# Topological algebras on Boolean spaces as dual spaces and applications in formal language theory

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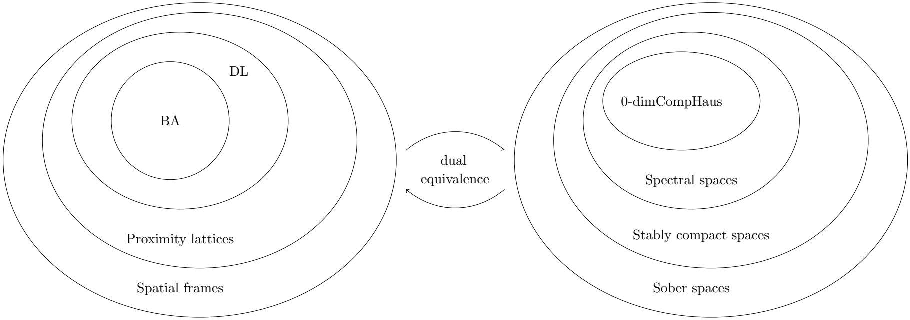
# Stone duality

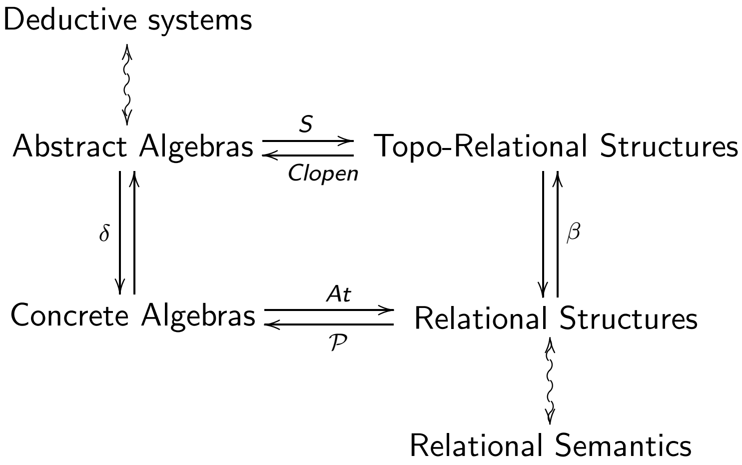



$A \twoheadrightarrow B$ quotient
$A \hookrightarrow B$ subalgebra
$A \oplus B$ coproduct
$A \times B$ product
$f : A^n \to A$ operation

$X \hookleftarrow Y$ subspace
$X \twoheadleftarrow Y$ quotient space
$X \times Y$ product
$X \uplus Y$ sum
$R \subseteq X \times X^n$ dual relation*

* Jonsson-Tarski 1951 for BAOs

# Duality theory in semantics I



# Duality theory in semantics II

- $\lambda$-calculus (a functional calculus allowing self application)

  $\lambda x.xx \quad \overset{\text{semantics???}}{\longrightarrow} \quad$ Scott's model

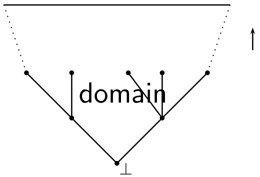
  Scott's models are Stone dual spaces

- Domain theory

  $$\frac{(M, N) : \sigma \times \tau}{M : \sigma, N : \tau}$$

  program logic
  for program specification

  $\overset{\text{semantics???}}{\longrightarrow}$




Abramsky: Domains are obtainable as dual spaces of program logics

# Duality theory in logic and computer science

Duality theory has been very successful in semantics. It often plays a role in:

- Completeness: Duality helps in obtaining semantics
- Decidability: Sometimes the dual of a problem is easier to solve.

So far, there have been very few applications of duality theory in complexity theory

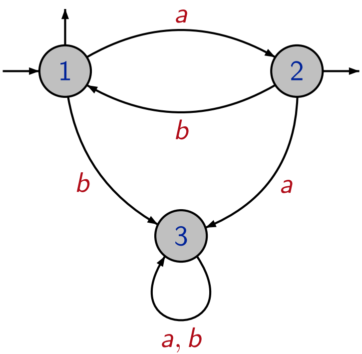
# Duality theory in complexity theory

In complexity theory computing machines are studied, e.g., through corresponding formal languages

Typical problems are decidability, separation, and comparison of complexity classes

In joint work with Serge Grigorieff and Jean-Eric Pin we have shown that duality theory is responsible for the standard tool for proving decidability results in automata theory

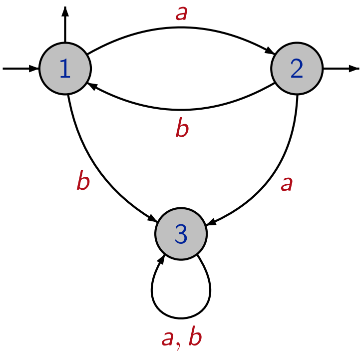
# A finite automaton




The states are $\{1, 2, 3\}$.
The initial state is 1, the final states are 1 and 2.
The alphabet is $A = \{a, b\}$ The transitions are

$$1 \cdot a = 2 \qquad 2 \cdot a = 3 \qquad 3 \cdot a = 3$$
$$1 \cdot b = 3 \qquad 2 \cdot b = 1 \qquad 3 \cdot b = 3$$

# Recognition by automata




Transitions extend to words: $1 \cdot aba = 2$, $1 \cdot abb = 3$.

The language recognized by the automaton is the set of words $u$ such that $1 \cdot u$ is a final state. Here:

$$L(\mathcal{A}) = (ab)^* \cup (ab)^* a$$

where $*$ means arbitrary iteration of the product.

# Rational and recognizable languages

A language is recognizable provided it is recognized by some finite automaton.

A language is rational provided it belongs to the smallest class of languages containing the finite languages which is closed under union, product and star.

Theorem: [Kleene '54] A language is rational iff it is recognizable.

Example: $L(\mathcal{A}) = (ab)^* \cup (ab)^* a$.

# Logic on words

To each non-empty word $u$ is associated a structure

$$\mathcal{M}_u = (\{1, 2, \ldots, |u|\}, <, (\mathbf{a})_{a \in A})$$

where **a** is interpreted as the set of integers $i$ such that the $i$-th letter of $u$ is an $a$, and $<$ as the usual order on integers.

Example:

Let $u = abbaab$ then

$$\mathcal{M}_u = (\{1, 2, 3, 4, 5, 6\}, <, (\mathbf{a}, \mathbf{b}))$$

where $\mathbf{a} = \{1, 4, 5\}$ and $\mathbf{b} = \{2, 3, 6\}$.

# Some examples

The formula $\phi = \exists x\ \mathbf{a}x$ interprets as:

> *There exists a position $x$ in $u$ such that*
> *the letter in position $x$ is an $a$.*

This defines the language $L(\phi) = A^* a A^*$.

The formula $\exists x\ \exists y\ (x < y) \wedge \mathbf{a}x \wedge \mathbf{b}y$ defines the language $A^* a A^* b A^*$.

The formula $\exists x\ \forall y\ [(x < y) \vee (x = y)] \wedge \mathbf{a}x$ defines the language $aA^*$.

# Defining the set of words of even length

Macros:

$$(x < y) \vee (x = y) \quad \text{means } x \leqslant y$$

$$\forall y \; x \leqslant y \quad \text{means } x = 1$$

$$\forall y \; y \leqslant x \quad \text{means } x = |u|$$

$$x < y \;\wedge\; \forall z \; (x < z \to y \leqslant z) \quad \text{means } y = x + 1$$

Let $\phi \;=\; \exists X \;\; (1 \notin X \;\wedge\; |u| \in X \;\wedge\; \forall x \; (x \in X \leftrightarrow x + 1 \notin X))$

Then $1 \notin X$, $2 \in X$, $3 \notin X$, $4 \in X$, ..., $|u| \in X$. Thus

$$L(\phi) = \{u \mid |u| \text{ is even}\} = (A^2)^*$$

# Monadic second order

Only second order quantifiers over unary predicates are allowed.

Theorem: (Büchi '60, Elgot '61)

Monadic second order captures exactly the recognizable languages.

Theorem: (McNaughton-Papert '71)

First order captures star free languages

(star free = the ones that can be obtained from the alphabet using the Boolean operations on languages and lifted concatenation product only).

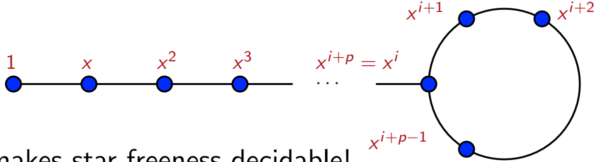
How does one decide the complexity of a given language???

# Algebraic theory of automata

Theorem: [Myhill '53, Rabin-Scott '59] There is an effective way of associating with each finite automaton, $\mathcal{A}$, a finite monoid, $(M_{\mathcal{A}}, \cdot, 1)$.

Theorem: [Schützenberger '65] Star free languages correspond to aperiodic monoids, i.e., $M$ such that there exists $n > 0$ with $x^n = x^{n+1}$ for each $x \in M$.

Submonoid generated by $x$:




This makes star freeness decidable!

# An example

$L = (ab)^*$ $\longrightarrow$ $\mathcal{M}(L) =$

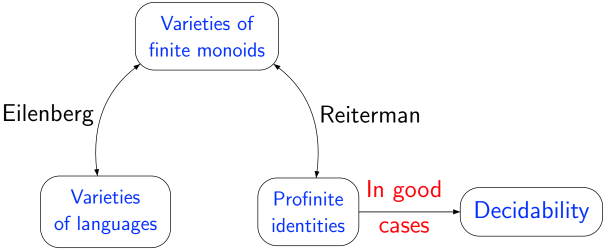
| $\cdot$ | 1 | $a$ | $ba$ | $b$ | $ab$ | 0 |
|---|---|---|---|---|---|---|
| 1 | 1 | $a$ | $ba$ | $b$ | $ab$ | 0 |
| $a$ | $a$ | 0 | $a$ | $ab$ | 0 | 0 |
| $ba$ | $ba$ | 0 | $ba$ | $b$ | 0 | 0 |
| $b$ | $b$ | $ba$ | $b$ | 0 | $b$ | 0 |
| $ab$ | $ab$ | $a$ | 0 | 0 | $ab$ | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |

Syntactic monoid

This monoid is aperiodic since $1 = 1^2$, $a^2 = 0 = a^3$, $ba = ba^2$, $b^2 = 0 = b^3$, $ab = ab^2$, and $0 = 0^2$

Indeed, $L$ is star-free since $L^c = bA^* \cup A^*a \cup A^*aaA^* \cup A^*bbA^*$ and $A^* = \emptyset^c$

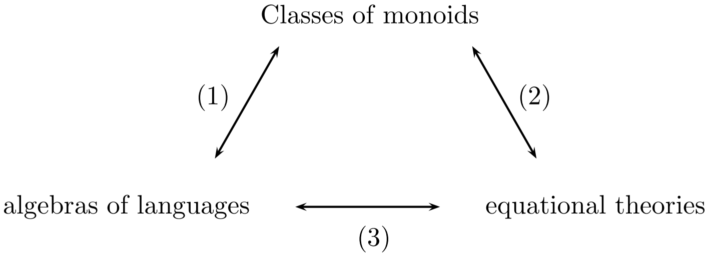
# Eilenberg-Reiterman theory




A variety of monoids here means a class of finite monoids closed under homomorphic images, submonoids, and finite products

Various generalisations: [Pin 1995], [Pin-Weil 1996], [Pippenger 1997], [Polák 2001], [Esik 2002], [Straubing 2002], [Kunc 2003]

# Eilenberg, Reiterman, and Stone




(1) Eilenberg theorems
(2) Reiterman theorems
(3) extended Stone/Priestley duality

(3) allows generalisation to non-varieties and even to non-regular languages

# Connection between duality and Eilenberg-Reiterman I

- The syntactic monoid of a language $L$ is the dual of a certain BAO generated by $L$ in $\mathcal{P}(A^*)$
- The free profinite monoid, $\widehat{A^*}$, the dual of $\mathrm{Rec}(A^*)$ equipped with certain residuation operations
- Sublattices of $\mathrm{Rec}(A^*)$ correspond via duality to quotients of $\widehat{A^*}$ (and hence equations/pairs in $\widehat{A^*} \times \widehat{A^*}$)

# Connection between duality and Eilenberg-Reiterman II

- The dual of a continuous operation

$$\cdot : X \times X \to X$$

should be a coalgebraic structure

$$h : A \to A \oplus A$$

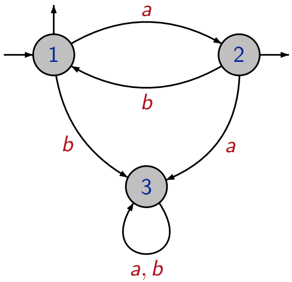
(this is the approach in classical algebra; see also Steinberg and Rhodes)

- It turns out that in an order theoretic setting, the residuals of the product encode this algebra giving and algebraic dual to a topological algebra

- From a lattices and order point of view, residuals are generalised implications, and the pertinent structures are closely related to nuclei.

# The residuals of the concatenation product

Consider a finite state automato




The language recognized by $\mathcal{A}$ is $L(\mathcal{A}) = (ab)^* \cup (ab)^* a$

Quotient operations on languages:

$$a^{-1}L = \{u \in A^* \mid au \in L\} = (ba)^* b \cup (ba)^*$$

$$La^{-1} = \{u \in A^* \mid ua \in L\} = (ab)^*$$

$$b^{-1}L = \{u \in A^* \mid bu \in L\} = \emptyset$$

All recognised by the same underlying machine!

# Capturing the underlying machine

Given a recognizable language $L$ the underlying machine is captured by the Boolean algebra $\mathcal{B}(L)$ of languages generated by

$$\{\ x^{-1}Ly^{-1} \mid x, y \in A^*\}$$

NB! This generating set is finite since all the languages are recognized by the same machine with varying sets of initial and final states.

NB! $\mathcal{B}(L)$ is closed under quotients since the quotient operations commute will all the Boolean operations.

# The residuation ideal generated by a language

Since $\mathcal{B}(L)$ is finite it is also closed under residuation. That is, for $M \in \mathcal{B}(L)$

$$S \backslash M = \bigcap_{u \in S} u^{-1} M \in \mathcal{B}(L)$$

$$M/S = \bigcap_{u \in S} M u^{-1} \in \mathcal{B}(L)$$

These are the upper adjoints in the left and right coordinate of the lifted product on $\mathcal{P}(A^*)$

$$KL \subseteq M \iff L \subseteq K \backslash M \iff K \subseteq M/L$$

$(\mathcal{B}(L), \backslash, /)$ is a Boolean Algebra with additional Operations (BAO)

# The syntactic monoid of a recognizable language

[G-Grigorieff-Pin 2008]

The relation dual to $\backslash$ and $/$ on $\mathcal{B}(L)$ is a function

$$f : X \times X \to X$$

Theorem: The dual space of the BAO $(\mathcal{B}(L(\mathcal{A})), \backslash, /)$ is the syntactic monoid of $L(\mathcal{A})$

# Boolean topological algebras

We call a topological algebra of some algebraic type $\tau$ Boolean provided the underlying topological space is Boolean

Theorem: Let $X$ be a Boolean space, $f : X^n \to X$ any function, and $R \subseteq X^n \times X$ its graph. The the following are equivalent:

- $R$ is a dual relation with $i$ as the output coordinate for some (and then for all) $1 \leqslant i \leqslant n$
- $f$ is continuous

Corollary: All Boolean topological algebras are dual spaces of certain residuation algebras (as are all Priestley topological algebras)

# Duals of topological algebra morphisms...

...are different (and incomparable) to residuation algebra morphisms in general

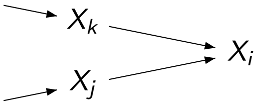
A special well behaved case: the dual of a Boolean topological algebra quotient is a Boolean residuation ideal:

$C \hookrightarrow B$ Boolean residuation subalgebra with $b \backslash c$ and $c/b \in C$ for all $b \in B$ and $c \in C$

# Characterization of profinite algebras

The inverse limit system $\mathcal{F}$

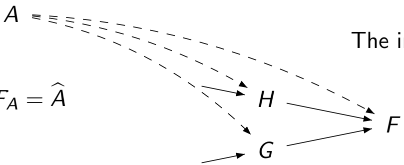
$$\varprojlim \mathcal{F} = X$$




All the $X_i$'s are finite topological algebra quotients, so by duality the dual Boolean residuation algebra is a directed union of finite Boolean residuation ideals

<u>Theorem:</u> A Boolean topological algebra $X$ is profinite iff each finitely generated Boolean residuation ideal of the dual algebra is finite

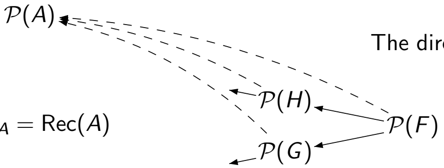
# Profinite completions




The inverse limit system $\mathcal{F}_A$

$\varprojlim \mathcal{F}_A = \widehat{A}$

is dual to




The direct limit system $\mathcal{G}_A$

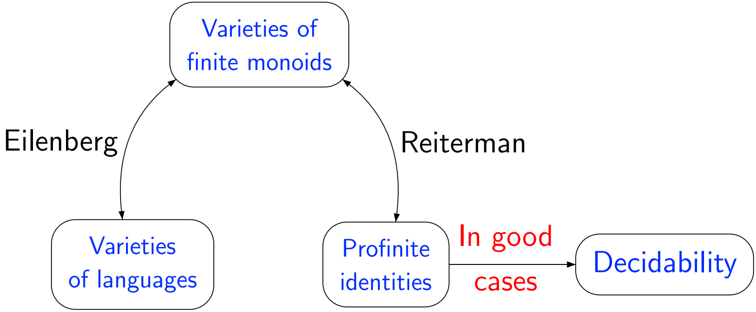
$\varinjlim \mathcal{G}_A = \mathrm{Rec}(A)$

$$\begin{aligned}\varinjlim \mathcal{G}_A &= \bigcup \{\varphi^{-1}(\mathcal{P}(F)) \mid \varphi : A \twoheadrightarrow F \text{ finite quotient}\} \\ &= \{\varphi^{-1}(P) \mid \varphi : A \twoheadrightarrow F \text{ finite quotient and } P \subseteq F\} = \mathrm{Rec}(A)\end{aligned}$$

# Profinite completions

Theorem: [G-Grigorieff-Pin 2008] The profinite completion of ANY algebra is the dual space of the BAO Rec($A$) with the residuals of the lifted operations

# Eilenberg-Reiterman theory




[Eilenberg76] + [Reiterman82]

# Characterizing subclasses of languages

[G-Grigorieff-Pin 2008]

subalgebras $\longleftrightarrow$ quotient structures

$\mathcal{C}$ a class of recognizable languages closed under $\cap$ and $\cup$

$$\mathcal{C} \hookrightarrow Rec(A^*)$$

DUALLY

$$X_{\mathcal{C}} \twoheadleftarrow \widehat{A^*}$$

That is, $\mathcal{C}$ is described dually by EQUATING elements of $\widehat{A^*}$.

This is a general form of Eilenberg-Reiterman theorem

# A Galois connection for subsets of an algebra

Let $B$ be a Boolean algebra, $X$ the dual space of $B$.

The maps $\mathcal{P}(B) \leftrightarrows \mathcal{P}(X \times X)$ given by

$$S \mapsto \approx_S = \{(x,y) \in X \mid \forall b \in S \quad (b \in y \iff b \in x)\}$$

and

$$E \mapsto B_E = \{b \in B \mid \forall (x,y) \in E \quad (b \in y \iff b \in x)\}$$

establish a Galois connection whose Galois closed sets are the Boolean equivalence relations and the Boolean subalgebras, respectively.

# Example

[Schützenberger 1965]

Star-free languages $\leqslant$ $\mathrm{Rec}(A^*)$

The equivalence relation on $\widehat{A^*}$ dual to this residuation ideal is generated in the Galois connection of the previous slide by the set

$$\{(ux^{\omega+1}v, ux^{\omega}v) \mid x, u, v \in \widehat{A^*}\}$$

That is, it is given by ONE pair, $(a^{\omega+1}, a^{\omega})$, when closing under:

- substitution
- monoid congruence
- Stone duality subalgebra-quotient adjunction

# References

The following are the papers from which the research discussed at the two talks I gave at AAA88 came:


1. Mai Gehrke, Serge Grigorieff, and Jean-Éric Pin, Duality and Equational Theory of Regular Languages, *LNCS* (ICALP) **5125** (2008), 246–257.

2. Mai Gehrke, Stone duality, topological algebra, and recognition, preprint. See, `http://hal.archives-ouvertes.fr/hal-00859717`

3. Mai Gehrke, Andreas Krebs, and Jean-Éric Pin, From ultrafilters on words to the expressive power of a fragment of logic, to appear in *Proceedings of the 16th International Workshop on Descriptional Complexity of Formal Systems*, 2014.