Sample Solutions

PHYSICS ADMISSIONS TEST
Wednesday, 30 October 2019

Time allowed: 2 hours

For candidates applying to Physics, Physics and Philosophy, Engineering, or Materials Science

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Total 24 questions [100 Marks]

Answers should be written on the question sheet in the spaces provided, and you are encouraged to show your working.
You should attempt as many questions as you can.

No tables, or formula sheets may be used.

Answers should be given exactly and in simplest terms unless indicated otherwise.

Indicate multiple-choice answers by circling the best answer.
Partial credit may be given for correct workings in multiple choice questions.

The numbers in the margin indicate the marks expected to be assigned to each question. You are advised to divide your time according to the marks available.

You may take the gravitational field strength on the surface of Earth to be $g \approx 10\,\mathrm{m\,s^{-2}}$

Do NOT turn over until told that you may do so.

1. What is the next number in the sequence? -972, 324, -108, 36, -12 [2]

A	B	C	D	E
-4	-3	3	4	9

$$-972 \xrightarrow{\div -3} 324 \xrightarrow{\div -3} -108 \xrightarrow{\div -3} 36 \xrightarrow{\div -3} -12$$

$$\frac{-12}{-3} = 4 \Rightarrow D$$

2. Which values of x and y solve the following equations simultaneously: [2]

$$\log x + 2\log y = \log 32$$
$$\log x - \log y = -\log 2$$

A	B	C	D	E
$x = 2$ $y = 4$	$x = -2$ $y = -4$	$x = 2$ $y = -4$	$x = -2$ $y = 4$	no solution exists

$$\log x + 2\log y = \log 32 \Rightarrow xy^2 = 32 \quad (1)$$
$$\log x - \log y = -\log 2 \Rightarrow \frac{x}{y} = \frac{1}{2} \quad (2)$$

$(2) \Rightarrow x = \frac{y}{2}$

Inserting this into (1) gives $\frac{y^3}{2} = 32$

$$\Rightarrow y^3 = 64$$
$$\Rightarrow y = 4$$

Hence $x = \frac{y}{2} = 2$

3. Consider a system of many interacting particles. Let each particle have a potential energy $V(r)$ with respect to any other particle, where $V(r) \propto r^n$ where r is the distance to another particle and n is an integer. For such systems the Virial Theorem relates the time averaged total kinetic energy of all particles $\langle T_{tot} \rangle$ to the time averaged total potential energy $\langle V_{tot} \rangle$ as follows:

$$2\langle T_{tot} \rangle = n\langle V_{tot} \rangle$$

If the particles in our system interact only via gravity, what is the time averaged total energy $\langle E_{tot} \rangle$ of the system? [2]

A	B	C	D	E
$\langle E_{tot} \rangle = 0$	$\langle E_{tot} \rangle = 2\langle V_{tot} \rangle$	$\langle E_{tot} \rangle = \langle V_{tot} \rangle / 2$	$\langle E_{tot} \rangle = -\langle V_{tot} \rangle$	$\langle E_{tot} \rangle = -2\langle V_{tot} \rangle$

Gravitational potential energy is proportional to $\frac{1}{r} = r^{-1}$

Hence $n = -1$

Thus the Virial Theorem implies

$$\langle T_{tot} \rangle = -\frac{1}{2} \langle V_{tot} \rangle$$

The total energy is the sum of potential and kinetic energy:

$$\begin{aligned} \langle E_{tot} \rangle &= \langle V_{tot} \rangle + \langle T_{tot} \rangle \\ &= \langle V_{tot} \rangle - \frac{1}{2} \langle V_{tot} \rangle \\ &= \frac{1}{2} \langle V_{tot} \rangle \end{aligned}$$

4. The acceleration g due to gravity on a spherical planet in any universe is given by:

$$g = \frac{GM}{R^2}$$

where M is the mass, R the radius of the planet and G is the gravitational constant in that planet's universe.

In a different universe the gravitational constant is G' and has twice the value of the gravitational constant in our Universe G.

Find the ratio $\frac{g_{planet}}{g_{Earth}}$ for a planet in the different universe which has half the radius and twice the density of the Earth. [2]

A	B	C	D	E
$\frac{g_{\text{planet}}}{g_{\text{Earth}}} = 2$	$\frac{g_{\text{planet}}}{g_{\text{Earth}}} = 1$	$\frac{g_{\text{planet}}}{g_{\text{Earth}}} = \frac{1}{2}$	$\frac{g_{\text{planet}}}{g_{\text{Earth}}} = 4$	$\frac{g_{\text{planet}}}{g_{\text{Earth}}} = \frac{1}{4}$

$G' = 2G$

$R_{planet} = \frac{1}{2} R_{Earth}$

$\rho_{planet} = 2 \rho_{Earth}$

In general, $g = \frac{GM}{R^2}$

$= \frac{G \times \frac{4}{3}\pi R^3 \rho}{R^2}$ since $M = V \times \rho$

$= \frac{4}{3}\pi R^3 \rho$

$= \frac{4}{3} G \pi \rho R$

Hence for Earth, $g_{Earth} = \frac{4}{3} G \pi \rho_{Earth} R_{Earth}$

For planet, $g_{planet} = \frac{4}{3} G' \pi \rho_{Planet} R_{planet}$

$= \frac{4}{3} (2G) \pi (2\rho_{Earth})(\frac{1}{2} R_{Earth})$

$= 2 \times \frac{4}{3} G \pi \rho_{Earth} R_{Earth}$

$= 2\, g_{Earth}$

5. In which range of α does the following equation have real solutions? [2]

$$\sec^2\theta + \alpha\tan\theta = 0$$

A	B	C	D	E
$\alpha \le -2$ or $\alpha \ge 2$	$\alpha \le -2$	$\alpha \ge 2$	$\alpha \ge -0$	$\alpha \le 0$

Trig identity: $\sec^2\theta - 1 = \tan^2\theta$

$\sec^2\theta + \alpha\tan\theta = 0$

$\Rightarrow \sec^2\theta - 1 + \alpha\tan\theta + 1 = 0$

$\Rightarrow \tan^2\theta + \alpha\tan\theta + 1 = 0$ by trig identity

This is quadratic in $\tan\theta$

Solution: $\tan\theta = \dfrac{-\alpha \pm \sqrt{a^2-4}}{2}$ which requires $a^2 \ge 4$

6. A bag contains b blue balls and r red balls. If two balls are picked at random and removed from the bag, what is the probability P that they are different colours? [2]

A	B	C	D	E
$\frac{2br}{(b+r)(b+r-1)}$	$\frac{br}{(b+r)(b+r-1)}$	$\frac{br}{(b+r)^2}$	$\frac{2br}{(b+r)^2}$	$2br$

Different colours $\Rightarrow$ blue then red <u>or</u> red then blue

$\downarrow$ $\downarrow$

$P(br)$ $P(rb)$

The probability is the sum of the probabilities for the two possible outcomes, which are br and rb

$P = P(br) + P(rb)$

$= \left(\frac{b}{b+r}\right)\left(\frac{r}{b+r-1}\right) + \left(\frac{r}{b+r}\right)\left(\frac{b}{b+r-1}\right)$

$= \dfrac{2br}{(b+r)(b+r-1)}$

7. We wish to represent integer numbers by using our ten fingers. A finger is assumed to be either stretched out or curled up. How many different integers can we represent with our fingers?

[2]

A	B	C	D	E
10	512	1000	20	1024

Let each finger represent a binary digit

Then ten fingers can represent the digits of a 10 digit binary no.

Hence $N_{max} = 2^{10} = 1024$

8. Without explicit calculation state which integrals are non-zero.

$$I_1 = \int_{-3\pi}^{3\pi} x^2 \sin(x) dx \tag{1}$$

$$I_2 = \int_{-\infty}^{\infty} e^{-x^2} dx \tag{2}$$

$$I_3 = \int_{3\pi/2}^{-3\pi/2} \cos^2(x) dx \tag{3}$$

$$I_4 = \int_{-\infty}^{\infty} xe^{-x^2} dx \tag{4}$$

[2]

A	B	C	D	E
2 and 3	1 and 4	1 and 3	2 and 4	all

The integrands of (2) and (3) are even and strictly non-negative functions, integrated over a symmetric interval around zero, and hence yield a non-zero integral.

9. A long, thin, straight wire carrying an electric current I causes a magnetic field of flux density B at a perpendicular distance r from the wire. The magnitude of this flux density is given by the following relation:

$$B = \frac{\alpha I}{r}$$

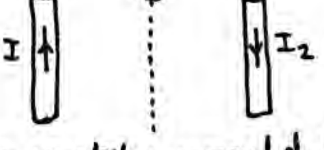
where α is a constant. The magnetic field points circumferentially around the wire. A second, identical wire is placed parallel to the first one at a distance D. Find the current I_2 that has to flow in the second wire if the flux density at a line half way between and parallel to the wires is to double, compared to the flux density from only one wire at current I. [2]

A	B	C	D	E
$I_2 = I$	$I_2 = 2I$	$I_2 = -2I$	$I_2 = -I$	$I_2 = -I/2$

The question does not state which orientation the B field takes, but it must change its direction when the current changes its direction(*).

If the fields from both wires are to point in the same direction at the midpoint (which is required for the field to double) then the orientations of the circular fields needs to be opposite and this demands that the currents need to be in opposite directions.

Both wires will contribute the same field at the middle line

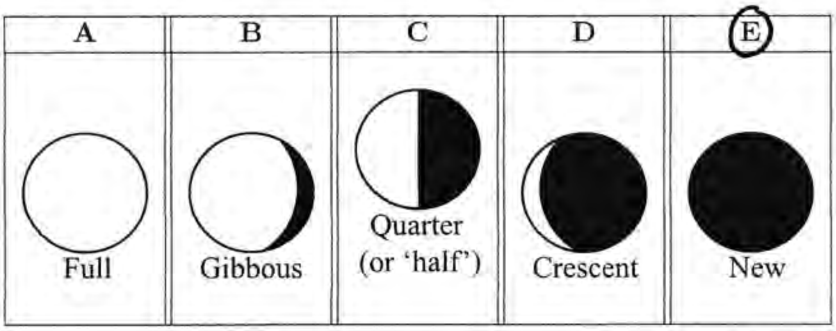


$\Rightarrow I_2 = -I$

(*) or else some other vector-like quantity would have to determine the field orientation, but current is the only one.

10. When the phase of the Moon as seen from the Earth is Full, what phase of the Earth is seen by an observer on the Moon?

[2]



The symbols above show phases of the Earth as seen from the Moon

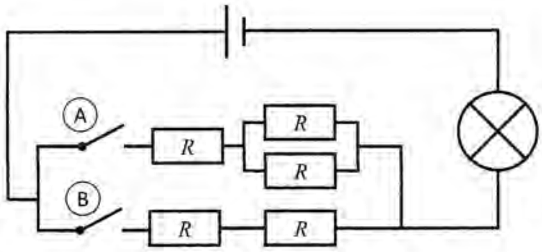
Phase of Moon as seen from Earth is Full

⇒ Moon is on far side of Earth as seen from the Sun, so fully illuminated Moon face is visible.

⇒ Sun is on far side of Earth when seen from Moon, producing a "New" Earth

(not to scale)

11. In the circuit shown below all resistors have the same resistance R and the light bulb has a fixed resistance. You wish to change the state of the switches so that the brightness of the bulb increases from its minimum to its maximum. Which sequence of switch states will achieve this? [2]



	A	B	C	D	E
1. 2. 3.	both closed only A closed only B closed	both closed only B closed only A closed	only B closed only A closed both closed	only A closed only B closed both closed	all states give the same brightness

To find the brightnesses in the different cases, need to calculate equivalent resistance of circuit in all switch configurations.

A: The branch active when switch A alone is closed has equivalent resistance of $R_A = R + \frac{R}{2} = \frac{3}{2}R$

B: The branch active when switch B alone is closed has equivalent resistance of $R_B = 2R$

both: When both switches are closed, the equivalent resistance is $R_{both} = \dfrac{1}{\frac{1}{R_A} + \frac{1}{R_B}} = \dfrac{1}{\frac{2}{3R} + \frac{1}{2R}}$

$$= \frac{6}{7}R$$

Since the circuit is powered by a constant voltage source, the maximum brightness is reached when the total resistance in series with the bulb is minimal. Hence max brightness with both closed, and min brightness with only B closed.

12. An organ pipe is open at one end and closed at the other. The lowest note you can play on this pipe has frequency f_{min}. If the speed of sound in the pipe is v, what is the length L of the pipe? [2]

A	B	C	D	E
$L = \frac{v}{2f_{min}}$	$L = \frac{v}{4f_{min}}$	$L = \frac{v}{f_{min}}$	$L = \frac{2v}{f_{min}}$	$L = \frac{4v}{f_{min}}$

Standing wave in the pipe must have:

- node at closed end
- maximum at open end
- no other nodes in between (λ_{max})

So length of pipe L:

Whole wave: 

$$L = \frac{\lambda_{max}}{4}$$

Dispersion relation of sound waves relates wavelength to frequency:

$$v = f\lambda = f_{min}\lambda_{max} \quad \Rightarrow \quad \lambda_{max} = \frac{v}{f_{min}}$$

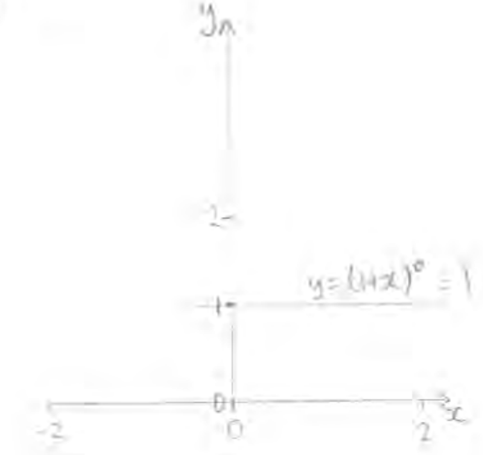
So $L = \dfrac{v}{4 f_{min}}$

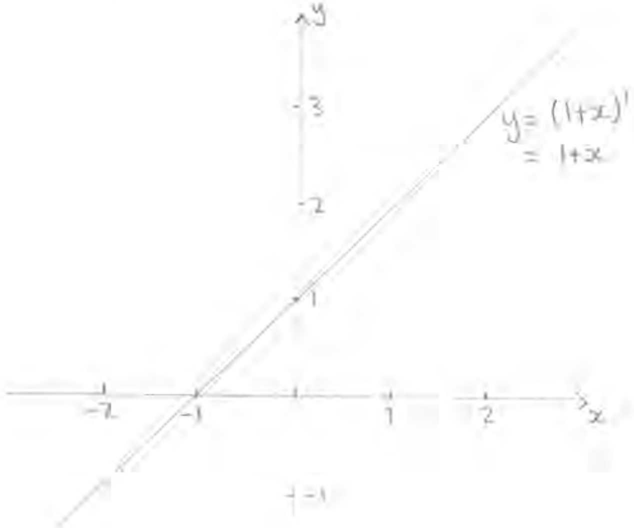
13.

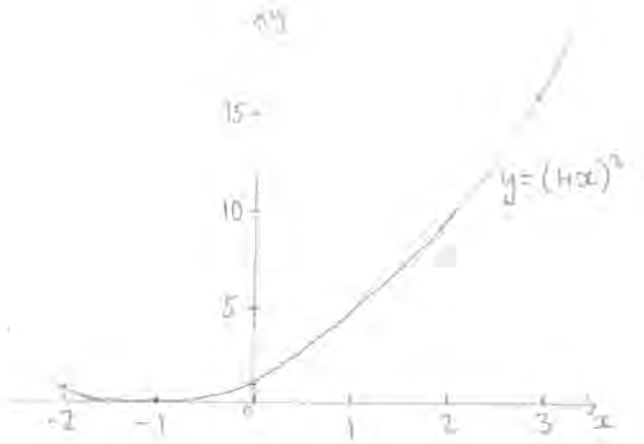
(a) Sketch the graphs of $y = (1 + x)^n$ for integer values of n from 0 to 3, each on a separate set of axes. Which point(s) are common to all the graphs ? [4]

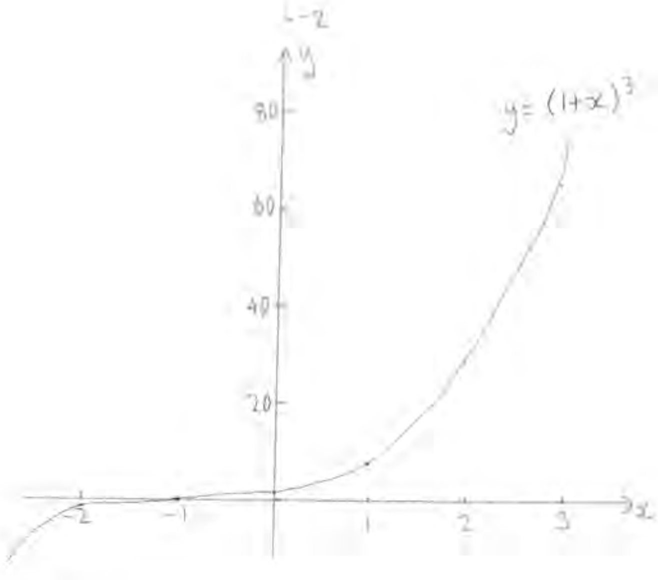
(b) Describe two of the further features common to the graphs for integer $n > 1$. [2]

(a)









The common point on all the graphs is $(x=0,\ y=1)$

(b)
- For $n>1$, a further common point is $(x=-1,\ y=0)$
- For $n>1$, the single stationary point is found from $\frac{dy}{dx} = n(1+x)^{n-1} = 0$ which is satisfied for $x=-1$ for $n \geq 2$. So there is only one stationary point and it is always at $x=-1$.
- All graphs diverge to infinity as x goes to infinity
- All graphs are continuous
- None of the graphs lie in the 4th quadrant

14. A radioactive sample contains two isotopes, A and B. Isotope A decays to isotope B with a half life of $t_{1/2}$. Isotope B is stable.

(a) The number of atoms of A left after a time t is given by:

$$N_A(t) = N_{A0}\, e^{-\lambda t}$$

where N_{A0} is the initial number of atoms of A. Derive an expression for λ in terms of $t_{1/2}$. [2]

(b) Initially the number of B atoms in the sample is $N_B(t=0) = N_{B0}$. Let $N_B(t)$ be the time dependent number of B atoms in the sample. Write down an expression for $N_B(t)$ in terms of λ, N_{B0}, N_{A0} and t. [1]

(c) At the start there are x times as many A atoms in the sample as there are B atoms. How long does it take until this ratio is reversed? [3]

(a) The half life $t_{1/2}$ is defined as the time after which half of the initial material has decayed.

$$N_A(t_{1/2}) = \frac{1}{2} N_{A0} = N_{A0}\, e^{-\lambda t_{1/2}}$$

$$\ln\left(\tfrac{1}{2}\right) = -\lambda t_{1/2}$$

$$\lambda = \frac{\ln 2}{t_{1/2}}$$

(b) $$N_B(t) = N_{B0} + N_{A0}\left(1 - e^{-\lambda t}\right)$$

(c) Given that $N_{A0} = x N_{B0}$, find a time t_i at which

$$N_B(t_i) = x\, N_A(t_i) \quad ①$$

$$N_{B0} + N_{A0}\left(1 - e^{-\lambda t_i}\right) = x\, N_{A0}\, e^{-\lambda t_i}$$

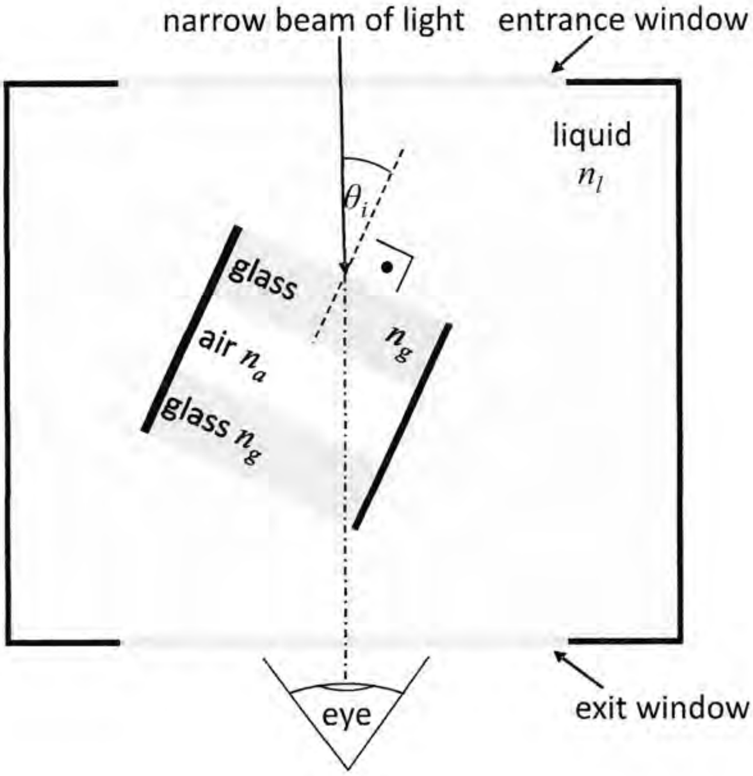
$$\frac{N_{A0}}{x} + N_{A0}\left(1 - e^{-\lambda t_i}\right) = x\, N_{A0}\, e^{-\lambda t_i} \quad \text{by ①}$$

$$\frac{x+1}{x} = (x+1)\, e^{-\lambda t_i}$$

$$\ln(x) = \lambda t_i$$

$$t_i = \frac{\ln x}{\ln 2}\, t_{\frac{1}{2}}$$

15. The diagram below shows an air-cell refractometer.

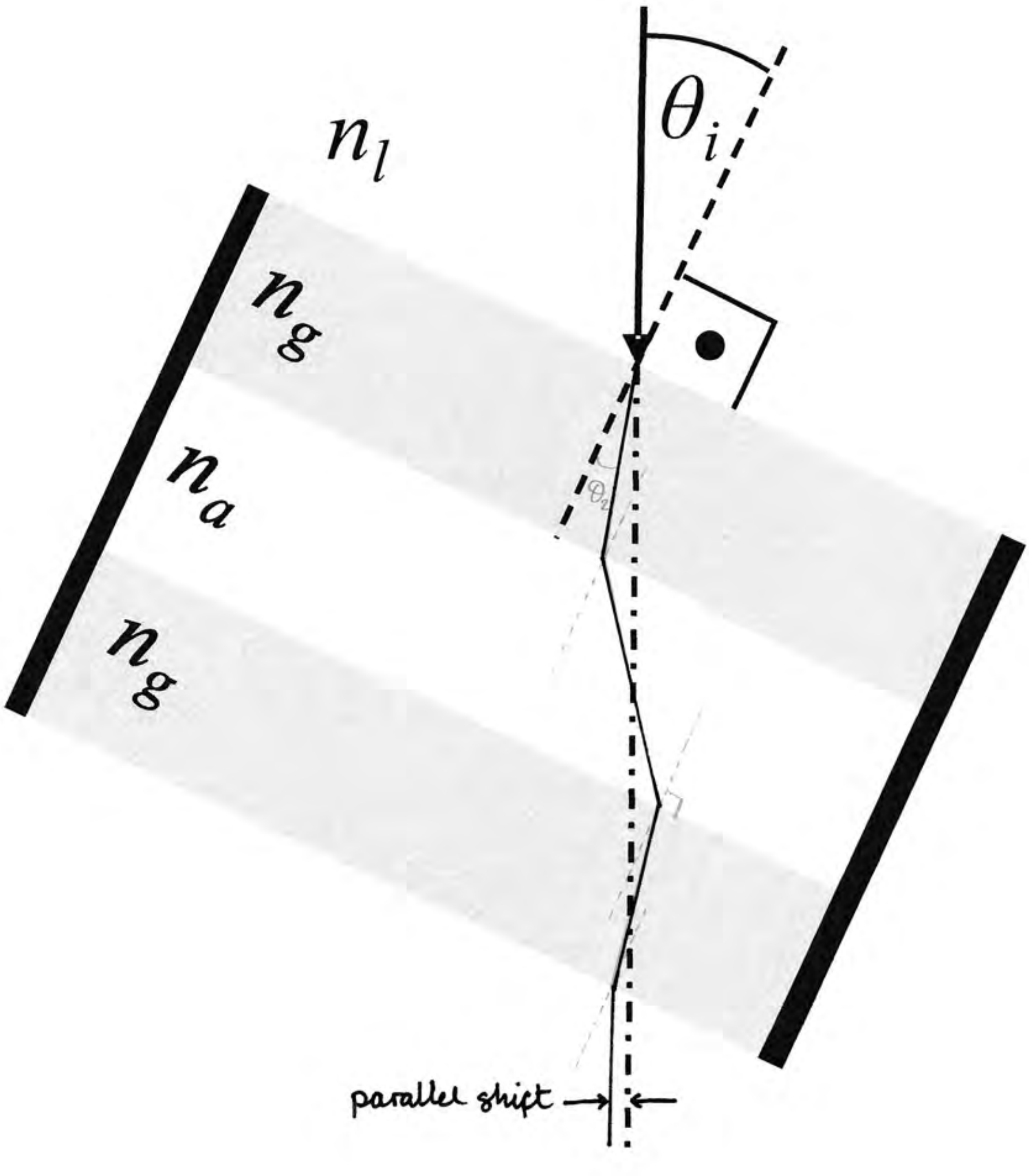


A narrow beam of light enters the entrance window at normal incidence and you can observe the light leaving the exit window by eye. The outer box is filled with a liquid of unknown refractive index n_l. The glass of the air cell has refractive index n_g and the cell is filled with air of refractive index n_a. The air-cell in the liquid filled vessel makes angle θ_i with respect to the incoming beam of light, as shown in the diagram. This angle can be precisely adjusted and measured.

(a) On the diagram provided on the next page, draw the refracted path of the beam through the air cell. For this diagram you should assume $n_g > n_l > n_a$. [3]

(b) Describe qualitatively what you will observe at the exit window as you increase θ_i from zero and hence explain how this instrument could be used to determine the refractive index of the liquid n_l in the chamber. Find the relation between n_l and a special value of θ_i. [3]

(c) Suggest with reasons, a way to modify the apparatus or its use to improve the measurement. [2]



(a) The angles at the glass-air and air-glass interfaces are bigger than those at the liquid-air and air-liquid interfaces.

The ray suffers a parallel shift.

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15 (b) As θ_i is increased from zero, beyond a certain θ_i, the ray leaving the glass into the air will suffer total internal reflection (TIR) and no ray will be visible at the exit window.

In addition, as θ_i is increased, the beam is increasingly parallel shifted. Also, the beam will continuously become dimmer as the reflections on the interfaces increase with angle.

To use the instrument:

- Continuously observe the outgoing light while increasing θ_i
- Record the angle at which the output goes dark

Snell's law on the first interface gives

$$n_l \sin(\theta_i) = n_g \sin(\theta_2)$$

Snell's law on the second interface and demanding to be at the critical angle gives

$$n_g \sin(\theta_2) = n_a \sin(90) = n_a$$

Combining the two gives

$$n_l \sin(\theta_i) = n_a$$

$$n_l = \frac{n_a}{\sin(\theta_i)}$$

(c) Possible improvements are:

- measure angle in both directions and average results to eliminate offset in angle scale.
- replace air with vacuum to accurately get a known constant refractive index
- record intensity with a quantitative detector rather than the eye.
- fit point of zero intensity from an intensity vs. angle graph.
- use monochromatic light to avoid dispersion as angle increases.

16. The energy levels of the electron in a hydrogen atom are characterised by a quantum number n:

$$E_n = -\frac{hcR}{n^2}$$

where h is Planck's constant, c is the speed of light and R is the Rydberg constant.

(a) State a formula that relates the wavelength of light λ to h, c and E which is the energy of a photon. [1]

(b) Let p and q be the quantum numbers of the upper and lower energy levels of an electron transition in hydrogen. Find a formula that relates the wavelength of light emitted in such transitions to p and q. [2]

(c) For each of the three hydrogen emission line sets shown in the table below, identify the quantum number of its lower energy level q . Each set (column) of five emission lines has the same lower energy level. The first column shows the quantum number of the upper energy level q relative to the lower level.

p	Set-A λ [nm]	Set-B λ [nm]	Set-C λ [nm]
$q+1$	121.57	4051	7460
$q+2$	102.57	2625	4654
$q+3$	97.254	2166	3741
$q+4$	94.974	1944	3297
$q+5$	93.780	1817	3039

You may assume that $p < 6$ and $R = 10973731.6\,\mathrm{m}^{-1}$ [4]

(a) $\lambda = \frac{hc}{E}$

(b) $E = E_p - E_q$

$= hcR\left(\frac{1}{q^2} - \frac{1}{p^2}\right)$

Hence

$\frac{1}{\lambda} = R\left(\frac{1}{q^2} - \frac{1}{p^2}\right)$

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16 (c) Find $\frac{1}{R\lambda}$ for some of the energy levels for the three sets

Set A

p	λ [nm]	$\frac{1}{R\lambda}$
$q+1$	121.57	0.7496
$q+2$	102.57	0.8884
$q+3$	97.254	0.9370

Set B

p	λ [nm]	$\frac{1}{R\lambda}$
$q+1$	4051	0.02249
$q+2$	2625	0.03471
$q+3$	2166	0.04207

Set C

p	λ [nm]	$\frac{1}{R\lambda}$
$q+1$	7460	0.01222
$q+2$	4654	0.01958
$q+3$	3741	0.02436

Calculate $\frac{1}{R\lambda} = \frac{1}{q^2} - \frac{1}{p^2}$ for combinations of q & p to see what matches the values of the three sets above.

q	p	$\frac{1}{q^2}$	$\frac{1}{p^2}$	$\frac{1}{R\lambda}$	
1	2	1	$\frac{1}{4}$	$\frac{3}{4} = 0.75$	Set A, $q=1$
1	3	1	$\frac{1}{9}$	$\frac{8}{9} = 0.889$	
2	3	$\frac{1}{4}$	$\frac{1}{9}$	$\frac{5}{36} = 0.139$	
3	4	$\frac{1}{9}$	$\frac{1}{16}$	$\frac{7}{144} = 0.0486$	
4	5	$\frac{1}{16}$	$\frac{1}{25}$	$\frac{9}{400} = 0.0225$	Set B, $q=4$
4	6	$\frac{1}{16}$	$\frac{1}{36}$	$\frac{5}{144} = 0.0347$	
5	6	$\frac{1}{25}$	$\frac{1}{36}$	$\frac{11}{900}$ 25 $= 0.0122$	Set C, $q=5$
5	7	$\frac{1}{25}$	$\frac{1}{49}$	$\frac{24}{1225} = 0.0196$	

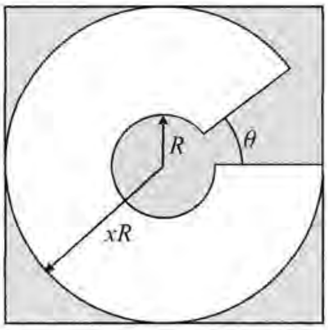
17.

(a) Find an expression for the angle θ for which the grey area A_g is f times the area of the outer square A_S. Your expression for θ should take the form:

$$\theta = B - \frac{C(f)x^2}{x^2 - 1}$$

where B is a constant and $C(f)$ is a function of f. State B and $C(f)$ explicitly. You may assume that $x > 1$, $f < 1$ and $f > 0$. The value of xR indicates the radius of the outer circle. [6]

(b) Find the numerical value for θ to five significant figures when $x = 3$ and $f = 1/2$. [1]



(a) Area of C-shape A_c is a fraction of the difference of the areas of the outer and inner circles proportional to the fractional angle covered by the C-shape.

$$A_c = \pi\left((xR)^2 - R^2\right)\left(1 - \frac{\theta}{2\pi}\right)$$

$$= \pi R^2 (x^2 - 1)\left(1 - \frac{\theta}{2\pi}\right)$$

The outer square has a side length of $2xR$ and its area is

$$A_S = (2xR)^2$$

The grey area A_g is the difference of the area of the outer square and the C-shape

$$A_g = 4x^2R^2 - \pi R^2 (x^2 - 1)\left(1 - \frac{\theta}{2\pi}\right)$$

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17 (a) CONT. The requested ratio of areas $A_g = f A_s$ is then

$$f(2xR)^2 = (2xR)^2 - \left(1 - \frac{\theta}{2\pi}\right)\left[\pi (xR)^2 - \pi R^2\right]$$

$$(1-f)\, 4x^2 = \pi \left(1 - \frac{\theta}{2\pi}\right)\left[x^2 - 1\right]$$

$$(1-f)\, 4 = \pi \left(1 - \frac{\theta}{2\pi}\right)\left[1 - \frac{1}{x^2}\right]$$

$$\frac{(1-f)4}{\pi\left[1-\frac{1}{x^2}\right]} = \left(1 - \frac{\theta}{2\pi}\right)$$

$$\theta = 2\pi - \frac{(1-f)\,8x^2}{x^2-1}$$

$\Rightarrow B = 2\pi \quad (=360°)$

$C(f) = 8(1-f)$

(b) $x = 3, \quad f = \frac{1}{2}$

$\Rightarrow$ numerical value of θ is

$$\theta = 2\pi - \frac{(1-\frac{1}{2}) \times 8 \times 3^2}{3^2 - 1}$$

$$= 2\pi - \frac{9}{2}$$

$$= 1.7832 \text{ rad}$$

$(= 102.17°)$

18. Solve the following equation for x:

$$\frac{e^x + 9}{e^{-x} + 5} = 2$$

[3]

$$\frac{e^x + 9}{e^{-x} + 5} = 2$$

$$e^x + 9 = 2e^{-x} + 10$$

$$e^x - 1 - 2e^{-x} = 0$$

$$e^{2x} - e^x - 2 = 0$$

$$(e^x - 2)(e^x + 1) = 0$$

$$e^x = 2 \text{ or } -1$$

$$e^x \neq -1$$

$$\text{So } e^x = 2$$

$$x = \ln 2$$

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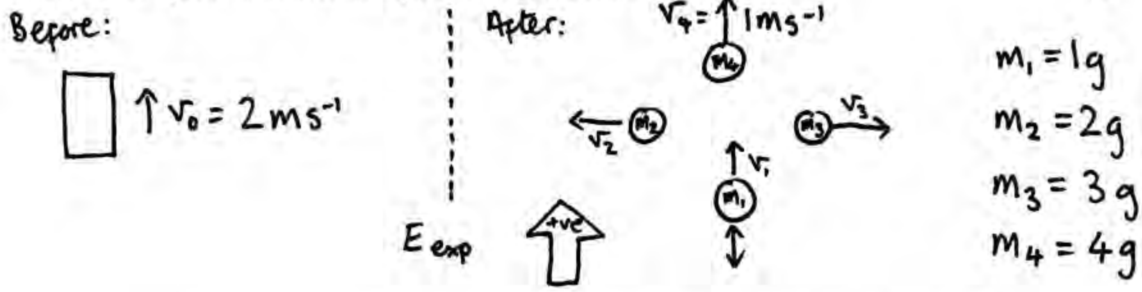
19. A firework rocket is launched vertically. At the moment of explosion it is moving with a vertical speed of $v_0 = 2\ \mathrm{m\,s^{-1}}$ upwards. The explosion releases an energy of $E_{exp} = 1$ J and the rocket bursts into four pieces with masses of $m_1 = 1\,\mathrm{g}$, $m_2 = 2\,\mathrm{g}$, $m_3 = 3\,\mathrm{g}$ and $m_4 = 4\,\mathrm{g}$. The piece with mass m_4 moves vertically upwards with a speed of $v_4 = 1\,\mathrm{m\,s^{-1}}$. The pieces of mass m_3 and m_2 move horizontally and the piece of mass m_1 moves vertically. All velocities and directions in this question are given relative to the ground and your answer should do the same.

(a) Obtain the speeds of all the pieces after the explosion. [5]

(b) Higher speed pieces can be obtained if the directions of movement of the pieces are different from those in part (a). Under which choice of directions would the maximum speed of one of the pieces be achieved? [3]

(a)



Momentum conservation – the rocket pieces immediately after the explosion must still have the same momentum as before:

$$P_0 = m_{tot}\, v_0 = 10g \times 2\,\mathrm{ms^{-1}}$$

$$= +20\,g\,\mathrm{ms^{-1}}$$ where positive sign indicates we count upward-going velocities as positive

Energy conservation – total kinetic energy of all the pieces after the explosion must be less than or equal to the sum of the explosion energy and the kinetic energy of the rocket before the explosion:

$$E_{tot} = E_{exp} + E_0$$

$$= E_{exp} + \tfrac{1}{2} m_{tot} v_0^2$$

$$= 1\,\mathrm{J} + \tfrac{1}{2} \times 10g \times (2\,\mathrm{ms^{-1}})^2$$

$$= \frac{51}{50}\,\mathrm{J}$$

$$= \sum_{i=1}^{4} \tfrac{1}{2} m_i v_i^2$$

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19 (a) CONT

To obtain maximum speeds for pieces, assume all chemical energy from explosion is converted to kinetic energy of pieces.

m_1 will fly vertically

m_2 will fly horizontally back-to-back with m_3

All pieces fly either horizontally or vertically

$\Rightarrow$ equation for momentum conservation separates into two equations

- momenta of horizontal pair must add to zero
- momenta of vertical pair must add to initial vertical momentum

Horizontal: $m_3 v_3 + m_2 v_2 = 0$ (2)

Vertical: $m_4 v_4 + m_1 v_1 = P_0$

Three equations for three unknown speeds

$$\Rightarrow v_1 = \frac{P_0 - m_4 v_4}{m_1} = \frac{20 - 4 \times 1}{1} = 16\,m/s$$

By (2), $v_2 = -\dfrac{m_3 v_3}{m_2}$

Hence
$$E_{tot} = \frac{1}{2}\left(m_1 v_1^2 + m_4 v_4^2 + m_2\left(\frac{m_3 v_3}{m_2}\right)^2 + m_3 v_3^2\right)$$
$$= \frac{1}{2}\left(m_1 v_1^2 + m_4 v_4^2 + \left(\frac{m_3}{m_2} + 1\right) m_3 v_3^2\right)$$

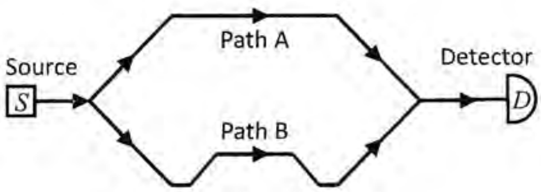
This is a quadratic in v_3 with solutions
$$v_3 = \pm\sqrt{\frac{(2E_{tot} - m_1 v_1^2 - m_4 v_4^2)m_2}{m_3(m_3 + m_2)}} = 15.406\,m/s$$

$\Rightarrow v_2 = 23.108$

(b) Highest speed configuration is achieved if three heaviest pieces (m_4, m_3, m_2) all recoil against lightest piece m_1.

To maximise speed of m_1 further, it should fly vertically up and hence gain speed of centre of mass in addition.

20. The diagram below shows an interferometer with two paths (Path A and Path B) which a wave can take from its source S to a detector D.



The lengths of the paths differ by an amount L which can change with time. The intensity at the detector I is measured and varies as a function of L as follows:

$$I = I_p + I_q \cos(kL)$$

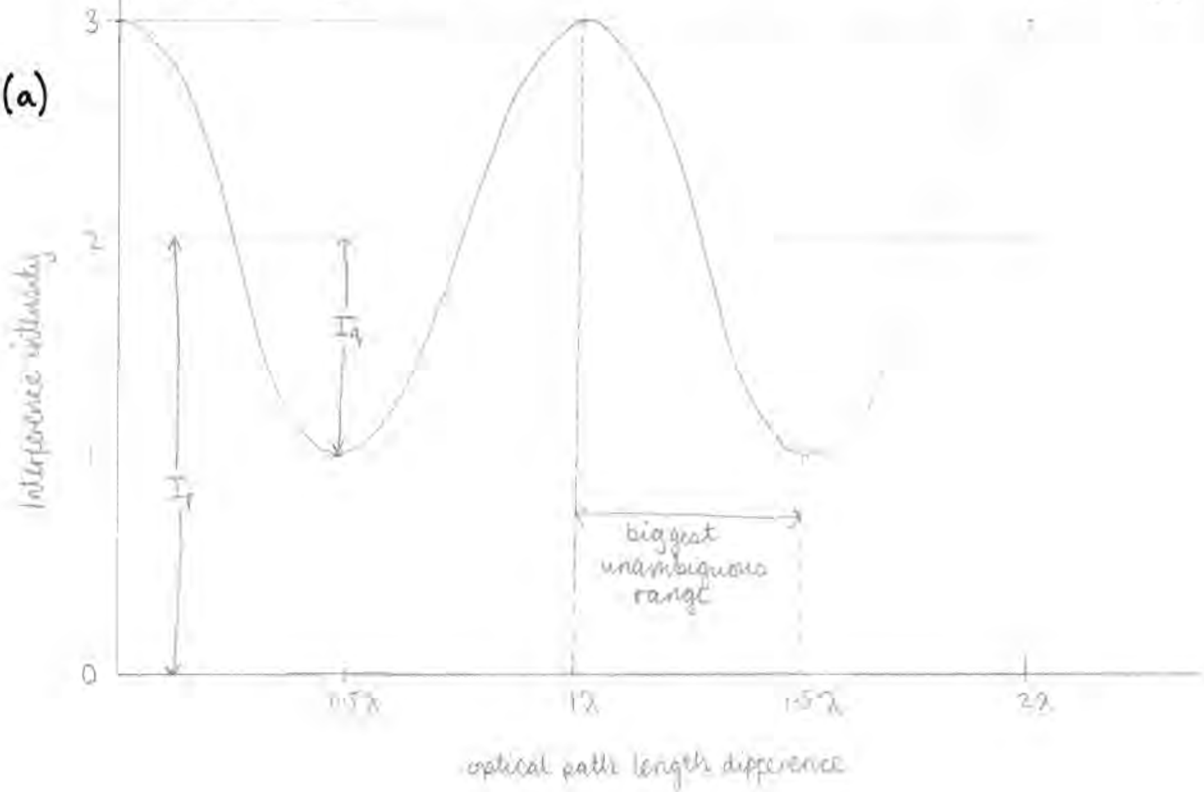
In the above k is the wavenumber of the wave which relates to the wavelength λ via $k = 2\pi/\lambda$. I_p and I_q are constants.

(a) Sketch the intensity as a function of L in the range from 0 to 2λ. Label both axes and identify I_p and I_q in the sketch. [3]

We wish to use the interferometer to measure how the path length difference L changes with time by measuring the intensity at the detector as a function of time. The change in path length difference is ΔL.

(b) Indicate on your sketch the biggest ΔL you can infer unambiguously from a measurement of intensity. [2]

(a)



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20 (b) The biggest unambiguous range over which changes in path length difference can be measured is equal to the biggest range in which the function I(L) can be inverted to give L(I).

This range is at most $\frac{\lambda}{2}$ and would stretch from one stationary point to the next.

- See sketch.

21. You wish to build an adjustable delay line using electrical switches as shown in the diagram below.

Its purpose is to adjust the delay of an electrical signal through the delay line by switching different amounts of delay into the signal path.

The delay line should use the minimum amount of switches.

The delay line should have a minimal delay of L_{min} and a maximal delay of $L_{max} \leq L_{min} + \Delta L$. We refer to ΔL as the delay range.

The delay should be adjustable in increments of δL so that the line can achieve an evenly spaced set of delays between L_{min} and L_{max} with a resolution of δL.

For all segments in the line you have to determine a common, small delay l which is active when its switch is in the lower position.

You further have to determine a larger, individual delay of $L_i + l \gg l$ for each segment which is active when its switch is in the upper position.

For a line with n segments to satisfy the demands on the minimum number of switches, minimum delay L_{min}, delay range ΔL and delay resolution δL:

(a) Find a value for l in terms of L_{min} and n. [1]

(b) Find the delays of each segment L_i in terms of δL. [3]

(c) Find the minimum necessary n in terms of ΔL and δL [3]

(a) The minimum delay in a chain of n switches is

$$L_{min} = n\,l$$

This is achieved if all switches are in the lower position.

This demands that $l = \dfrac{L_{min}}{n}$

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21 (b) The accuracy of adjustment is given by the smallest change in delay that any of the segments can generate. In order to generate all intermediate delays with the same granularity δL the sequence of increasingly long delays has to continuously double. I.e. the delays have to be powers of two multiplied by δL

$$L_i = 2^{i-1}\,\delta L$$

In this case any binary numbered delay from 0 to $(2^n-1)\,\delta L$ can be represented.

This stems from the fact that each switch can take on two positions just like each digit in a binary number can take on two states. The delay that the n^{th} switch (or stage) contributes is $\delta L\, 2^{n-1}$, just like the n^{th} digit in a binary number contributes 2^{n-1} its value. (Our switches and digits are counted from 1 in this formulation.)

(c) The delay range of the chain can be written down mathematically as the sum of all the long delays, and it must be larger or equal to the desired delay range ΔL:

$$\Delta L \geq \sum_{i=1}^{n} L_i = \delta L \sum_{i=1}^{n} 2^{n-1}$$

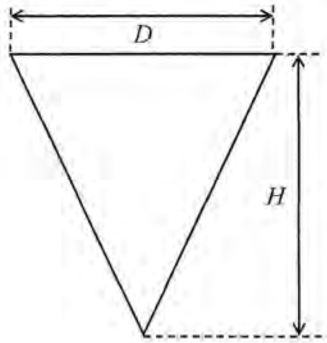
This is a finite geometric sum, which can be reformulated and evaluated and the inequality solved for n as follows:

$$\Delta L \geq \delta L \sum_{j=0}^{n} 2^{j}$$

$$\geq \delta L \;\frac{1-2^n}{1-2}$$

$$n \geq \log_2\left(\frac{\Delta L}{\delta L} + 1\right)$$

22. A conical cup has dimensions as shown in the diagram of its cross-section below. The cup can hold a maximum volume V when filled to its full depth H. Find an expression for the depth h to which you have to fill the cup so that it contains a volume of liquid equal to $V/2$. Your expression for h should only depend on the dimensions of the cup. [4]



Volume of a cone: $V = \pi \left(\frac{D}{2}\right)^2 \frac{H}{3}$

We want to fill the cup to a level h at which the diameter of the filled fractional cone is d.

Volume of filled cone should be half of max volume.

Hence: $\pi \left(\frac{d}{2}\right)^2 \frac{h}{3} = \frac{1}{2} \pi \left(\frac{D}{2}\right)^2 \frac{H}{3}$ ①

Need to relate diameter at a given height to the full height, via the opening half-angle of the cone, α.

$\Rightarrow$ similar triangles between full and partially filled cups as they have same half-angle

$$\tan(\alpha) = \frac{D}{2H}$$

$$\tan(\alpha) = \frac{d}{2h}$$

$$\Rightarrow d = \frac{2hD}{2H} = D\frac{h}{H} \quad ②$$

So $\pi \left(\frac{Dh}{2H}\right)^2 \frac{h}{3} = \frac{1}{2} \pi \left(\frac{D}{2}\right)^2 \frac{H}{3}$ by inserting ② into ①

$$\Rightarrow h = \frac{H}{\sqrt[3]{2}}$$

23. In an imaginary water filtration process a fraction of $1/n$ of an impurity is removed in the first pass of the water through the system. In each succeeding pass, the amount of impurity removed is $1/n$ of the amount removed in the preceding pass. Show that if $n = 2$ the water can be made arbitrarily pure but if $n = 3$, at least half of the impurity will remain. [5]

Let F_m be the fraction of the initial impurity that is remaining after m passes through the system.

$$F_m(n) = 1 - \frac{1}{n} - \frac{1}{n^2} - \frac{1}{n^3} - \dots - \frac{1}{n^m}$$

$$= 1 - \frac{1}{n}\left(1 + \frac{1}{n} + \frac{1}{n^2} + \cdots + \frac{1}{n^{m-1}}\right)$$

$$= 1 - \frac{1}{n}\sum_{k=0}^{m-1}\frac{1}{n^k}$$

$$\lim_{m\to\infty} F_m(n) = 1 - \frac{1}{n}\left(\frac{1}{1-\frac{1}{n}}\right) = F(n)$$

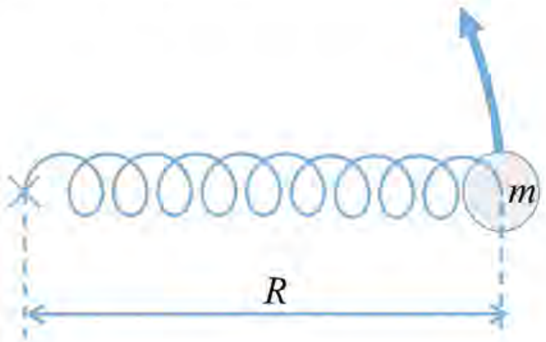
$$F(n) = \frac{n-2}{n-1}$$

Evaluating F for $n=2$ and $n=3$

$$F(2) = 0$$

$$F(3) = \frac{1}{2}$$

24. The sketch below shows a ball of mass m on a spring of unextended length R_0 and spring constant k. The spring is pivoted on the left on a central axis marked with a cross. The axis is perpendicular to the plane of the sketch. The ball and spring rotate around the central axis on a smooth horizontal table as indicated by the arrow in the sketch. The spring will break if it is stretched with a force larger than F_{max}.



(a) Find the equilibrium extension R of the system when it rotates with angular frequency ω. [3]

(b) Find the equilibrium angular frequency ω_c at which the spring will break. [1]

(c) Sketch ω_c against F_{max} in the range from zero to one Newton for the following parameters $m = 1\,\mathrm{kg}$, $R_0 = 1\,\mathrm{m}$, $k = 1\,\mathrm{N\,m^{-1}}$. Label your axes. [2]

(d) Under some conditions the system can only achieve a maximum angular frequency $\omega_i < \omega_c$. Find a relationship between k, m and ω_i and explain what happens to the system as the angular frequency increases to ω_i. [4]

(a) Centripetal force needed to keep ball on radius R at rotational speed ω is

$$F_c = m\omega^2 R$$

Centripetal force is provided by spring, which must obey Hooke's law

$$F_s = k(R-R_0)$$

Hence $\quad k(R-R_0) = m\omega^2 R$

So for a given ω,

$$R = \frac{k}{k-m\omega^2} R_0$$

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24 (b)

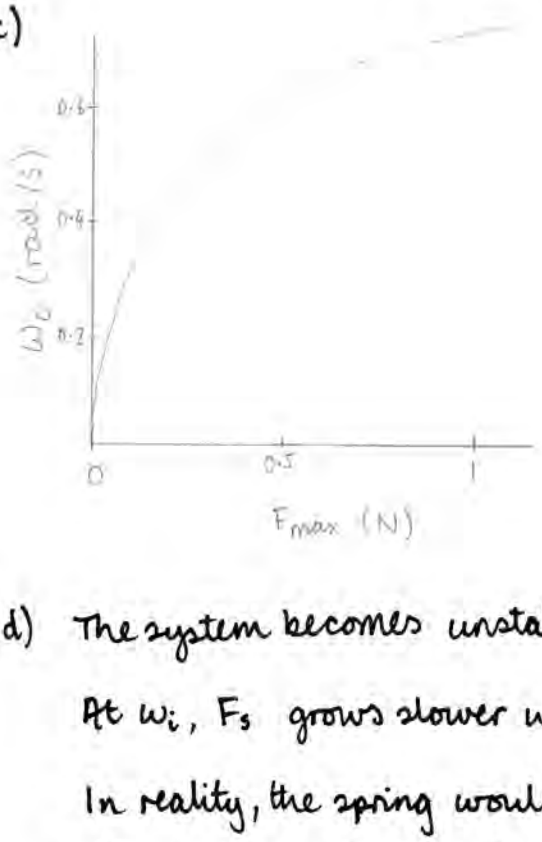
If we insert $R = \frac{k}{k - m\omega^2} R_0$ into equation for centripetal force:

$$F_c = \frac{m\,\omega_c^2\, k}{k - m\omega_c^2} R_0 < F_{max}$$

Solving the inequality for ω_c:

$$\omega_c = \sqrt{\frac{F_{max}\, k}{m(kR_0 + F_{max})}}$$

(c)



(d) The system becomes unstable when $k \leq m\,\omega_i^2$.

At ω_i, F_s grows slower with R than F_c and R diverges as $R = \frac{k}{k - m\omega^2} R_0$.

In reality, the spring would stretch beyond its linear regime (F_s grows faster than linear with R), equilibrium is reached and ω can continue to grow if F_{max} is outside the linear regime

→ ie. a fully extended spring is a straight wire with k being the wire's elastic modulus. It breaks when the wire is overstretched.

If F_{max} lies in the linear regime and $\omega_i < \omega_c$, then, at ω_i, the spring will extend rapidly to its breaking length and break at ω_i.