Topological Quantum Matter, 7.5 ECTS

physics.gu.se/~tfkhj/topomatter.html

Part 1: 10 double hour lectures 12/9 - 3/10, 27/10 - 31/10 (tentatively)
Part 2: 5 guest lectures 10/11 - 12/12

Examination: homework problems on part 1, project based on one of the guest lectures

Literature: Lecture notes downloadable from the course homepage.
Additional text/references will be made available during the course.

Masters students: To get credit for the course, please contact Bengt-Erik Mellander, f5xrk@chalmers.se, after completion of the examination.

Concepts in Topological Quantum Matter, 4.5 ECTS

physics.gu.se/~tfkhj/topomatter/

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What is topological quantum matter & why do we care?

Before the era of topology in physics...

Standard paradigm of emergent order:
spontaneous symmetry breaking

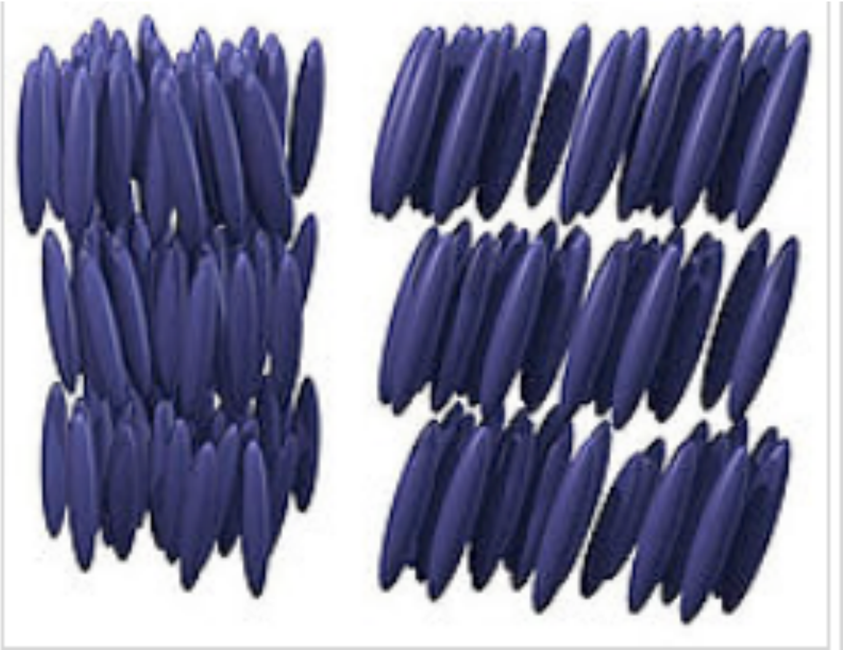
What is topological quantum matter
& why do we care?

Some examples:

Crystals break the translational and rotational symmetry of free space

What is topological quantum matter
& why do we care?

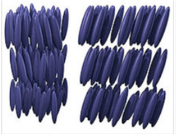
Crystals



Liquid crystals break rotational but not translational symmetry

What is topological quantum matter & why do we care?

Crystals

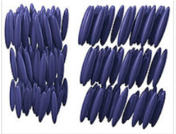


Liquid crystals

Magnets break time-reversal symmetry
and the rotational symmetry of spin space

What is topological quantum matter & why do we care?


Crystals

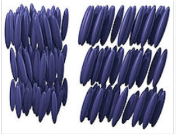

Liquid crystals


Magnets


Superconductors break a gauge symmetry

What is topological quantum matter
& why do we care?

Crystals



Liquid crystals

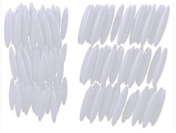
Magnets

Superconductors

... and many more examples

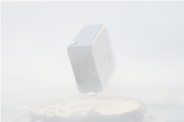
What is topological quantum matter
& why do we care?

Crystals



Liquid crystals

Magnets



Superconductors

... and many more examples

At high temperature, entropy dominates and leads to a disordered state.
At low temperature, energy dominates and leads to an ordered state.

The ordered state breaks one or more symmetries "spontaneously".

Lev Landau

What is topological quantum matter & why do we care?

In 1980, the ”Landau paradigm” was challenged by the discovery of the (integer) *quantum Hall effect* (von Klitzing et al.)

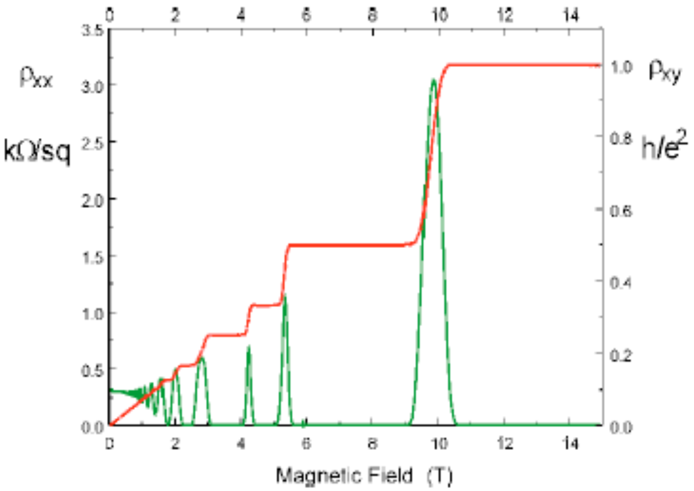
Electrons confined to a plane and in a strong magnetic field show, at low enough temperature, plateaus in the “Hall conductance”:

$$\sigma_{xy} = n\frac{e^2}{h}$$

to a precision of at least 9 decimal places!

There is no symmetry breaking. What type of order causes this precise quantization?

TOPOLOGICAL ORDER!



Topological quantum matter

”Symmetry-protected topological matter”

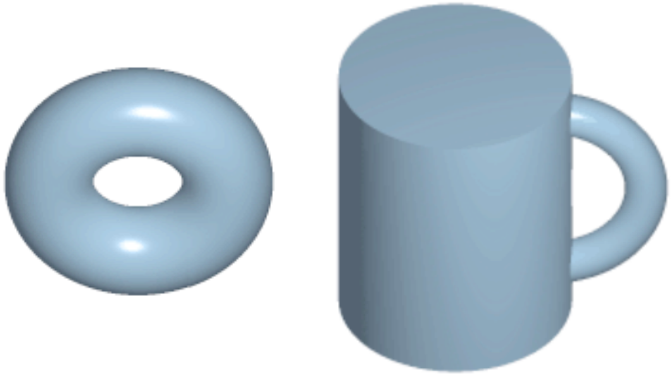
Unique groundstate protected by a topological invariant (Chern number, Z_2-index,...), ordinary electron excitations, short-range quantum entanglement:

integer quantum Hall effect,
topological insulators,
Chern insulators,
topological superconductors,...

”Topologically ordered matter” (proper)

Groundstate degeneracies on higher-genus manifolds, fractionalized excitations, long-range quantum entanglement:

fractional quantum Hall effect,
quantum spin liquids,...



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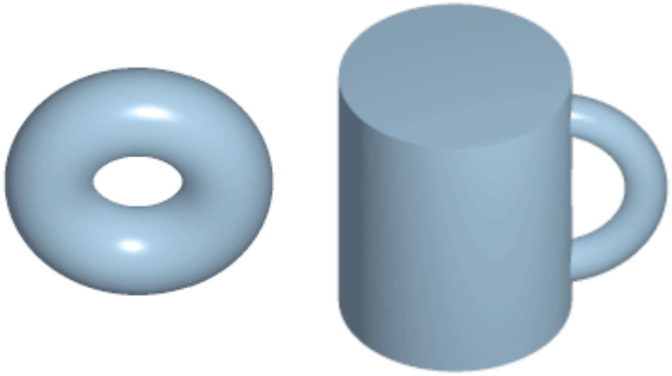


The topological insulators

provide the best ”port of entry” to the study of topological quantum matter.
This is how we shall go about it in the course!

”An **electrical insulator** is a material whose internal electric charges do not flow freely... examples include glass, paper, Teflon, plastics, ceramic materials,... ”

?

$$\int_M e(M) = \chi(M)$$

$$e(M) = \mathrm{Pf}(\mathcal{R}/2\pi) = \frac{(-1)^l}{(4\pi)^l l!}\sum_P \mathrm{sgn}(P)\mathcal{R}_{P(1)P(2)}\ldots\mathcal{R}_{P(2l-1)P(2l)}$$

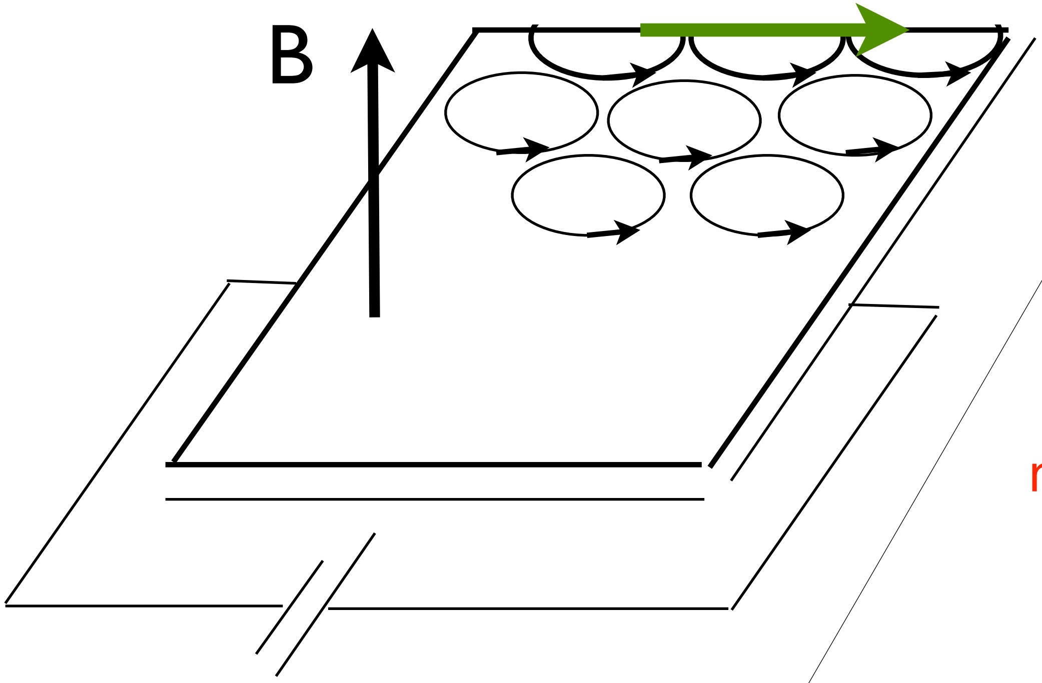
?

Ordinary insulators have nothing to do with topology, but *topological insulators* do!

2D topological insulators...

taking off from the quantum Hall effect

quantum Hall effect



“skipping current”

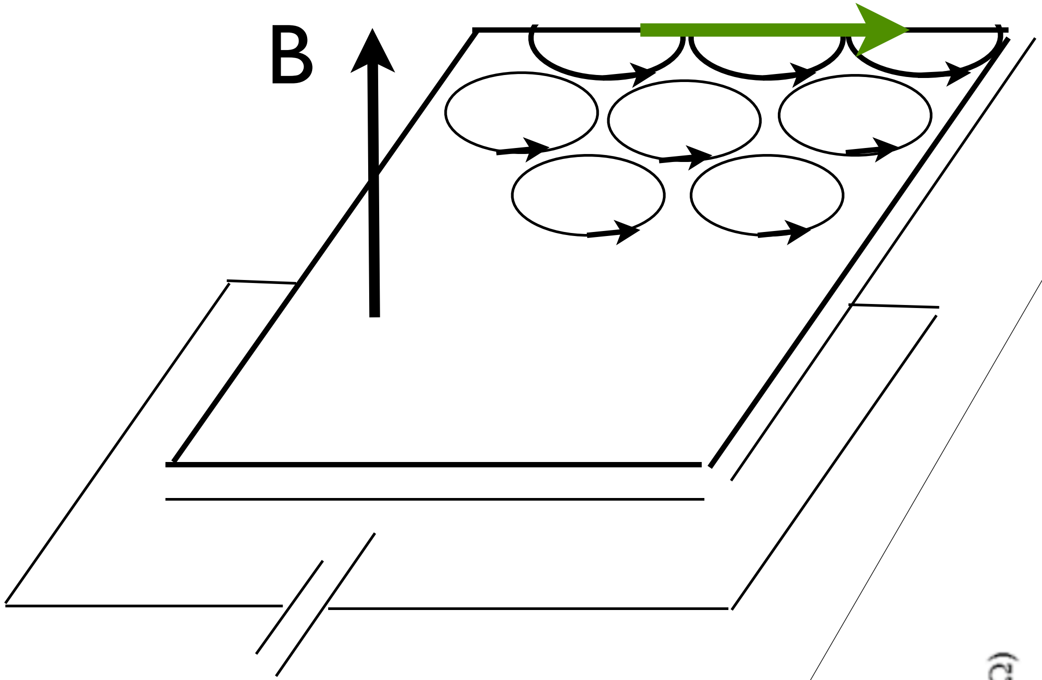
quantization

chiral edge states

no channel for backscattering

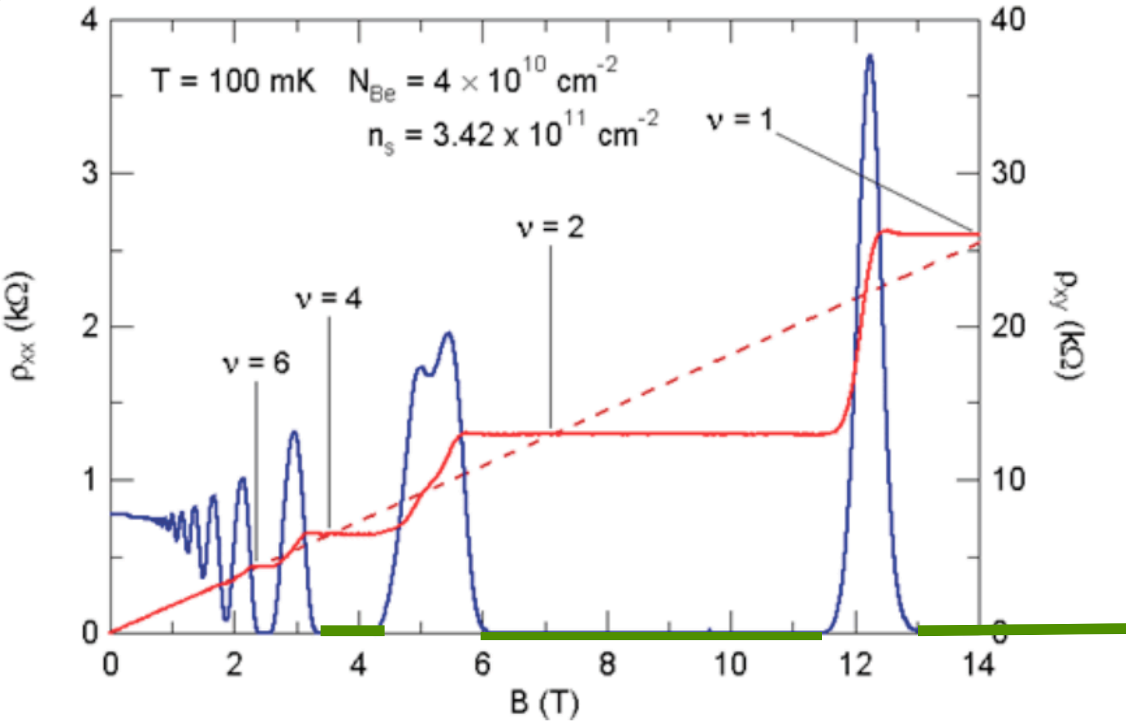
perfect conductance along the edge

quantum Hall effect



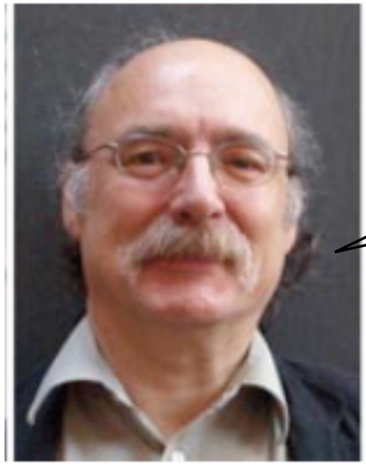
perfect conductance along the edge

von Klitzing *et al.*, PRL (1980)



a bulk insulator with perfectly
conducting edge states

Is this kind of physics possible without a magnetic field?


Duncan Haldane

Well..., at least one doesn't need a *net* magnetic field... *PRL, 1988*


Charlie Kane

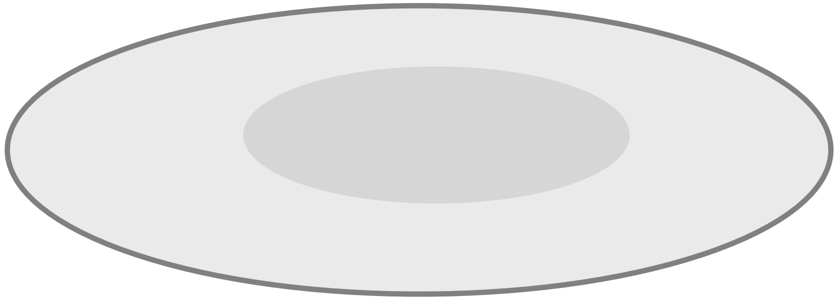

Gene Mele

In fact, one can do away with the magnetic field altogether! *PRLs, 2005*

To see how this is possible, consider a Gedanken experiment...

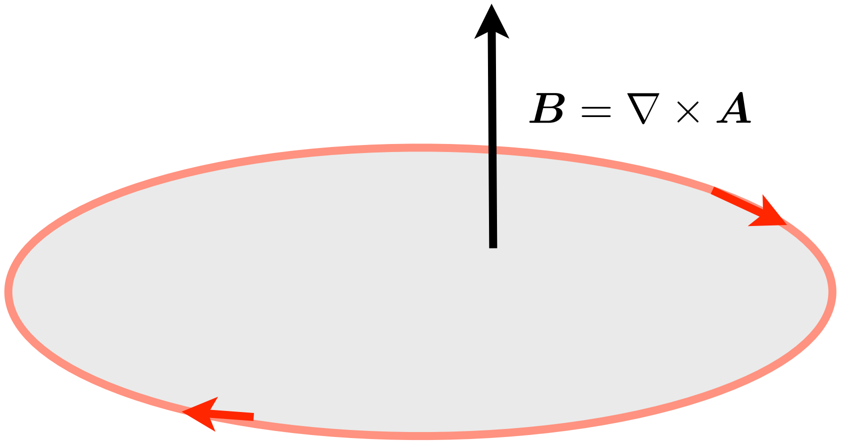
Bernevig & Zhang, PRL (2006)

To see how this is possible, consider a Gedanken experiment...



spin-orbit interaction

$$(\boldsymbol{E} \times \boldsymbol{k}) \cdot \boldsymbol{\sigma} = E\sigma^z(k_y x - k_x y)$$



compare with an integer quantum Hall system

Lorentz force

$$\boldsymbol{A} \cdot \boldsymbol{k} \sim eB(k_y x - k_x y)$$

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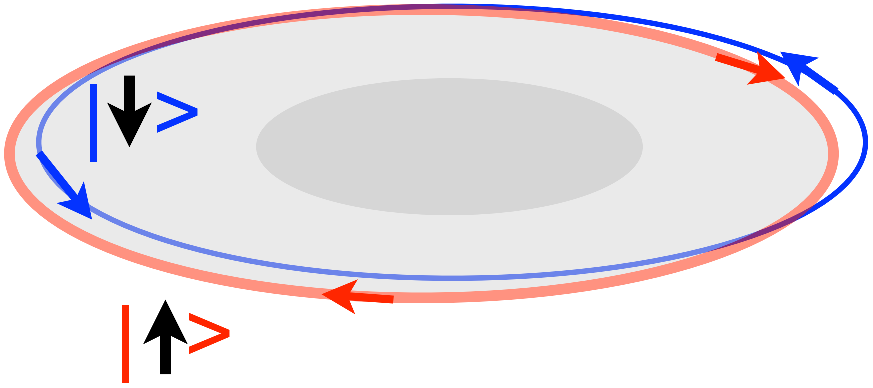

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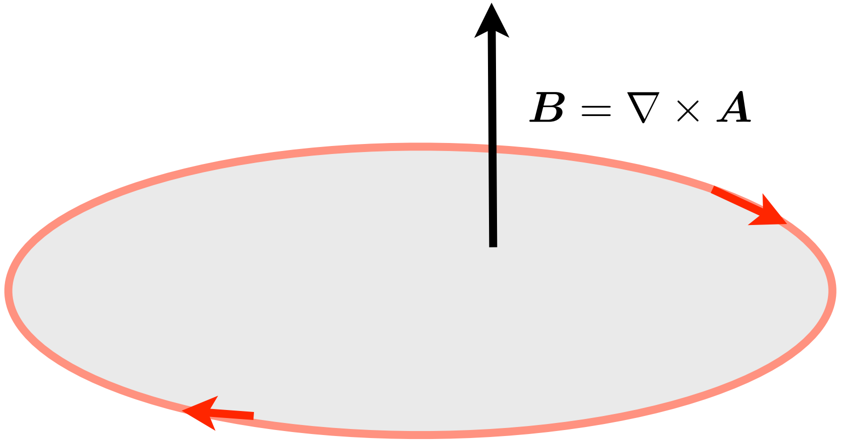

compare with an integer quantum Hall system

Lorentz force

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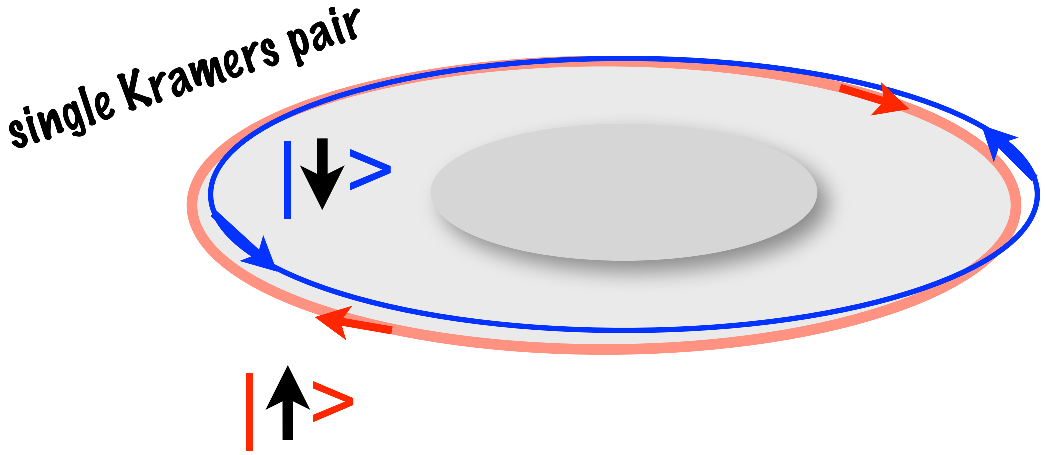
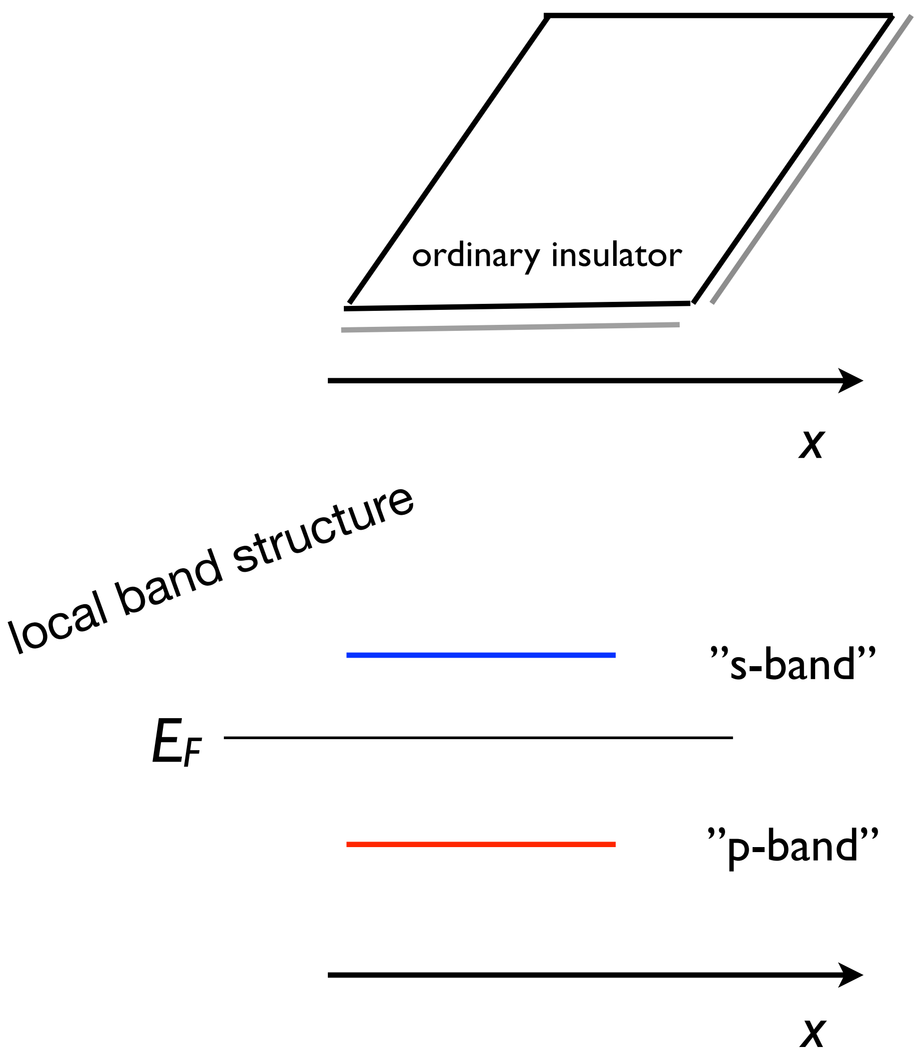
2D topological insulator

Two copies of a quantum Hall system, bulk insulator with **helical edge states**

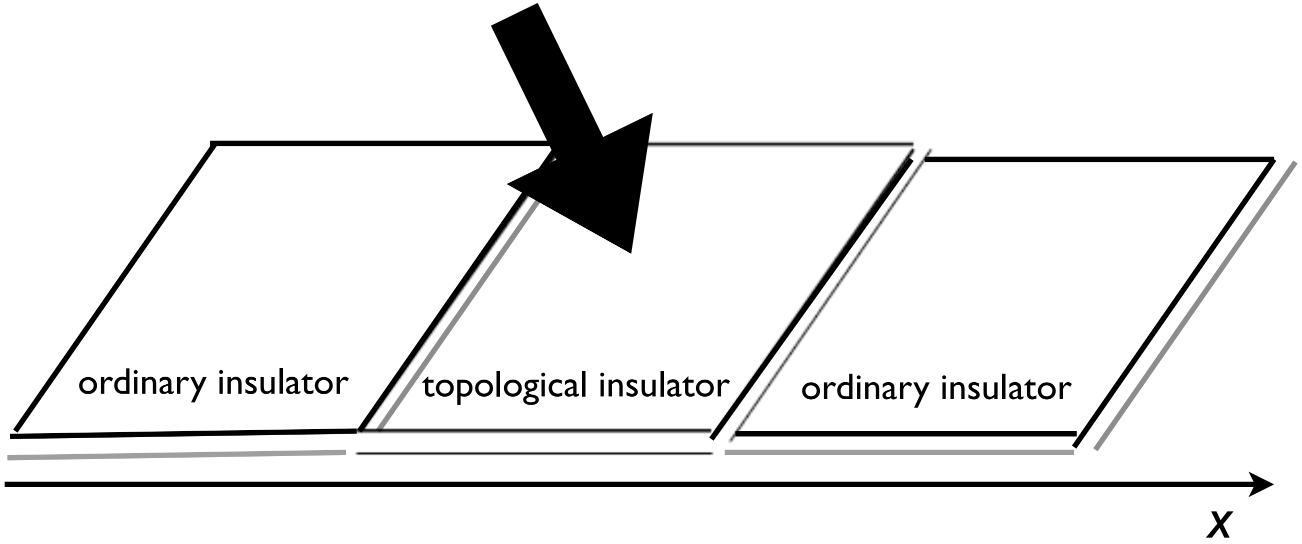


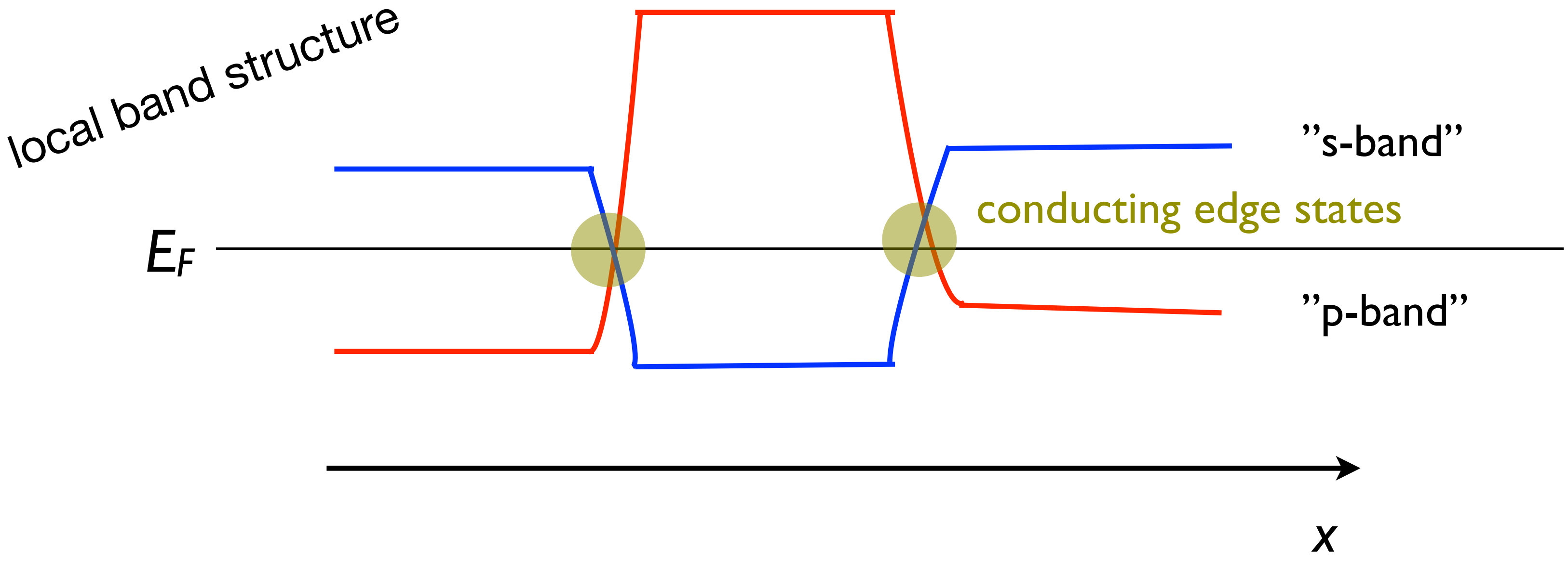
How does Nature do it?
Also by spin-orbit interactions!



How does Nature do it?

Strong atomic spin-orbit interactions!





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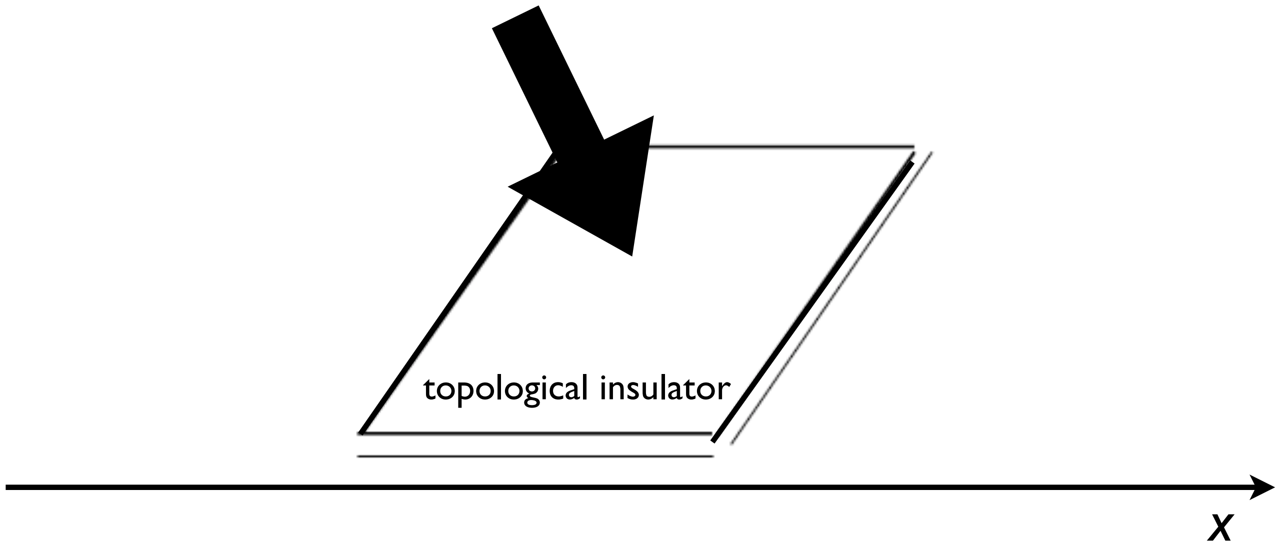
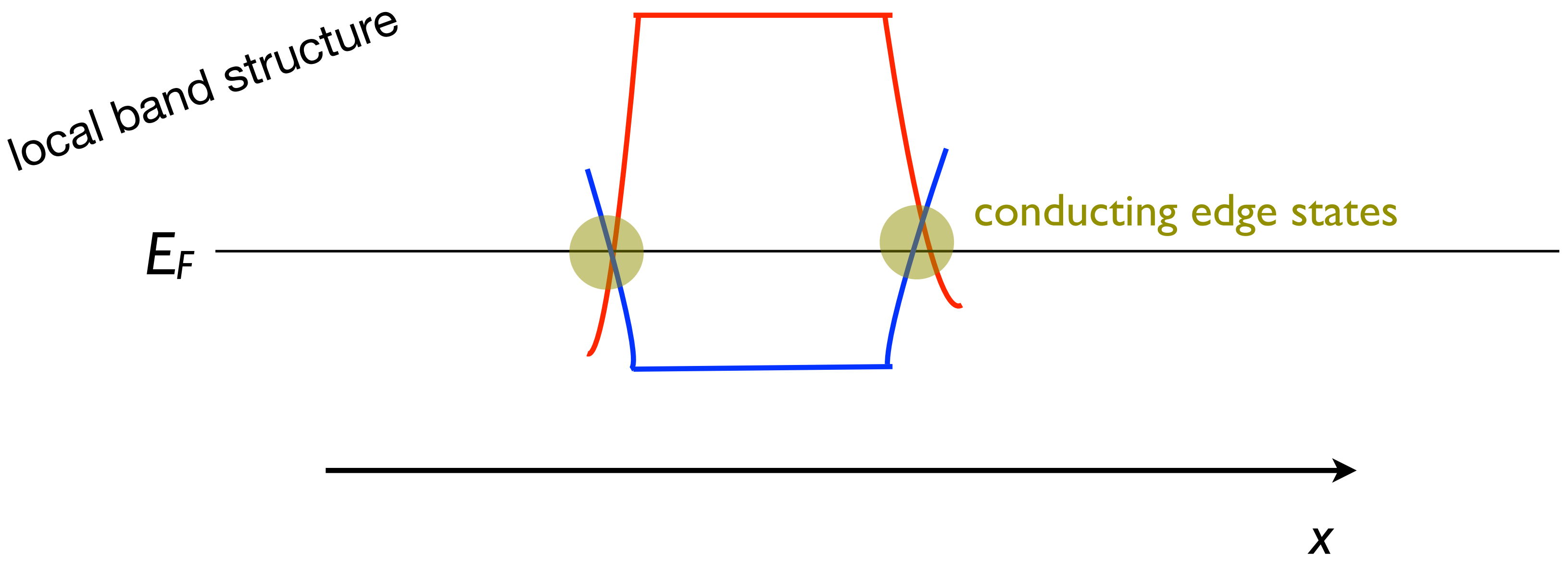
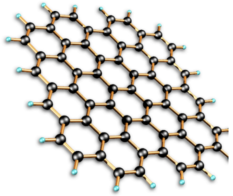


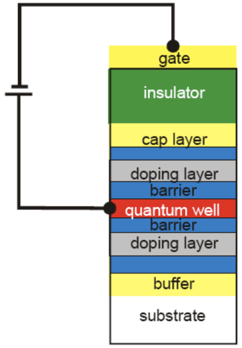


Experimental realizations...



First proposed by Kane and Mele for graphene (2005)

C. L. Kane and E. J. Mele, PRL **95**, 226801 (2005)

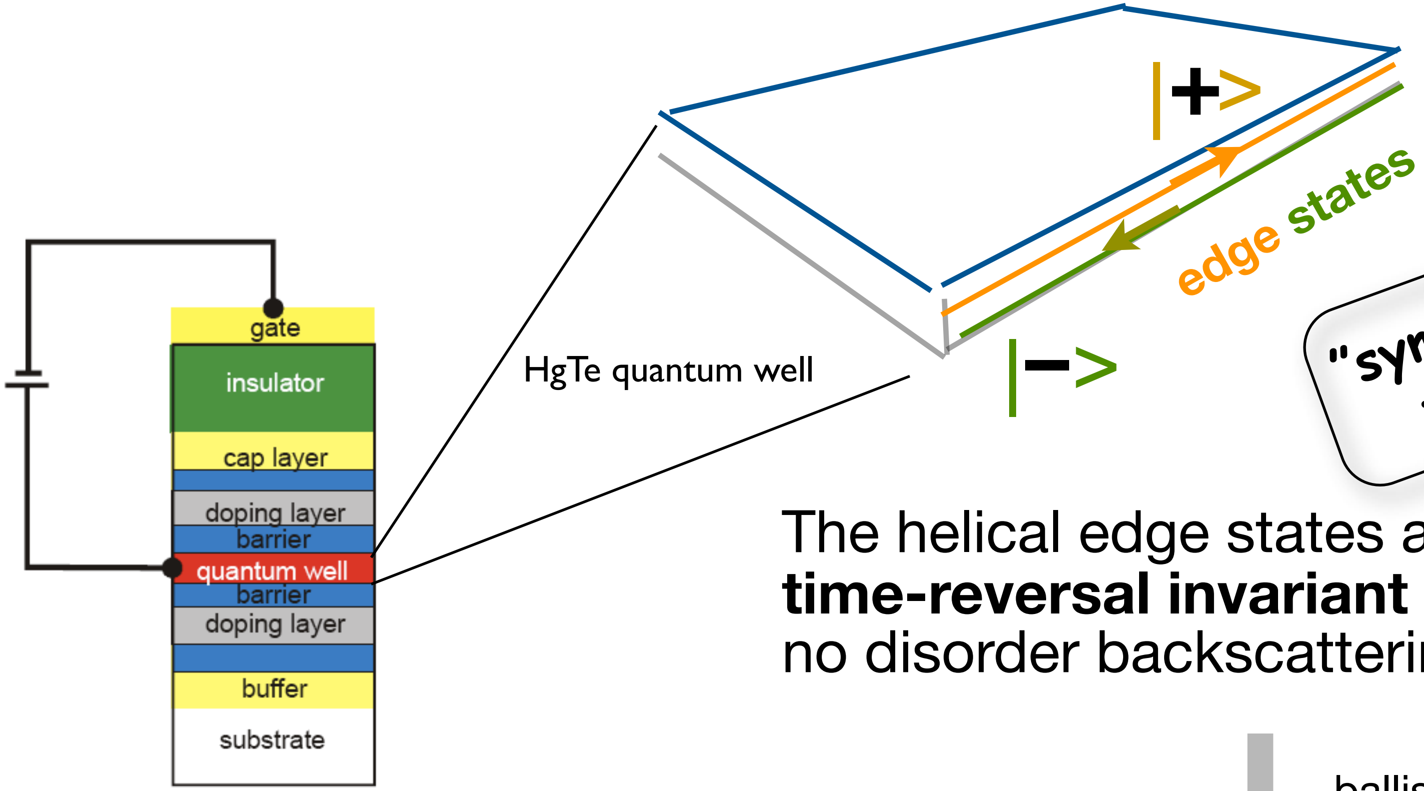


Bernevig et al. proposal for HgTe quantum wells (2006)

B. A. Bernevig, T. A. Hughes, and S. C. Zhang, Science **314**, 1757 (2006)

Experimental observation by König *et al.* (2007)

M. König *et al.*, Science **318**, 766 (2007)



"symmetry-protected" topological phase

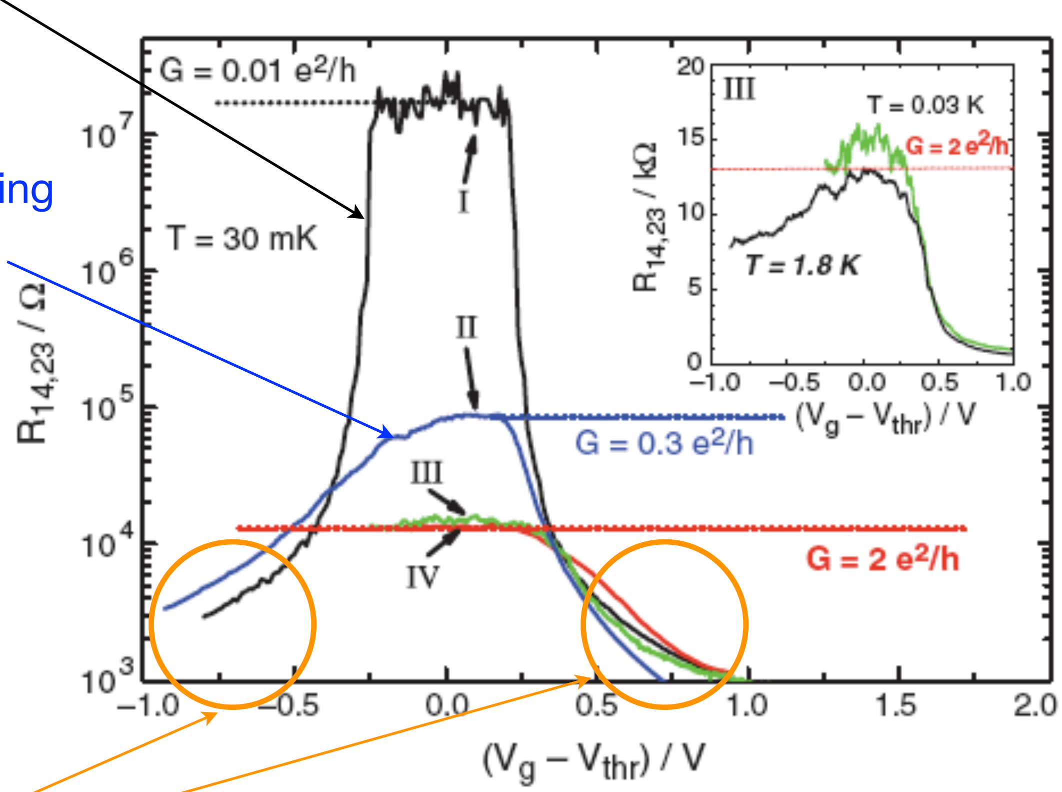
The helical edge states are stable against **time-reversal invariant perturbations,** no disorder backscattering

↓ ballistic transport

$$G = \frac{2e^2}{h}$$

normal band gap

large samples (inelastic scattering from the bulk)



Fermi level *not* inside the inverted gap

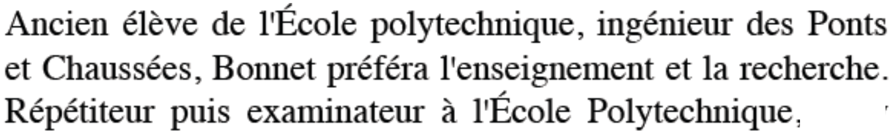
König *et al.* (2007)

What is ”topological” about a topological insulator?

Warm up... *Gauss-Bonnet theorem*

$$\frac{1}{2\pi}\int_{\mathcal{M}} K\,dS = 2(1-g)$$

Pierre Ossian Bonnet, 1819-1892



Ancien élève de l'École polytechnique, ingénieur des Ponts et Chaussées, Bonnet préféra l'enseignement et la recherche. Répétiteur puis examinateur à l'École Polytechnique,

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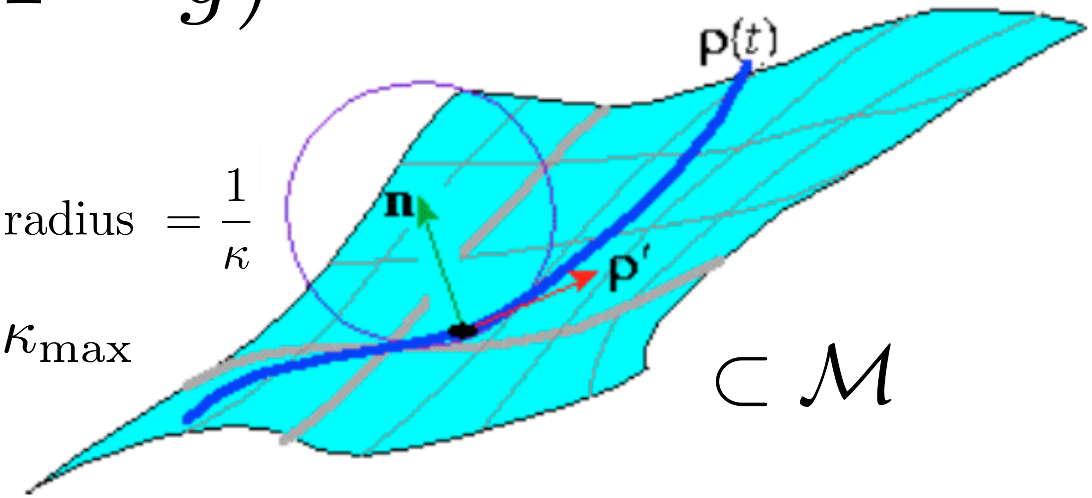


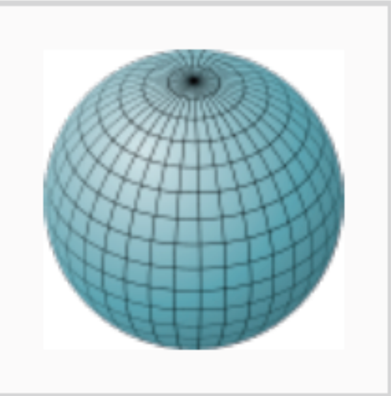
$$\text{radius} \ = \frac{1}{\kappa}$$

$$K = \kappa_{\text{min}} \cdot \kappa_{\text{max}}$$

$$\subset \mathcal{M}$$

What is ”topological” about a topological insulator?

$$\frac{1}{2\pi}\int_{\mathcal{M}} K dS = 2(1-g)$$



$g = 0$

$g = 1$

$g = 2$

$g = 3$

What is ”topological” about a topological insulator?

”Chern theorem”

Generalized *Gauss-Bonnet theorem*

Berry curvature

$$\frac{1}{2\pi}\int_{\mathcal{M}} \mathcal{F}\cdot d\boldsymbol{S} = C$$

Chern number

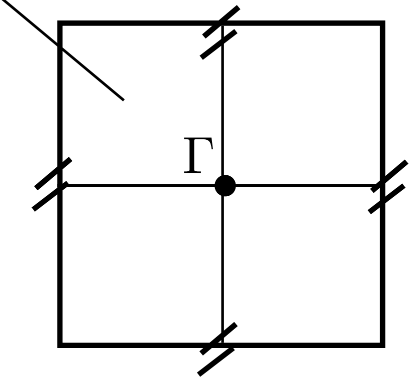
Shiing-Shen Chern, 1911-2004

$$\frac{1}{2\pi}\int_{\mathcal{M}} \mathcal{F}\cdot d\boldsymbol{S} = C$$

Electron wave function in a crystal

$$\psi_{n,\boldsymbol{k}}(\boldsymbol{r}) = u_{n,\boldsymbol{k}}(\boldsymbol{r})\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{r}}$$

band index



Brillouin zone (BZ)

$$\frac{1}{2\pi}\int_{\mathcal{M}} \mathcal{F}\cdot d\boldsymbol{S} = C$$

Electron wave function in a crystal

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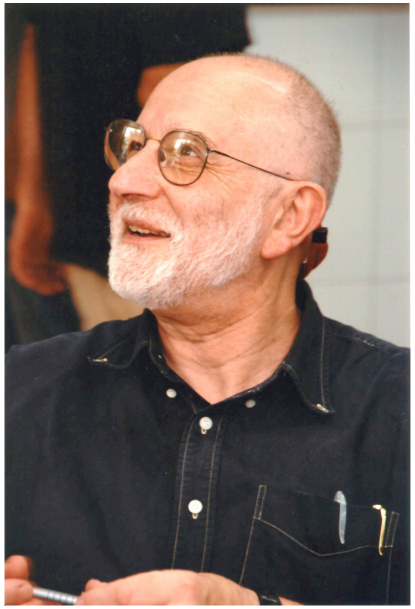
Berry curvature Berry, RSPSA (1984)

$$\boldsymbol{\mathcal{F}}_n(\boldsymbol{k}) = -i\nabla_{\boldsymbol{k}} \times \langle u_{n,\boldsymbol{k}} \mid \nabla_{\boldsymbol{k}} \mid u_{n,\boldsymbol{k}}\rangle$$



Michael Berry

$$\frac{1}{2\pi}\int_{\mathcal{M}} \mathcal{F} \cdot d\boldsymbol{S} = C$$

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$$\frac{1}{2\pi}\sum_{n=1}^{\sharp\ \text{filled bands}} \int_{\mathcal{M}=\mathrm{BZ}} \mathcal{F}_{n,z}(\boldsymbol{k})\, dk_x\, dk_y = C$$

Electron wave function in a crystal

$$\psi_{n,\boldsymbol{k}}(\boldsymbol{r}) = u_{n,\boldsymbol{k}}(\boldsymbol{r})\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{r}}$$

odd under
time reversal

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vanishing
Chern number

broken time-reversal symmetry (like in the quantum Hall effect)

Electron wave function in a crystal

$$\psi_{n,\boldsymbol{k}}(\boldsymbol{r}) = u_{n,\boldsymbol{k}}(\boldsymbol{r})\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{r}}$$

~~odd under time reversal~~

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integer quantum Hall effect: ”TKNN invariant”

Thouless *et al.*, PRL (1982)

David Thouless

What if we identify time-reversed points in the BZ?

Electron wave function in a crystal

$$\psi_{n,\boldsymbol{k}}(\boldsymbol{r}) = u_{n,\boldsymbol{k}}(\boldsymbol{r})\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{r}}$$

odd under
time reversal

Berry curvature Berry, RSPSA (1984)

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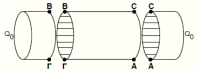
$$\frac{1}{2\pi}\sum_{n}\int \boldsymbol{\mathcal{F}}_n(\boldsymbol{k})\cdot d\boldsymbol{k} = C$$



The *parity* of the Chern number is unique!

Kane & Mele, PRL (2005)

$$C = \begin{cases} 0 \text{ mod } 2 & \text{ordinary insulator} \\ 1 \text{ mod } 2 & \text{topological insulator} \end{cases}$$

”Z_2 topological invariant”, counts the number of Kramers pairs at the edge of the topological insulator

(bulk-boundary correspondence)

Why all the hoopla?

“Who here is familiar with the concept of Topological Insulators?”

TV-Series: „The Big Bang Theory“,
https://www.youtube.com/watch?v=HBuLMrzgbgM

BREAKTHROUGH OF THE YEAR
ELECTRONS TAKE A NEW SPIN

STAR MATERIAL

A new class of materials is poised to take condensed-matter physics by storm. **Geoff Brumfiel** looks at what is making topological insulators all the rage.

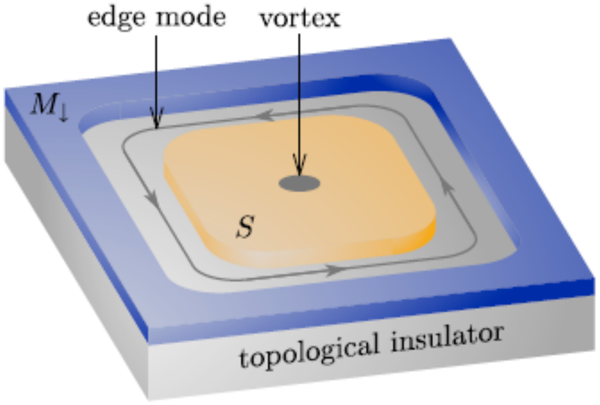
Why all the hoopla?

- Future electronics/spintronics:

Toward
dissipationless
spin transport
in semiconductors

- New physics in hybrid structures:

  Majorana fermions, magnetic monopoles, "dyons",...

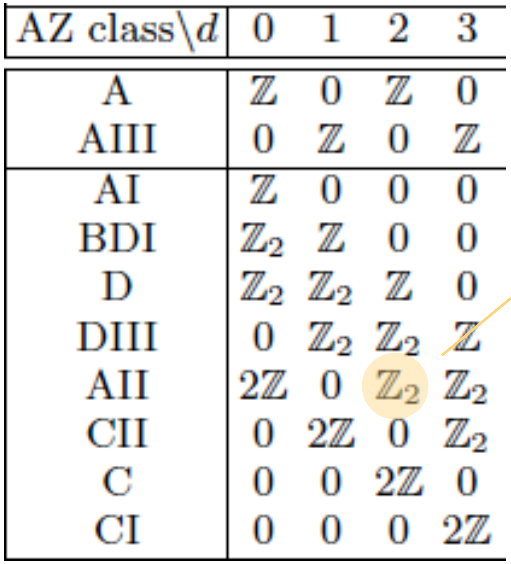


- An inroad to the general study of

  TOPOLOGICAL QUANTUM MATTER

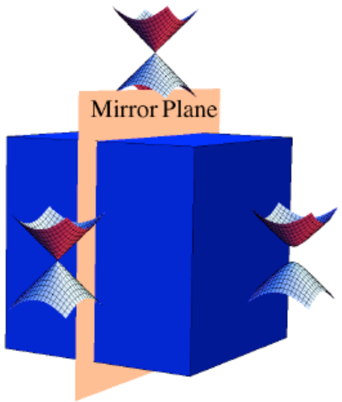
An inroad to the general study of **TOPOLOGICAL QUANTUM MATTER…**

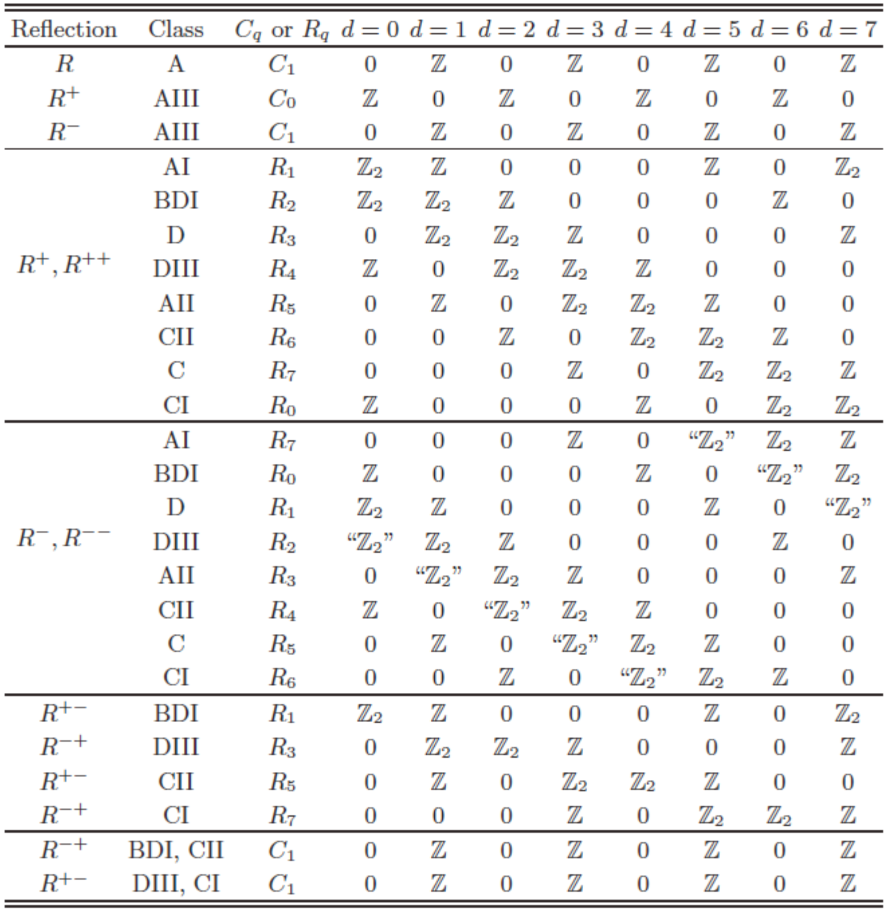
”Periodic tables” for symmetry-protected topological matter:



AZ class\\d	0	1	2	3
A	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AIII	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AI	$\mathbb{Z}$	0	0	0
BDI	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
D	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
DIII	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
AII	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
CII	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$
C	0	0	$2\mathbb{Z}$	0
CI	0	0	0	$2\mathbb{Z}$

2D topological insulator





Reflection	Class	C_q or R_q	$d=0$	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$
R	A	C_1	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
R^+	AIII	C_0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
R^-	AIII	C_1	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
R^+, R^{++}	AI	R_1	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
	BDI	R_2	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
	D	R_3	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
	DIII	R_4	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
	AII	R_5	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
	CII	R_6	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
	C	R_7	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
	CI	R_0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
R^-, R^{--}	AI	R_7	0	0	0	$\mathbb{Z}$	0	“$\mathbb{Z}_2$”	$\mathbb{Z}_2$	$\mathbb{Z}$
	BDI	R_0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	“$\mathbb{Z}_2$”	$\mathbb{Z}_2$
	D	R_1	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	“$\mathbb{Z}_2$”
	DIII	R_2	“$\mathbb{Z}_2$”	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
	AII	R_3	0	“$\mathbb{Z}_2$”	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
	CII	R_4	$\mathbb{Z}$	0	“$\mathbb{Z}_2$”	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
	C	R_5	0	$\mathbb{Z}$	0	“$\mathbb{Z}_2$”	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
	CI	R_6	0	0	$\mathbb{Z}$	0	“$\mathbb{Z}_2$”	$\mathbb{Z}_2$	$\mathbb{Z}$	0
R^{+-}	BDI	R_1	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
R^{-+}	DIII	R_3	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
R^{+-}	CII	R_5	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
R^{-+}	CI	R_7	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
R^{-+}	BDI, CII	C_1	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
R^{+-}	DIII, CI	C_1	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$

Now for some serious work on the blackboard!