

# Transient mixed thermo-elastohydrodynamic lubrication in multi-speed transmissions

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## Abstract

Noise, vibration and harshness (NVH) refinement as well as thermo-mechanical efficiency are the key design attributes of modern compact multi-speed transmissions. Therefore, unlike simple gear pair models, a full transmission model is required for a simultaneous study. The prominent NVH concern is transmission rattle, dominated by the intermittent unintended meshing of several lightly loaded unselected loose gear pairs arising from the system compactness. These gear pairs are subject to hydrodynamic impacts. The thermo-mechanical efficiency is dominated by the engaged gears, with simultaneous meshing of teeth pairs subject to thermo-elastohydrodynamic regime of lubrication, with often quite thin films, promoting asperity interactions. Therefore, a full transmission model is presented, comprising system dynamics, lubricated contacts, asperity interactions and thermal balance. Generic multi-physics models of this type are a prerequisite for in-depth analysis of transmission efficiency and operational refinement. Hitherto, such an approach has not been reported in literature.

**Keywords:** Multi-speed transmissions, Mixed thermo-elastohydrodynamics, Loose gear rattle

## Nomenclature

$b$	-	Hertzian contact half width .....	$[m]$
$C$	-	Gear teeth and retaining bearings' clearance .....	$[m]$
$C_p$	-	Specific heat capacity of lubricant.....	$[J.kg^{-1}.K^{-1}]$
$E$	-	Young's modulus of elasticity .....	$[Pa]$
$E^*$	-	Reduced modulus of elasticity .....	$[Pa]$
$F$	-	Friction .....	$[N]$
$h$	-	Lubricant film thickness.....	$[m]$
$h_o$	-	Undeformed rigid gap .....	$[m]$
$I$	-	Mass moment of Inertia .....	$[kg.m^2]$
$k$	-	Thermal conductivity of lubricant .....	$[W.K^{-1}.m^{-1}]$
$\ell$	-	Contact length .....	$[m]$
$P$	-	Pressure .....	$[Pa]$
$P_c$	-	Cavitation pressure.....	$[Pa]$
$P_H$	-	Hertzian pressure .....	$[Pa]$
$P_m$	-	Mean Hertzian pressure .....	$[Pa]$

$Q_s^*$	-	Non-dimensional side leakage flow .....	$[-]$
$r, R$	-	Radius .....	$[m]$
$r_x$	-	Equivalent contact radius in the $zx$ plane.....	$[m]$
$t$	-	Time .....	$[s]$
$T_D$	-	Resistive torque .....	$[N.m]$
$u_{av} = u$	-	Speed of entraining motion .....	$[m.s^{-1}]$
$v_{ppitch}$	-	Pinions' pitch velocity .....	$[m.s^{-1}]$
$v_{wpitch}$	-	Wheels' pitch velocity .....	$[m.s^{-1}]$
$W$	-	Contact load .....	$[N]$

### Greek characters

$\alpha_o$	-	Pressure-viscosity coefficient at ambient pressure and temperature .....	$[Pa^{-1}]$
$\alpha_n$	-	Normal pressure angle .....	$[rad]$
$\alpha_t$	-	Transverse pressure angle.....	$[rad]$
$\beta$	-	Bulk modulus of the lubricant.....	$[Pa]$
$\beta'$	-	viscosity-temperature coefficient .....	$[^{\circ}C^{-1}]$
$\beta_b$	-	Helix angle .....	$[rad]$
$\gamma$	-	Slope of the oil film limiting shear stress-pressure relationship	$[-]$
$\delta$	-	Localised deflection .....	$[m]$
$\Delta u$	-	Sliding speed .....	$[m.s^{-1}]$
$\eta$	-	Effective viscosity .....	$[Pa.s]$
$\eta_o$	-	Viscosity at ambient conditions .....	$[Pa.s]$
$\mathcal{G}$	-	Coefficient of thermal expansion .....	$[-]$
$\theta$	-	Fractional film content .....	$[-]$
$\Theta$	-	Lubricant temperature .....	$[K]$
$\kappa$	-	Average asperity density .....	$[m^{-2}]$
$\varsigma$	-	Mean asperity tip radius.....	$[m]$
$\rho$	-	Density .....	$[kg.m^{-3}]$
$\rho_c$	-	Density at cavitation pressure .....	$[kg.m^{-3}]$
$\rho_o$	-	Ambient density of lubricant .....	$[kg.m^{-3}]$
$\tau$	-	Shear stress .....	$[Pa]$
$\nu$	-	Poisson's ratio.....	$[-]$
$\varphi$	-	Torsional displacement .....	$[rad]$

$\psi$  - Instantaneous lubricant temperature ..... [ $^{\circ}C$ ]

**Subscripts**

1...7 - 1<sup>st</sup> to 6<sup>th</sup> gear pair, 7 refers to reverse gear pair ..... [-]  
*brg* - Gear retaining needle bearing ..... [-]  
*bp* - Gear pinion base circle ..... [-]  
*bw* - Gear wheel base circle ..... [-]  
*n',m'* - Number of simultaneous teeth pairs in mesh ..... [-]  
*P* - Denotes Petrov friction ..... [-]  
*pp* - Pinions' pitch location ..... [-]  
*pw* - Wheels' pitch location ..... [-]  
*PR* - Pinion of the reverse gear ..... [-]  
*os* - Output shaft ..... [-]

**Superscripts**

' - First time derivative ..... [ $s^{-1}$ ]  
'' - Second time derivative ..... [ $s^{-2}$ ]

**1- Introduction**

Gears have played a significant role in industrial progress almost from the outset, with the recorded history bearing witness to their use as far back as 3129 BC in a potter's wheel discovered in Ur, Mesopotamia. From the very early days, friction, wear and lubrication of contacting surfaces have been the main concerns in design and use of gears. Another problem has been gear vibration, most pertinently in recent years when it has become a quality issue. This is a major preoccupation for transmission engineers, who search for palliation for a plethora of onomatopoeically named noise phenomena such as gear rattle, whine and grunt. Therefore, accurate methods of prediction for both tribological and dynamic conditions have always been sought. Of course dynamics and tribology of gears are closely intertwined.

Use of low shear strength media to reduce friction of contacting surfaces was clearly intuitive right from the dawn of transportation, such as application of animal or vegetable fats in the rudimentary chariot wheels. However, a fundamental understanding of the underlying mechanism of elastohydrodynamic lubrication (EHL) only emerged 70 years ago with the postulate of Ertel and Grubin (1949).

Prediction of conditions in tribology of gears has evolved with the improving understanding of fundamentals of elastohydrodynamic contacts, with numerical solutions for line contact configuration first obtained by Dowson and Higginson (1959) and for point contacts by Cameron and Gohar (1966) under assumed steady state conditions. These formed the basis for gear tribology among other concentrated counterforming contacts.

Thin elastohydrodynamic films predicted for gear meshing teeth are usually insufficient to guard against the interaction of asperities on the contiguous surfaces. Therefore, mixed regime of lubrication is often prevalent (Snidle and Evans, 1997). Under these circumstances both shear of the lubricant film and boundary interactions contribute to generated friction (De la Cruz *et al*, 2010). Additionally, for such thin films viscous shear often exceeds the limiting Eyring shear stress of the lubricant and non-Newtonian behaviour ensues. To take these into account a number of methods have been devised, either to determine an effective coefficient of friction according to the prevailing regime of traction (Evans and Johnson, 1986, and Gohar and Rahnejat, 2008) or combined mixed regime of lubrication (Greenwood and Tripp, 1971, and De la Cruz *et al*, 2010). The generated friction causes shear thinning of the lubricant, with the effective viscosity obtained because of a corresponding rise in the contact temperature. In turn, the reduced effective viscosity of the lubricant further decreases the film thickness.

Under isothermal conditions, often assumed to somewhat reduce the complexity of numerical solutions, a combined solution of Reynolds and elasticity potential equation is sought. The solutions proposed, for example, by Li and Kahraman (2010) and Wang and Cheng (1981a and 1981b) for spur gear pairs have included the effect of boundary interactions as previously described. However, the effect of temperature is significant, requiring a mixed thermo-elastohydrodynamic analysis. Hence, the inclusion of energy equation is necessary in the aforementioned simultaneous solution of Reynolds and elasticity potential equations. The computational task, therefore, becomes quite arduous. However, it has been shown that in thin film conjunctions heat is conducted away through the bounding solid surfaces, and any convection cooling through lubricant flow may be neglected (Johnson and Greenwood, 1980). This finding provides the opportunity for an analytical solution of energy equation to obtain the average rise in contact temperature based on viscous shear heating of the lubricant and conduction cooling through the solid boundaries. Gohar and Rahnejat (2008) showed the validity of such assumptions for thin elastohydrodynamic films, where a linear temperature gradient across the lubricant film may be assumed. However, one should be cautious that such an analytical solution disregards heat convection from the bounding surfaces at the inlet to the conjunction, which in turn affects the inlet temperature of the lubricant as shown by Olver and Spikes (1998). A simplifying assumption is to consider the inlet temperature to be that of the air-oil mist within the transmission under steady state condition, which itself is assumed to be in thermal equilibrium with the transmission bath of lubricant.

With an assumed Gaussian asperity distribution, only a small area of asperity contact often exists and any temperature rise due to asperity interactions is also deemed to be localised and may thus be neglected, at least in the first instance.

These simplifying assumptions enable the solution of mixed thermo-elastohydrodynamics of gear teeth pair. It should be noted that adoption of such assumptions is crucial under transient conditions, where a number of teeth pair are in simultaneous contact during the gear meshing process as shown by Hua and Khonsari (1995).

Whilst the approach expounded here deals with the meshing of a single gear pair, it does not address analysis of modern transmission systems, where a multitude of unselected (unengaged) gear pairs also interact simultaneously. This is as a direct result of a down-sizing trend in

transmissions, where pinion teeth impact their loose gear wheel conjugates. The impact/contact loads are light, thus rendering an *improper* meshing of teeth pairs (Brancati *et al*, 2005 , Tangasawi *et al*, 2007 and De la Cruz *et al*, 2010). These lightly loaded teeth pairs are often treated as being subject to thermo-hydrodynamic regime of lubrication or more crudely as dead band intermittent interactions within the confine of their backlash. The inclusion of loose gear pair interactions is essential in the study of transmission refinement, when dealing with noise and vibration issues such as gear rattle. De la Cruz *et al* (2010) describe a 7-speed transmission model which includes an engaged and a number of loose gear pairs, where the engaged teeth pairs are treated under quasi-steady thermo-elastohydrodynamic conditions, using Grubin's extrapolated oil film thickness equation.

The current paper is a tribo-dynamic study of transmission systems, one which combines dynamics of the gear train with tribology of the interacting gear teeth pairs under transient conditions. The engaged (torque transmitting) gear teeth pairs are subject to mixed thermo-elastohydrodynamic regime of lubrication, often subject to non-Newtonian Eyring shear, whereas the loose gear pairs are subject to complex lightly loaded thermo-hydrodynamics in their interacting teeth pairs and Petrov-type friction in loose gear wheel to the retaining shaft conjunctions. This is a comprehensive approach to transmission modelling, not hitherto reported in literature.

## 2- The tribo-dynamics model

A 7-speed (including reverse gear) transaxle transmission (figure 1) is investigated here. It comprises an input shaft, upon which all the pinion gears are mounted. It utilises two output shafts, chiefly in a quest to reduce the required package space. The driven gear wheels are mounted onto these output shafts in the configuration shown in the figure. The study reported here corresponds to an engaged gear pair (transmission in 2<sup>nd</sup> gear). The backlash between the other unselected gear pairs allows for repetitive impacts of their conjugate teeth pairs. This is fairly common in modern transmissions. The effect of these lightly loaded impacts is a phenomenon commonly referred to as transmission rattle (De la Cruz *et al*, 2010; Tangasawi *et al*, 2007 and Brancati *et al*, 2005, among others). This paper is not concerned specifically with the rattle phenomenon, rather the tribological study of various loaded conjunctions, particularly those of engaged (torque transmitting) pairs under mixed thermo-elastohydrodynamic regime of lubrication. However, dynamics of the system and tribology of various conjunctions are inexorably linked and a realistic tribological assessment necessarily dictates the solution of the tribo-dynamics problem as a whole.

The equations of motion for this transaxle transmission with second gear pair engaged are (De la Cruz *et al*, 2010):

For the 1<sup>st</sup> gear wheel (unselected):

$$(I_1 + I_{PR})\ddot{\phi}_1 = \sum_{j=1, n'} W_{1,j} r_{bw1} - \sum_{k=1, m'} W_{7,k} r_{bw7} - \sum_{j=1, n'} F_{1,j} r_{x1,j} - \sum_{k=1, m'} F_{7,k} r_{x7,k} - F_{P1} r_{brg1} \quad (1)$$

For the 2<sup>nd</sup> gear wheel, selected (engaged) pair:

$$(I_2 + I_{os})\ddot{\phi}_2 = \sum_{k=1,n'} W_{2,k} r_{bw2} - \sum_{i=1,3,4,7} \sum_{j=1,n} F_{i,j} r_{xi,j} - \sum_{k=1,n'} F_{2,k} r_{x2,k} - T_{D2} \quad (2)$$

And for the loose unselected pinion-wheel pairs;  $i \in 3-6$ :

$$I_i \ddot{\phi}_i = \sum_{j=1,n'} W_{i,j} r_{bwi} - \sum_{j=1,n'} F_{i,j} r_{xi,j} - F_{Pi} r_{brgi} \quad (3)$$

For the reverse gear wheel, denoted by  $i = 7$ ; an unselected pair:

$$I_7 \ddot{\phi}_7 = \sum_{k=1,m'} W_{7,k} r_{bw7} - \sum_{k=1,m'} F_{7,k} r_{x7,k} - F_{P7} r_{brg7} \quad (4)$$

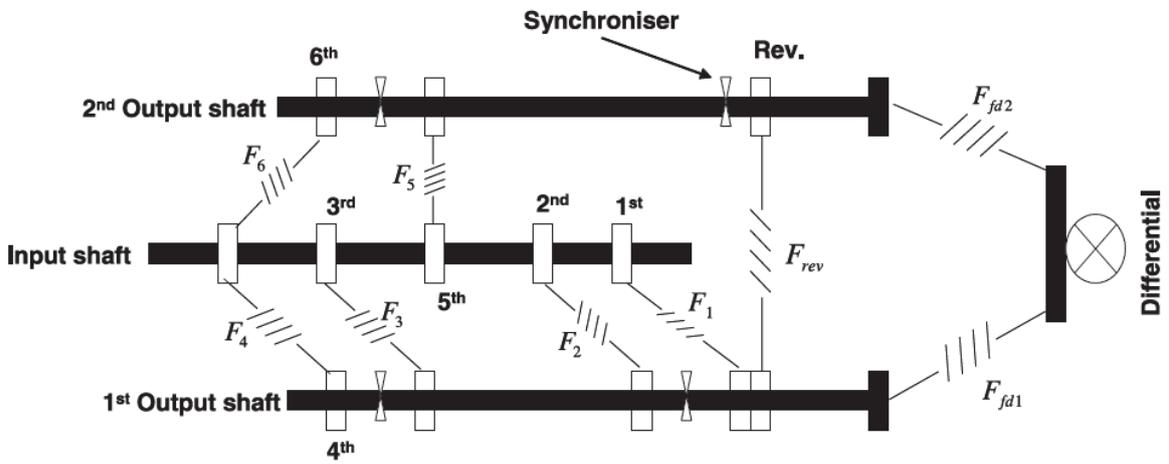


Figure 1: A 7-speed transaxle transmission

The contact forces and the conjunctional frictions are, therefore, necessary to determine the overall dynamic response. In turn, the dynamics of meshing is essential to predict the prevailing tribological performance of various conjunctions. The former is an essential requirement for noise and vibration refinement of a transmission system, whilst the latter determines transmission efficiency and emission characteristics. These attributes can often lead to a contradiction, and a degree of technical pragmatism should normally be exercised.

## 2.2- Lubricated Conjunctions

There are a number of lubricated conjunctions in a transmission system. One can classify these into three types:

(a) – Loaded counterformal contact of teeth pairs of the engaged pinion-gear wheel (in the current example; the 2<sup>nd</sup> gear): these are the meshing gear teeth, with a number of pairs involved simultaneously with different load shares. In the current example 1-2 such teeth pairs are in simultaneous mesh. These conjunctions are subject to moderate to high loads. The lubricant film thickness is fairly thin, thus a mixed thermo-elastohydrodynamic regime of lubrication would

usually be expected. One can refer to this class of gear teeth contacts within a transmission system as the engaged gear pairs.

(b) - Lightly loaded counterformal contact of gear teeth of unselected pairs: these teeth are in *unintended* (improper) mesh, due to their proximate locations and their repetitive impacts. Light loads and relatively thick films can result in thermo-hydrodynamic regime of lubrication.

(c) – Lightly loaded conformal contact of loose wheels, rolling and sliding upon their retaining shafts: the gear wheels are usually mounted upon linear or full complement needle bearings. The conjunction may be regarded as a Petrov bearing with zero eccentricity (unity Petrov multiplier).

The conjunction types (b) and (c) may be referred to as the loose gear pairs.

### 2.2.1- The engaged gear pair

The teeth pair contacts of the engaged gears are subject to moderate to high loads. For the transmission system studied here, at any instant of time there are 1-2 teeth pairs in simultaneous mesh. These contacting pairs experience engine loading, transmitted through the conjugate engaged pinion, as well as the resistive load torque of the road-wheels and the drive train system. Therefore, unlike the loose gear pairs, the lubricated contacts here encounter elastohydrodynamic regime of lubrication. The film thickness is usually quite thin. This is because the heat generated because of friction reduces the lubricant viscosity and causes shear thinning of the film. In fact, often asperity interactions take place. Thus, a mixed thermo-elastohydrodynamic regime of lubrication is prevalent. At high loads, significant sliding motion and an insufficient supply of lubricant (modern transmissions have a low depth of oil bath) cavitation can also play a role, resulting in reduced load carrying capacity and paradoxically reduced friction.

Under transient conditions the contact load for any teeth pair conjunction is the integrated pressure distribution. The helical gears of the transmission yield an elastic line contact footprint, which may be assumed to be a thin rectangular strip. Thus:

$$W = \int_{x_i}^{x_e} p dx \quad (5)$$

where, as an initial guess the inlet position  $x_i$  may be assumed to be far ahead of the centre of the contact (fully flooded condition). Determination of the position of film rupture,  $x_e$  is important in the calculation of contact load. A realistic exit boundary condition, taking into account the effect of cavitation is proposed by Elrod (1981).

Assuming a long line contact and no side-leakage of the lubricant, Reynolds' equation becomes:

$$\frac{\partial}{\partial x} \left[ \frac{\rho h^3}{\eta} \cdot \frac{\partial p}{\partial x} \right] = 12 \left\{ u_{av} \frac{\partial}{\partial x} (\rho h) + \frac{d}{dt} (\rho h) \right\} \quad (6)$$

where, the speed of entraining motion of the lubricant into the contact is:  $u_{av} = \frac{1}{2}(u_p + u_w)$ .

Note that the corresponding sliding velocities ( $u_p, u_w$ ):

$$u_p = v_{pitch} \left( \sin \alpha_t + \frac{l_p}{r_{pp}} \right) \cos \beta_b, \quad u_w = v_{pitch} \left( \sin \alpha_t - \frac{l_p}{r_{pw}} \right) \cos \beta_b \quad (7)$$

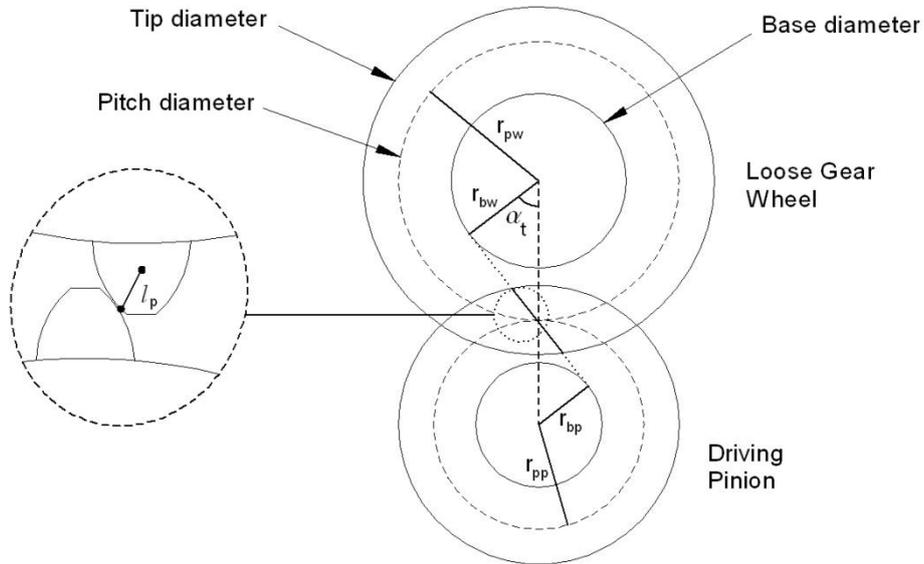
The non-holonomic constraint function at the pitch point determines the pitch velocities as:  $v_{pitch} = r_{pp} \dot{\phi}_p = r_{pw} \dot{\phi}_i$  for  $i = 1-7$ . Also:  $l_p = r_{pw} \sin \alpha_t - r_{wx}$  (figure 2).

If the instantaneous radii of curvature of a pinion gear tooth and its conjugate gear wheel tooth are  $r_{px}$  and  $r_{wx}$  respectively, then the reduced radius of the concentrated counterformal contact (that of

an equivalent ellipsoidal rigid solid against a semi-infinite elastic half-space) is:  $r_x = \frac{r_{px} r_{wx}}{r_{px} + r_{wx}}$ , and

the reduced elastic modulus of the semi-infinite solid under plane strain condition is:  $E' = \frac{E}{1-\nu^2}$

where:  $E_p = E_w = E$  and  $\nu_p = \nu_w = \nu$ .



**Figure 2: Gear meshing sequence**

The form of Reynolds' equation stated above assumes the formation of a coherent lubricant film in the conjunction, prior to a film rupture point. In reality, as the generated pressures fall below the vapour pressure of the lubricant in the region of contact exit, cavitation takes place. This can reduce the load carrying capacity of the contact, as well as affect the underlying mechanisms that contribute to friction. It is, therefore, more instructive to account for this effect. A solution to this problem was first obtained by considering a full film region, followed by a cavitated region and film reformation beyond it by Jakobsson and Floberg (1957) and Olsson (1965), who assumed a set of exit boundary conditions, which has come to be known as the JFO boundary conditions. On the other hand, Reynolds' equation in the form stated above is suited to the region of full film, and is often used in conjunction with the Swift-Stieber boundary conditions, which determine the oil film rupture position,  $x_e$ . The solution does not extend beyond this point, where continuity of flow

conditions, based only on the Couette flow has to be satisfied. An approximate solution to the rather computationally intensive approach of Jakobsson and Floberg (1957) and Olsson (1965) was first proposed by Elrod (1981), which is based on a fractional film content  $\theta$ , where any value equating unity or in its excess implies the presence of a coherent lubricant film (this being a film sustained above the vaporisation cavitation pressure of the lubricant). Consequently, a value of  $0 < \theta < 1$  suggests the presence of vapour bubbles within a film of lubricant. Hence:

$$p = g\beta \ln \theta + p_c \quad (8)$$

where  $g$  may be regarded as a switching function:  $g = 1$ ,  $\theta \geq 1$  (Full film) and  $g = 0$ ,  $0 < \theta < 1$  (partial film: cavitation). Reynolds' equation can now be rewritten in terms of  $\theta$  as:

$$\frac{\partial}{\partial x} \left[ \frac{\rho h^3}{\eta} g\beta \frac{\partial \theta}{\partial x} \right] = 12 \left\{ u_{av} \frac{\partial}{\partial x} (\rho \theta h) + \frac{\partial}{\partial t} (\rho \theta h) \right\} \quad (9)$$

It can be seen that in the cavitated region the flow takes place due to Couette flow only (on the right hand side of the equation), as well as film *memory* represented by squeeze film effect (the ultimate term on the right hand side of the equation). Clearly, under this condition,  $\rho = \rho_c$  at  $p = p_c$ . Hence:

$$\left\{ u_{av} \frac{\partial}{\partial x} + \frac{\partial}{\partial t} \right\} (\rho_c \theta h) = 0 \quad (10)$$

The elastic film shape is:

$$h = h_0 + h_s + \delta \quad (11)$$

The localised contact deflection is obtained by solution of elasticity potential equation (Gohar and Rahnejat, 2008). For an elastic line contact this simplifies to:

$$\delta_m = \frac{1}{\pi E'} \int \frac{p_k}{(x_m - x_k)} dx_k \quad (12)$$

where, the deflection at a point  $x_m$  results from all the generated pressures at points  $x_k$ . When a one dimensional computational grid is made with the pressure distribution,  $p_k$ , this equation can be stated as:

$$\delta_m = \sum_k D_m^k p_k \quad (13)$$

where,  $\delta_m$  is the vector of contact deflections and  $D_m^k$  contains the influence coefficients (see Appendix A1).

Now simultaneous solution of equations (9) or (10) for full film or cavitation region with (11) and (13) yields values of  $p, \delta, h$  and  $\theta$  for an isothermal iso-viscous analysis. However, it is important to take into account the effect of lubricant rheological state  $(\rho, \eta)$ .

Lubricant density is adjusted for pressure (Dowson and Higginson, 1959) :

$$\bar{\rho} = \frac{\rho}{\rho_o} = 1 + \frac{0.6 \times 10^{-9} p}{1 + 1.7 \times 10^{-9} p} \quad (14)$$

Any significant change in lubricant density will be noted in the cavitation region. In the region of full film, under elastohydrodynamic conditions, the lubricant is almost incompressible. The main lubricant rheological change is its viscosity under the prevalent thermo-elastohydrodynamic conditions. The lubricant viscosity is, therefore, adjusted using Houpert's (1985) equation, which assumes Newtonian behaviour of the lubricant:

$$\eta = \eta_o e^{\alpha^* p} \quad (15)$$

$$\text{where: } \alpha^* = \frac{1}{p} [\ln(\eta_o) + 9.67] \left\{ \left( \frac{\Theta - 138}{\Theta_o - 138} \right)^{-s_0} \left[ \left( 1 + \frac{p}{1.98 \times 10^8} \right)^z - 1 \right] \right\} \quad (16)$$

Note,  $\Theta$  is temperature in  $^{\circ}K$ , thus:  $\Theta = \psi + 273$ .

$$\text{Also: } Z = \frac{\alpha_o}{5.1 \times 10^{-9} [\ln(\eta_o) + 9.67]} \text{ and } s_0 = \frac{\beta' (\Theta_o - 138)}{\ln(\eta_o) + 9.67}$$

The elastohydrodynamic conjunctions of loaded gear teeth pairs are subject to contact kinematics which undergoes rolling, sliding and normal approach and separation (squeeze film action). The sliding motion, in particular, causes viscous shear of the lubricant which results in a corresponding rise in its temperature. This, in turn affects the lubricant viscosity. In order to account for this, the solution to the above set of equations should be accompanied by the energy equation:

$$\rho u_{av} \psi \frac{\partial p}{\partial x} + \eta \left( \frac{\partial u_{av}}{\partial z} \right)^2 = \rho u_{av} C_p \frac{\partial \psi}{\partial x} - k \frac{\partial^2 \psi}{\partial z^2} \quad (17)$$

For a thin rectangular contact footprint, as in a concentrated elastic line contact with a thin elastohydrodynamic film, Gohar and Rahnejat (2008) show that an order of magnitude analysis indicates the dominance of conduction cooling, thus the penultimate term in the above equation, being due to convection cooling may be neglected. The contrary is true for loose gear teeth pairs, where lightly loaded hydrodynamic conditions dominate, as described in section 2.2.2. If viscous shear heating (the second term in (17)) is assumed to dominate the heating of the lubricant (at high shear), then the energy equation simplifies by ignoring the first term due to compressive heating. Assuming that the temperature rises linearly across the film (with one surface assumed stationary

and the other moving with relative velocity  $\Delta u$ ), then the average temperature rise of the lubricant within the hydrodynamic contact becomes (De la Cruz *et al*, 2010):

$$\Delta\psi = \frac{4b\eta\Delta u}{h^2\rho C_p} \quad (18)$$

where, the sliding velocity is:  $\Delta u = u_p - u_w$ .

Using the principle of superposition to also account for compressive heating (EHL), the average temperature rise in a narrow rectangular contact footprint was obtained by Karthikeyan *et al* (2010) as:

$$\Delta\psi = \left\{ \frac{\mathcal{G}u_{av}\psi_i hp_m + 2b\eta\Delta u^2/h}{bk/h - \mathcal{G}u_{av} hp_m} \right\} \quad (19)$$

where  $\psi = \psi_i + \Delta\psi$  is the average lubricant temperature in the contact for an inlet temperature of  $\psi_i$ .  $p_m$  is the mean pressure peak in an elastohydrodynamic pressure distribution which is in fact, for all intent and purposes, the Hertzian mean pressure. Thus, the half-width of the rectangular contact strip,  $b$  may be obtained from the classical Hertzian theory as:  $b = 2\left(\frac{Wr_x}{\pi\ell E'}\right)^{1/2}$ , where  $\ell$  is the contact length (Tangasawi *et al*, 2007). Appendix A2 shows typical contact length variation for a teeth pair through mesh.

Now the effective lubricant viscosity can be obtained from equation (15), and the simultaneous solution of this equation with (9) or (10), (11), (13) and (14) yields the pressure distribution and the corresponding film thickness, usually sought in tribological studies.

To solve equation (2), flank friction terms  $F_{i,k}$ ,  $i = 2$  (engaged gear pair) and  $k \in 1,2$  (number of teeth pairs in simultaneous contact) are required. Thermo-elastohydrodynamic films,  $h$ , may be of insufficient thickness to guard against asperity interactions on the contiguous surfaces. Therefore, the overall flank friction is given as a combination of viscous friction,  $F_v$  and boundary,  $F_b$  interactions as:  $F = F_v + F_b$ . These individual contributions are determined according to the lubricant shear stress,  $\tau = \pm \frac{h}{2} \frac{dp}{dx} + \frac{\eta\Delta u}{h}$ .

If  $\tau < \tau_0$ , being the Eyring shear stress (Evans and Johnson, 1986), then:

$$F_v = \tau(A - A_e) \quad (20)$$

where, the Eyring shear stress for the transmission fluid used is 4.5 MPa, and  $A_e$  is the real contact area, as opposed to the apparent contact area, determined by the Hertzian theory,  $A$  (Greenwood and Tripp, 1971):

$$A_e = \pi^2 (\zeta \kappa \sigma)^2 A F_2(\lambda) \quad (21)$$

where,  $A = 2lb$ .  $\lambda = \frac{h}{\sigma}$  is the Stribeck (1902) oil film parameter, with  $\sigma$  being the composite root mean square roughness of the contiguous surfaces in contact.

If, on the other hand,  $\tau \geq \tau_0$ , then the shear stress is adjusted to:  $\tau = \tau_0 + \eta p^*$ , where:

$$p^* = \frac{W - W_e}{A_e} \quad (22)$$

with  $W$  given by equation (5) and the load share of asperities ( $W_e$ ) is obtained as (Greenwood and Tripp, 1971, and Gohar and Rahnejat, 2008):

$$W_e = \frac{8\sqrt{2}}{15} \pi (\zeta \kappa \sigma)^2 \sqrt{\frac{\sigma}{\kappa}} E' A F_{5/2}(\lambda) \quad (23)$$

The variables  $F_2$  and  $F_{5/2}$  are statistical functions introduced to match the assumed Gaussian distribution of asperities considered. Teodorescu *et al* (2003) and (2005) propose a polynomial fit to describe these functions (figure 3). Now the boundary contribution to friction is obtained as:

$$F_b = \tau_0 A_e + \xi W_e \quad (24)$$

where  $\xi$  is the pressure coefficient for boundary shear strength for direct asperity interactions of ground heat treated high alloy steel, approximated to 0.17 for this study (Teodorescu *et al*, 2005 and De la Cruz *et al*, 2010).

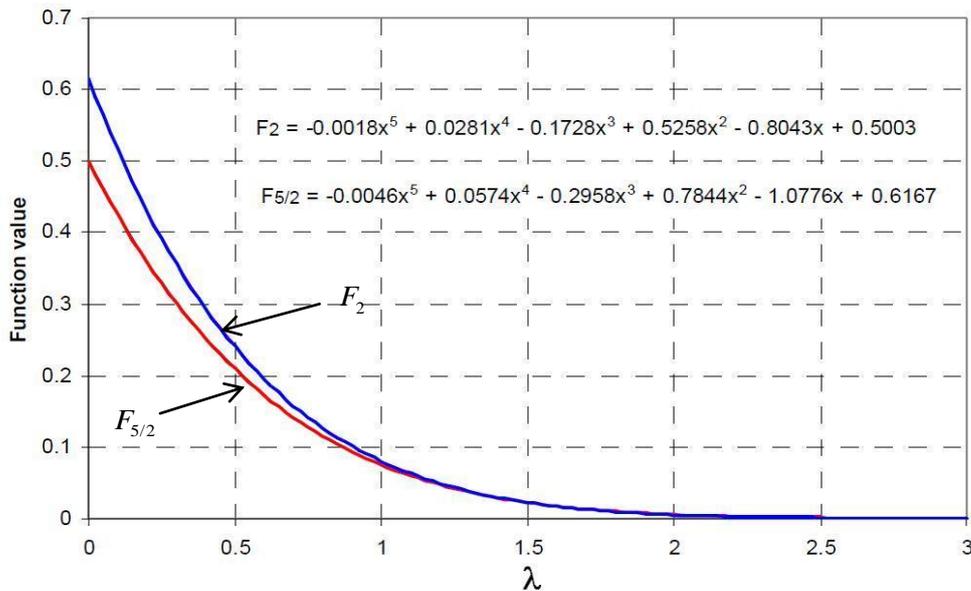


Figure 3: 5th order fit for the statistical functions (after Teodorescu *et al*, 2003)

### 2.2.2- The loose gear pair

As noted in section 2.2 there are two types of conjunctions in the loose gear pairs' interactions. These are the teeth pair flank contacts and those for loose wheels-to-their retaining output shafts. For the counterformal contact of the former an analytical one dimensional solution to Reynolds' equation (6) can be obtained with no side-leakage and use of half-Sommerfeld boundary conditions (Sasaki *et al*, 1962 and Rahnejat, 1985). Therefore, the integrated pressure distribution for any pair of lightly loaded teeth, rolling with squeeze film effect is obtained as the lubricant reaction according to equation (5) as:

$$W = \frac{2\ell u_{av} \eta r_x}{h} - \frac{3\pi\ell\eta}{\sqrt{2}} \left(\frac{r_x}{h}\right)^{3/2} \frac{\partial h}{\partial t} \quad (25)$$

where the squeeze film velocity is obtained as a first order approximation by  $\frac{\partial h}{\partial t} = \frac{h_i - h_{i-1}}{\Delta t}$ , where the film thickness change corresponds to two successive simulation time steps. Note that when  $\frac{\partial h}{\partial t} < 0$ , the bounding surfaces approach each other, resulting in an enhanced load carrying capacity (equation (25)). On the contrary, when  $\frac{\partial h}{\partial t} > 0$ , separation occurs, and the second term in equation (25) is omitted.

For the iso-viscous rigid conditions assumed here, the lubricant film thickness is as the result of the difference in the nominal clearance of the teeth pairs and their mutual approach, thus:

$$h = \begin{cases} C - \frac{r_{bpi}\varphi_p - r_{bwi}\varphi_i}{\cos \alpha_n \cos \beta_b}, & r_{bpi}\varphi_p > r_{bwi}\varphi_i \\ C + \frac{r_{bpi}\varphi_p - r_{bwi}\varphi_i}{\cos \alpha_n \cos \beta_b}, & r_{bpi}\varphi_p < r_{bwi}\varphi_i \end{cases} \quad i \in 1 \rightarrow 7 \notin 2 \quad (26)$$

Note that the motion of all the pinions are specified by the transmission input shaft (i.e.  $\varphi_p$  is known *a priori*). This motion is measured from the transmission input shaft using a 2 beam laser vibrometer and is an input to the simulation study. It is a function of engine speed and all its harmonics, known as engine orders (Rahnejat, 1998). For an assumed thin rectangular footprint of length  $\ell$ , the flank friction is obtained as (Gohar, 2001):

$$F = \begin{cases} \frac{\ell\pi\eta\Delta u}{\sqrt{2h}} \sqrt{r_x}, & \text{when } \Delta u \geq 0 \\ -\frac{\ell\pi\eta\Delta u}{\sqrt{2h}} \sqrt{r_x}, & \text{when } \Delta u < 0 \end{cases} \quad (27)$$

The loose gear wheels, when impacted upon, rotate relative to their supporting shaft. As already noted, these conjunctions are treated as Petrov bearings, thus (De la Cruz, 2010):

$$F_p = \frac{2\pi\eta u_{brg} r_{brg} l_{brg}}{c} \quad (28)$$

where, the speed of lubricant entrainment is given as:  $u_{brg} = \frac{1}{2} \left( r_{brg} \dot{\phi}_i + \frac{1}{2\pi} r_{os} \dot{\phi}_{os} \right)$   $i \in 1-7, \neq 2$ .

For both loose gear pair conjunctions analytical solutions to the energy equation (17) are obtained. In the case of teeth flanks, lubricant heating is assumed to be as the result of viscous shear, with negligible compressive heating, whilst the heat is taken away from such conjunctions by convection cooling due to relatively thick films of at least several micrometres. A linear temperature distribution is assumed along the rectangular strip footprint of width  $2b$  as:  $\partial\psi/\partial x = \Delta\psi/2b$ . Implementing these assumptions into the energy equation (17) and after integration:

$$\Delta\psi = \frac{8\eta b \Delta u}{h^2 \rho C_p} \quad (29)$$

The Petrov conjunctions are essentially journal bearings with zero eccentricity (unity Petrov multiplier). For relatively thick hydrodynamic films, one can employ the procedure developed for journal bearings by Gohar and Rahnejat (2008) to obtain the temperature rise as:

$$\Delta\psi = \left( \frac{2K_1 W_{brg}}{\rho C_p r_{brg} l_{brg}} \right) \left( \frac{\mu^*}{Q_s^*} \right) \quad (30)$$

where:  $\mu^* = \frac{\pi\eta u_{brg} l_{brg}}{W_{brg}} \left( \frac{r_{brg}}{c} \right)^2$  and  $Q_s^* = \frac{Q_s}{\pi N r_{brg} l_{brg} c} = 2\varepsilon$  and the coefficient  $K_1$  admits that not all the frictional power is lost through convection.

The temperature rise is then used to obtain the effective viscosity of the lubricant in the same manner as in the case of loaded gear teeth pairs.

### 3- Method of solution

The solution method used is low relaxation effective influence Newton-Raphson (EIN) method with Gauss-Seidel iterations for the thermo-elastohydrodynamic conjunctions at each step of time (Jalali-Vahid *et al*, 2001 for point contacts and Teodorescu *et al*, 2007 for line contact and Chong *et al*, 2011 for line contact with cavitation boundary). The equations of motion are solved using the linear acceleration method, described by Newmark (1959) and Timoshenko *et al* (1974) with the adoption method for the case of lubricated contacts, outlined by Rahnejat (1985). The overall procedure is highlighted by Gohar and Rahnejat (2008).

For the sake of numerical stability the governing equations for lubricated contacts are non-dimensionalised. Thus, Reynolds' equation (9) with Elrod modification becomes (see Appendix 3 for non-dimensional variables):

$$\frac{\partial}{\partial X} \left[ \frac{\bar{\rho} H^3}{\bar{\eta}} \frac{\partial}{\partial X} g(\theta-1) \right] = Y \left\{ \frac{\partial}{\partial X} (\theta \bar{\rho} H) + \frac{r_x}{b} S \bar{\rho} \theta \right\} \quad (31)$$

where,  $S = \frac{1}{u_{av}} \frac{\partial h}{\partial t}$  is the squeeze-roll ratio and  $\Upsilon = \frac{12\eta_0 u_{av}}{\beta b} \left(\frac{r_x}{b}\right)^2$ , also  $g \frac{d\theta}{dX} = \frac{dg(\theta-1)}{dX}$ , which retains the physical meaning of equation (8) (Vijayaraghavan and Keith, 1989) in the subsequent use of finite differences.

$$H = H_o + H_s + \Delta \quad (32)$$

where, the non-dimensional form of deflection is:

$$\Delta = D(X) \frac{g\beta(\theta-1) + p_c}{P_H} \quad (33)$$

Now the above equations are put in finite difference form (Appendix A3) and solved in each step of time during simulation in the form:

$$\sum_{k=1}^{n-1} J_m^k \Delta \theta_k = -\bar{F}_m^R \quad (34)$$

where  $J_m^k$  and  $\bar{F}_m^R$  are the Jacobian and residual terms respectively.

There are two convergence criteria. One is for the computation of fraction film content (Gauss-Seidel iterations):

$$\frac{\sum_m |\theta_m^k - \theta_m^{k-1}|}{\sum_m \theta_m^k} \leq \varepsilon_\theta \quad (35)$$

where, typically:  $2.5 \times 10^{-7} \leq \varepsilon_\theta \leq 3.5 \times 10^{-7}$

If the above criterion is not satisfied, then under-relaxation is used in the form:

$$(g\theta)_m^k = (g\theta)_m^{k-1} + \Omega \Delta \theta_m^k \quad (36)$$

where, the under-relaxation factor is typically  $\Omega = 5 \times 10^{-5}$ .

The other convergence criterion is for system dynamics based on the Newmark's linear acceleration method:

$$\left| \frac{\ddot{\phi}_l^j - \ddot{\phi}_l^{j-1}}{\ddot{\phi}_l^j} \right| \times 100 \leq 0.01 \quad \text{for } l \in 1-7 \quad (37)$$

Here  $l$  refers to a gear pair. If the criterion is not met, then the conjunctural gaps (rigid separations) are adjusted as:

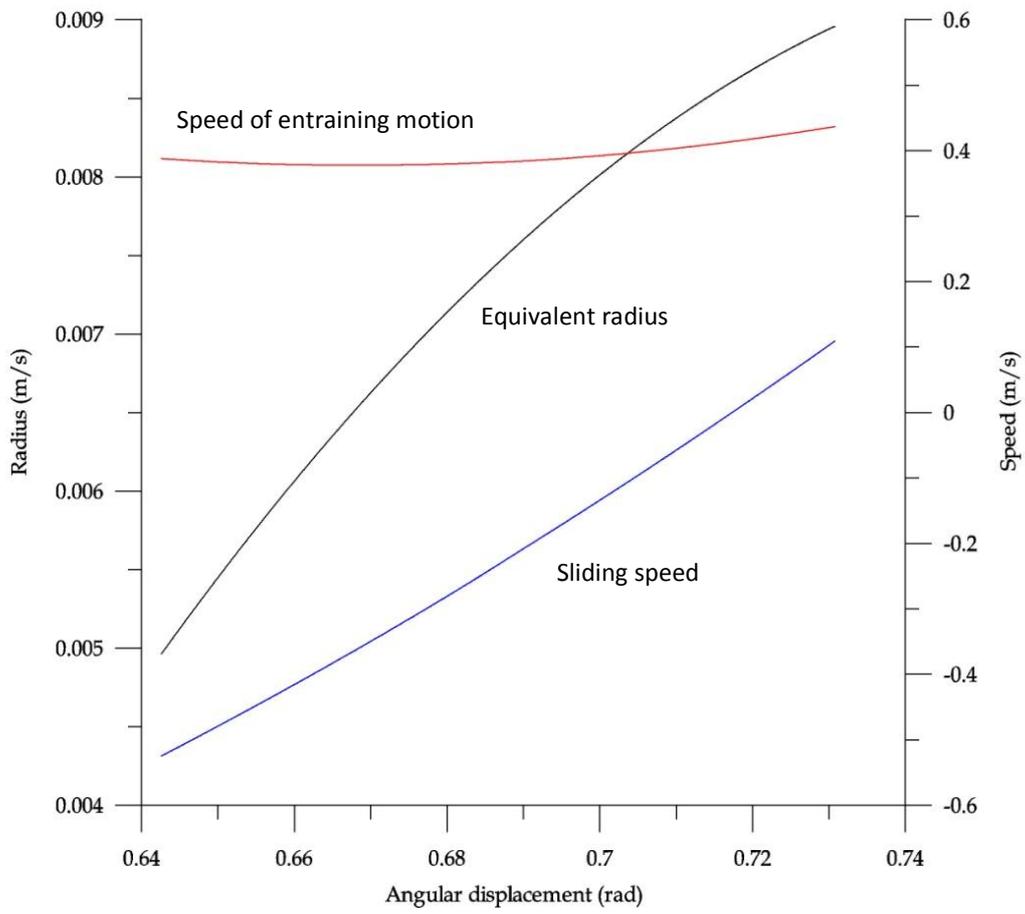
$$H_0^k = H_0^{k-1} \left( \frac{W^k}{W^{k-1}} \right)^\zeta \quad (38)$$

where, the damping factor is typically:  $\zeta = 0.05$ .

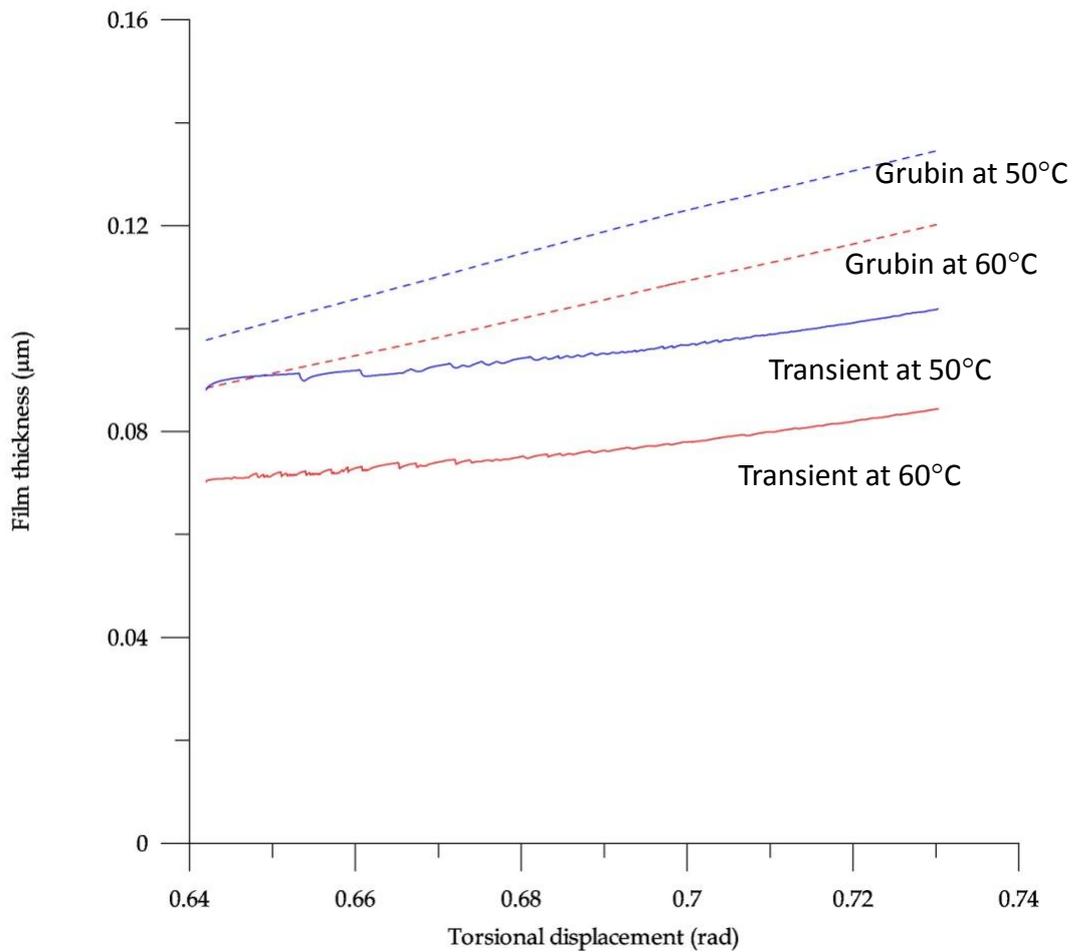
#### 4- Results and Discussion

A number of simulation studies of the entire transmission model are reported here for the bulk oil temperatures of 50° C and 60° C respectively. Note that the inlet oil temperature is assumed to be that of the bath of oil. All simulation studies are for the case of 2<sup>nd</sup> gear pair selected and after the initial transient response has elapsed. The time-step of simulation in all cases is set to 1  $\mu$ s. The spatial discretisation for the solution of Reynolds' equation is based on 500 elements in the direction of entraining motion.

Figure 4 shows the equivalent reduced radius of contact of any meshing teeth pair of the 2<sup>nd</sup> gear set. The abscissa in the figure refers to the angular position of the gear wheel in a typical meshing cycle. The position of the pitch point is at  $\varphi_w = 0.714$  radians. At this point the contact is subject to pure rolling motion (no sliding). The speed of entraining motion changes only marginally through mesh. This means that the film thickness changes only slightly through mesh (being mainly dependent on the speed of entraining motion under EHL conditions, Figure 5). However, the shear stress  $\tau$  alters due to the sliding speed, away from the pitch point (sliding speed changes direction). This is the main cause of viscous friction and generated heat, thus altering the effective lubricant viscosity. Hence, the results are for the transient mixed thermo-elastohydrodynamic analysis of the 2<sup>nd</sup> gear teeth contacting pairs through mesh.

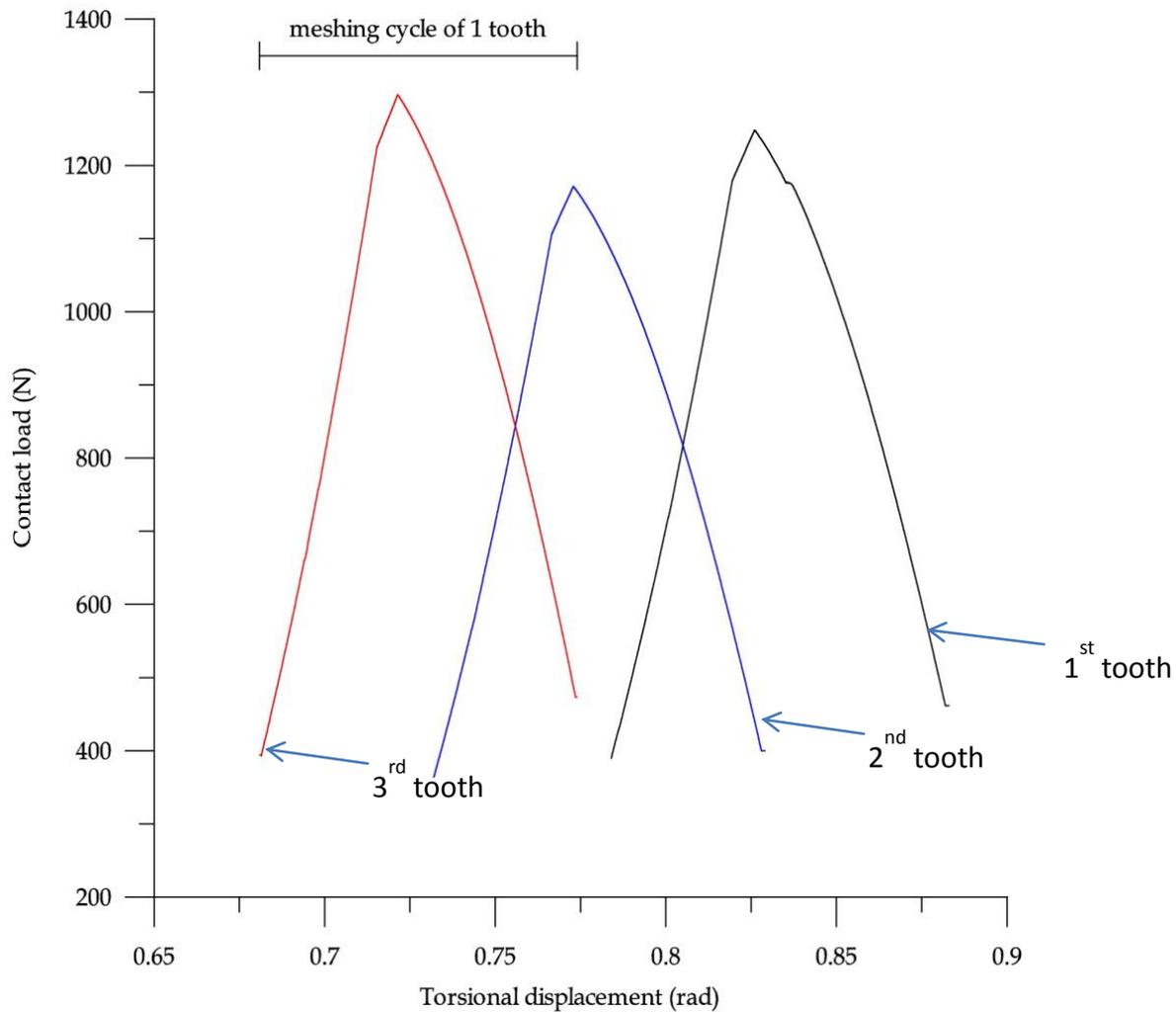


**Figure 4: Variation of contact geometry and kinematics**



**Figure 5: Transient history of central oil film thickness of typical loaded gear teeth pair**

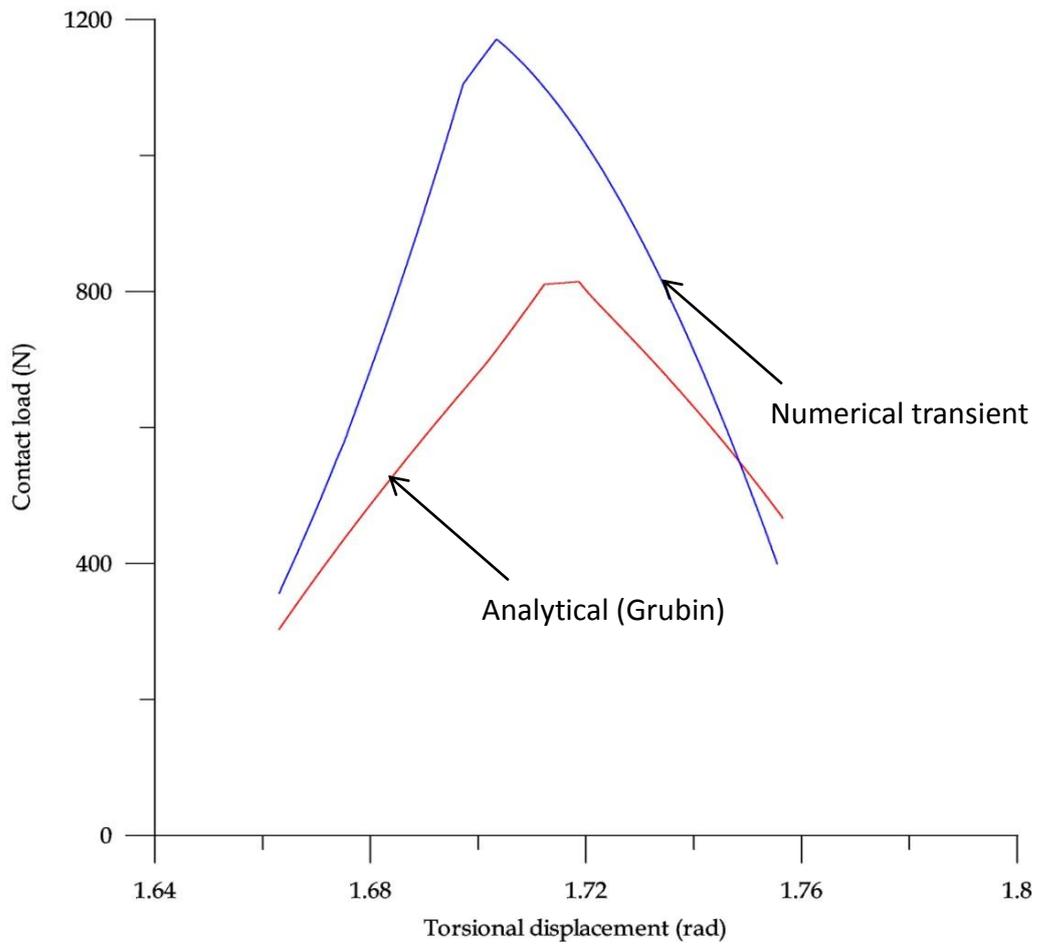
Figure 5 shows the predicted central oil film thickness at 50° C and 60° C inlet bulk oil temperatures respectively. It can be seen that with the increased inlet temperature, the lubricant film has reduced by nearly 15%. The results also show that the film is significantly thinner than the root mean square roughness of the contiguous surfaces in contact, being 0.5 µm. At best, this points to a mixed regime of lubrication. The corresponding load variation through a meshing cycle for the transient analysis is shown in Figure 6, indicating the load shared per teeth pair contact during meshing. As noted previously between 1-2 teeth pairs are in simultaneous mesh. Taking into account figures 5 and 6 together, the important observation is that the lubricant film thickness remains almost insensitive to load, which is indicative of prevailing elastohydrodynamic conditions.



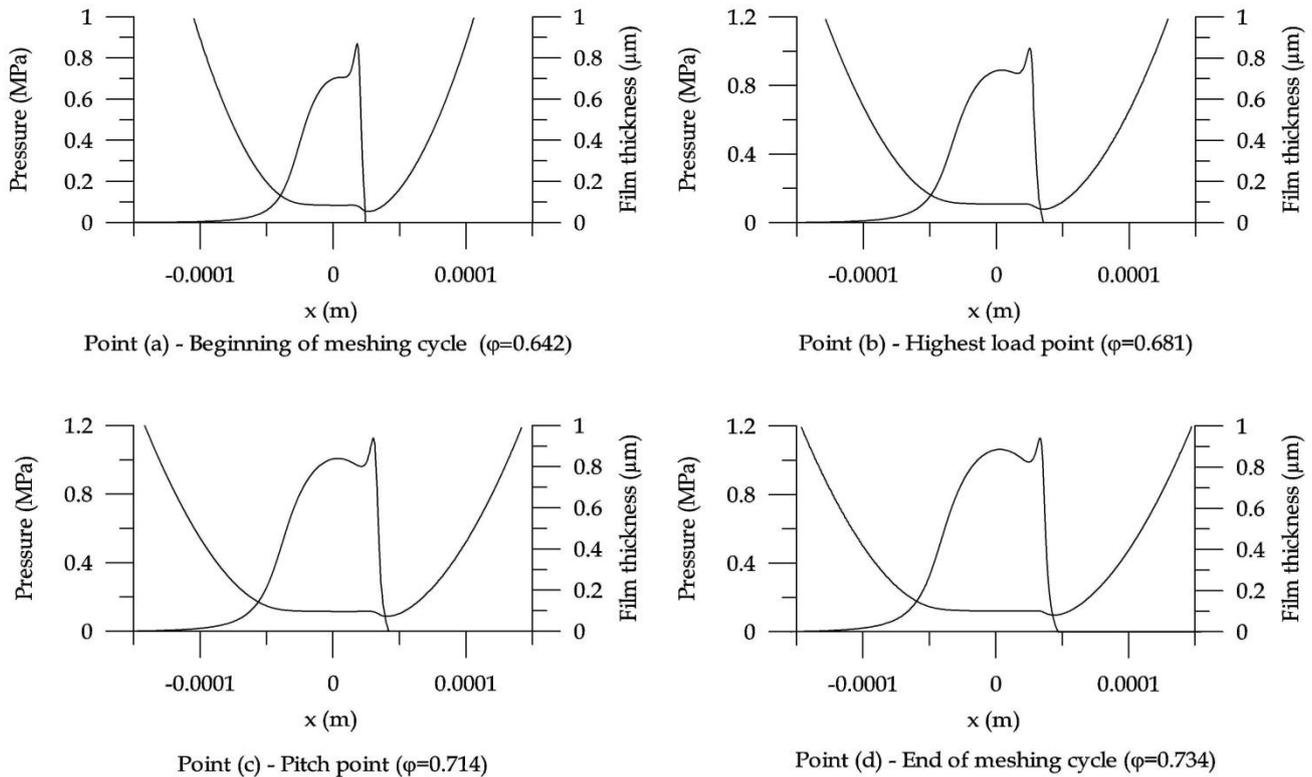
**Figure 6: Teeth pair load share through mesh (transient at 60°C)**

Figure 5 also includes the film thickness predicted for the same bulk oil temperatures using a quasi-static solution, based on the lubricant film thickness equation of Grubin (1949) with adjustment made for viscosity variation with temperature (De la Cruz *et al*, 2010). There is reasonable agreement considering that the results of quasi-static (analytical Grubin) solution correspond to a lubricant reaction around 2-30% lower than the current transient analysis (Figure 7). This shows that a higher load carrying capacity is achieved under transient conditions because of the contribution from squeeze film action. One conclusion from this comparison is that the use of much simpler quasi-static analysis may be regarded as a reasonable first approximation with a considerably reduced computation time (a few CPU seconds for analytical solution versus several hours for the transient analysis).

As can be observed in figure 6, the contact load (lubricant reaction) rises by threefold in any teeth pair contact during mesh. Figure 8 shows a series of EHL pressure distributions with their corresponding elastic film shapes. These locations are for instants of contact of a meshing teeth pair, prior to, at, and after the pitch point.



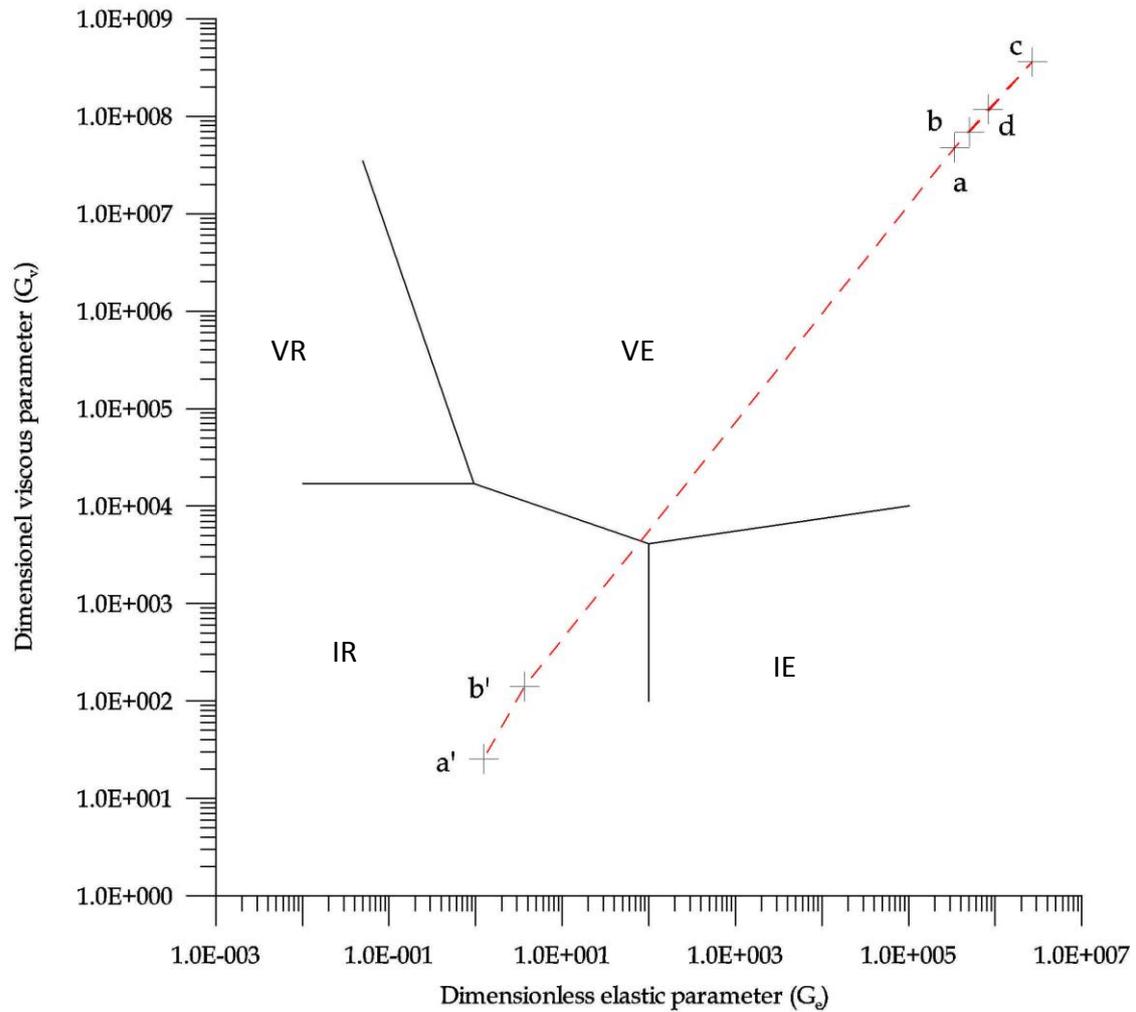
**Figure 7: Comparison of load per EHL conjunction under transient and analytical quasi-static conditions (transient at 60°C)**



**Figure 8: Pressure distribution and film profile (transient at 60°C)**

The contact conditions remain within EHL (viscous-elastic) at all times. This is shown in the Greenwood chart (Figure 9) for regimes of fluid film lubrication. The predicted conditions in figure 8 are depicted on the chart (a to d). The points indicated by  $a'$  and  $b'$  on the chart are typical conditions related to a pair of meshing teeth of the 5<sup>th</sup> loose gear set. These are subject to the lightly loaded iso-viscous rigid (hydrodynamic) regime of lubrication.

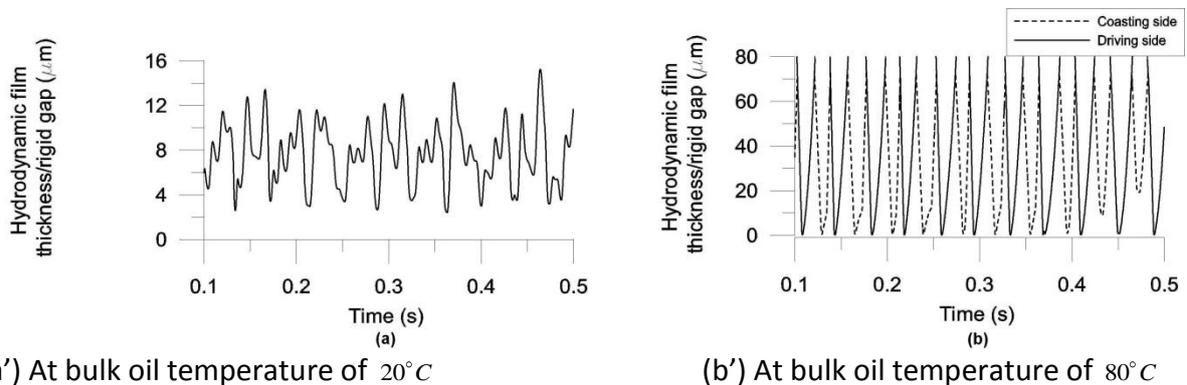
To observe rattle conditions as is usually perceived in accord with high level of annoyance, double-sided teeth impact in loose gears should be promoted. In practice such conditions are noted for bulk oil temperatures exceeding 70 °C or 80 °C. Figures 10(a') and (b') for loose gear pairs correspond to the points  $a'$  and  $b'$  on the Greenwood chart and are for bulk oil temperatures of 20° C and 80° C respectively. It is noted that the film thickness far exceeds that predicted for any engaged gear teeth pair conjunction (by up to 2 orders of magnitude) and fluctuates due to the impacting nature of these lightly loaded conditions. This may be regarded as repetitive mutual approach and separation of teeth pairs, which is exacerbated by the increased bulk lubricant temperature. In the case of Figure 10 ( $b'$ ), the approach and separation of teeth spans the nominal backlash, corresponding to double-sided impact of loose gear teeth pairs. This promotes the conditions commonly perceived and referred to as rattle, described in some detail in De la Cruz *et al* (2010).



**Figure 9: Greenwood chart**

$$(G_e = W^{8/3}/U^{*2}, G_v = G^*W^3/U^{*2}, \text{ see Appendix 3})$$

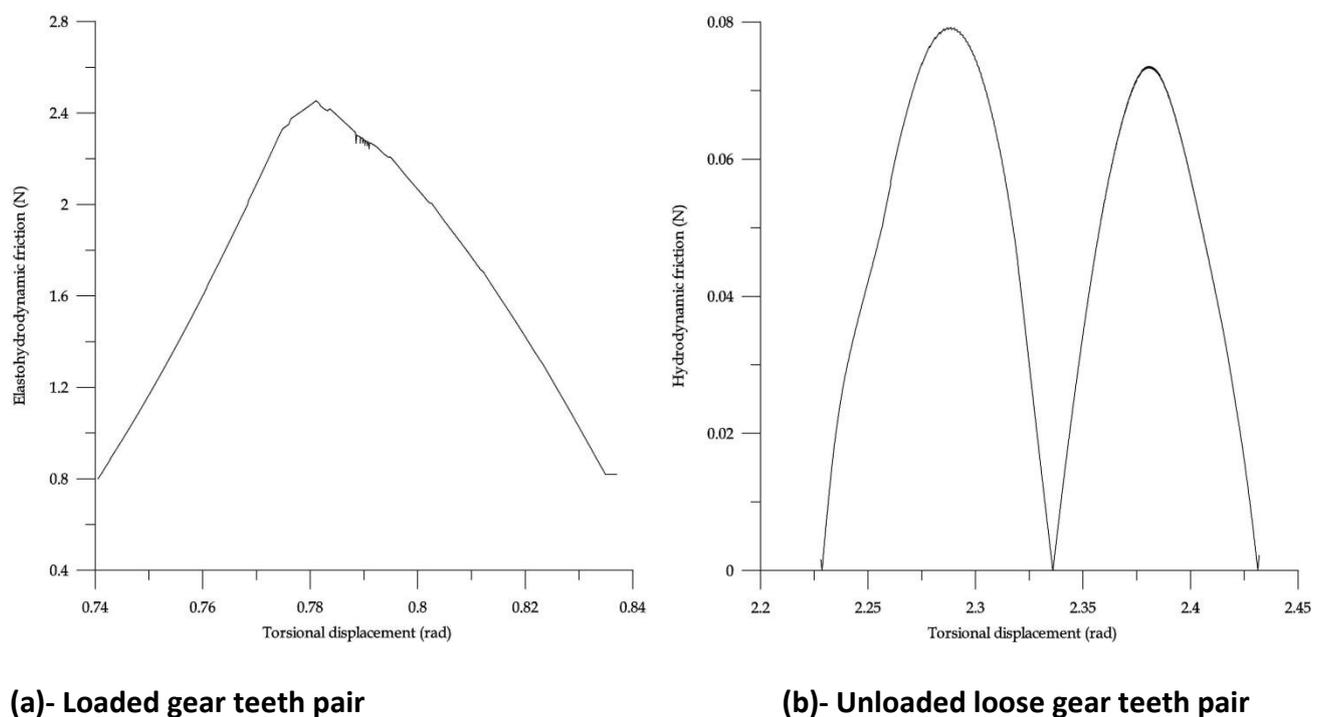
IR=Iso-viscous Rigid, IE=Iso-viscous Elastic, VR=Viscous Rigid, VE=Viscous Elastic (EHL)



**Figure 10: Fluctuations in film thickness in lightly loaded conjunctions of loose gear pair**

It is clear that larger variations in film thickness in successive contact separation and squeeze effect in the lightly loaded contacts would induce pressure perturbations that contribute to noise

propagation from the impact sites. This effect is minimal in loaded elasto-hydrodynamic conjunctions due to almost constant film thickness through mesh. However, in practice transmission error can induce momentary losses of contact that induce impulsive actions. This is more common in the cases of hypoid gears of differential systems. The resulting emanated noise is referred to as axle whine (Koronias *et al*, 2011). Therefore, noise and vibration refinement is mainly governed by the tribo-dynamics of unengaged loose gear pairs. On the contrary transmission efficiency because of frictional and thermal losses is determined by the interactions of engaged gear pairs. Figure 11 (a) shows the generated friction in the elasto-hydrodynamic conjunction of a pair of teeth of the engaged 2<sup>nd</sup> gear pair, whilst its counterpart for a pair of teeth of the loose 5<sup>th</sup> gear pair is shown in figure 11 (b). Note the insignificant generated friction in the case of the latter. There is no viscous friction at the pitch point due to instantaneous pure rolling motion of the teeth pair there. The same does not arise in the loaded conjunction, because of high boundary friction at the pitch point.



**Figure 11: Friction in loaded and loose gear pair conjunctions through a meshing cycle**

## 5- Conclusion

Modern transmission engineering is concerned with efficiency, emission and noise and vibration refinement. Progressively, analysis plays an important and an integral part of design and development. Therefore, models should take all the aforementioned key concerns into account. This constitutes development of multi-speed transmission tribo-dynamic models, rather than the traditional gear pair models. Such a model is described in this paper, comprising transient mixed thermo-elasto-hydrodynamics of loaded gear teeth pairs as well as thermo-hydrodynamics of loose (unengaged) gear pair conjunctions. The results show that whilst the loose gear pairs are the main sources of noise and vibration response of a transmission, frictional and thermal losses, thus emission characteristics of the transmission are affected by the engaged gear pair conjunctions.

The reported model can be used to study a plethora of noise and vibration concerns as well as assessment of likely palliative actions. It can also be employed to study the effect of tribological parameters on transmission efficiency such as lubricant rheology, surface topography, gear teeth form and transmission error.

## 6- Acknowledgements

The authors wish to express their gratitude to the Engineering and Physical Sciences Research Council (EPSRC) for its financial support of the research project: Automotive Transmission Rattle and Ford Motor Company under its University Research Program (URP). Thanks are also due to other project research partners; Getrag Ford, GKN and AVL.

## 7- Appendices:

### Appendix A1

$$D_{m(i,l)} = -\frac{1}{2\pi} \left( a_3 K + a_4 \frac{M}{2} \right)$$

where:

$$a_3 = \frac{2}{(X_i - X_{i-1})(X_i - X_{i+1})}$$

$$a_4 = -\frac{(X_{i+1} - X_{i-1})}{(X_i - X_{i-1})(X_i - X_{i+1})} + \frac{2}{(X_i - X_{i-1})(X_i - X_{i+1})} X_i$$

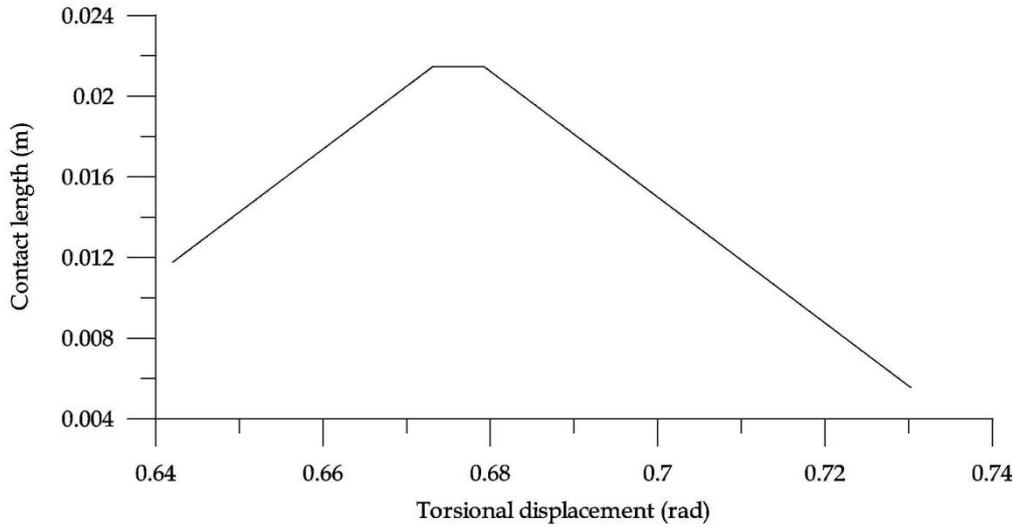
$$M = (X_l - X_{i-1})^2 (\ln(X_l - X_{i-1})^2 - 3) - (X_i - X_{i+1})^2 (\ln(X_l - X_{i+1})^2 - 3)$$

$$N = (X_l - X_{i+1})^3 (\ln|X_l - X_{i+1}|^3 - 4) - (X_l - X_{i-1})^3 (\ln|X_l - X_{i-1}|^3 - 4)$$

$$K = M \frac{X_l - X_i}{2} + \frac{2N}{9}$$

### Appendix A2

The length of the elastic line contact footprint along the flank of a pair of meshing teeth is shown in figure A2 below.



**Figure A2: Variation of thin rectangular strip length in meshing of a gear teeth pair (the instantaneous contact width,  $b$  is obtained through Hertzian theory, section 2.2.1)**

### Appendix A3

The non-dimensional terms are:

Parameters	Dimensionless form	Expression
$x(m)$	$X$	$X = \frac{x}{b}$
$\rho \left( \frac{kg}{m^3} \right)$	$\bar{\rho}$	$\bar{\rho} = \frac{\rho}{\rho_o}$
$\eta \left( \frac{Ns}{m^2} \right)$	$\bar{\eta}$	$\bar{\eta} = \frac{\eta}{\eta_o}$
$h(m)$	$H$	$H = \frac{hr_x}{b^2}$
$p(Pa)$	$P$	$P = \frac{p}{p_H}$
$t(s)$	$\bar{t}$	$\bar{t} = \frac{ut}{r_x}$
$W(s)$	$W^*$	$W^* = \frac{W}{E' r_x \ell}$
$u \left( \frac{m}{s} \right)$	$U^*$	$U^* = \frac{u \eta_o}{E' r_x}$
$\beta \left( \frac{N}{m^2} \right)$	$\bar{\beta}$	$\bar{\beta} = \frac{\beta r_x}{\eta_o u}$
$E' (N / m^2)$	$G^*$	$G^* = E' \alpha^*$

The residual term in equation (34) is obtained as:

$$F_i = \frac{1}{2\Delta X^2} \left\{ \begin{aligned} & \left[ \left( \frac{\bar{\rho}H^3}{\bar{\eta}} \right)_{i+1} + \left( \frac{\bar{\rho}H^3}{\bar{\eta}} \right)_i \right] [g(\theta-1)]_{i+1} \\ & - \left[ \left( \frac{\bar{\rho}H^3}{\bar{\eta}} \right)_{i+1} + 2 \left( \frac{\bar{\rho}H^3}{\bar{\eta}} \right)_i + \left( \frac{\bar{\rho}H^3}{\bar{\eta}} \right)_{i-1} \right] [g(\theta-1)]_i \\ & + \left[ \left( \frac{\bar{\rho}H^3}{\bar{\eta}} \right)_i + \left( \frac{\bar{\rho}H^3}{\bar{\eta}} \right)_{i-1} \right] [g(\theta-1)]_{i-1} \end{aligned} \right\} \\ - \frac{\Upsilon}{\Delta X} \left\{ (1-\Phi) [(\theta\bar{\rho}H)_{i+1} - (\theta\bar{\rho}H)_i] + (\Phi) [(\theta\bar{\rho}H)_i - (\theta\bar{\rho}H)_{i-1}] \right\} - \Upsilon \frac{r_x(\theta\bar{\rho})_i}{b} S$$

where,  $0.5 \leq \Phi \leq 1$ , with the upper bound corresponding to forward differencing and the lower bound representing backward differences.

This expression is normally solved in EHL problems by means of the modified Newton-Raphson method (Jalali *et al*, 2001).  $F_i$  denotes the approximate solution of the equation, whereas  $\bar{F}_i$  denotes the exact solution, obtained by applying Taylor's series.

$$\bar{F}_i = f(\bar{\theta}_{i-1}, \bar{\theta}_i, \bar{\theta}_{i+1}) = 0$$

$$F_i = f(\theta_{i-1}, \theta_i, \theta_{i+1}) \neq 0$$

And:

$$\bar{F}_i = F_i + \frac{\partial F_i}{\partial [g\theta]_{i+1}} \{ [g\bar{\theta}]_{i+1} - [g\theta]_{i+1} \} + \frac{\partial F_i}{\partial [g\theta]_i} \{ [g\bar{\theta}]_i - [g\theta]_i \} + \frac{\partial F_i}{\partial [g\theta]_{i-1}} \{ [g\bar{\theta}]_{i-1} - [g\theta]_{i-1} \} + e_i$$

$$\bar{F}_i = F_i + \frac{\partial F_i}{\partial [g\theta]_{i+1}} [g_{i+1}(\bar{\theta}_{i+1} - \theta_{i+1})] + \frac{\partial F_i}{\partial [g\theta]_i} [g_i(\bar{\theta}_i - \theta_i)] + \frac{\partial F_i}{\partial [g\theta]_{i-1}} [g_{i-1}(\bar{\theta}_{i-1} - \theta_{i-1})] + e_i$$

If the truncation error ( $e_i$ ) is assumed to be negligible and  $\Delta\theta_n = g_n(\bar{\theta}_n - \theta_n)$ , then:

$$-F_i = \frac{\partial F_i}{\partial [g\theta]_{i+1}} \Delta\theta_{i+1} + \frac{\partial F_i}{\partial [g\theta]_i} \Delta\theta_i + \frac{\partial F_i}{\partial [g\theta]_{i-1}} \Delta\theta_{i-1}$$

Now, the Jacobian terms ( $J$ ) can be expressed as:

$$J_i = \frac{\partial F_i}{\partial [g\theta]_i}$$

Therefore:

$$-F_i = J_{i,i+1}\Delta\theta_{i+1} + J_{i,i}\Delta\theta_i + J_{i,i-1}\Delta\theta_{i-1}$$

A Gauss-Seidel iteration technique can be used to solve this, where the super-script  $k$  denotes the loop count within the fractional film's iterative process:

$$\Delta\theta_i^k = \frac{-F_i - J_{i,i-1}\Delta\theta_{i-1}^k - J_{i,i+1}\Delta\theta_{i+1}^k}{J_{i,i}}$$

$$\Delta\theta_i^k = \frac{-J[3] - J[0]\Delta\theta_{i-1}^k - J[2]\Delta\theta_{i+1}^k}{J[1]}$$

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