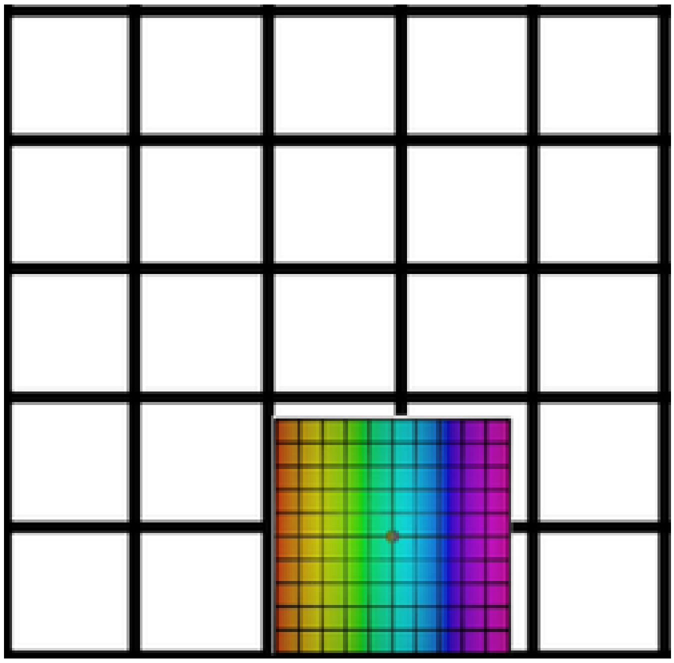
Transformations of the complex plane

A: the z-plane



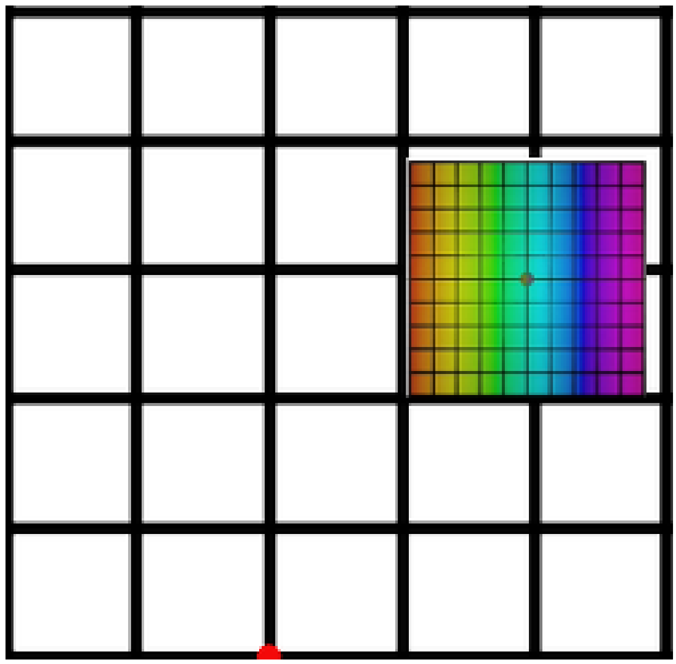
For a *complex* function of a *complex* variable w=f(z), we can't draw a graph, because we'd need four dimensions and four axes (real part of z, imaginary part of z, real part of w, imaginary part of w). So we get a picture of the function by sketching the shapes in the w-plane produced from familiar shapes in the z-plane.

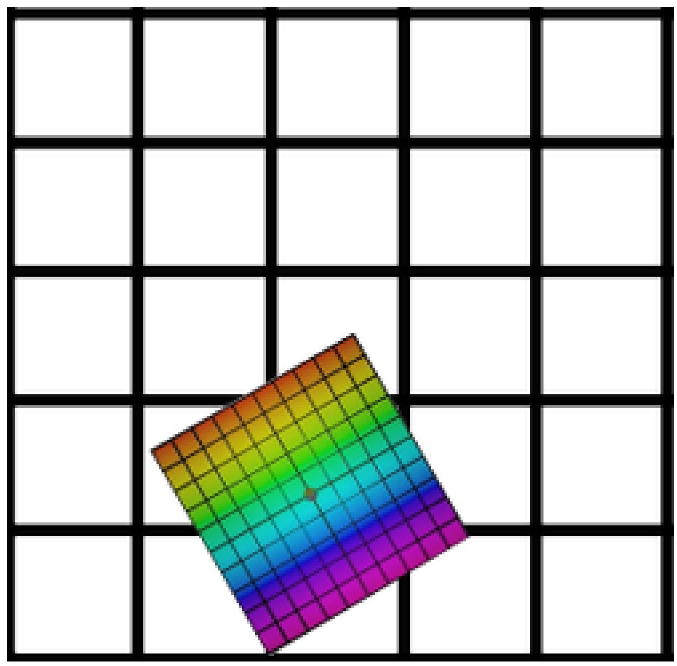
ACTIVITY 1: if picture A represents the z-plane, what sort of function w=f(z) would be represented by the w-pictures on the right? Geometrically: translation, rotation, and enlargement (or shrinking)? And algebraically:

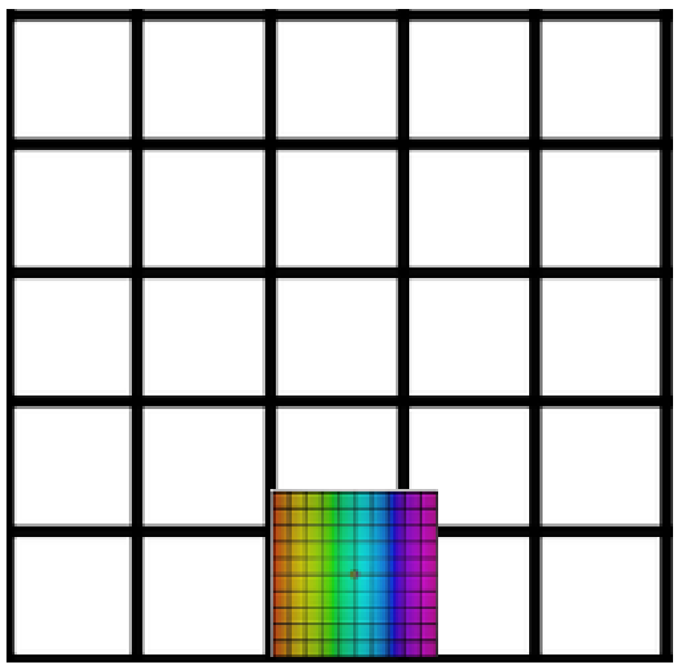
Top w-picture:

Middle w-picture:

Bottom w-picture:



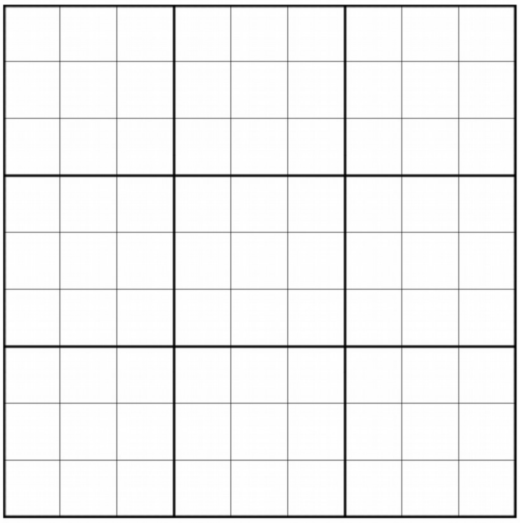




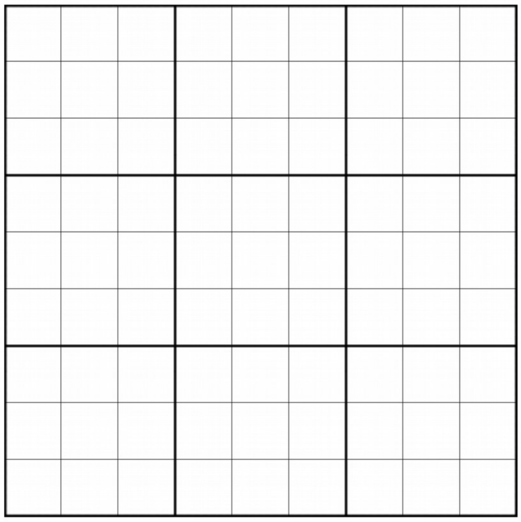
ACTIVITY 2: Cut out a Molly Alice shape of the right size and stick it in the correct positions in a w-plane to represent these transformations.

w= z+1+i

z-plane

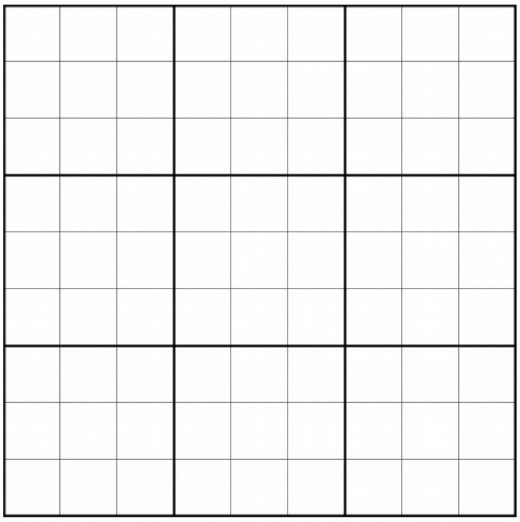


w-plane

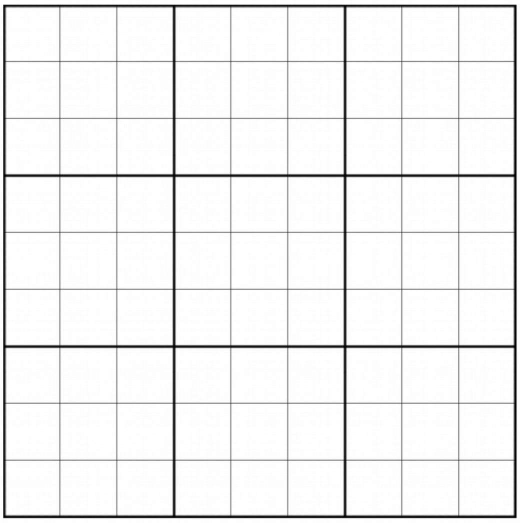


w=2iz+3

z-plane



w-plane



With complex variables, mathematicians spend more time studying what would be very simple functions with real variables (because it's all harder with complex variables). The next stage more complicated than combining

w=z+k and **w=pz**

is combining those sorts of functions with

w=1/z

ACTIVITY 3: In the algebra of complex numbers:

adding a number, or translating, is: **w=z+k**

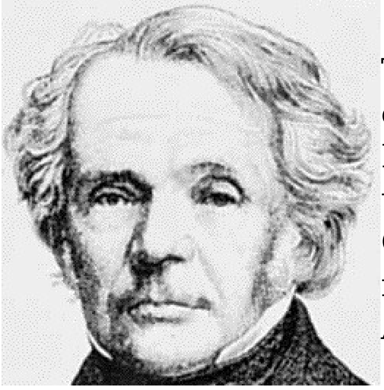
multiplying by a number, or enlarging/shrinking and rotating is: **w=pz**

taking the reciprocal (called **inversion**) is: **w=1/z**

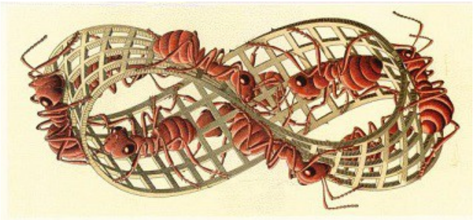
If $w=\frac{az+b}{cz+d}=\frac{\frac{a}{c}z+\frac{b}{c}}{z+\frac{d}{c}}=\frac{a}{c}+\frac{\frac{bc-ad}{c^2}}{z+\frac{d}{c}}$, break down this transformation into four transformations, one after the other: (1) a translation; (2) an inversion; (3) an enlarging/ shrinking/ rotating; (4) a translation.

FACT

Transformations of the form $w=\frac{az+b}{cz+d}$ are called Möbius transformations, and are combinations of translations, inversions, and enlargings/ shrinkings/ rotatings.



There is no special connection with the Möbius strip, other than that the same German mathematician, August Möbius, discovered both.



The inverse of every Möbius transformation is also a Möbius transformation.

We will learn:

- **why Möbius transformations w=f(z) always map z-circles into w-circles**

- **how to work out the w-circle produced from each z-circle**

You may end up learning a lot more about Möbius transformations at uni or at work. They are the *only* complex functions which have inverse functions and can be differentiated, well, not quite everywhere, but everywhere except a few points. They are useful for studying non-Euclidean geometry. They are useful in the theory of relativity: the transformations of space-time which convert one observer's description of an event to another observer's description are equivalent to Möbius transformations of the light-rays emitted by that event as coded in complex numbers. They are used in electrical engineering (the "Smith chart").

Möbius transformations w=f(z) always map z-circles into w-circles

To see this:

1. Define the **"centre of inversion"** or **"pole"** of a Möbius transformation. It is the point at which it "blows up", so the formula for the transformation asks you to divide by zero, and w goes to infinity. For w=(az+b)/(cz+d), the "centre of inversion" or "pole" is z=−d/c.

2. Do a little detour by way of the function w=1/z*

ACTIVITY 4: If z=r cis θ, what is 1/z? what is 1/z*, which = (1/z)*?

If z = 4 = 4 cis 0, what is 1/z*?

If z = 4 cis 30°, what is 1/z*?

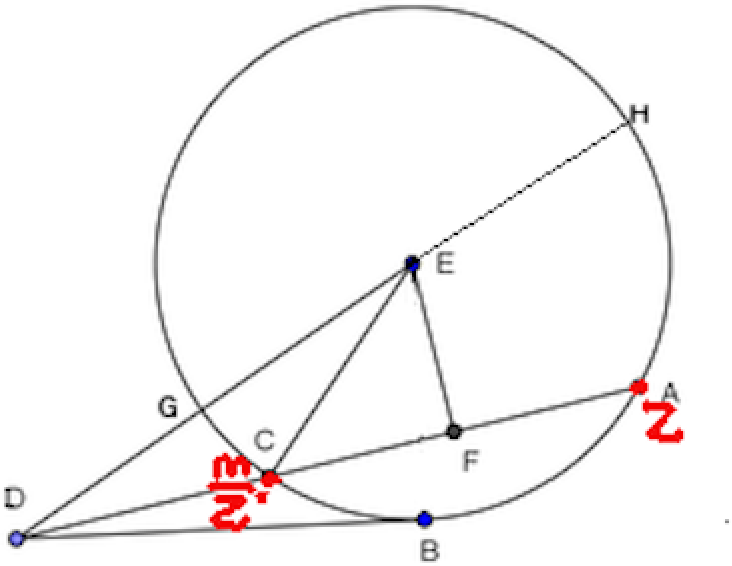
If z = ¼ cis 30°, what is 1/z*?

So the *direction* of 1/z* (its "θ") has what relation to the "θ" of z?

When |z| is big, |1/z*| is big? or small?

When |z| is small, |1/z*| is big? or small?

So the transformation z ↦ 1/z* maps points close to the origin into points far from the origin, but in the same direction; it maps points far from the origin into points close to the origin, but in the same direction.



If D is the origin, and $m=\sqrt{DB}$, then $z \mapsto m/z^*$ maps a point like A into a point like C, and a point like C into a point like A. It maps the whole circle onto *itself*, and it also maps the diameter GH onto itself (except swapped round: GH $\mapsto$ HG).

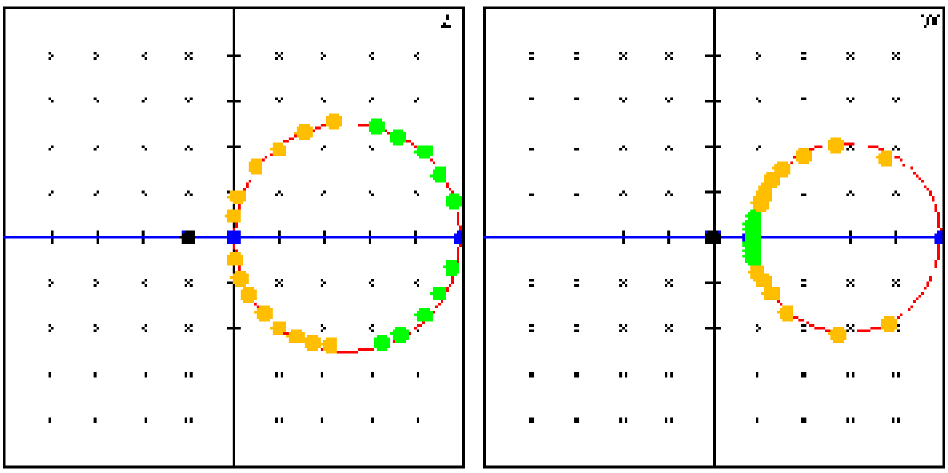
So when the path (locus) of z is a circle, $z \mapsto m/z^*$ produces the same circle, and the diameter of the circle in line with $z=0 \mapsto$ a diameter

$z \mapsto 1/z^*$ must produce a circle (of different size), and the diameter of the z-circle in line with $z=0 \mapsto$ a diameter of the $1/z^*$ circle.

The path (locus) of $1/z$ is just the path (locus) of $1/z^*$ reflected in the real axis, so $z \mapsto 1/z$ must also produce a circle, and the diameter of the z-circle in line with $z=0 \mapsto$ a diameter of the $1/z$ circle.

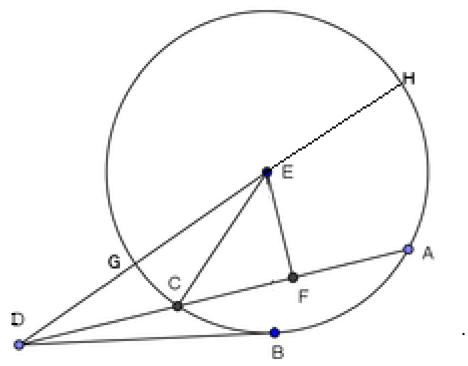
Translations and enlargements/ shrinkings/ rotations map circles to circles, and diameters of circles to diameters of circles. So every combination of inversions, translations, and enlargements/ shrinkings/ rotations - which means, every Möbius transformation - maps a circle to a circle, and the diameter of the z-circle in line with the centre of inversion $\mapsto$ a diameter of the image circle. The picture below shows how one circle is transformed by $z \mapsto 5/(z-1)$. The blue diameter $\mapsto$ a diameter.

Activities 5, 6, and 7 are deleted from this edition.



CLAIM: if z moves on a circle, the tangent from the origin to the circle has length k, and $m=k^2$, then $w=m/z^*$ moves on the *same circle*, and the diameter of the z-circle ⟼ diameter of the w-circle.

PROOF:



A is any point on the circle; E is the centre; F is the midpoint of CA, so angle CFE is a right angle. D is the origin.

Because DFE and CFE are both right-angled triangles

Pythagoras ⇒ $DE^2-DF^2=EF^2=CE^2-CF^2$

⇒ $DE^2-CE^2=DF^2-CF^2$ [*]

But DC.DA = (DF−CF).(DF+FA) = (DF−CF).(DF+CF) = DF^2-CF^2

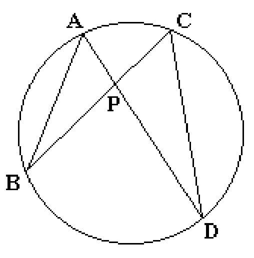
and DG.DH = (DE−GE).(DE+EH) = (DE−CE).(DE+CE) = DE^2-CE^2

since GE, CE, and EH are all radii of the circle

So equation [*] ⇒ DC.DA = DG.DH. Since this is true wherever A is on the circle, it is also true when A is at B, and so DC.DA = DG.DH = $DB^2 = k^2$

So DC is in the same direction as DA, but of length (or magnitude, or modulus) equal to $k^2/|DA|$. If A is represented by the complex number z, C is represented by the complex number k^2/z^*

Since the whole transformation is symmetrical around the line DH, the diameter GH of the z-circle ⟼ the diameter HG of the w-circle (though in general other diameters of the z-circle do not ⟼ diameters of the w-circle). □



We've assumed the origin is *outside* the z-circle. Another bit of geometry proves that if the origin is *inside* the z-circle, then again $w=m/z^*$ moves on the same circle. And the diameter AD of the z-circle ⟼ a diameter of the w-circle. Here m=AP.PD if AD is a diameter. (What if the origin is *on* the circle? Leave that aside for now).

https://mathsmartinthomas.wordpress.com/2015/08/18/images-of-loci-of-complex-numbers/

To transform from m/z^* to $1/z$ just means shrinking by a factor $1/m$ and reflecting in the x-axis. Circles stay circles when you shrink and reflect them, and every diameter of the original circle ⟼ a diameter of the shrunk-and-reflected circle. So, if $w=1/z$, then a z-circle ⟼ a w-circle, and *the diameter* of the *z-circle* in line with the origin ⟼ *a diameter* of the *w-circle*.

Circles obviously also stay circles when you translate them or rotate them, and every diameter of the original circle maps into a diameter of the translated-and-rotated circle.

All Möbius transformations w=(az+b)/(cz+d) are combinations of *inversion* (w=1/z), translation, and rotation. So:

With all Möbius transformations w=(az+b)/(cz+d)

every z-circle ⟼ a w-circle;

The diameter of the z-circle in line with the centre of inversion (also called pole) z=−d/c ⟼ diameter of the w-circle.

You don't have to remember the geometric proofs of these rules. The rules are simple, and they make the job of calculating Möbius transformations ten times easier.

ACTIVITY 8: Use the rules to do the following examples in your book. *Draw a diagram* for each question.

Example 36

The transformation T from the z-plane, where $z = x + iy$, to the w-plane where $w = u + iv$, is given by $w = \frac{3z - 2}{z + 1}$, $z \neq -1$.

Show that the image, under T, of the circle with equation $x^2 + y^2 = 4$ in the z-plane is a circle C in the w-plane. State the centre and radius of C.

13 The transformation T from the z-plane, where $z = x + iy$, to the w-plane where $w = u + iv$, is given by $w = \frac{4z - 3i}{z - 1}$, $z \neq 1$.

Show that the circle $|z| = 3$ is mapped by T onto a circle C.
Find the centre and radius of C.

15 The transformation T from the z-plane, where $z = x + iy$, to the w-plane where $w = u + iv$, is given by $w = \frac{3iz + 6}{1 - z}$, $z \neq 1$.

Show that the circle $|z| = 2$ is mapped by T onto a circle C. State the centre of C and show that the radius of C can be expressed in the form $k\sqrt{5}$ where k is an integer to be determined.

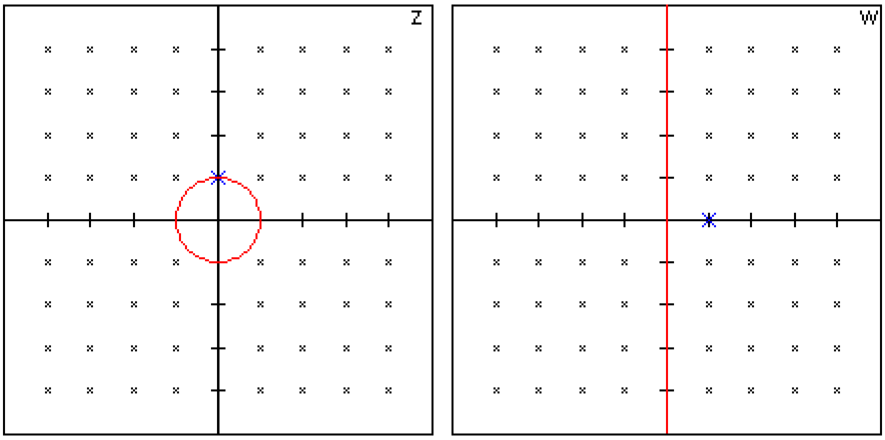
Remember we left aside the case of w=1/z when the origin (which is the centre of inversion, or pole) is *on* the z-circle?

If the centre of inversion (pole) of a Möbius transformation is *on* the z-circle, then the w-path is a *line*. Proof: https://mathsmartinthomas.wordpress.com/2015/08/18/images-of-loci-of-complex-numbers/

The diameter-ends of the z-circle in line with the centre of inversion (pole) ⟼ the point on the line closest to the pole (or, in other words, the foot of the perpendicular ⊥ from the pole to the line), and ∞.

(Activity 9 is deleted from this edition)

Example: z ↦ (z−i)/(z+i) transforms the circle on the left into the line on the right, because the centre of inversion (pole), z=−i, is on the z-circle |z|=1.



Since the inverse of every Möbius transformation is a Möbius transformation:

If the z-path is a *line*, and the pole is not on z, then the w-path is a *circle*.

The point on the z-line closest to the pole (which is the foot of the perpendicular from the pole to the line) ⟼ one diameter-end of the w-circle, and ∞ on the z-line ⟼ the other end of that diameter of the w-circle.

What if the z-path is a line, and the pole is on z? Well, what if the pole is z=0 and the line is z=r cis θ (r varying, θ fixed)? Then w=1/z=(1/r) cis (−θ). In other words, w goes along another line through the pole, the reflection of the z-line in the real axis.

The Earth looks flat to us small people. A big circle would look like a straight line to an ant crawling along the circumference. So, in one way of looking at it, a straight line is just a very, very big circle. It is a circle with centre at infinity and infinite radius.

Define a **generalised circle** to mean an ordinary circle *or* a "circle at infinity" (a line), and the "diameter-ends" of a line to be ∞ and the point on the line closest to the centre of inversion (pole), and then the rules we had before still hold:

every z-circle ⟼ a w-circle;

The diameter of the z-circle in line with the centre of inversion (also called pole) z=−d/c ⟼ diameter of the w-circle.

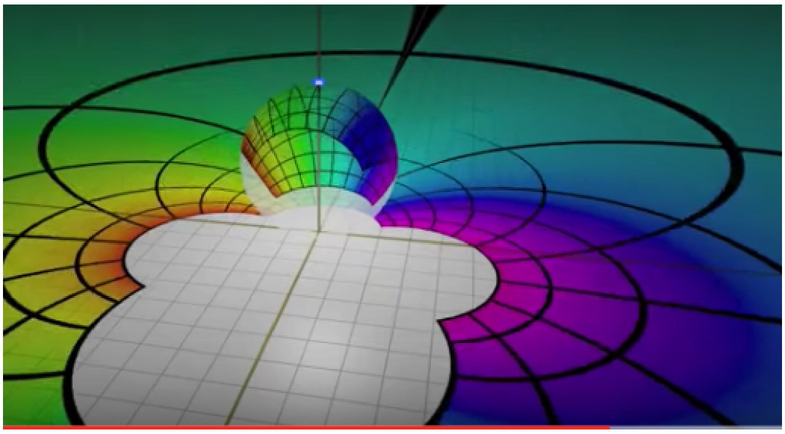
We can see straight off whether the w-"circle" is a line by seeing if the pole z=−d/c is on the z-"circle". If it is, then z=−d/c ⟼ w=∞, so the w-"circle" goes all the way to infinity. So it's a line rather than an ordinary circle.

Using the rules to calculate Möbius transformations is still easy. *In fact, sometimes easier*. If the w-"circle" is a line, then just two points are enough to fix it. We only need to find the w-values for *any* two easy z-values we like (never mind about diameters), and we're done.

If the z-"circle" is a line, you need to find the foot of ⊥ from the pole to the z-line, It may be obvious. Otherwise you can do it by writing the z-line in y=mx+c form; then finding the line through the pole of form y=(−1/m)x+k, and solving the simultaneous equations for x and y.

This short video sums it all up:

https://www.youtube.com/watch?v=0z1fIsUNhO4



ACTIVITY 10: Use the rules to do these examples, in your book.

Example 37

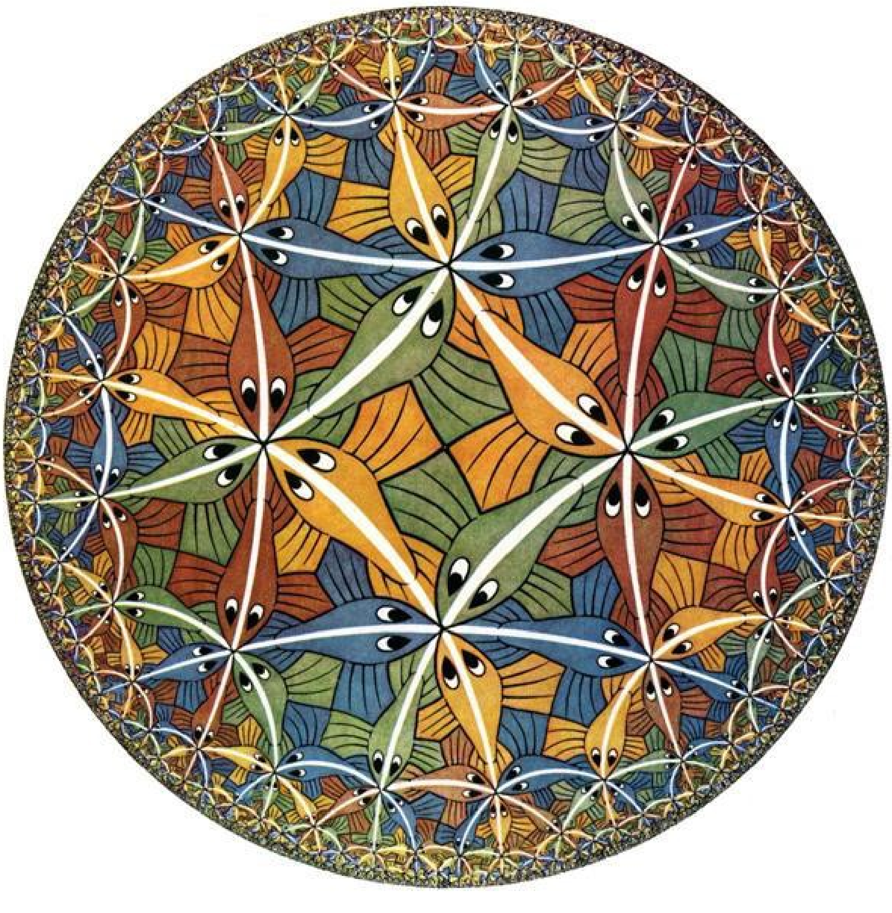
A transformation T of the z-plane to the w-plane is given by $w = \frac{iz - 2}{1 - z}$, $z \neq 1$.

Show that as z lies on the real axis in the z-plane, then w lies on the line l in the w-plane. Sketch l on an Argand diagram.

9 For the transformation $w = \frac{1}{2 - z}$, $z \neq 2$, show that the image, under T, of the circle centre O, radius 2 in the z-plane is a line l in the w-plane. Sketch l on an Argand diagram.

10 The transformation T from the z-plane, where $z = x + iy$, to the w-plane where $w = u + iv$, is given by $w = \frac{16}{z}$, $z \neq 0$.

a The transformation T maps the points on the circle $|z - 4| = 4$, in the z-plane, to points on a line l in the w-plane. Find the equation of l.



A Möbius transformation will transform the inside or outside of a z-"circle" into the inside or outside of a w-"circle". If the z-"circle" (or the w-"circle") is a line, interpret "inside" and "outside" to mean half-plane above the line, and half-plane below it.

In particular, the Möbius transformation w=(z−i)/(z+i) transforms the upper half-plane (all complex numbers with Im(z)>0) into the disc |z|<1, i.e. the inside of the circle |z|=1. This transformation is used in art (as in M C Escher's "Circle Limit", previous page), and in non-Euclidean geometry.

To work out which it is, inside or outside, take one easy z-point in the z-region you want to know about, and see where the corresponding w-point is. Often the easiest z-point to take is z=∞.

ACTIVITY 11.

12 The transformation T from the z-plane, where $z = x + iy$, to the w-plane where $w = u + iv$, is given by $w = \frac{-iz + i}{z + 1}$, $z \neq -1$.

a The transformation T maps the points on the circle with equation $x^2 + y^2 = 1$ in the z-plane, to points on a line l in the w-plane. Find the equation of l.

b Hence, or otherwise, shade and label on an Argand diagram the region R of the w-plane which is the image of $|z| \leqslant 1$ under T.

c Show that the image, under T, of the circle with equation $x^2 + y^2 = 4$ in the z-plane is a circle C in the w-plane. Find the equation of C.

ACTIVITY 12: In your book, draw a diagram of the points z for which |z−1|=|z−3|. Write down the general rule for the path of z if |z−a|=|z−b|. Work this out in algebra. (See examples 24 and 25 on pages 42 and 43 of the textbook).

ACTIVITY 13: |z|=r is the circle with centre at the origin and radius r. What is the path of z if |z−a−ib|=r? If z=x+iy, use the fact that |x+iy−a−ib|=|(x−a)+i(y−b)| to write an equation in x and y for the path of z. (See example 22 on page 41 of the textbook).

ACTIVITY 14

Example 30

If $\arg\left(\frac{z}{z - 4i}\right) = \frac{\pi}{2}$, sketch the locus of $P(x, y)$ which is represented by z on an Argand diagram.

Explanation: the angle in a semicircle is π/2. So if arg[z/z−4i]=π/2, which means that the line from 0 to z is always at right angles to the line from 4i to z, then the triangle with corners at 0, z, and 4i is always a triangle in a semicircle. z moves on a semicircle with diameter-ends at 0 and 4i. Draw a diagram! Arg[z/z−4i]=π/2 also means that the line from 0 to z is π/2 *anticlockwise* from the line from 4i to z, which tells you *which* semicircle, the right-hand one or the left.

Example 29

If $\arg\left(\dfrac{z-6}{z-2}\right) = \dfrac{\pi}{4}$,

a sketch the locus of $P(x, y)$ which is represented by z on an Argand diagram,

b find the Cartesian equation of this locus.

Explanation: the angle subtended by a chord is the same at every point on the circumference of a circle. Therefore z is on the circumference of a circle which has a chord between 2 and 6. And if the angle at the circumference is π/4, then the angle subtended by the chord at the centre of the circle is... what?

ACTIVITY 15.

Example 28

If $\arg(z + 3 + 2\mathrm{i}) = \dfrac{3\pi}{4}$, sketch the locus of z on an Argand diagram.

Find the Cartesian equation of this locus.

HOW TO WRITE ANSWERS FOR EDEXCEL

To explain your work to the marker, you must write words at the start of your answer.

"By the circle inversion theorems, the w-locus is either a circle or a line".

Either: **"Since z=[value of −d/c] is not on the z-locus and so w↛∞, the w-locus is a circle"**.
Or: **"Since z=[value of −d/c] is not on the z-locus and so w→∞, the w-locus is a line".**

Either: **"By symmetry, the diameter of the z-circle in line with the centre of inversion (pole) ⟼ a diameter of the w-circle".**
Or: **"By symmetry, z=∞ and the foot of ⊥ from the centre of inversion (pole) to the z-line ⟼ a diameter of the w-circle"**. If the w-locus is a line, you don't need this third lot of words: you can just find two w-points to define the line.