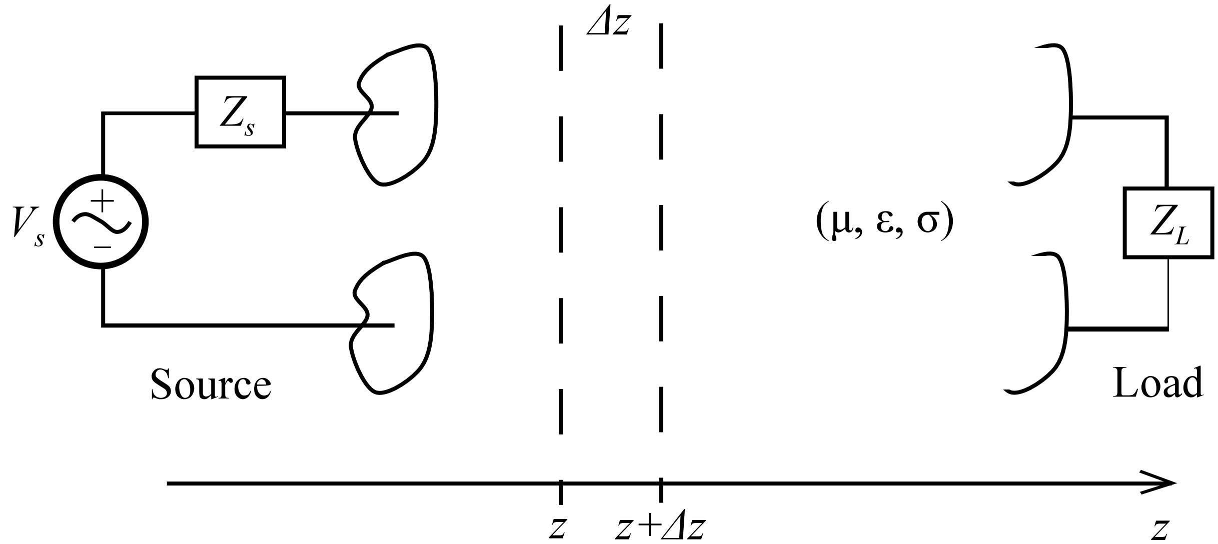
Transmission Lines

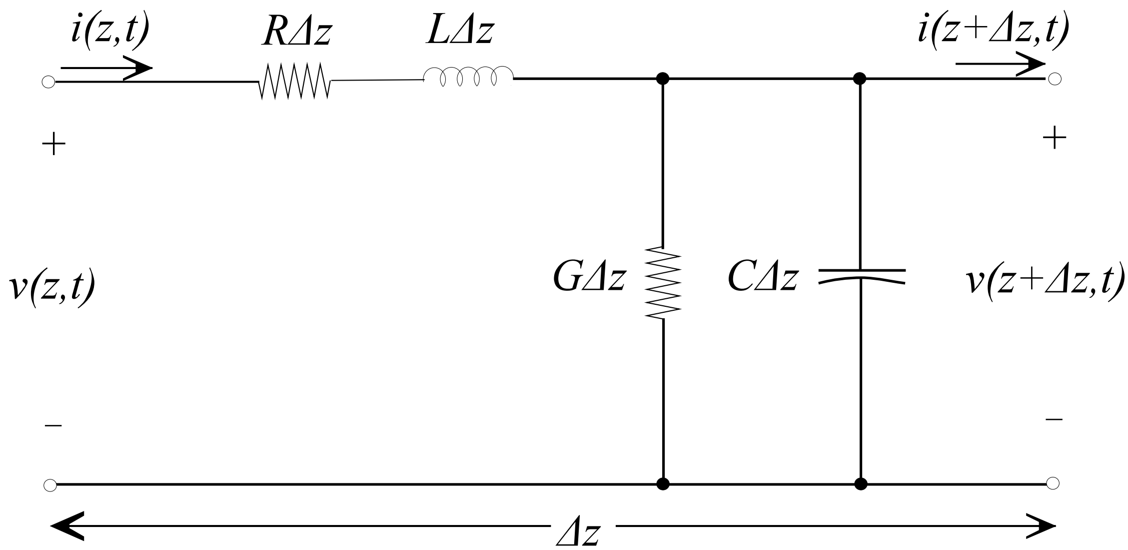
The problem of plane waves propagating in air represents an example of *unguided wave propagation*. Transmission lines and waveguides offer an alternative way of transmitting signals in the form of *guided wave propagation*. Transmission lines are typically electrically large (several wavelengths) such that we cannot accurately describe the voltages and currents along the transmission line using a simple *lumped-element* equivalent circuit. We must use a *distributed-element* equivalent circuit which describes each short segment of the transmission line by a lumped-element equivalent circuit.

Consider a simple uniform two-wire transmission line with its conductors parallel to the z-axis as shown below.

Uniform transmission line - conductors and insulating medium maintain the same cross-sectional geometry along the entire transmission line.



The equivalent circuit of a short segment Δz of the two-wire transmission line may be represented by simple lumped-element equivalent circuit.



Equivalent circuit for a segment of two-wire transmission line.

R = series resistance per unit length (Ω/m) of the transmission line conductors.

L = series inductance per unit length (H/m) of the transmission line conductors (internal plus external inductance).

G = shunt conductance per unit length (S/m) of the media between the transmission line conductors.

C = shunt capacitance per unit length (F/m) of the transmission line conductors.

We may relate the values of voltage and current at z and $z+\Delta z$ by writing KVL and KCL equations for the equivalent circuit.

KVL

$$v(z,t) - R\Delta z\, i(z,t) - L\Delta z \frac{\partial i(z,t)}{\partial t} = v(z+\Delta z,t)$$

KCL

$$i(z,t) - G\Delta z\, v(z+\Delta z,t) - C\Delta z \frac{\partial v(z+\Delta z,t)}{\partial t} = i(z+\Delta z,t)$$

Grouping the voltage and current terms and dividing by Δz gives

$$-Ri(z,t) - L\frac{\partial i(z,t)}{\partial t} = \frac{v(z+\Delta z,t) - v(z,t)}{\Delta z}$$

$$-Gv(z+\Delta z,t) - C\frac{\partial v(z+\Delta z,t)}{\partial t} = \frac{i(z+\Delta z,t) - i(z,t)}{\Delta z}$$

Taking the limit as $\Delta z \rightarrow 0$, the terms on the right hand side of the equations above become partial derivatives with respect to z which gives

$$\frac{\partial v(z,t)}{\partial z} = -Ri(z,t) - L\frac{\partial i(z,t)}{\partial t}$$

$$\frac{\partial i(z,t)}{\partial z} = -Gv(z,t) - C\frac{\partial v(z,t)}{\partial t}$$

Time-domain transmission line equations (coupled PDE's)

For time-harmonic signals, the instantaneous voltage and current may be defined in terms of phasors such that

$$v(z,t) = \mathrm{Re}\left\{V(z)e^{j\omega t}\right\}$$

$$i(z,t) = \mathrm{Re}\left\{I(z)e^{j\omega t}\right\}$$

The derivatives of the voltage and current with respect to time yield $j\omega$ times the respective phasor which gives

$$\frac{dV(z)}{dz} = -[R + j\omega L]\,I(z)$$

$$\frac{dI(z)}{dz} = -[G + j\omega C]\,V(z)$$

Frequency-domain (phasor) transmission line equations (coupled DE's)

Note the similarity in the functional form of the time- and frequency-domain transmission line equations to the respective source-free Maxwell's equations (curl equations). Even though these equations were derived without any consideration of the electromagnetic fields associated with the transmission line, remember that circuit theory is based on Maxwell's equations.

Given the similarity of the phasor transmission line equations to Maxwell's equations, we find that the voltage and current on a transmission line satisfy *wave equations*. These voltage and current wave equations are derived using the same techniques as the electric and magnetic field wave equations. Beginning with the phasor transmission line equations, we take derivatives of both sides with respect to z.

$$\frac{d^2 V(z)}{dz^2} = -[R + j\omega L]\,\frac{dI(z)}{dz}$$

$$\frac{d^2 I(z)}{dz^2} = -[G + j\omega C]\,\frac{dV(z)}{dz}$$

We then insert the first derivatives of the voltage and current found in the original phasor transmission line equations.

$$\frac{d^2 V(z)}{dz^2} = [R + j\omega L][G + j\omega C]\,V(z) = \gamma^2 V(z)$$

$$\frac{d^2 I(z)}{dz^2} = [G + j\omega C][R + j\omega L]\,I(z) = \gamma^2 I(z)$$

The voltage and current wave equations may be written as

$$\frac{d^2 V(z)}{dz^2} - \gamma^2 V(z) = 0$$

$$\frac{d^2 I(z)}{dz^2} - \gamma^2 I(z) = 0$$

where γ is the complex propagation constant of the wave on the transmission line given by

$$\gamma = \alpha + j\beta = \sqrt{(R + j\omega L)(G + j\omega C)}$$

Just as with unguided waves, the real part of the propagation constant (α) is the attenuation constant while the imaginary part (β) is the phase constant. The general equations for α and β in terms of the per-unit-length transmission line parameters are

$$\alpha = \frac{1}{\sqrt{2}}\sqrt{RG - \omega^2 LC + [(R^2 + \omega^2 L^2)(G^2 + \omega^2 C^2)]^{1/2}}$$

$$\beta = \frac{1}{\sqrt{2}}\sqrt{-RG + \omega^2 LC + [(R^2 + \omega^2 L^2)(G^2 + \omega^2 C^2)]^{1/2}}$$

The general solutions to the voltage and current wave equations are

$$V(z) = V_o^+ e^{-\gamma z} + V_o^- e^{\gamma z}$$

$$I(z) = \underbrace{I_o^+ e^{-\gamma z}}_{+z\text{-directed waves}} + \underbrace{I_o^- e^{\gamma z}}_{-z\text{-directed waves}}$$

The current equation may be written in terms of the voltage coefficients through the original phasor transmission line equations.

$$\frac{dV(z)}{dz} = -[R + j\omega L]\, I(z)$$

$$-\gamma V_o^+ e^{-\gamma z} + \gamma V_o^- e^{\gamma z} = -[R + j\omega L]\, I(z)$$

$$I(z) = \frac{\gamma}{R + j\omega L}[V_o^+ e^{-\gamma z} - V_o^- e^{\gamma z}] = \frac{1}{Z_o}[V_o^+ e^{-\gamma z} - V_o^- e^{\gamma z}]$$

The complex constant Z_o is defined as the transmission line characteristic impedance and is given by

$$Z_o = \frac{R + j\omega L}{\gamma} = \frac{R + j\omega L}{\sqrt{(R + j\omega L)(G + j\omega C)}} = \sqrt{\frac{R + j\omega L}{G + j\omega C}}$$

The transmission line equations written in terms of voltage coefficients only are

$$V(z) = V_o^+ e^{-\gamma z} + V_o^- e^{\gamma z}$$

$$I(z) = \frac{1}{Z_o} [\, V_o^+ e^{-\gamma z} - V_o^- e^{\gamma z} \,]$$

The complex voltage coefficients may be written in terms of magnitude and phase as

$$V_o^+ = |\, V_o^+ \,|\, e^{j\varphi^+}$$

$$V_o^- = |\, V_o^- \,|\, e^{j\varphi^-}$$

The instantaneous voltage becomes

$$\begin{aligned}
v(z,t) &= Re\left\{ V(z)\, e^{j\omega t} \right\} \\
&= Re\left\{ [\, V_o^+ e^{-\gamma z} + V_o^- e^{\gamma z}] e^{j\omega t} \right\} \\
&= Re\left\{ [\, |\, V_o^+ \,|\, e^{j\varphi^+} e^{-\alpha z} e^{-j\beta z} + |\, V_o^- \,|\, e^{j\varphi^-} e^{\alpha z} e^{j\beta z}] e^{j\omega t} \right\} \\
&= |\, V_o^+ \,|\, e^{-\alpha z} \cos(\omega t - \beta z + \varphi^+) \\
&\qquad + |\, V_o^- \,|\, e^{\alpha z} \cos(\omega t + \beta z + \varphi^-)
\end{aligned}$$

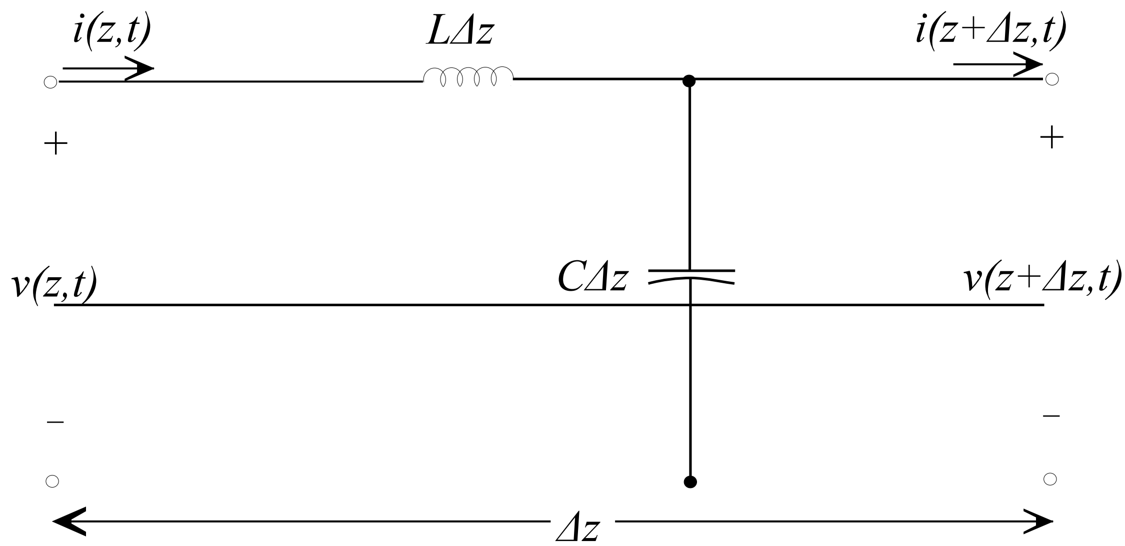
The wavelength and phase velocity of the waves on the transmission line may be found using the points of constant phase as was done for plane waves.

$$v_p = \frac{\omega}{\beta} = \lambda f$$

$$\lambda = \frac{2\pi}{\beta}$$

Lossless Transmission Line

If the transmission line loss is neglected ($R = G = 0$), the equivalent circuit reduces to



Equivalent circuit for a segment of lossless two-wire transmission line.

Note that for a true lossless transmission line, the insulating medium between the conductors is characterized by a zero conductivity ($\sigma = 0$), and real-valued permittivity ε and permeability μ ($\varepsilon'' = \mu'' = 0$). The propagation constant on the lossless transmission line is

$$\gamma = \alpha + j\beta = j\omega\sqrt{LC}$$

$$\alpha = 0 \qquad\qquad \beta = \omega\sqrt{LC}$$

Given the purely imaginary propagation constant, the transmission line equations for the lossless line are

$$V(z) = V_o^+ e^{-j\beta z} + V_o^- e^{j\beta z}$$

$$I(z) = \frac{1}{Z_o}\left[V_o^+ e^{-j\beta z} - V_o^- e^{j\beta z} \right]$$

The characteristic impedance of the lossless transmission line is purely real and given by

$$Z_o = \sqrt{\frac{L}{C}}$$

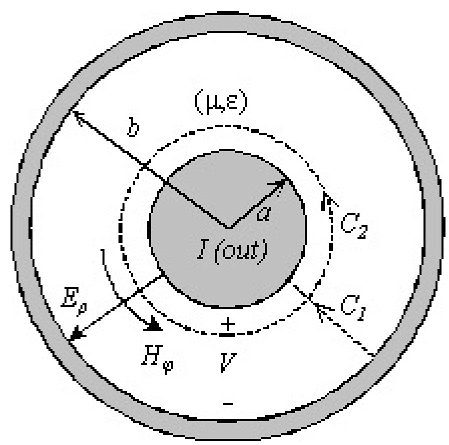
The phase velocity and wavelength on the lossless line are

$$v_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}}$$

$$\lambda = \frac{2\pi}{\beta} = \frac{1}{f\sqrt{LC}} = \frac{v_p}{f}$$

<u>Example</u> (Lossless coaxial transmission line)

The dominant mode on a coaxial transmission line is the TEM (transverse electromagnetic) mode defined by $E_z = H_z = 0$. Due to the symmetry of the coaxial transmission line, the transverse fields are independent of φ. Thus, we may write



$$\boldsymbol{E} = E_\rho(\rho,z)\,\boldsymbol{a}_\rho$$

$$\boldsymbol{H} = H_\varphi(\rho,z)\,\boldsymbol{a}_\varphi$$

Between the conductors, the fields must satisfy the source-free Maxwell's equations:

$$\nabla\times\boldsymbol{E} = -j\omega\mu\boldsymbol{H} \qquad \Rightarrow \qquad \frac{\partial E_\rho(\rho,z)}{\partial z} = -j\omega\mu H_\varphi(\rho,z)$$

$$\nabla\times\boldsymbol{H} = j\omega\varepsilon\boldsymbol{E} \qquad \Rightarrow \qquad \frac{\partial H_\varphi(\rho,z)}{\partial z} = -j\omega\varepsilon E_\rho(\rho,z)$$

The field components E_ρ and H_φ are related to the transmission line voltage and current by

$$V = -\int_{C_1} \boldsymbol{E} \cdot d\boldsymbol{l} = \int_{C_1} E_\rho \, d\rho$$

$$I = \int_{C_2} \boldsymbol{H} \cdot d\boldsymbol{l} = \int_{C_2} H_\varphi \, d\varphi$$

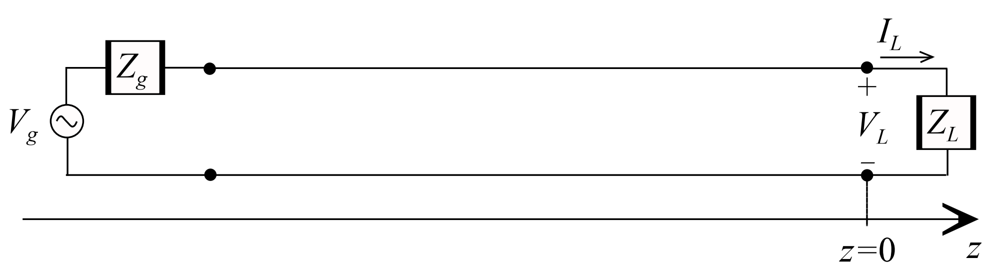
If we integrate the Faraday's law equation along the path C_1 and integrate the Ampere's law equation along the path C_2,

$$\frac{\partial V}{\partial z} = \int_{C_1} \frac{\partial E_\rho}{\partial z} d\rho = -j\omega \int_{C_1} B_\varphi d\rho = -j\omega L I$$

$$\frac{\partial I}{\partial z} = \int_{C_2} \frac{\partial H_\varphi}{\partial z} d\varphi = -j\omega \int_{C_2} D_\rho \, d\varphi = -j\omega C V$$

which are the lossless transmission line equations.

<u>Terminated Lossless Transmission Line</u>



If we choose our reference point (z = 0) at the load termination, then the lossless transmission line equations evaluated at z = 0 give the load voltage and current.

$$V(z) = V_o^+ e^{-j\beta z} + V_o^- e^{j\beta z} \qquad V_L = V(0) = V_o^+ + V_o^-$$

$$I(z) = \frac{1}{Z_o}[\, V_o^+ e^{-j\beta z} - V_o^- e^{j\beta z}\,] \qquad I_L = I(0) = \frac{1}{Z_o}[V_o^+ - V_o^-]$$

The ratio of voltage to current at $z = 0$ must equal the load impedance.

$$Z_L = Z_o \frac{V_o^+ + V_o^-}{V_o^+ - V_o^-}$$

Solving this equation for the voltage coefficient of the $-z$ traveling wave gives

$$V_o^- = \frac{Z_L - Z_o}{Z_L + Z_o} V_o^+ = \Gamma V_o^+$$

where Γ is the reflection coefficient which defines the ratio of the reflected wave to the incident wave.

$$\Gamma = \frac{Z_L - Z_o}{Z_L + Z_o}$$

Note that the reflection coefficient is in general complex with $0 \leq |\Gamma| \leq 1$. If the reflection coefficient is zero ($Z_L = Z_o$), there is no reflected wave and the load is said to be *matched* to the transmission line. If $Z_L \neq Z_o$, the magnitude of the reflection coefficient is non-zero (there is a reflected wave). The presence of forward and reverse traveling waves on the transmission line produces *standing waves*.

We may rewrite the transmission line equations in terms of the reflection coefficient as

$$V(z) = V_o^+ \left[e^{-j\beta z} + \frac{V_o^-}{V_o^+} e^{j\beta z} \right] = V_o^+ [e^{-j\beta z} + \Gamma e^{j\beta z}]$$

$$I(z) = \frac{V_o^+}{Z_o} \left[e^{-j\beta z} - \frac{V_o^-}{V_o^+} e^{j\beta z} \right] = \frac{V_o^+}{Z_o} [e^{-j\beta z} - \Gamma e^{j\beta z}]$$

or

$$V(z) = V_o^+ e^{-j\beta z} [\, 1 + \Gamma e^{j2\beta z}]$$

$$I(z) = \frac{V_o^+}{Z_o} e^{-j\beta z} [\, 1 - \Gamma e^{j2\beta z}]$$

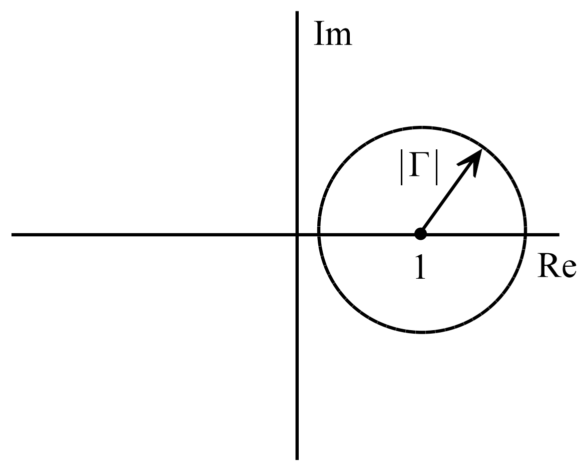
The magnitude of the transmission line voltage may be written as

$$|V(z)| = |V_o^+| \; |1 + \Gamma e^{j2\beta z}|$$

The maximum and minimum voltage magnitudes are

$$|V(z)|_{\max} = |V_o^+| \, [1 + |\Gamma|\,]$$

$$|V(z)|_{\min} = |V_o^+| \, [1 - |\Gamma|\,]$$



The ratio of maximum to minimum voltage magnitudes defines the *standing wave ratio* (s).

$$s = \frac{|V(z)|_{\max}}{|V(z)|_{\min}} = \frac{1 + |\Gamma|}{1 - |\Gamma|}$$

Note that the standing wave ratio is real with $1 \le s \le \infty$.

The time-average power at any point on the transmission line is given by

$$P_{av} = \frac{1}{2} Re\left\{VI^*\right\}$$

$$= \frac{1}{2} Re\left\{V_o^+[e^{-j\beta z} + \Gamma e^{j\beta z}]\frac{V_o^{+*}}{Z_o^*}[e^{j\beta z} - \Gamma^* e^{-j\beta z}]\right\}$$

$$= \frac{1}{2}\frac{|V_o^+|^2}{Z_o} Re\left\{1 - |\Gamma|^2 + \underbrace{\Gamma e^{j2\beta z} - \Gamma^* e^{-j2\beta z}}_{\substack{\text{Purely imaginary} \\ A - A^* = 2j\,\text{Im}\{A\}}}\right\}$$

$$P_{av} = \frac{1}{2}\frac{|V_o^+|^2}{Z_o}[1 - |\Gamma|^2]$$

$$= \frac{|V_o^+|^2}{2Z_o} - |\Gamma|^2\frac{|V_o^+|^2}{2Z_o}$$

$$= P_i - P_r$$

P_i = incident power

P_r = reflected power

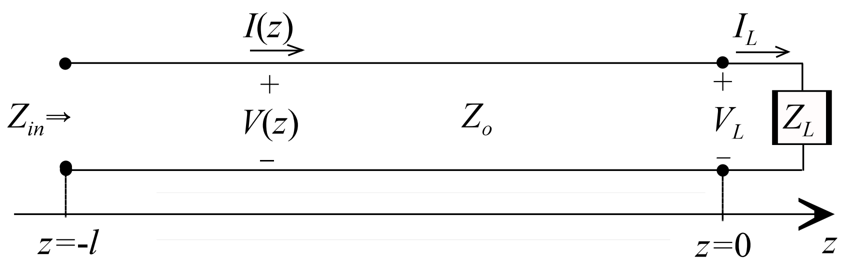
The *return loss* (*RL*) is defined as the ratio of incident power to reflected power. The return loss in dB is

$$RL = 10\log_{10}\left(\frac{P_i}{P_r}\right) = 10\log_{10}\left(\frac{1}{|\Gamma|^2}\right) = 10\log_{10}(|\Gamma|)^{-2}$$

$$= -20\log_{10}(\Gamma) \quad \text{(dB)}$$

Matched load	$\lvert\Gamma\rvert = 0$	$s = 1$	$RL = \infty$ (dB)
Total reflection	$\lvert\Gamma\rvert = 1$	$s = \infty$	$RL = 0$ (dB)

<u>Transmission Line Impedance</u>



The impedance at any point on the transmission line is given by

$$Z(z) = \frac{V(z)}{I(z)} = \frac{V_o^+[e^{-j\beta z} + \Gamma e^{j\beta z}]}{\dfrac{V_o^+}{Z_o}[e^{-j\beta z} - \Gamma e^{j\beta z}]} = Z_o \frac{e^{-j\beta z} + \Gamma e^{j\beta z}}{e^{-j\beta z} - \Gamma e^{j\beta z}}$$

$$= Z_o \frac{e^{-j\beta z} + \dfrac{Z_L - Z_o}{Z_L + Z_o} e^{j\beta z}}{e^{-j\beta z} - \dfrac{Z_L - Z_o}{Z_L + Z_o} e^{j\beta z}}$$

$$= Z_o \frac{(Z_L + Z_o)\, e^{-j\beta z} + (Z_L - Z_o)\, e^{j\beta z}}{(Z_L + Z_o)\, e^{-j\beta z} - (Z_L - Z_o)\, e^{j\beta z}}$$

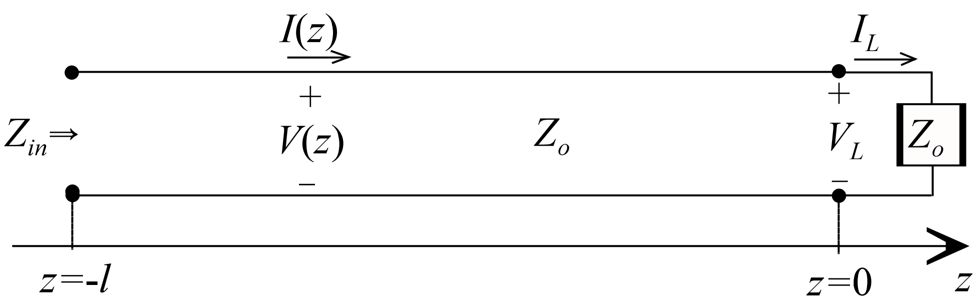
$$= Z_o \frac{Z_L(2\cos\beta z) + Z_o(-j2\sin\beta z)}{Z_o(2\cos\beta z) - Z_L(j2\sin\beta z)}$$

$$= Z_o \frac{Z_L - jZ_o \tan\beta z}{Z_o - jZ_L \tan\beta z}$$

The impedance at the input of a transmission line of length l terminated with an impedance Z_L is

$$Z_{in} = Z(-l) = Z_o \frac{Z_L + jZ_o \tan \beta l}{Z_o + jZ_L \tan \beta l}$$

Lossless Transmission Line with Matched Load ($Z_L = Z_o$)



$$\Gamma = 0 \qquad s = 1$$

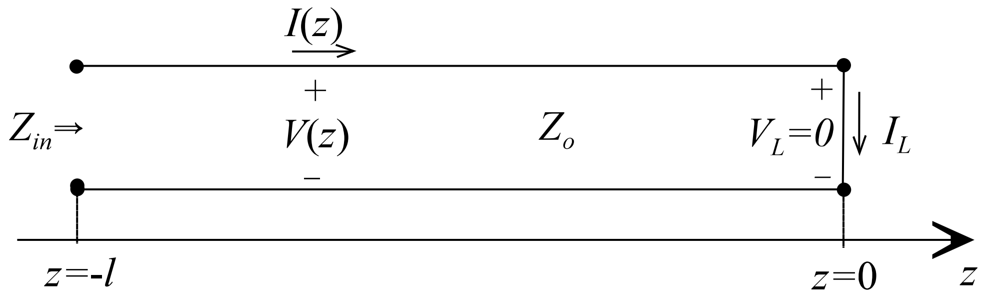
$$Z_{in} = Z_o \qquad \text{(independent of line length)}$$

$$V(z) = V_o^+ e^{-j\beta z} \qquad |V(z)| = |V_o^+|$$

$$I(z) = \frac{V_o^+}{Z_o} e^{-j\beta z} \qquad |I(z)| = \frac{|V_o^+|}{Z_o}$$

Note that the input impedance of the lossless transmission line terminated with a matched impedance is independent of the line length. Any mismatch in the transmission line system will cause standing waves and make the input impedance dependent on the length of the line.

Short-Circuited Lossless Transmission Line ($Z_L = 0$)



$$\Gamma = -1 \qquad s = \infty$$

$$Z_{in} = jZ_o \tan \beta l \qquad \text{(purely reactive, dependent on line length)}$$

$$V(z) = V_o^+ [e^{-j\beta z} - e^{j\beta z}] = -2jV_o^+ \sin \beta z$$

$$I(z) = \frac{V_o^+}{Z_o} [e^{-j\beta z} + e^{j\beta z}] = \frac{2V_o^+}{Z_o} \cos \beta z$$

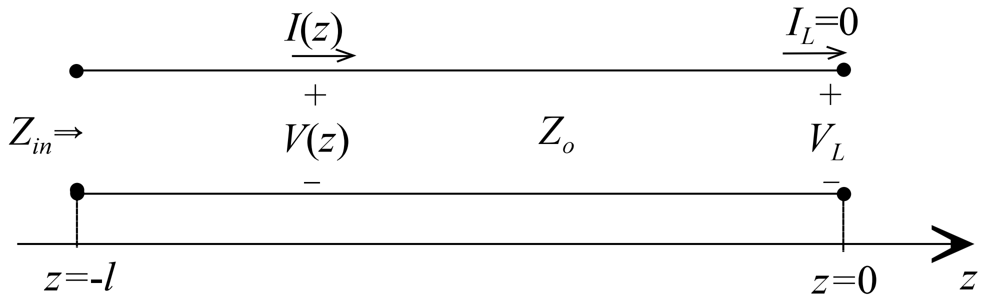
[See Figure 2.6 (p.61)]

The input impedance of a short-circuited lossless transmission line is purely reactive and can take on any value of capacitive or inductive reactance depending on the line length. The incident wave is totally reflected (with inversion) from the load setting up standing waves with $|V|_{min} = 0$ and $|V|_{max} = 2|V_o^+|$.

$$l = n\frac{\lambda}{2} \qquad |Z_{in}| = 0 \qquad n = 0, 1, 2, \ldots$$

$$l = (2n-1)\frac{\lambda}{4} \qquad |Z_{in}| = \infty \qquad n = 1, 2, \ldots$$

Open-Circuited Lossless Transmission Line ($Z_L = \infty$)



$\Gamma = -1 \qquad\qquad s = \infty$

$Z_{in} = -jZ_o \cot\beta l$ (purely reactive, dependent on line length)

$$V(z) = V_o^+ [e^{-j\beta z} + e^{j\beta z}] = 2V_o^+ \cos\beta z$$

$$I(z) = \frac{V_o^+}{Z_o}[e^{-j\beta z} - e^{j\beta z}] = \frac{-2jV_o^+}{Z_o}\sin\beta z$$

[See Figure 2.8 (p.62)]

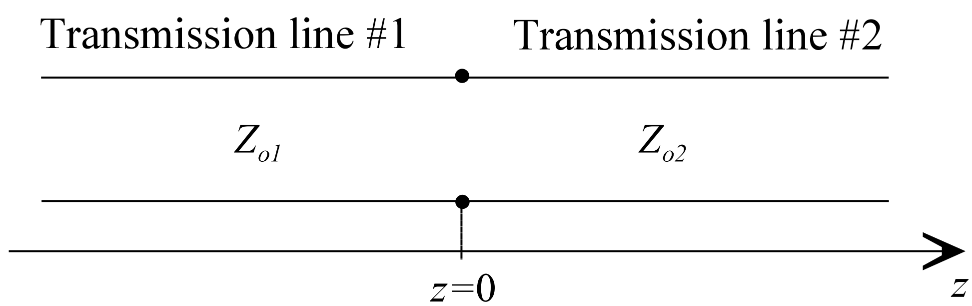
The input impedance of an open-circuited lossless transmission line is purely reactive and can take on any value of capacitive or inductive reactance depending on the line length. The incident wave is totally reflected (without inversion) from the load setting up standing waves with $|V|_{min} = 0$ and $|V|_{max} = 2|V_o^+|$.

$$l = n\frac{\lambda}{2} \qquad |Z_{in}| = \infty \qquad n = 0, 1, 2, ...$$

$$l = (2n-1)\frac{\lambda}{4} \qquad |Z_{in}| = 0 \qquad n = 1, 2, ...$$

Transmission Line Connections

The analysis of a connection between two distinct transmission lines can be performed using the same techniques used for plane wave transmission/reflection at a material interface. Consider two lossless transmission lines of different characteristic impedances connected as shown below.



Assume that a source is connected to transmission line #1 and transmission line #2 is terminated with a matched impedance ($Z_{L2} = Z_{o2}$). Transmission line #1 is then effectively terminated with a load impedance of Z_{o2} which constitutes a mismatch. At the transmission line connection, a portion of the incident wave on transmission line #1 is transmitted onto transmission line #2 while the remainder is reflected back on transmission line #1. Thus, we may write the voltages on the two transmission lines as

$$V(z) = V_o^+ (e^{-j\beta z} + \Gamma e^{j\beta z}) \qquad (z < 0)$$

$$V(z) = V_o^+ T e^{-j\beta z} \qquad (z > 0)$$

where Γ is the reflection coefficient on transmission line #1 and T is the transmission coefficient on transmission line #2. Equating the voltages at the transmission line connection point (z = 0) gives

$$1 + \Gamma = T \qquad \Rightarrow \qquad T = 1 + \frac{Z_{o2} - Z_{o1}}{Z_{o2} + Z_{o1}} = \frac{2Z_{o2}}{Z_{o2} + Z_{o1}}$$

Note the similarity between the equations for plane wave (unguided wave) transmission/reflection at a material interface and the guided wave transmission/reflection at a transmission line connection.

$$\textit{Plane waves} \qquad \Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} \qquad T = \frac{2\eta_2}{\eta_2 + \eta_1}$$

$$t\textit{-line guided waves} \qquad \Gamma = \frac{Z_{o2} - Z_{o1}}{Z_{o2} + Z_{o1}} \qquad T = \frac{2Z_{o2}}{Z_{o2} + Z_{o1}}$$

Insertion Loss

Strictly speaking, insertion loss is the ratio of power absorbed by a load before and after a network is inserted into the line. For the previous example connection between two transmission lines, we consider the matched case for transmission line #1 and the case with transmission line #2 inserted into the system.

$$P_{av} = P_i = \frac{|V_o^+|^2}{2Z_o} \qquad (\textit{matched case})$$

$$P_{av} = P_t = |T|^2 \frac{|V_o^+|^2}{2Z_o} \qquad (\textit{connected } t\textit{-lines})$$

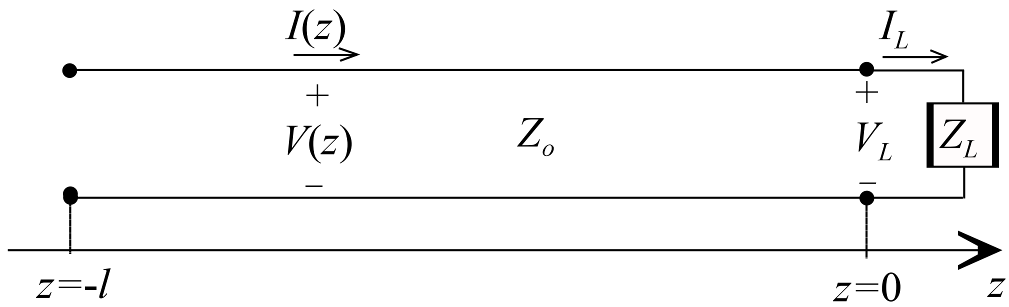
The *insertion loss* (*IL*) is then

$$IL = 10\log_{10}\left(\frac{P_i}{P_t}\right) = 10\log_{10}\left(\frac{1}{|T|^2}\right) = 10\log_{10}(|T|)^{-2}$$

$$= -20\log_{10}(T) \quad \text{(dB)}$$

Smith Chart

The Smith chart is a useful graphical tool used to calculate the reflection coefficient and impedance at various points on a transmission line system. The Smith chart is actually a polar plot of the complex reflection coefficient $\Gamma(z)$ overlaid with the corresponding impedance $Z(z)$.



The voltage at any point on the transmission line is

$$V(z) = V_o^+ e^{-j\beta z} + V_o^- e^{j\beta z}$$

The reflection coefficient at any point on the transmission line is defined as

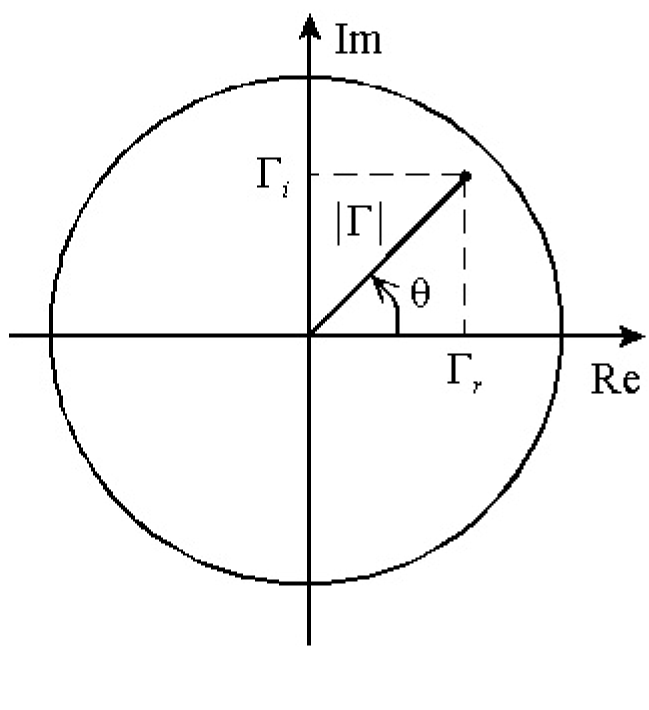
$$\Gamma(z) = \frac{V_o^- e^{j\beta z}}{V_o^+ e^{-j\beta z}} = \frac{V_o^-}{V_o^+} e^{j2\beta z} = \Gamma e^{j2\beta z}$$

where Γ is the reflection coefficient at the load (z = 0).

$$\Gamma = |\Gamma| e^{j\theta} = \Gamma_r + j\Gamma_i$$

Smith chart center ⇒ $|\Gamma| = 0$
(no reflection - matched)

Outer circle ⇒ $|\Gamma| = 1$
(total reflection)



The reflection coefficient for a lossless transmission line of characteristic impedance Z_o, terminated with an impedance Z_L is given by

$$\Gamma = \frac{Z_L - Z_o}{Z_L + Z_o} = \frac{(Z_L / Z_o) - 1}{(Z_L / Z_o) + 1} = \frac{z_L - 1}{z_L + 1} \tag{1}$$

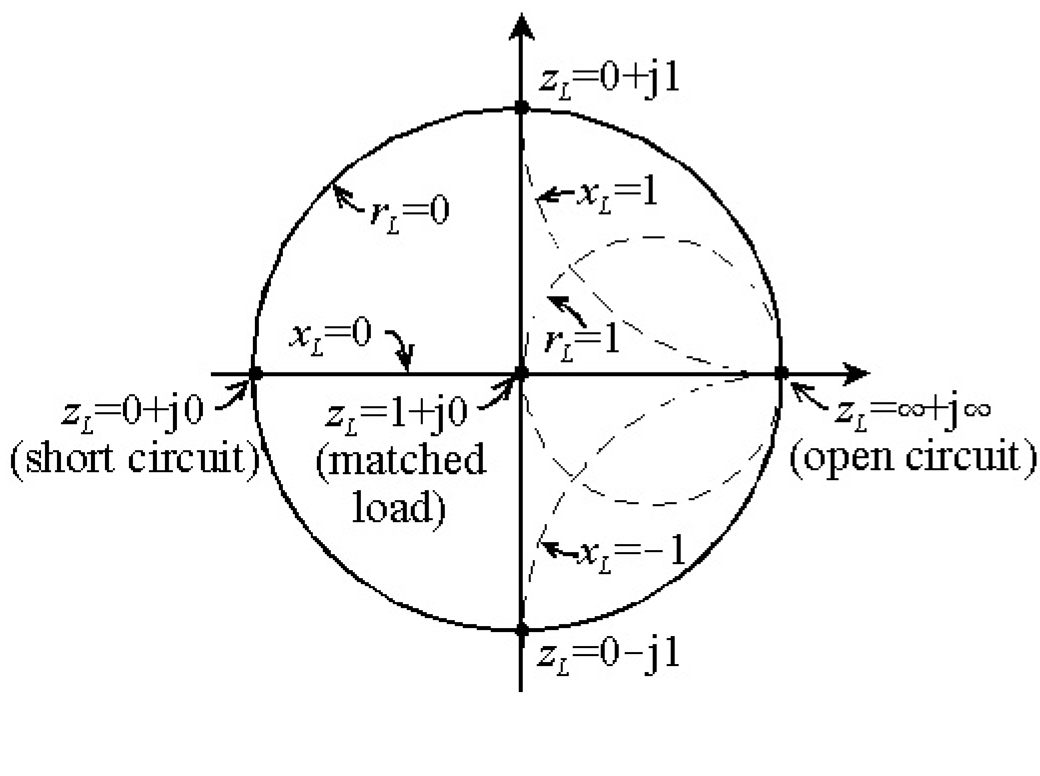
where z_L is the normalized load impedance. If we solve (1) for z_L, we find

$$z_L = \frac{Z_L}{Z_o} = r_L + jx_L = \frac{1 + \Gamma}{1 - \Gamma} = \frac{1 + \Gamma_r + j\Gamma_i}{1 - \Gamma_r - j\Gamma_i} \tag{2}$$

where r_L and x_L are the normalized load resistance and reactance, respectively. Equation (2) shows that the reflection coefficient on the Smith chart corresponds to a specific normalized load impedance for the given transmission line / load combination. Solving (2) for the resistance and reactance gives equations for the "resistance" and "reactance" circles:

$$r_L = \frac{1 - \Gamma_r^2 - \Gamma_i^2}{(1 - \Gamma_r)^2 + \Gamma_i^2} \qquad \text{center} = \left(\frac{r_L}{1 + r_L}, 0 \right) \qquad \text{radius} = \frac{1}{1 + r_L}$$

$$x_L = \frac{2\Gamma_i}{(1 - \Gamma_r)^2 + \Gamma_i^2} \qquad \text{center} = \left(1, \frac{1}{x_L} \right) \qquad \text{radius} = \frac{1}{x_L}$$



In a similar fashion, the general impedance at any point along the length of the transmission line may be written as

$$Z(z) = Z_o \frac{e^{-j\beta z} + \Gamma e^{j\beta z}}{e^{-j\beta z} - \Gamma e^{j\beta z}} = Z_o \frac{1 + \Gamma e^{j2\beta z}}{1 - \Gamma e^{j2\beta z}} = Z_o \frac{1 + \Gamma(z)}{1 - \Gamma(z)}$$

The normalized value of the impedance $z_n(z)$ is

$$z_n(z) = \frac{Z(z)}{Z_o} = \frac{1 + \Gamma(z)}{1 - \Gamma(z)} = r(z) + jx(z) \qquad (3)$$

Note the similarity between Equations (2) and (3). The magnitude of the reflection coefficient is constant on a lossless line. Thus, Equation (3) shows that once we locate the normalized load impedance on the Smith chart, we simply rotate through an angle of $2\beta z$ on the $|\Gamma|$ circle to find the impedance at a given point on the transmission line. Evaluating the reflection coefficient at the transmission line input ($z = -l$) gives

$$\Gamma(-l) = \Gamma e^{-j2\beta l} = |\Gamma| e^{j(\theta - 2\beta l)}$$

which defines a negative phase shift moving toward the generator from the load. Thus, in general

CW rotation $\Rightarrow$ toward the generator

CCW rotation $\Rightarrow$ toward the load

One complete revolution on the Smith chart occurs for

$$2\beta l = 2\pi \qquad \Rightarrow \qquad \frac{4\pi l}{\lambda} = 2\pi \qquad \Rightarrow \qquad l = \frac{\lambda}{2}$$

Thus, each revolution on the Smith chart represents a movement of one-half wavelength along the transmission line ($\lambda = 720^\circ$).

Once an impedance point is located on the Smith chart, the equivalent admittance point is found by rotating 180° from the impedance point on the constant reflection coefficient circle.

<u>Example</u> Problem 2.19 (using equations and Smith chart)

(b.)
$$\Gamma = \frac{Z_L - Z_o}{Z_L + Z_o} = \frac{(60 + j50) - 50}{(60 + j50) + 50} = \frac{10 + j50}{110 + j50} = 0.422\angle 54^o$$

(a.)
$$s = \frac{1 + |\Gamma|}{1 - |\Gamma|} = \frac{1 + 0.422}{1 - 0.422} = 2.46$$

(c.)
$$Y_L = \frac{1}{Z_L} = \frac{1}{60 + j50} = (9.84 - j8.2)\ \text{mS}$$

(d.)
$$Z_{in} = Z_o \frac{Z_L + jZ_o \tan\beta l}{Z_o + jZ_L \tan\beta l} \qquad \beta l = \frac{2\pi}{\lambda}\frac{4}{10}\lambda = \frac{4\pi}{5}$$

$$= 50\frac{(60 + j50) + j50\tan\left(\dfrac{4\pi}{5}\right)}{50 + j(60 + j50)\tan\left(\dfrac{4\pi}{5}\right)} = (24.5 + j\,20.3)\ \Omega$$

(e.)
$$|V(z)| = |V_o^+|\,|1 + \Gamma e^{j2\beta z}| \qquad \Gamma e^{j2\beta z} = |\Gamma|\,e^{j\theta}e^{j2\beta z}$$

$$|V(z)|_{\max} = |V_o^+|\,[1 + |\Gamma|] \qquad \theta + 2\beta z_{\max} = n\pi \quad (\text{even } n)$$

$$|V(z)|_{\min} = |V_o^+|\,[1 - |\Gamma|] \qquad \theta + 2\beta z_{\min} = n\pi \quad (\text{odd } n)$$

$$z_{\min} = \frac{n\pi - \theta}{2\beta} = \frac{n\pi - \left(\dfrac{54\pi}{180}\right)}{4\pi}\lambda = \frac{n - 0.3}{4}\lambda \qquad (\text{odd } n)$$

$$n = 1 \quad \Rightarrow \quad z_{\min} = \frac{0.7}{4}\lambda = 0.175\lambda$$

$$n = -1 \quad \Rightarrow \quad z_{\min} = -\frac{1.3}{4}\lambda = -0.325\lambda \quad \Rightarrow \quad l_{\min} = 0.325\lambda$$

(f.)

$$z_{max} = \frac{n\pi - \theta}{2\beta} = \frac{n\pi - \left(\frac{54\pi}{180}\right)}{4\pi}\lambda = \frac{n - 0.3}{4}\lambda \qquad \text{(even } n\text{)}$$

$$n = 0 \qquad \Rightarrow \qquad z_{max} = -\frac{0.3}{4}\lambda = -0.075\lambda \qquad \Rightarrow \qquad l_{max} = 0.075\lambda$$

$$n = 2 \qquad \Rightarrow \qquad z_{max} = \frac{1.7}{4}\lambda = 0.425\lambda$$

Lossy Transmission Lines

The general transmission line equations (defined in terms of a complex propagation constant) must be used when dealing with lossy lines. The general transmission line equations are

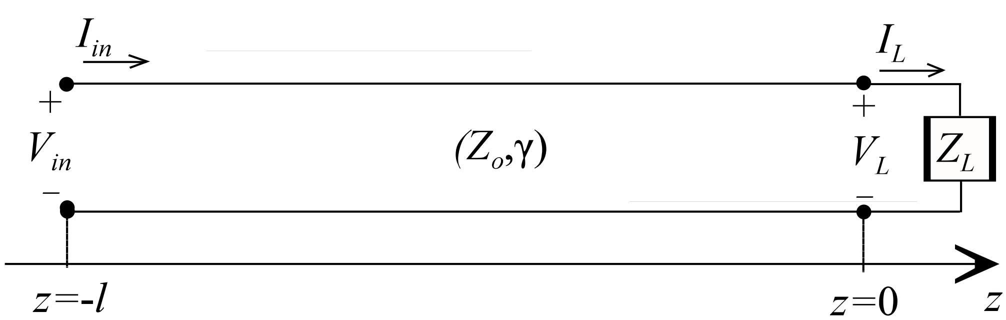
$$V(z) = V_o^+ e^{-\gamma z} + V_o^- e^{\gamma z} = V_o^+ [e^{-\gamma z} + \Gamma e^{\gamma z}]$$

$$I(z) = \frac{1}{Z_o}\left[V_o^+ e^{-\gamma z} - V_o^- e^{\gamma z}\right] = \frac{V_o^+}{Z_o}[e^{-\gamma z} - \Gamma e^{\gamma z}]$$

The reflection coefficient for a lossy transmission line may be determined by replacing each $j\beta$ term with γ which gives

$$\Gamma(z) = \frac{V_o^- e^{\gamma z}}{V_o^+ e^{-\gamma z}} = \frac{V_o^-}{V_o^+} e^{2\gamma z} = \Gamma e^{2\gamma z} = |\Gamma|\, e^{2\alpha z} e^{j2\beta z}$$

In a similar fashion, the input impedance of a terminated lossy transmission line is



$$Z_{in} = Z_o \frac{Z_L + jZ_o \tan(-j\gamma l)}{Z_o + jZ_L \tan(-j\gamma l)} = Z_o \frac{Z_L + Z_o \tanh(\gamma l)}{Z_o + Z_L \tanh(\gamma l)}$$

Most transmission lines are designed with materials which produce small losses (*low loss lines*). The general expressions for the characteristic impedance and propagation constant of a lossy transmission line may be simplified somewhat for a low-loss line.

$$\text{Lossless line} \quad \Rightarrow \quad R = G = 0$$

$$\text{Low loss line} \quad \Rightarrow \quad R << \omega L, \;\; G << \omega C$$

Thus, for the equivalent circuit of a low loss line, the reactance terms must be much larger than the resistance terms at all frequencies of operation.

The general equation for the propagation constant on a lossy transmission line may be approximated as follows for a low loss line.

$$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)}$$

$$= \sqrt{(j\omega L)\left(1 + \frac{R}{j\omega L}\right)(j\omega C)\left(1 + \frac{G}{j\omega C}\right)}$$

$$= j\omega\sqrt{LC}\sqrt{1 - j\left(\frac{R}{\omega L} + \frac{G}{\omega C}\right) - \frac{RG}{\omega^2 LC}}$$

$$\approx j\omega\sqrt{LC}\sqrt{1 - j\left(\frac{R}{\omega L} + \frac{G}{\omega C}\right)}$$

Note that we still include the dominant loss terms (even though they are small for a low loss line). Using a Taylor series expansion for the square root term above yields

$$\sqrt{1 + x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \ldots \approx 1 + \frac{x}{2} \qquad (x << 1)$$

$$\gamma \approx j\omega\sqrt{LC}\left[1 - \frac{j}{2}\left(\frac{R}{\omega L} + \frac{G}{\omega C}\right)\right]$$

$$= \frac{1}{2}\left(R\sqrt{\frac{C}{L}} + G\sqrt{\frac{L}{C}}\right) + j\omega\sqrt{LC}$$

$$= \alpha + j\beta$$

$$\alpha \approx \frac{1}{2}\left(R\sqrt{\frac{C}{L}} + G\sqrt{\frac{L}{C}} \right) \qquad \beta \approx \omega\sqrt{LC} \qquad \text{(low loss line)}$$

Using the same approximations for the characteristic impedance, we find

$$Z_o = \sqrt{\frac{R + j\omega L}{G + j\omega C}} \approx \sqrt{\frac{L}{C}} \qquad \text{(low loss line)}$$

The power level delivered to the load is lower than that delivered to the transmission line input due to the line losses. The line losses can be determined by calculating the difference in these power levels. The voltage and current at the load are

$$V_L = V(0) = V_o^+[1 + \Gamma]$$

$$I_L = I(0) = \frac{V_o^+}{Z_o}[1 - \Gamma]$$

The power delivered to the load is

$$P_L = \frac{1}{2} Re\left\{ V_L I_L^* \right\}$$

$$= \frac{1}{2} Re\left\{ V_o^+[1 + \Gamma] \frac{V_o^{+*}}{Z_o}[1 - \Gamma^*] \right\}$$

$$= \frac{|V_o^+|^2}{2Z_o} Re\left\{ 1 - |\Gamma|^2 + \Gamma - \Gamma^* \right\}$$

$$= \frac{|V_o^+|^2}{2Z_o} \left\{ 1 - |\Gamma|^2 \right\}$$

The voltage and current at the input to the transmission line are

$$V_{in} = V(-l) = V_o^+[e^{\gamma l} + \Gamma e^{-\gamma l}]$$

$$I_{in} = I(-l) = \frac{V_o^+}{Z_o}[e^{\gamma l} - \Gamma e^{-\gamma l}]$$

The power delivered to the input of the transmission line

$$P_{in} = \frac{1}{2} Re\left\{V_{in} I_{in}^*\right\}$$

$$= \frac{1}{2} Re\left\{V_o^+[e^{\alpha l}e^{j\beta l} + \Gamma e^{-\alpha l}e^{-j\beta l}]\frac{V_o^{+*}}{Z_o}[e^{\alpha l}e^{-j\beta l} - \Gamma^* e^{-\alpha l}e^{j\beta l}]\right\}$$

$$= \frac{|V_o^+|^2}{2Z_o} Re\left\{e^{2\alpha l} - |\Gamma|^2 e^{-2\alpha l} + \Gamma e^{-j2\beta l} - \Gamma^* e^{j2\beta l}\right\}$$

$$= \frac{|V_o^+|^2}{2Z_o}\left\{e^{2\alpha l} - |\Gamma|^2 e^{-2\alpha l}\right\}$$

The power lost in the transmission line is

$$P_{loss} = P_{in} - P_L = \frac{|V_o^+|^2}{2Z_o}\left[(e^{2\alpha l} - 1) + |\Gamma|^2(1 - e^{-2\alpha l})\right]$$

Distortionless Transmission Line

On a lossless transmission line, the propagation constant is purely imaginary and given by

$$\gamma = j\beta = j\omega\sqrt{LC}$$

The phase velocity on the lossless line is

$$v_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}}$$

Note that the phase constant which varies linearly with frequency produces a constant phase velocity (independent of frequency) so that all frequencies propagate along the lossless transmission line at the same velocity. From Fourier theory, we know that any time-domain signal may be represented as a weighted sum of sinusoids. Thus, signals transmitted along a lossless transmission line will suffer no distortion since all of the frequency components propagate at the same velocity. When the phase velocity of a transmission line is a function of frequency, signals will become distorted as different components of the signal arrive at different times. This effect is called *dispersion*.

For the low-loss line, using the appropriate approximations, we found

$$\gamma = \alpha + j\beta \approx \alpha + j\omega\sqrt{LC}$$

which implies that the phase velocity on a low loss line is near constant. However, the small variations in the phase velocity on a low loss line may produce significant distortion if the line is very long.

There is a special case of lossy line with the linear phase constant that produces a *distortionless line*. A transmission line is a distortionless line if the per-unit-length parameters satisfy

$$\frac{R}{L} = \frac{G}{C} \qquad (\text{distortionless line})$$

Inserting the per-unit-length parameter relationship into the general equation for the propagation constant on a lossy line gives

$$\begin{aligned}\gamma &= \sqrt{(R + j\omega L)(G + j\omega C)} \\ &= \sqrt{R\left(1 + j\omega\frac{L}{R}\right) G\left(1 + j\omega\frac{C}{G}\right)} \\ &= \sqrt{RG\left(1 + j\omega\frac{L}{R}\right)^2} \\ &= \sqrt{RG}\left(1 + j\omega\frac{L}{R}\right) = \alpha + j\beta\end{aligned}$$

$$\alpha = \sqrt{RG} \qquad\qquad \beta = \omega\sqrt{RG}\,\frac{L}{R} = \omega\sqrt{\frac{G}{R}}L = \omega\sqrt{\frac{C}{L}}L = \omega\sqrt{LC}$$

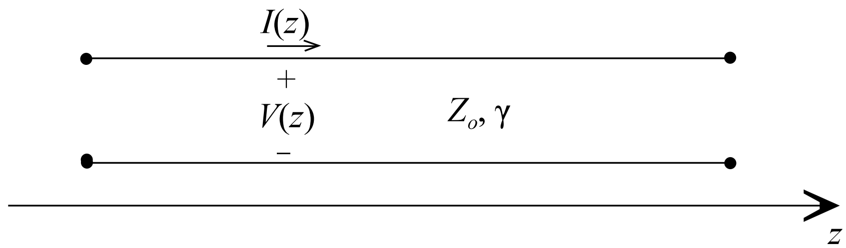
Although the shape of the signal is not distorted, the signal will suffer attenuation as the wave propagates along the line since the distortionless line is a lossy transmission line. Note that the attenuation constant for a distortionless transmission line is also independent of frequency. If this were not true, the signal would suffer distortion due to different frequencies being attenuated by different amounts.

In the previous derivation, we have assumed that the per-unit-length parameters of the transmission line are independent of frequency. This is also an approximation that depends on the spectral content of the propagating signal. For very wideband signals, the attenuation and phase constants will, in general, be functions of frequency.

For most practical transmission lines, we find that $RC > GL$. In order to satisfy the distortionless line requirement, series loading coils are typically placed periodically along the line to increase L.

Perturbation Method for Determining Attenuation

Given only a forward wave propagating along a low loss transmission line, the voltage and current of the wave may be written as



$$V(z) = V_o^+ e^{-\gamma l} = V_o^+ e^{-\alpha z} e^{-j\beta z}$$

$$I(z) = \frac{V_o^+}{Z_o} e^{-\gamma l} = \frac{V_o^+}{Z_o} e^{-\alpha z} e^{-j\beta z}$$

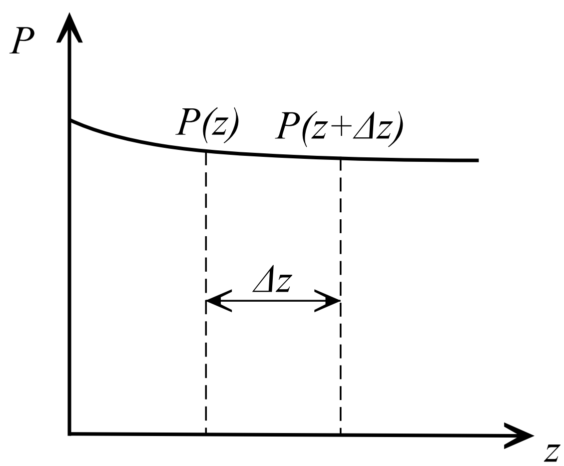
The power flow as a function of position along the transmission line is given by

$$P(z) = \frac{1}{2} Re\left\{V(z)\, I^*(z)\right\}$$

$$= \frac{1}{2} Re\left\{\left(V_o^+ e^{-\alpha z} e^{-j\beta z}\right)\left(\frac{V_o^{+*}}{Z_o} e^{-\alpha z} e^{j\beta z}\right)\right\}$$

$$= \frac{|V_o^+|^2}{2Z_o} e^{-2\alpha z}$$

$$= P_o e^{-2\alpha z}$$



The power loss per unit length along the transmission line [$P_l(z)$] may be written as

$$P_l(z) = \lim_{\Delta z \to 0} \left[\frac{P(z) - P(z + \Delta z)}{\Delta z} \right]$$

$$= -\frac{\partial P(z)}{\partial z}$$

$$= -\left(-2\alpha P_o e^{-2\alpha z} \right)$$

$$= 2\alpha P(z)$$

Solving for the attenuation constant gives

$$\alpha = \frac{P_l(z)}{2P(z)}$$

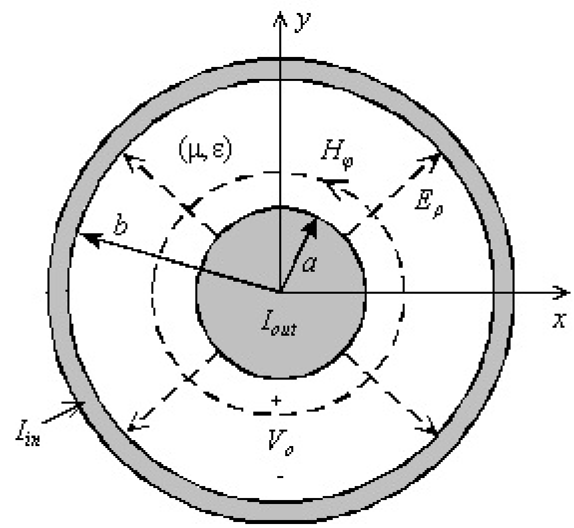
Since the fields of a low loss transmission line are very close to those of a lossless line, we may use the lossless line fields to calculate the power loss per unit length (*perturbation method*). Note that power $P(z)$ and the power loss per unit length $P_l(z)$ may be evaluated at any point on the transmission line. The perturbation method allows for the calculation of the attenuation constant using the transmission line fields rather than using the per-unit-length parameters in the general propagation constant formula.

Example (Perturbation method - coaxial line attenuation constant)

The fields within a lossless coaxial line are

$$\boldsymbol{E} = \frac{V_o}{\rho \ln(b/a)} e^{-j\beta z} \boldsymbol{a}_\rho$$

$$\boldsymbol{H} = \frac{V_o}{2\pi\rho Z_o} e^{-j\beta z} \boldsymbol{a}_\varphi$$



The attenuation constant, according to the perturbation method, is

$$\alpha = \frac{P_l(z)}{2P(z)}$$

The power flow at any point on the transmission line may be found by integrating the Poynting vector over the surface S where the fields are located [S is defined by ($a \le \rho \le b$) and ($0 \le \varphi \le 2\pi$)].

$$P(z) = \frac{1}{2} Re \int_S (\boldsymbol{E} \times \boldsymbol{H}^*) \cdot d\boldsymbol{s}$$

$$= \frac{1}{2} Re \int_0^{2\pi} \int_a^b \left[\left(\frac{V_o}{\rho \ln(b/a)} e^{-j\beta z} \boldsymbol{a}_\rho \right) \times \left(\frac{V_o^*}{2\pi\rho Z_o} e^{j\beta z} \boldsymbol{a}_\varphi \right) \right] \cdot \rho\, d\rho\, d\varphi\, \boldsymbol{a}_z$$

$$= \frac{|V_o|^2}{4\pi \ln(b/a) Z_o} \int_0^{2\pi} \int_a^b \frac{d\rho}{\rho} d\varphi$$

$$= \frac{|V_o|^2}{4\pi \ln(b/a) Z_o} (2\pi) \ln(b/a) = \frac{|V_o|^2}{2Z_o}$$

Assuming that the dielectric and magnetic losses are negligible, the power loss per unit length in the conductors of the coaxial line is given by

$$P_l(z) = \left[\frac{R_{si}}{2} \int_{S_i} |J_{si}|^2 \, ds + \frac{R_{so}}{2} \int_{S_o} |J_{so}|^2 \, ds \right] / \, l$$

where J_{si} and J_{so} are the surface currents on the inner and outer conductors while R_{si} and R_{so} are the surface resistances of the inner and outer conductors. The surface S_i on the inner conductor is defined by $\rho = a$, $0 \leq \varphi \leq 2\pi$, and $0 \leq z \leq l$ while the surface S_o on the outer conductor is defined by $\rho = b$, $0 \leq \varphi \leq 2\pi$, and $0 \leq z \leq l$. Using the surface impedance approx-imation, the surface current on a good conductor is approximately that on a PEC which may be written as

$$\boldsymbol{J_s} = \boldsymbol{n} \times \boldsymbol{H}$$

Thus, we may replace the surface currents in the power loss per unit length equation by the surface magnetic fields associated with a lossless coaxial line. The power loss per unit length is then

$$P_l(z) = \left[\frac{R_{si}}{2} \int_{S_i} |H_{ti}|^2 \, ds + \frac{R_{so}}{2} \int_{S_o} |H_{to}|^2 \, ds \right] / \, l$$

where H_{ti} and H_{to} are the tangential magnetic fields on the surface of the inner and outer conductors. If the inner and outer conductors are the same material, then

$$R_{si} = R_{so} = R_s$$

Evaluation of the integrals in the power loss per unit length expression yields

$$P_l(z) = \frac{R_s}{2l}\left[\int_o^l\int_0^{2\pi}|H_\varphi(\rho=a)|^2\, a\,d\varphi\,dz + \int_o^l\int_0^{2\pi}|H_\varphi(\rho=b)|^2\, b\,d\varphi\,dz\right]$$

$$= \frac{R_s}{2l}\left[\int_o^l\int_0^{2\pi}\frac{|V_o|^2}{4\pi^2 a^2 Z_o^2}\, a\,d\varphi\,dz + \int_o^l\int_0^{2\pi}\frac{|V_o|^2}{4\pi^2 b^2 Z_o^2}\, b\,d\varphi\,dz\right]$$

$$= \frac{R_s}{2l}\left[\frac{|V_o|^2}{4\pi^2 a Z_o^2}(l)(2\pi) + \frac{|V_o|^2}{4\pi^2 b Z_o^2}(l)(2\pi)\right]$$

$$= \frac{R_s|V_o|^2}{4\pi Z_o^2}\left[\frac{1}{a} + \frac{1}{b}\right]$$

The attenuation constant due to conductor loss (α_c) on the coaxial line becomes

$$\alpha_c = \frac{P_l(z)}{2P(z)} = \frac{\dfrac{R_s|V_o|^2}{4\pi Z_o^2}\left[\dfrac{1}{a} + \dfrac{1}{b}\right]}{2\dfrac{|V_o|^2}{2Z_o}} = \frac{R_s}{4\pi Z_o}\left[\frac{1}{a} + \frac{1}{b}\right]$$

The surface resistance (R_s) of the conductors is related to the skin depth (δ_s) by

$$R_s = \frac{1}{\sigma\delta_s} = \sqrt{\frac{\omega\mu}{2\sigma}}$$

For RG-59 coaxial cable ($Z_o = 75\ \Omega$, copper conductors, $\sigma = 5.8 \times 10^7$ ℧/m, $a = 0.292$ mm, $b = 1.854$ mm, $\mu = \mu_o$) at 500 MHz,

$$R_s = 5.83 \times 10^{-3}\ \Omega,\ \alpha_c = 0.0245 \text{ Np/m}$$

Manufacturers normally specify the transmission line attenuation factor in units of dB/m as opposed to Np/m. The conversion factor between the two units is determined below.

With α defined in Np/m (α_{nep}), the transmission line voltage attenuation is

$$V(z) = V_o e^{-\alpha_{nep} z}$$

With α defined in dB/m (α_{dB}), the transmission line voltage attenuation is

$$V(z) = V_o \left[10^{-(\alpha_{dB}/20)z} \right]$$

Equating the two expressions yields

$$e^{-\alpha_{nep} z} = 10^{-(\alpha_{dB}/20)z}$$

$$-\alpha_{nep} z = \ln[10^{-(\alpha_{dB}/20)z}] = -\frac{\alpha_{dB} z}{20} \ln(10)$$

$$\alpha_{nep} = \frac{\ln(10)}{20} \alpha_{dB} = 0.115 \alpha_{dB}$$

$$or \qquad \alpha_{dB} = 8.686 \alpha_{nep}$$

For the RG-59 coaxial line,

$$\alpha(dB/m) = 8.686 \times 0.0245\ Np/m = 0.213\ dB/m$$

Wheeler Incremental Inductance Rule

By noting that the conductor loss in a transmission line can be related to the small change in the transmission line inductance due to the penetration of the fields into the conductors, Wheeler derived an equation for the transmission line conductor loss in terms of the change in the characteristic impedance. The result of the Wheeler incremental inductance rule may be written as

$$\alpha_c = \frac{R_s}{2Z_o\eta}\frac{dZ_o}{dl}$$

where is R_s the surface resistance of the conductor, Z_o is the characteristic impedance of the transmission line assuming perfect conductors, η is the intrinsic impedance of the dielectric between the conductors, and l defines the direction into the conductors. The characteristic impedance of the lossless coaxial transmission line is given by

$$Z_o = \frac{\eta}{2\pi}\ln\left(\frac{b}{a}\right)$$

The derivative term in the attenuation factor expression is

$$\begin{aligned}
\frac{dZ_o}{dl} &= \left[\frac{dZ_o}{dl}\right]_{inner} + \left[\frac{dZ_o}{dl}\right]_{outer} \\
&= \frac{\eta}{2\pi}\left\{-\frac{d}{da}\left(\ln\frac{b}{a}\right) + \frac{d}{db}\left(\ln\frac{b}{a}\right)\right\} \\
&= \frac{\eta}{2\pi}\left\{-\frac{d}{da}(\ln b - \ln a) + \frac{d}{db}(\ln b - \ln a)\right\} \\
&= \frac{\eta}{2\pi}\left[\frac{1}{a} + \frac{1}{b}\right]
\end{aligned}$$

The attenuation factor due to conductor loss in the coaxial transmission line becomes

$$\alpha_c = \frac{R_s}{2Z_o\eta}\frac{dZ_o}{dl} = \frac{R_s}{4\pi Z_o}\left[\frac{1}{a} + \frac{1}{b}\right]$$

which is identical to the result found using the perturbation method.