TRANSIENT ANALYSIS OF ELECTRIC POWER CIRCUITS HANDBOOK

Transient Analysis of Electric Power Circuits Handbook

by

ARIEH L. SHENKMAN

Holon Academic Institute of Technology,
Holon, Israel

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Printed in the Netherlands.

This book is dedicated to my dearest wife Iris
and wonderful children
Daniel, Elana and Joella

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FOREWORD

Every now and then, a good book comes along and quite rightfully makes itself a distinguished place among the existing books of the electric power engineering literature. This book by Professor Arieh Shenkman is one of them.

Today, there are many excellent textbooks dealing with topics in power systems. Some of them are considered to be classics. However, many of them do not particularly address, nor concentrate on, topics dealing with transient analysis of electrical power systems.

Many of the fundamental facts concerning the transient behavior of electric circuits were well explored by Steinmetz and other early pioneers of electrical power engineering. Among others, *Electrical Transients in Power Systems* by Allan Greenwood is worth mentioning. Even though basic knowledge of transients may not have advanced in recent years at the same rate as before, there has been a tremendous proliferation in the techniques used to study transients. The application of computers to the study of transient phenomena has increased both the knowledge as well as the accuracy of calculations.

Furthermore, the importance of transients in power systems is receiving more and more attention in recent years as a result of various blackouts, brownouts, and recent collapses of some large power systems in the United States, and other parts of the world. As electric power consumption grows exponentially due to increasing population, modernization, and industrialization of the so-called third world, this topic will be even more important in the future than it is at the present time.

Professor Arieh Shenkman is to be congratulated for undertaking such an important task and writing this book that singularly concentrates on the topics related to the transient analysis of electric power systems. The book successfully fills the long-existing gap in such an important area.

Turan Gonen, Ph.D., Fellow IEEE
Professor and Director
Electric Power Educational Institute
California State University, Sacramento

PREFACE

Most of the textbooks on electrical and electronic engineering only partially cover the topic of transients in simple *RL*, *RC* and *RLC* circuits and the study of this topic is primarily done from an electronic engineer's viewpoint, i.e., with an emphasis on low-current systems, rather than from an electrical engineer's viewpoint, whose interest lies in high-current, high-voltage power systems. In such systems a very clear differentiation between steady-state and transient behavior of circuits is made. Such a division is based on the concept that steady-state behavior is normal and transients arise from the faults. The operation of most electronic circuits (such as oscillators, switch capacitors, rectifiers, resonant circuits etc.) is based on their transient behavior, and therefore the transients here can be referred to as "desirable". The transients in power systems are characterized as completely "undesirable" and should be avoided; and subsequently, when they do occur, in some very critical situations, they may result in the electrical failure of large power systems and outages of big areas. Hence, the Institute of Electrical and Electronic Engineers (IEEE) has recently paid enormous attention to the importance of power engineering education in general, and transient analysis in particular.

It is with the belief that transient analysis of power systems is one of the most important topics in power engineering analysis that the author proudly presents this book, which is wholly dedicated to this topic.

Of course, there are many good books in this field, some of which are listed in the book; however they are written on a specific technical level or on a high theoretical level and are intended for top specialists. On the other hand, introductory courses, as was already mentioned, only give a superficial knowledge of transient analysis. So that there is a gap between introductory courses and the above books.

The present book is designed to fill this gap. It covers the topic of transient analysis from simple to complicated, and being on an intermediate level, this book therefore is a link between introductory courses and more specific technical books. In the book the most important methods of transient analysis, such as the classical method, Laplace and Fourier transforms and state variable analysis

are presented; and of course, the emphasis on transients in three-phase systems and transmission lines is made.

The appropriate level and the concentration of all the topics under one cover make this book very special in the field under consideration. The author believes that this book will be very helpful for all those specializing in electrical engineering and power systems. It is recommended as a textbook for specialized undergraduate and graduate curriculum, and can also be used for master and doctoral studies. Engineers in the field may also find this book useful as a handbook and/or resource book that can be kept handy to review specific points. Theoreticians/researchers who are looking for the mathematical background of transients in electric circuits may also find this book helpful in their work.

The presentation of the covered material is geared to readers who are being exposed to (a) the basic concept of electric circuits based on their earlier study of physics and/or introductory courses in circuit analysis, and (b) basic mathematics, including differentiation and integration techniques

This book is composed of eight chapters. The study of transients, as mentioned, is presented from simple to complicated. Chapters 1 and 2 are dedicated to the classical method of transient analysis, which is traditional for many introductory courses. However, these two chapters cover much more material giving the mathematical as well as the physical view of transient behavior of electrical circuits. So-called incorrect initial conditions and two generalized commutation laws, which are important for a better understanding of the transient behavior of transformers and synchronous machines, are also discussed in Chapter 2.

Chapters 3 and 4 give the transform methods of transient analysis, introducing the Laplace as well as the Fourier transforms. What is common between these two methods and the differences are emphasized. The theoretical study of the transform methods is accompanied by many practical examples.

The state variable method is presented in Chapter 5. Although this method is not very commonly used in transient analysis, the author presumes that the topic of the book will not be complete without introducing this essential and interesting method. It should be noted that the state variable method in its matrix notation, which is given here, is very appropriate for transient analysis using computers.

Naturally, an emphasis and a great amount of material are dedicated to transients in three-phase circuits, which can be found in Chapter 6. As power systems are based on employing three-phase generators and transformers, the complete analysis of their behavior under short-circuit faults at both steady-state and first moment operations is given. The overvoltages following switching-off in power systems are also analyzed under the influence of the electric arc, which accompanies such switching.

In Chapter 7 the transient behavior of transmission lines is presented. The transmission line is presented as a network with distributed parameters and subsequently by partial differential equations. The transient analysis of such lines is done in two ways: as a method of traveling waves and by using the

Laplace transform. Different engineering approaches using both methods are discussed.

Finally, in Chapter 8 an overview of the static and dynamic stability of power systems is given. Analyzing system stability is done in traditional ways, i.e., by solving a swing equation and by using an equal area criterion.

Throughout the text, the theoretical discussions are accompanied by many worked-out examples, which will hopefully enable the reader to get a better understanding of the various concepts.

The author hopes that this book will be helpful to all readers studying and specializing in power system engineering, and of value to professors in the educational process and to engineers who are concerned with the design and R&D of power systems.

Last but not least, my sincere appreciation goes to my wife, Iris, who prodigiously supported and aided me throughout the writing of this book. I am also extremely grateful for her assistance in editing and typing in English.

Chapter #1

CLASSICAL APPROACH TO TRANSIENT ANALYSIS

1.1 INTRODUCTION

Transient analysis (or just transients) of electrical circuits is as important as steady-state analysis. When transients occur, the currents and voltages in some parts of the circuit may many times exceed those that exist in normal behavior and may destroy the circuit equipment in its proper operation. We may distinguish the transient behavior of an electrical circuit from its steady-state, in that during the transients all the quantities, such as currents, voltages, power and energy, are changed in time, while in steady-state they remain invariant, i.e. constant (in d.c. operation) or periodical (in a.c. operation) having constant amplitudes and phase angles.

The cause of transients is any kind of changing in circuit parameters and/or in circuit configuration, which usually occur as a result of switching (commutation), short, and/or open circuiting, change in the operation of sources etc. The changes of currents, voltages etc. during the transients are not instantaneous and take some time, even though they are extremely fast with a duration of milliseconds or even microseconds. These very fast changes, however, cannot be instantaneous (or abrupt) since the transient processes are attained by the interchange of energy, which is usually stored in the magnetic field of inductances or/and the electrical field of capacitances. Any change in energy cannot be abrupt otherwise it will result in infinite power (as the power is a derivative of energy, $p = dw/dt$), which is in contrast to physical reality. All transient changes, which are also called transient responses (or just responses), vanish and, after their disappearance, a new steady-state operation is established. In this respect, we may say that the transient describes the circuit behavior between two steady-states: an old one, which was prior to changes, and a new one, which arises after the changes.

A few methods of transient analysis are known: the classical method, The Cauchy-Heaviside (C-H) operational method, the Fourier transformation

method and the Laplace transformation method. The C-H operational or symbolic (formal) method is based on replacing a derivative by symbol s ($(d/dt) \leftrightarrow s$) and an integral by $1/s$

$$\left(\int dt \leftrightarrow \frac{1}{s} \right).$$

Although these operations are also used in the Laplace transform method, the C-H operational method is not as systematic and as rigorous as the Laplace transform method, and therefore it has been abandoned in favor of the Laplace method. The two transformation methods, Laplace and Fourier, will be studied in the following chapters. Comparing the classical method and the transformation method it should be noted that the latter requires more knowledge of mathematics and is less related to the physical matter of transient behavior of electric circuits than the former.

This chapter is concerned with the classical method of transient analysis. This method is based on the determination of differential equations and splitting the solution into two components: natural and forced responses. The classical method is fairly complicated mathematically, but is simple in engineering practice. Thus, in our present study we will apply some known methods of steady-state analysis, which will allow us to simplify the classical approach of transient analysis.

1.2 APPEARANCE OF TRANSIENTS IN ELECTRICAL CIRCUITS

In the analysis of an electrical system (as in any physical system), we must distinguish between the stationary operation or steady-state and the dynamical operation or transient-state.

An electrical system is said to be in **steady-state** when the variables describing its behavior (voltages, currents, etc.) are either invariant with time (d.c. circuits) or are periodic functions of time (a.c. circuits). An electrical system is said to be in **transient-state** when the variables are changed non-periodically, i.e., when the system is not in steady-state. The transient-state vanishes with time and a new steady-state regime appears. Hence, we can say that the transient-state, or just transients, is usually the transmission state from one steady-state to another.

The parameters L and C are characterized by their ability to store energy: *magnetic energy* $w_L = \frac{1}{2}\psi i = \frac{1}{2}Li^2$ (since $\psi = Li$), in the magnetic field and *electric energy* $w_C = \frac{1}{2}qv = \frac{1}{2}Cv^2$ (since $q = Cv$), in the electric field of the circuit. The voltage and current sources are the elements through which the energy is supplied to the circuit. Thus, it may be said that an electrical circuit, as a physical system, is characterized by certain energy conditions in its steady-state behavior. Under steady-state conditions the energy stored in the various inductances and capacitances, and supplied by the sources in a d.c. circuit, are constant; whereas in an a.c. circuit the energy is being changed (transferred between the magnetic and electric fields and supplied by sources) *periodically*.

When any sudden change occurs in a circuit, there is usually a redistribution of energy between L-s and C-s, and a change in the energy status of the sources, which is required by the new conditions. **These energy redistributions cannot take place instantaneously, but during some period of time, which brings about the transient-state.**

The main reason for this statement is that an instantaneous change of energy would require infinite power, which is associated with inductors/capacitors. As previously mentioned, power is a derivative of energy and any abrupt change in energy will result in an infinite power. Since infinite power is not realizable in physical systems, the energy cannot change abruptly, but only within some period of time in which transients occur. Thus, from a physical point of view it may be said that the transient-state exists in physical systems while the energy conditions of one steady-state are being changed to those of another.

Our next conclusion is about the current and voltage. To change *magnetic energy* requires a change of current through inductances. Therefore, currents in inductive circuits, or inductive branches of the circuit, cannot change abruptly. From another point of view, the change of current in an inductor brings about the induced voltage of magnitude $L(di/dt)$. An instantaneous change of current would therefore require an infinite voltage, which is also unrealizable in practice. Since the induced voltage is also given as $d\psi/dt$, where ψ is a magnetic flux, the magnetic flux of a circuit cannot suddenly change.

Similarly, we may conclude that to change the *electric energy* requires a change in voltage across a capacitor, which is given by $v = q/C$, where q is the charge. Therefore, neither the voltage across a capacitor nor its charge can be abruptly changed. In addition, the rate of voltage change is $dv/dt = (1/C)\, dq/dt = i/C$, and the instantaneous change of voltage brings about infinite current, which is also unrealizable in practice. Therefore, we may summarize that **any change in an electrical circuit, which brings about a change in energy distribution, will result in a transient-state.**

In other words, by any switching, interrupting, short-circuiting as well as any rapid changes in the structure of an electric circuit, the transient phenomena will occur. Generally speaking, **every change of state leads to a temporary deviation from one regular, steady-state performance of the circuit to another one.** The redistribution of energy, following the above changes, i.e., the transient-state, theoretically takes infinite time. However, in reality the transient behavior of an electrical circuit continues a relatively very short period of time, after which the voltages and currents almost achieve their new steady-state values.

The change in the energy distribution during the transient behavior of electrical circuits is governed by the principle of energy conservation, i.e., the amount of supplied energy is equal to the amount of stored energy plus the energy dissipation. The rate of energy dissipation affects the time interval of the transients. The higher the energy dissipation, the shorter is the transient-state. Energy dissipation occurs in circuit resistances and its storage takes place in inductances and capacitances. In circuits, which consist of only resistances, and neither inductances nor capacitances, the transient-state will not occur at all

and the change from one steady-state to another will take place instantaneously. However, since even resistive circuits contain some inductances and capacitances the transients will practically appear also in such circuits; but these transients are very short and not significant, so that they are usually neglected.

Transients in electrical circuits can be recognized as either desirable or undesirable. In power system networks, the transient phenomena are wholly undesirable as they may bring about an increase in the magnitude of the voltages and currents and in the density of the energy in some or in most parts of modern power systems. All of this might result in equipment distortion, thermal and/or electrodynamics' destruction, system stability interferences and in extreme cases an outage of the whole system.

In contrast to these unwanted transients, there are desirable and controlled transients, which exist in a great variety of electronic equipment in communication, control and computation systems whose normal operation is based on switching processes.

The transient phenomena occur in electric systems either by intentional switching processes consisting of the correct manipulation of the controlling apparatus, or by unintentional processes, which may arise from ground faults, short-circuits, a break of conductors and/or insulators, lightning strokes (particularly in high voltage and long distance systems) and similar inadvertent processes.

As was mentioned previously, there are a few methods of solving transient problems. The most widely known of these appears in all introductory textbooks and is used for solving simpler problems. **It is called the classical method.** Other useful methods are Laplace (see Chap. 3) and Fourier (see Chap. 4) transformation methods. These two methods are more general and are used for solving problems that are more complicated.

1.3 DIFFERENTIAL EQUATIONS DESCRIBING ELECTRICAL CIRCUITS

Circuit analysis, as a physical system, is completely described by *integrodifferential equations* written for voltages and/or currents, which characterize circuit behavior. For linear circuits these equations are called linear differential equations with constant coefficients, i.e. in which every term is of the first degree in the dependent variable or one of its derivatives. Thus, for example, for the circuit of three basic elements: R, L and C connected in series and driven by a voltage source $v(t)$, Fig. 1.1, we may apply Kirchhoff's voltage law

$$v_R + v_L + v_C = v(t),$$

in which

$$v_R = Ri$$

$$v_L = L\frac{di}{dt}$$

$$v_C = \int i\,dt,$$

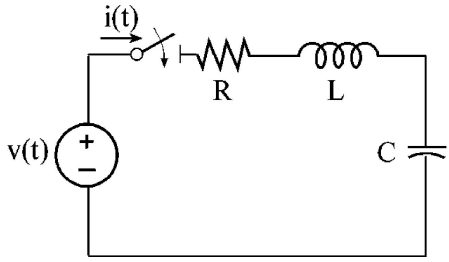

Figure 1.1 Series *RLC* circuit driven by a voltage source.

and then we have

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int i\,dt = v(t). \tag{1.1}$$

After the differentiation of both sides of equation 1.1 with respect to time, the result is a second order differential equation

$$L\frac{d^2t}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i = \frac{dv}{dt}. \tag{1.2}$$

The same results may be obtained by writing two simultaneous first order differential equations for two unknowns, i and v_C:

$$\frac{dv_C}{dt} = \frac{1}{C}i \tag{1.3a}$$

$$Ri + L\frac{di}{dt} + v_C = v(t). \tag{1.3b}$$

After differentiation equation 1.3b and substituting dv_C/dt by equation 1.3a, we obtain the same (as equation 1.2) second order singular equation. The solution of differential equations can be completed only if the initial conditions are specified. It is obvious that in the same circuit under the same commutation, but with different initial conditions, its transient response will be different.

For more complicated circuits, built from a number of loops (nodes), we will have a set of differential equations, which should be written in accordance with Kirchhoff's two laws or with nodal and/or mesh analysis. For example, considering the circuit shown in Fig. 1.2, after switching, we will have a circuit, which consists of two loops and two nodes. By applying Kirchhoff's two laws, we may write three equations with three unknowns, i, i_L and v_C,

$$C\frac{dv_C}{dt} + i_L - i = 0 \tag{1.4a}$$

$$L\frac{di_L}{dt} + R_1 i_L + Ri = 0 \tag{1.4b}$$

$$L\frac{di_L}{dt} + R_1 i_L - v_C = 0 \tag{1.4c}$$

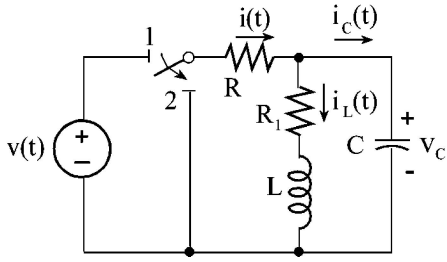

Figure 1.2 A two-loop circuit.

These three equations can then be redundantly transformed into a single second order equation. First, we differentiate the third equation of 1.4c once with respect to time and substitute dv_C/dt by taking it from the first one. After that, we have two equations with two unknowns, i_L and i. Solving these two equations for i_L (i.e. eliminating the current i) results in the second order homogeneous differential equation

$$LCR\frac{d^2i_L}{dt^2} + (L + CRR_1)\frac{di_L}{dt} + (R + R_1)i_L = 0. \tag{1.5}$$

As another example, let us consider the circuit in Fig. 1.3. Applying mesh analysis, we may write three *integro-differential equations* with three unknown mesh currents:

$$\begin{aligned} L\frac{di_1}{dt} - L\frac{di_2}{dt} + R_1 i_1 &= v(t) \\ L\frac{di_2}{dt} - L\frac{di_1}{dt} + (R_2 + R_3)i_2 - R_3 i_3 &= 0 \\ -R_3 i_2 + R_3 i_3 + \frac{1}{C}\int i_3 dt &= 0. \end{aligned} \tag{1.6}$$

In this case it is preferable to solve the problem by treating the whole set of equations 1.6 rather than reducing them to a single one (see further on).

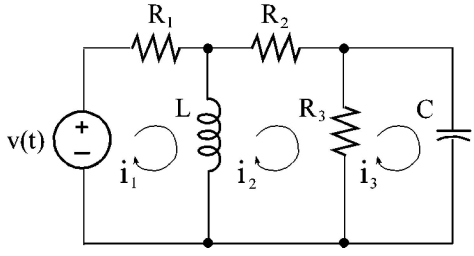

Figure 1.3 A three-loop circuit.

From mathematics, we know that there are a number of ways of solving differential equations. Our goal in this chapter is to analyze the transient behavior of electrical circuits from the physical point of view rather than applying complicated mathematical methods. (This will be discussed in the following chapters.) Such a way of transient analysis is in the formulation of differential equations in accordance with the properties of the circuit elements and in the *direct* solution of the obtained equations, using only the necessary mathematical rules. Such a method is called the **classical method or classical approach** in transient analysis. We believe that the classical method of solving problems enables the student to better understand the transient behavior of electrical circuits.

1.3.1 Exponential solution of a simple differential equation

Let us, therefore, begin our study of transient analysis by considering the simple series *RC* circuit, shown in Fig. 1.4. After switching we will get a source free circuit in which the precharged capacitor *C* will be discharged via the resistance *R*. To find the capacitor voltage we shall write a differential equation, which in accordance with Kirchhoff's voltage law becomes

$$Ri + v_C = 0, \quad \text{or} \quad RC\frac{dv_C}{dt} + v_C = 0. \tag{1.7}$$

A direct method of solving this equation is to write the equation in such a way that the variables are separated on both sides of the equation and then to integrate each of the sides. Multiplying by dt and dividing by v_C, we may arrange the variables to be separated.

$$\frac{dv_C}{v_C} = -\frac{1}{RC}dt. \tag{1.8}$$

The solution may be obtained by integrating each side of equation 1.8 and by adding a constant of integration:

$$\int \frac{dv_C}{v_C} = -\frac{1}{RC}\int dt + K,$$

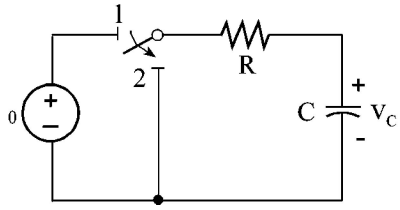

Figure 1.4 A series *RC* circuit.

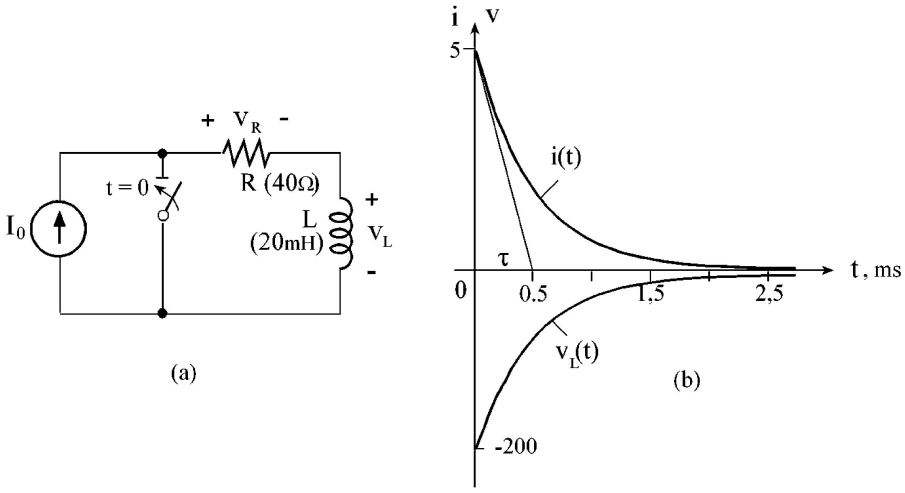

Figure 1.6 A circuit of Example 1.1 (a) and two plots of current and voltage (b).

Example 1.1

Consider a numerical example. The *RL* circuit in Fig. 1.6(a) is fed by a d.c. current source, $I_0 = 5\,\text{A}$. At instant $t = 0$ the switch is closed and the circuit is short-circuited. Find: 1) the current after switching, by separating the variables and applying the *definite integrals*, 2) the voltage across the inductance.

Solution

1) First, we shall write the differential equation:

$$v_L + v_R = L\frac{di}{dt} + Ri = 0,$$

or after separating the variables

$$\frac{di}{i} = \frac{R}{L}\,dt.$$

Since the current changes from I_0 at the instant of switching to $i(t)$, at any instant of t, which means that the time changes from $t = 0$ to this instant, we may perform the integration of each side of the above equation between the corresponding limits

$$\int_{I_0}^{i(t)} \frac{di}{i} = \int_0^t -\frac{R}{L}\,dt.$$

Therefore,

$$\ln i\,|_{I_0}^{i(t)} = -\frac{R}{L}\,t\,|_0^t$$

and

$$\ln i(t) - \ln I_0 = \frac{R}{L}t, \quad \text{or} \quad \ln \frac{i(t)}{I_0} = -\frac{R}{L}t,$$

which results in

$$\frac{i(t)}{I_0} = e^{-\frac{R}{L}t}$$

Thus,

$$i(t) = I_0 e^{-\frac{R}{L}t} = 5e\check{S}^{2000t},$$

or

$$i(t) = I_0 e^{-\frac{t}{\tau}} = 5e^{-\frac{t}{0.5\cdot 10^{-3}}}.$$

where

$$R/L = \frac{40}{20\cdot 10^{-3}} = 2000 \text{ s}^{-1},$$

which results in time constant

$$\tau = \frac{L}{R} = 0.5 \text{ ms}.$$

Note that by applying the definite integrals we avoid the step of evaluating the constant of the integration.

2) The voltage across the inductance is

$$v_L = L\frac{di}{dt} = L\frac{d}{dt}(5e^{-2000t}) = 20\cdot 10^{-3}\cdot 5\cdot(-2000)e^{-2000t} = -200e^{-\frac{t}{0.5}}, \text{ V}$$

(time in ms).

Note that the voltage across the resistance is $v_R = Ri = 40\cdot 5e^{-t/0.5} = 200e^{-t/0.5}$, i.e., it is equal in magnitude to the inductance voltage, but opposite in sign, so that the total voltage in the short-circuit is equal to zero. The plots of the current and voltage are shown in Fig. 1.6(b).

1.4 NATURAL AND FORCED RESPONSES

Our next goal is to introduce a general approach to solving differential equations by the classical method. Following the principles of mathematics we will consider the complete solution of any linear differential equation as composed of two parts: the complementary solution (or natural response in our study) and the particular solution (or forced response in our study). To understand these

principles, let us consider a first order differential equation, which has already been derived in the previous section. In a more general form it is

$$\frac{dv}{dt} + P(t)v = Q(t). \tag{1.14}$$

Here $Q(t)$ is identified as a forcing function, which is generally a function of time (or constant, if a d.c. source is applied) and $P(t)$, is also generally a function of time, represents the circuit parameters. In our study, however, it will be a constant quantity, since the value of circuit elements does not change during the transients (indeed, the circuit parameters do change during the transients, but we may neglect this change as in many cases it is not significant).

A more general method of solving differential equations, such as equation 1.14, is to multiply both sides by a so-called *integrating factor*, so that each side becomes an exact differential, which afterwards can be integrated directly to obtain the solution. For the equation above (equation 1.14) the integrating factor is $e^{\int P\,dt}$ or e^{Pt}, since P is constant. We multiply each side of the equation by this integrating factor and by dt and obtain

$$e^{Pt}dv + vPe^{Pt}dt = Qe^{Pt}dt.$$

The left side is now the exact differential of ve^{Pt} (indeed, $d(ve^{Pt}) = e^{Pt}dv + vPe^{Pt}dt$), and thus

$$d(ve^{Pt}) = Qe^{Pt}dt.$$

Integrating each side yields

$$ve^{Pt} = \int Qe^{Pt}dt + A, \tag{1.15}$$

where A is a constant of integration. Finally, the multiplication of both sides of equation 1.15 by e^{-Pt} yields

$$v(t) = e^{-Pt}\int Qe^{Pt}dt + Ae^{-Pt}, \tag{1.16}$$

which is the solution of the above differential equation. As we can see, this complete solution is composed of two parts. The first one, which is dependent on the forcing function Q, is the *forced response* (it is also called the steady-state response or the particular solution or the particular integral). The second one, which does not depend on the forcing function, but only on the circuit parameters P (the types of elements, their values, interconnections, etc) and on the initial conditions A, i.e., on the "nature" of the circuit, is the *natural response*. It is also called the solution of the homogeneous equation, which does not include the source function and has anything but zero on its right side.

Following this rule, we will solve differential equations by finding natural and forced responses separately and combining them for a complete solution. This principle of dividing the solution of the differential equations into two

components can also be understood by applying the superposition theorem. Since the differential equations, under study, are linear as well as the electrical circuits, we may assert that superposition is also applicable for the transient-state. Following this principle, we may subdivide, for instance, the current into two components

$$i = i' + i'',$$

and by substituting this into the set of differential equations, say of the form

$$\sum \left(L \frac{di}{dt} + Ri + \frac{1}{C} \int i \, dt \right) = \sum v_s,$$

we obtain the following two sets of equations

$$\sum \left(L \frac{di'}{dt} + Ri' + \frac{1}{C} \int i' \, dt \right) = \sum v_s,$$

$$\sum \left(L \frac{di''}{dt} + Ri'' + \frac{1}{C} \int i'' \, dt \right) = 0.$$

It is obvious that by summation (superimposition) of these two equations, the original equation will be achieved. This means that i'' is a natural response since it is the solution of a homogeneous equation with a zero on the right side and develops without any action of any source, and i' is a steady-state current as it develops under the action of the voltage sources v_s (which are presented on the right side of the equations).

The most difficult part in the classical method of solving differential equations is evaluating the particular integral in equation 1.16, especially when the forcing function is not a simple d.c. or exponential source. However, in circuit analysis we can use all the methods: node/mesh analysis, circuit theorems, the phasor method for a.c. circuits (which are all given in introductory courses on steady-state analysis) to find the forced response. In relation to the natural response, the most difficult part is to formulate the characteristic equation (see further on) and to find its roots. Here in circuit analysis we also have special methods for evaluating the characteristic equation simply by inspection of the analyzed circuit, avoiding the formulation of differential equations.

Finally, it is worthwhile to clarify the use of exponential functions as an integrating factor in solving linear differential equations. As we have seen in the previous section, such differential equations in general consist of the second (or higher) derivative, the first derivative and the function itself, each multiplied by a *constant factor*. If the sum of all these derivatives (the function itself might be treated as a derivative of order zero) achieves zero, it becomes a homogeneous equation. A function whose derivatives have the same form as the function itself is an exponential function, so it may satisfy these kinds of equations. Substituting this function into the differential equation, whose right side is zero (a homogeneous differential equation) the exponential factor in each member of the equation

might be simply crossed out, so that the remaining equation's coefficients will be only circuit parameters. Such an equation is called a characteristic equation.

1.5 CHARACTERISTIC EQUATION AND ITS DETERMINATION

Let us start by considering the simple circuit of Fig. 1.7(a) in which an *RL* in series is switching on to a d.c. voltage source.

Let the desired response in this circuit be current $i(t)$. We shall first express it as the sum of the natural and forced currents

$$i = i_n + i_f.$$

The form of the natural response, as was shown, must be an exponential function, $i_n = A e^{st}$(*). Substituting this response into the homogeneous differential equation, which is $L(di/dt) + Ri = 0$, we obtain $Ls\,e^{st} + R\,e^{st} = 0$, or

$$Ls + R = 0. \tag{1.17a}$$

This is a characteristic (or auxiliary) equation, in which the left side expresses the input impedance seen from the source terminals of the analyzed circuit.

$$Z_{in}(s) = Ls + R. \tag{1.17b}$$

We may treat s as the complex frequency $s = \sigma + j\omega$ (for more about complex frequencies see any introductory course to circuit analysis and further on in Chap. 3). Note that by equaling this expression of circuit impedance to zero, we obtain the *characteristic equation*. Solving this equation we have

$$s = -\frac{R}{L} \quad \text{and} \quad \tau = \frac{L}{R}. \tag{1.18}$$

Hence, the natural response is

$$i_n = Ae^{-\frac{R}{L}t}. \tag{1.19}$$

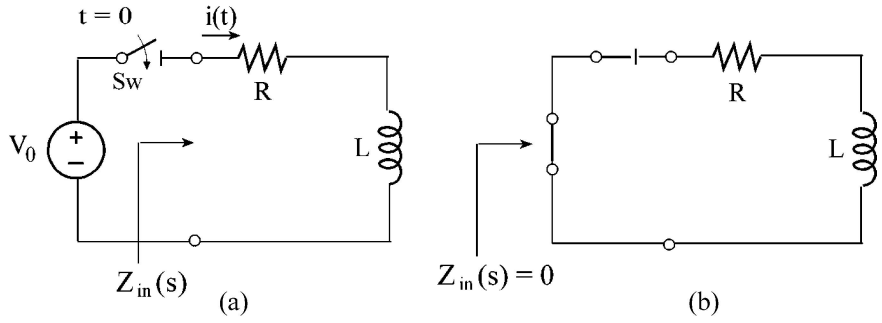

Figure 1.7 An *RL* circuit switching to a d.c. voltage source (a) and after "killing" the source (b).

(*) Here and in the future, we will use the letter s for the circuit parameters' dependent exponent.

Subsequently, the root of the characteristic equation defines the exponent of the natural response. The fact that the input impedance of the circuit should be equaled to zero can be explained from a physical point of view.(*) Since the natural response does not depend on the source, the latter should be "killed". i.e. short-circuited as shown in Fig. 1.7(b). This action results in short-circuiting the entire circuit, i.e. its input impedance.

Consider now a parallel *LR* circuit switching to a d.c. current source in which the desired response is $v_L(t)$, as shown in Fig. 1.8(a). Here, "killing" the current source results in open-circuiting, as shown in Fig. 1.8(b).

This means that the input admittance should be equaled to zero. Thus,

$$\frac{1}{R} + \frac{1}{sL} = 0,$$

or

$$sL + R = 0,$$

which however gives the same root

$$s = -\frac{R}{L} \quad \text{and} \quad \tau = \frac{L}{R}. \tag{1.20}$$

Next, we will consider a more complicated circuit, shown in Fig. 1.9(a). This circuit, after switching and short-circuiting the remaining voltage source, will be as shown in Fig. 1.9(b). The input impedance of this circuit "measured" at the switch (which is the same as seen from the "killed" source) is

$$Z_{in}(s) = R_1 + R_3 // R_4 // (R_2 + sL),$$

or

$$Z_{in}(s) = R_1 + \left(\frac{1}{R_3} + \frac{1}{R_4} + \frac{1}{R_2 + sL}\right)^{-1}.$$

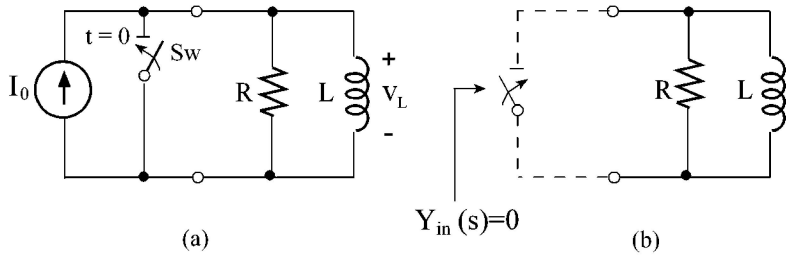

Figure 1.8 A parallel *RL* circuit switching to d.c. current source (a) and after " killing" the source (b).

(*)This fact is proven more correctly mathematically in Laplace transformation theory (see further on).

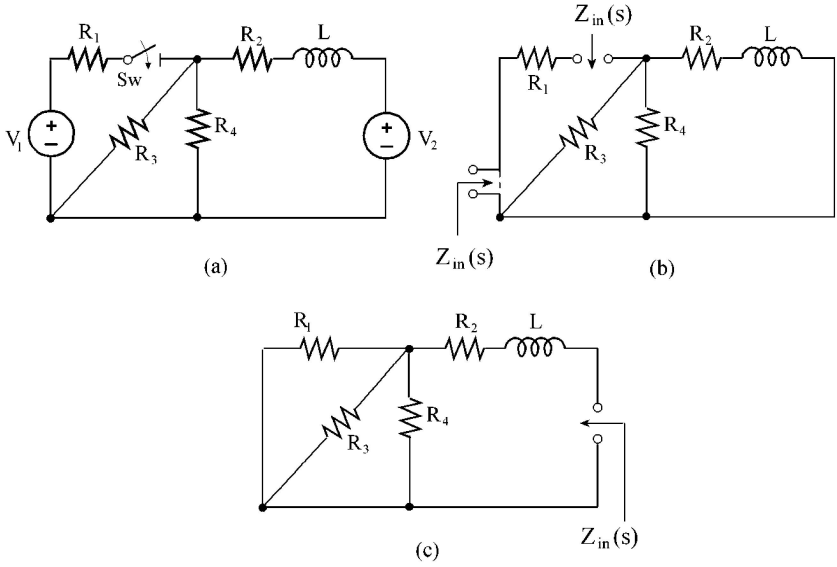

Figure 1.9 A given circuit (a), determining the input impedance as seen from the switch (b) and as seen from the inductance branch (c).

Evaluating this expression and equaling it to zero yields

$$(R_1R_3 + R_1R_4 + R_3R_4)(R_2 + sL) + R_1R_3R_4 = 0,$$

and the root is

$$s = -\frac{R_{eq}}{L}, \quad \text{where} \quad R_{eq} = \frac{R_1R_3R_4 + R_1R_2R_3 + R_1R_2R_4 + R_2R_3R_4}{R_1R_3 + R_1R_4 + R_3R_4}.$$

It is worthwhile to mention that the same results can be obtained if the input impedance is "measured" from the inductance branch, i.e. the energy-storing element, as is shown in Fig. 1.9(c).

The characteristic equation can also be determined by inspection of the differential equation or set of equations. Consider the second-order differential equation like in equation 1.2

$$L\frac{d^2i(t)}{dt} + R\frac{di(t)}{dt} + \frac{1}{C}i(t) = g(t). \tag{1.21}$$

Replacing each derivative by s^n, where n is the order of the derivative (the function by itself is considered as a zero-order derivative), we may obtain the characteristic equation:

$$Ls^2 + Rs^1 + \frac{1}{C}s^0 = 0, \quad \text{or} \quad s^2 + \frac{R}{L}s + \frac{1}{LC} = 0. \tag{1.22}$$