\infty} \exp(-t(1+\mathrm{i}\omega))\mathrm{d}t \\ &= \left[-\frac{\exp(-t(1+\mathrm{i}\omega))}{(1+\mathrm{i}\omega)}\right|_0^{\infty} \qquad = \quad \frac{1}{(1+\mathrm{i}\omega)} \end{aligned}$$

Hence the energy spectral density is

$$|F(\omega)|^2 = \frac{1}{1+\omega^2}$$

Integration over all frequency f (not ω remember!!) gives the total energy of

$$\int_{-\infty}^{\infty} |F(\omega)|^2 \mathrm{d}f = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{1+\omega^2}\mathrm{d}\omega$$

Substitute $\tan\theta = \omega$

$$\begin{aligned} \int_{-\infty}^{\infty} |F(\omega)|^2 \mathrm{d}f &= \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} \frac{1}{1+\tan^2\theta}\sec^2\theta\mathrm{d}\theta \\ &= \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} \mathrm{d}\theta \\ &= \frac{1}{2\pi}\pi \qquad = \quad \frac{1}{2} \quad \textit{which is nice} \end{aligned}$$

5.6 Correlation

Correlation is a tool for analysing whether processes considered random *a priori* are in fact related. In signal processing, *cross-correlation* R_{fg} is used to assess how similar two different signals $f(t)$ and $g(t)$ are. R_{fg} is found by multiplying one signal, $f(t)$ say, with time-shifted values of the other $g(t+\tau)$, then summing up the products. In the example in Figure 5.1 the cross-correlation will low if the shift $\tau = 0$, and high if $\tau = 2$ or $\tau = 5$.

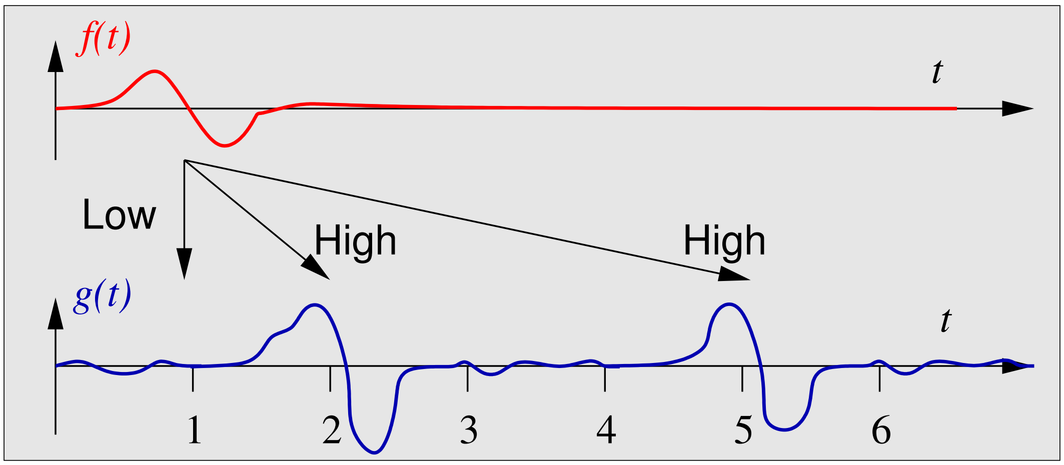


Figure 5.1: The signal $f(t)$ would have a higher cross-correlation with parts of $g(t)$ that look similar.

One can also ask how similar a signal is to *itself*. Self-similarity is described by the *auto-correlation* R_{ff}, again a sum of products of the signal $f(t)$ and a copy of the signal at a shifted time $f(t+\tau)$.

An auto-correlation with a high magnitude means that the value of the signal $f(t)$ at one instant signal has a strong bearing on the value at the next instant. Correlation can be used for both deterministic and random signals. We will explore random processes this in Lecture 6.

The cross- and auto-correlations can be derived for both finite energy and finite power signals, but they have different dimensions (energy and power respectively) and differ in other more subtle ways.

We continue by looking at the auto- and cross-correlations of finite energy signals.

5.7 The Auto-correlation of a finite energy signal

The auto-correlation of a finite energy signal is defined as follows. We shall deal with real signals f, so that the conjugate can be omitted.

The auto-correlation of a signal $f(t)$ of finite energy is defined

$$R_{ff}(\tau) = \int_{-\infty}^{\infty} f^*(t) f(t+\tau) \mathrm{d}t =_{\text{(for real signals)}} \int_{-\infty}^{\infty} f(t) f(t+\tau) \mathrm{d}t$$

The result is an **energy**.

There are two ways of envisaging the process, as shown in Figure 5.2. One is to shift a copy of the signal and multiply vertically (so to speak). For positive τ this is a shift to the "left". This is most useful when calculating analytically.

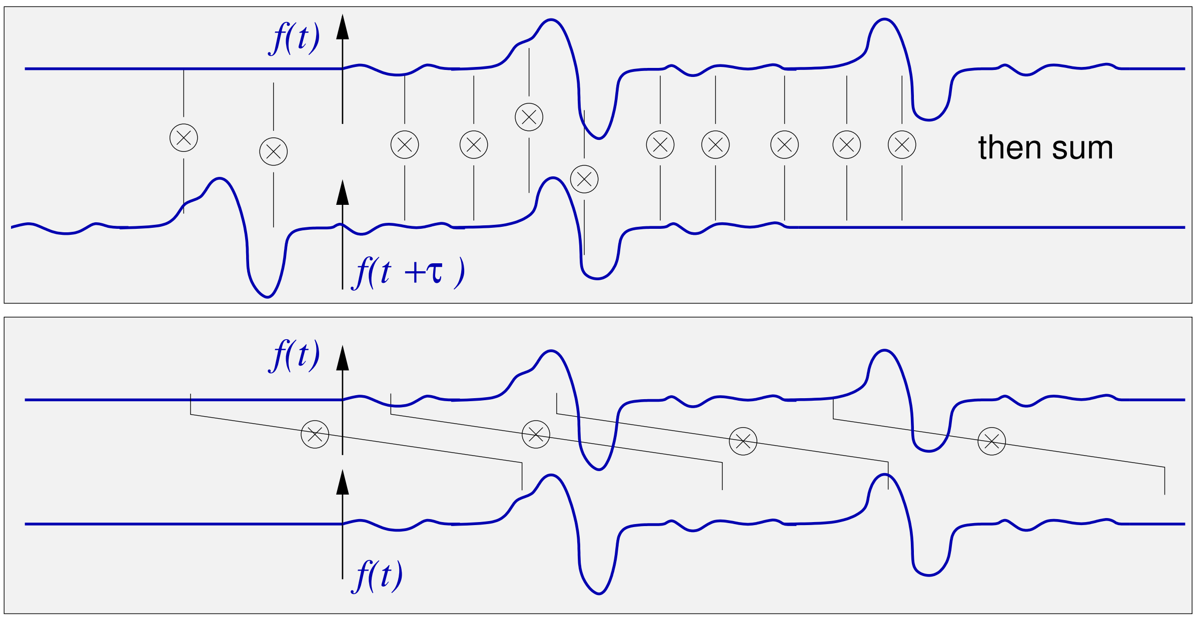


Figure 5.2: $g(t)$ and $g(t+\tau)$ for a positive shift τ.

5.7.1 Basic properties of auto-correlation

1. Symmetry. The auto-correlation function is an *even* function of τ:

$$R_{ff}(\tau) = R_{ff}(-\tau) \ .$$

Proof: Substitute $p = t + \tau$ into the definition, and you will get

$$R_{ff}(\tau) = \int_{-\infty}^{\infty} f(p-\tau) f(p) \mathrm{d}p \ .$$

But p is just a dummy variable. Replace it by t and you recover the expression for $R_{ff}(-\tau)$. (In fact, in some texts you will see the autocorrelation defined with a minus sign in front of the τ.)

2. For a non-zero signal, $R_{ff}(0) > 0$.

Proof: For any non-zero signal there is at least one instant t_1 for which $f(t_1) \neq 0$, and $f(t_1)f(t_1) > 0$. Hence $\int_{-\infty}^{\infty} f(t)f(t)\mathrm{d}t > 0$.

3. The value at $\tau = 0$ is largest: $R_{ff}(0) \geq R_{ff}(\tau)$.
Proof: Consider any pair of real numbers a_1 and a_2. As $(a_1 - a_2)^2 \geq 0$, we know that $a_1^2 + a_2^2 \geq a_1a_2 + a_2a_1$. Now take the pairs of numbers at random from the function $f(t)$. Our result shows that there is no rearrangement, random or ordered, of the function values into $\phi(t)$ that would make $\int f(t)\phi(t)\mathrm{d}t > \int f(t)^2\mathrm{d}t$. Using $\phi(t) = f(t+\tau)$ is an ordered rearrangement, and so for any τ

$$\int_{-\infty}^{\infty} f(t)^2\mathrm{d}t \geq \int_{-\infty}^{\infty} f(t)f(t+\tau)\mathrm{d}t$$

5.8 ♣ Applications

5.8.1 ♣ Synchronising to heartbeats in an ECG (DIY search and read)

5.8.2 ♣ The search for Extra Terrestrial Intelligence

For several decades, the SETI organization have been looking for extra terrestrial intelligence by examining the autocorrelation of signals from radio telescopes. One project scans the sky around nearby (200 light years) sun-like stars chopping up the bandwidth between 1-3 GHz into 2 billion channels each 1 Hz wide. (It is assumed that an attempt to communicate would use a single frequency, highly tuned, signal.) They determine the autocorrelation each channel's signal. If the channel is noise, one would observe a very low autocorrelation for all non-zero τ. (See white noise in Lecture 6.) But if there is, say, a repeated message, one would observe a periodic rise in the autocorrelation.

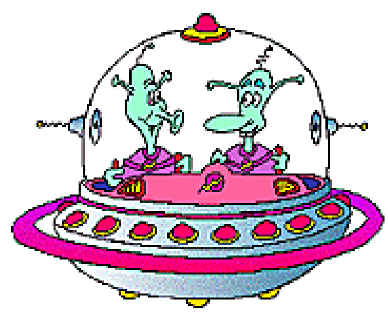


Figure 5.3: Chatty aliens

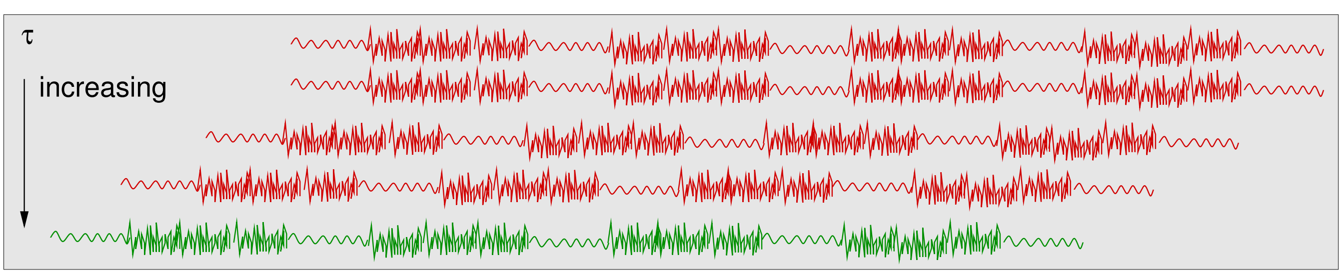


Figure 5.4: R_{ff} at $\tau = 0$ is always large, but will drop to zero if the signal is noise. If the messages align the autocorrelation with rise.

5.9 The Wiener-Khinchin Theorem

Let us take the Fourier transform of the *cross-correlation* $\int f(t)g(t+\tau)\mathrm{d}t$, then switch the order of integration,

$$\begin{aligned}\mathcal{FT}\left[\int_{-\infty}^{\infty} f(t)g(t+\tau)\mathrm{d}t\right] &= \int_{\tau=-\infty}^{\infty}\int_{t=-\infty}^{\infty} f(t)g(t+\tau)\;\mathrm{d}t\;\mathrm{e}^{-\mathrm{i}\omega\tau}\;\mathrm{d}\tau \\ &= \int_{t=-\infty}^{\infty} f(t)\int_{\tau=-\infty}^{\infty} g(t+\tau)\;\;\mathrm{e}^{-\mathrm{i}\omega\tau}\;\mathrm{d}\tau\mathrm{d}t\end{aligned}$$

Notice that t is a constant for the integration wrt τ (that's how $f(t)$ floated through the integral sign). Substitute $p = t+\tau$ into it, and the integrals become separable

$$\begin{aligned}\mathcal{FT}\left[\int_{-\infty}^{\infty} f(t)g(t+\tau)\mathrm{d}t\right] &= \int_{t=-\infty}^{\infty} f(t)\int_{p=-\infty}^{\infty} g(p)\;\;\mathrm{e}^{-\mathrm{i}\omega p}\mathrm{e}^{+\mathrm{i}\omega t}\;\mathrm{d}p\mathrm{d}t \\ &= \int_{-\infty}^{\infty} f(t)\mathrm{e}^{\mathrm{i}\omega t}\;\mathrm{d}t\int_{-\infty}^{\infty} g(p)\;\mathrm{e}^{-\mathrm{i}\omega p}\;\mathrm{d}p \\ &= F^*(\omega)G(\omega).\end{aligned}$$

If we specialize this to the auto-correlation, $G(\omega)$ gets replaced by $F(\omega)$. Then

For a finite energy signal
The Wiener-Khinchin Theorem[a] says that
The FT of the Auto-Correlation is the Energy Spectral Density

$$\mathcal{FT}\left[R_{ff}(\tau)\right] = |F(\omega)|^2 = \mathcal{E}_{ff}(\omega)$$

[a]Norbert Wiener (1894-1964) and Aleksandr Khinchin (1894-1959)

(This method of proof is valid only for finite energy signals, and rather trivializes the Wiener-Khinchin theorem. The fundamental derivation lies in the theory of stochastic processes.)

5.10 Corollary of Wiener-Khinchin

This corollary just confirms a result obtained earlier. We have just shown that $R_{ff}(\tau) \Leftrightarrow \mathcal{E}_{ff}(\omega)$. That is

$$R_{ff}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{E}_{ff}\omega)\mathrm{e}^{\mathrm{i}\omega\tau}\mathrm{d}\omega$$

where τ is used by convention. Now set $\tau = 0$

Auto-correlation at $\tau = 0$ is

$$R_{ff}(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{E}_{ff}\omega)\mathrm{d}\omega = E_{\mathrm{Tot}}$$

But this is exactly as expected! Earlier we defined the energy spectral density as

$$E_{\mathrm{Tot}} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{E}_{ff}\omega)\mathrm{d}\omega \ ,$$

and we know that for a finite energy signal

$$R_{ff}(0) = \int_{-\infty}^{\infty} |f(t)|^2 \mathrm{d}t = E_{\mathrm{Tot}} \ .$$

5.11 How is the ESD affected by passing through a system?

If $f(t)$ and $g(t)$ are in the input and output of a system with transfer function $H(\omega)$, then

$$G(\omega) = H(\omega)F(\omega) \ .$$

But $\mathcal{E}_{ff}(\omega) = |F(\omega)|^2$, and so

$$\mathcal{E}_{gg}(\omega) = |H(\omega)|^2|F(\omega)|^2 = |H(\omega)|^2\mathcal{E}_{ff}(\omega)$$

5.12 Cross-correlation

The cross-correlation describes the dependence between two different signals.

Cross-correlation

$$R_{fg}(\tau) = \int_{-\infty}^{\infty} f(t)g(t+\tau)\mathrm{d}t$$

5.12.1 Basic properties

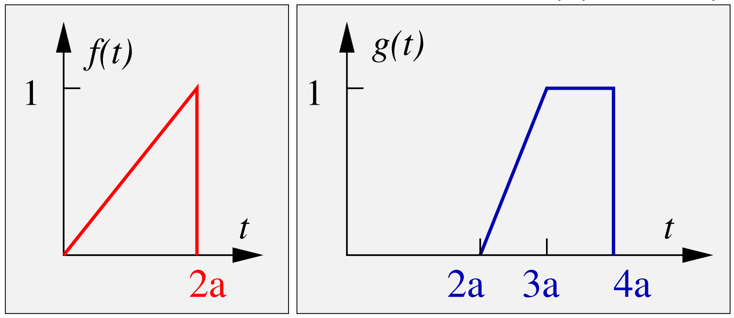
1. Symmetries The cross-correlation does not in general have a definite reflection symmetry. However, $R_{fg}(\tau) = R_{gf}(\tau)$.

2. Independent signals

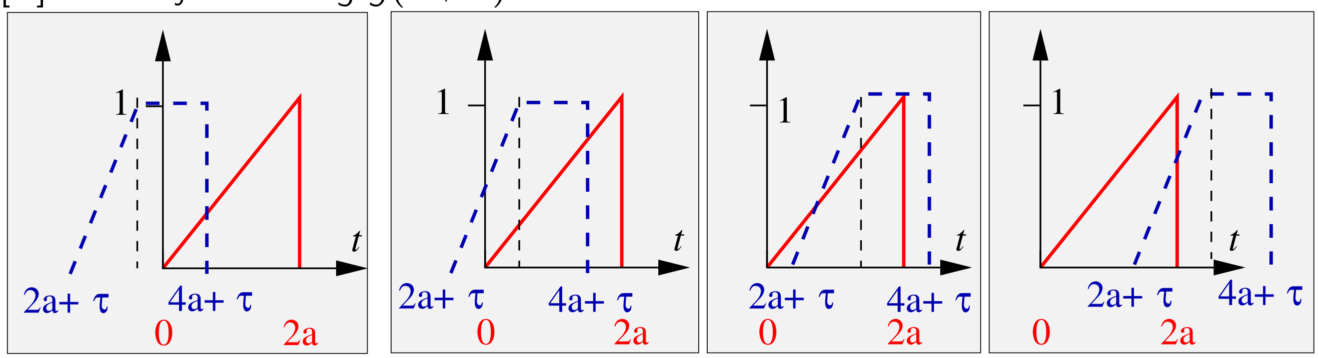
The auto-correlation of even white noise has a non-zero value at $\tau = 0$. This is not the case for the cross-correlation. If $R_{fg}(\tau) = 0$, the signal $f(t)$ and $g(t)$ have no dependence on one another.

5.13 ♣ Example and Application

[Q] Determine the cross-correlation of the signals $f(t)$ and $g(t)$ shown.



[A] Start by sketching $g(t+\tau)$ as function of t.



$f(t)$ is made of sections with $f=0$, $f=\frac{t}{2a}$, then $f=0$.
$g(t+\tau)$ is made of $g=0$, $g=\frac{t}{a}-\left(2+\frac{\tau}{a}\right)$, $g=1$, then $g=0$.
The left-most non-zero configuration is valid for $0 \le 4a+\tau \le a$, so that

- For $-4a \le \tau \le -3a$:

$$R_{fg}(\tau)=\int_{-\infty}^{\infty} f(t)g(t+\tau)\mathrm{d}t=\int_{0}^{4a+\tau}\frac{t}{2a}\cdot 1\ \mathrm{d}t=\frac{(4a+\tau)^2}{4a}$$

- For $-3a \le \tau \le -2a$:

$$R_{fg}(\tau)=\int_{-\infty}^{\infty} f(t)g(t+\tau)\mathrm{d}t=\int_{0}^{3a+\tau}\frac{t}{2a}\cdot\left(\frac{t}{a}-\left(2+\frac{\tau}{a}\right)\right)\ \mathrm{d}t+\int_{3a+\tau}^{4a+\tau}\frac{t}{2a}\cdot 1\ \mathrm{d}t$$

- For $-2a \le \tau \le -a$:

$$R_{fg}(\tau)=\int_{-\infty}^{\infty} f(t)g(t+\tau)\mathrm{d}t=\int_{2a+\tau}^{3a+\tau}\frac{t}{2a}\cdot\left(\frac{t}{a}-\left(2+\frac{\tau}{a}\right)\right)\ \mathrm{d}t+\int_{3a+\tau}^{2a}\frac{t}{2a}\cdot 1\ \mathrm{d}t$$

- For $-a \le \tau \le 0$:

$$R_{fg}(\tau)=\int_{-\infty}^{\infty} f(t)g(t+\tau)\mathrm{d}t=\int_{2a+\tau}^{2a}\frac{t}{2a}\cdot\left(\frac{t}{a}-\left(2+\frac{\tau}{a}\right)\right)\ \mathrm{d}t$$

Working out the integrals and finding the maximum is left as a DIY exercise.

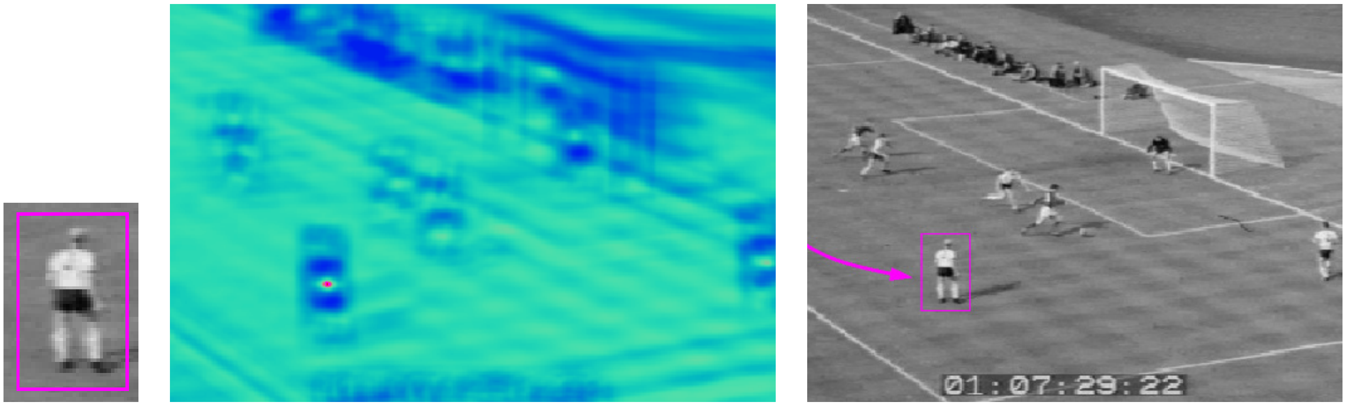


Figure 5.5:

5.13.1 Application

It is obvious enough that cross-correlation is useful for detecting occurences of a "model" signal $f(t)$ in another signal $g(t)$. This is a 2D example where the model signal $f(x, y)$ is the back view of a footballer, and the test signals $g(x, y)$ are images from a match. The cross correlation is shown in the middle.

5.14 Cross-Energy Spectral Density

The Wiener-Khinchin Theorem was actually derived for the cross-correlation. It said that

> **The Wiener-Khinchin Theorem shows that, for a finite energy signal, the FT of the Cross-Correlation is the Cross-Energy Spectral Density**
>
> $$\mathcal{FT}\left[R_{fg}(\tau)\right] = F^*(\omega)G(\omega) = \mathcal{E}_{fg}(\omega)$$

5.15 Finite Power Signals

Let us use $f(t) = \sin \omega_0 t$ to motivate discussion about finite power signals.

All periodic signals are finite power, infinite energy, signals. One cannot evaluate $\int_{-\infty}^{\infty} |\sin \omega_0 t|^2 \mathrm{d}t$.

However, by sketching the curve and using the notion of self-similarity, one would wish that the auto-correlation is positive, but decreasing, for small but increasing τ; then negative as the the curves are in anti-phase and dissimilar in an "organized" way, then return to being similar. The autocorrelation should have the same period as its parent function, and large when $\tau = 0$ — so R_{ff} proportional to $\cos(\omega_0 \tau)$ would seem right.

We define the autocorrelation as an average power. Note that for a periodic

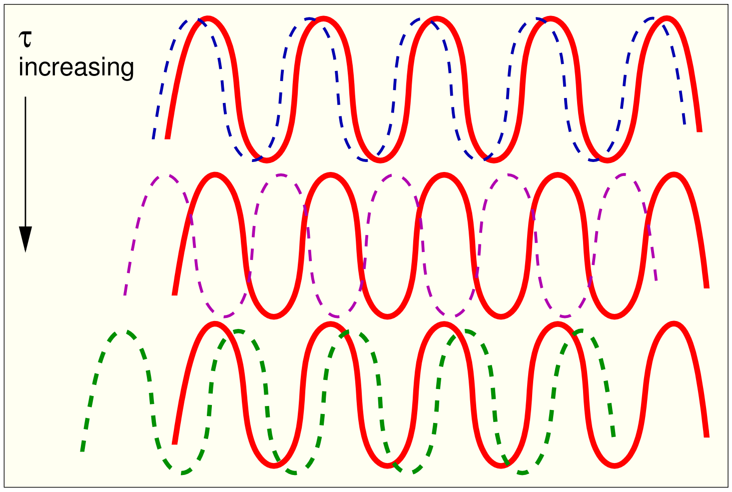


Figure 5.6:

function the limit over all time is the same as the value over a period T_0

$$\begin{aligned} R_{ff}(\tau) &= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \sin(\omega_0 t)\sin(\omega_0(t+\tau))\mathrm{d}t \\ &\to \frac{1}{2(T_0/2)}\int_{-T_0/2}^{T_0/2} \sin(\omega_0 t)\sin(\omega_0(t+\tau))\mathrm{d}t \\ &= \frac{\omega_0}{2\pi}\int_{-\pi/\omega_0}^{\pi/\omega_0} \sin(\omega_0 t)\sin(\omega_0(t+\tau))\mathrm{d}t \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin(p)\sin(p+\omega_0\tau))\mathrm{d}p \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left[\sin^2(p)\cos(\omega_0\tau) + \sin(p)\cos(p)\sin(\omega_0\tau)\right]\mathrm{d}p = \frac{1}{2}\cos(\omega_0\tau) \end{aligned}$$

For a finite energy signal, the Fourier Transform of the autocorrelation was the energy spectral density. What is the analogous result now? In this example,

$$\mathcal{FT}\left[R_{ff}\right] = \frac{\pi}{2}\left[\delta(\omega+\omega_0) + \delta(\omega-\omega_0)\right]$$

This is actually the *power* spectral density of $\sin\omega_0 t$, denoted $S_{ff}(\omega)$. The δ-functions are obvious enough, but to check the coefficient let us integrate over all frequency f:

$$\begin{aligned} \int_{-\infty}^{\infty} S_{ff}(\omega)\mathrm{d}f &= \int_{-\infty}^{\infty} \frac{\pi}{2}\left[\delta(\omega+\omega_0) + \delta(\omega-\omega_0)\right]\mathrm{d}f \\ &= \int_{-\infty}^{\infty} \frac{\pi}{2}\left[\delta(\omega+\omega_0) + \delta(\omega-\omega_0)\right]\frac{\mathrm{d}\omega}{2\pi} = \frac{1}{4}[1+1] = \frac{1}{2}\ . \end{aligned}$$

This does indeed return the average power in a sine wave. We can use Fourier *Series* to conclude that this results must also hold for any periodic function. It is also applicable to any infinite energy "non square-integrable" function. We will justify this a little more in Lecture 6[1].

To finish off, we need only state the analogies to the finite energy formulae, replacing Energy Spectral Density with Power Spectral Density, and replacing Total Energy with Average Power.

The autocorrelation of a finite power signal is defined as

$$R_{ff}(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(t)f(t+\tau)\mathrm{d}t \ .$$

The autocorrelation function and Power Spectral Density are a Fourier Transform Pair

$$R_{ff}(\tau) \Leftrightarrow S_{ff}(\omega)$$

The average power is

$$P_{\text{Ave}} = R_{ff}(0)$$

The power spectrum transfers across a system as

$$S_{gg}(\omega) = |H(\omega)|^2 S_{ff}(\omega)$$

This result is proved in the next lecture.

5.16 Cross-correlation and power signals

Two power signals can be cross-correlated, using a similar definition:

$$R_{fg}(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(t)g(t+\tau)\mathrm{d}t$$

$$R_{fg}(\tau) \Leftrightarrow S_{fg}(\omega)$$

5.17 Input and Output from a system

One very last thought. If one applies an finite power signal to a system, it cannot be converted into a finite energy signal — or vice versa.

[1] To really nail it would require us to understand Wiener-Khinchin in too much depth.