5

TRIGONOMETRIC AND CIRCULAR FUNCTIONS

IN CHAPTER 2 WE INTRODUCED the concept of functions. Chapters 3 and 4 explored important special functions, namely polynomial, exponential, and logarithmic functions. Equally important in both applications and theoretical mathematics are the trigonometric functions we introduce in this chapter.

Trigonometry (meaning "triangle measurement") has a long and remarkable history. Some of its roots and applications go back to antiquity, but it continues to find new applications through the space age and beyond. Trigonometry has provided tools for surveying and navigation for thousands of years. Today it is built into sophisticated devices that, for example, help satellites navigate among the planets or determine how fast the spreading ocean floor is pushing continents apart.

Partly because it has served so many different uses, trigonometry may appear somewhat schizophrenic in its presentation. Triangle and circle measurement commonly use degree measure, while all modern applications of trigonometry that describe periodic phenomena—from tides to orbiting satellites to the wave nature of quantum physics—require functions of real numbers, not degrees. In Section 5.1 we introduce both modes of angle measure because it is important to become familiar with both. In Section 5.2 the trigonometric functions are also defined in both modes.

5.1 ANGLES AND UNITS OF MEASURE

. . . mathematics, just as all other scientific branches, is developed in the process of examining, verifying, and modifying itself.

Yi Lin

I intended to take either physics or mathematics . . . and intended to become a high school teacher. I found myself very excited by a course called Physical Measurements. We kept measuring things to more and more decimal places by more and more ingenious methods.

Frederick Mosteller

The study of plane geometry considers all geometric figures as sets of points in a plane. An angle, for instance, is the union of two rays with a common endpoint. In trigonometry we talk about angles of a triangle as the union of two line segments that have a common endpoint. More critically, however, the *measure* of an angle involves the notion of rotation. For most purposes, we consider an angle as being generated by rotating a ray in the plane about its endpoint, from an initial position to a final position. The initial position is called the **initial side** and the final position is called the **terminal side** of the angle. The point about which the ray rotates is called the **vertex** of the angle. An angle is the union of two rays together with a rotation.

The measure of an angle is described by the amount of rotation. An angle has **positive measure** if the rotation is counterclockwise, and **negative measure** if the rotation is clockwise. For brevity, we say the angle is positive if its measure is positive. Figure 1 illustrates the labeling of angles and rotation. The curved arrow indicates the direction and amount of rotation. Angles A and B are positive while angle is negative. The rotation in angle B is greater than one revolution.

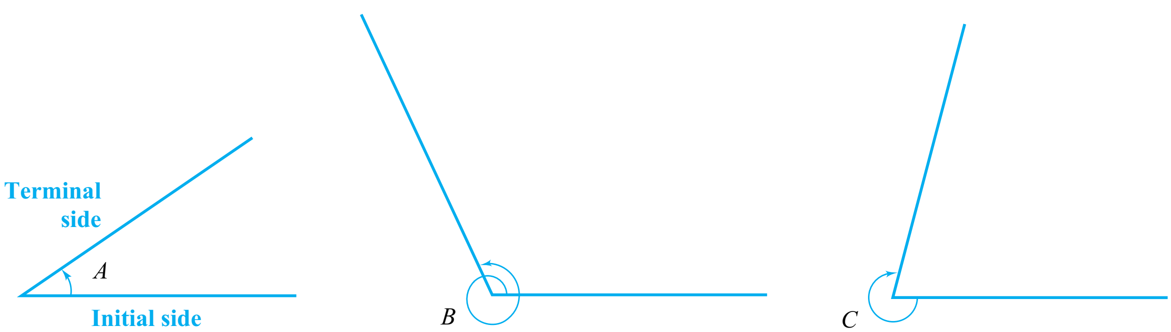


FIGURE 1

Units of Angular Measure

Your calculator operates both in *degree mode* and in *radian mode,* reflecting two different ways of measuring angles. The modes are related by their measures of one complete revolution. The measure of one revolution is 360 degrees, or 2π radians.

Degree measure. In geometry, angles are measured most often in degrees, minutes, and seconds, or decimal fractions of degrees. Degree measure is part of our legacy from Babylonian mathematics, with numeration based on multiples and fractions of 60. Units of time (hours, minutes, seconds) have the same historical basis.

Figure 2 illustrates several angles and their degree measures. For brevity, we write, for example, $A = 90°$ to denote "the measure of angle A is 90°."

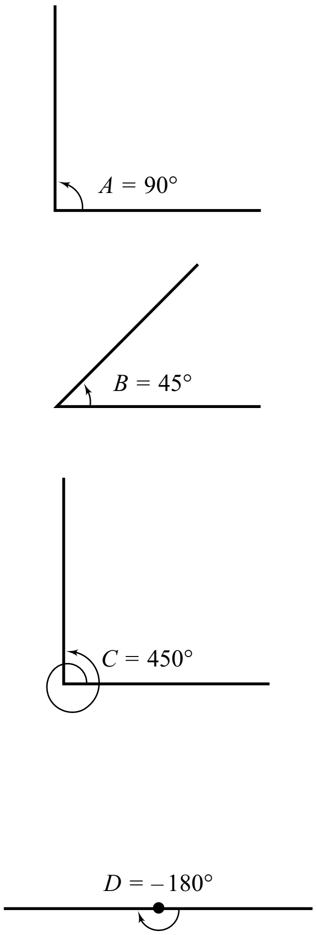


FIGURE 2

Degrees, minutes, and seconds are related by the following:

1 **degree,** written 1°, is $\frac{1}{360}$ of a complete rotation.

1 **minute,** written 1′, is $\frac{1}{60}$ of a degree.

1 **second,** written 1″, is $\frac{1}{60}$ of a minute, or $\frac{1}{3600}$ of a degree.

Your calculator may allow you to enter angles in degree–minute–second (DMS) form (see your instruction manual), but you can also use the relations above to change between DMS and decimal forms, as in the next example.

▶EXAMPLE 1 ***DMS to decimal form*** Express 36°16′23″ in decimal form and round the result to three decimal places.

Solution

16′ is $\frac{16}{60}$ of a degree, and 23″ is $\frac{23}{3600}$ of a degree, so we have

$$36°16'23'' = \left(36 + \frac{16}{60} + \frac{23}{3600}\right)^{\circ} \approx 36.273°. \quad ◀$$

▶EXAMPLE 2 ***Decimal form to DMS*** Express 64.24° in degrees, minutes, and seconds.

Solution

First convert the decimal part, 0.24°, into minutes. Since 1° is 60′,

$$0.24° = (0.24)(60') = 14.4'.$$

Next convert the 0.4′ into seconds.

$$0.4' = (0.4)(60'') = 24''.$$

Therefore, 64.24° is 64°14′24″. ◀

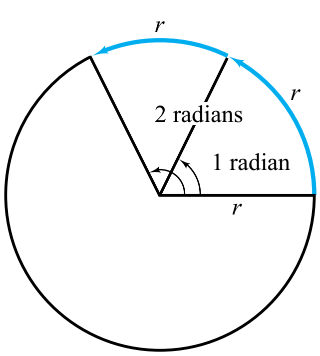


FIGURE 3
An arc length of one radius measures 1 radian

Radian measure. The radian measure of an angle is determined as a ratio of arc length to radius. That is, if we have a segment of length equal to the radius and lay it out along the circle, the central angle is **1 radian.** An arc length of two radii (or a diameter of the circle) measures a central angle of **2 radians.** See Figure 3. Since the total arc length of a circle (its circumference) is $2\pi r$, the *radian measure of one revolution is 2π radians.*

Greek letters such as θ (theta) and ϕ (phi) are often used to refer to angles. For an arbitrary angle θ, take a circle of a radius r with center at the vertex, with the initial side meeting the circle at A and the terminal side at B. Think of a point P moving around the circle from A to B. The directed distance s that P travels is the *directed arc length associated with* θ (see Figure 4.) For counterclockwise rotation, s is positive; for clockwise rotation, s is negative. If the rotation is greater than one revolution, the arc length is greater than $2\pi r$ (positive or negative), so s can be any real number. The radian measure of θ is defined as the *ratio of s to r*. Note that this definition is independent of any particular circle.

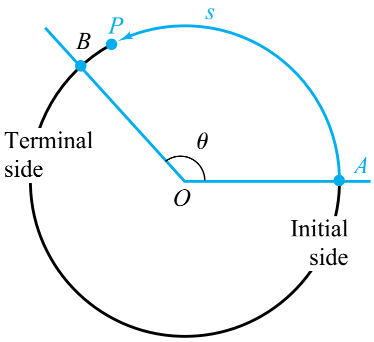


FIGURE 4
Directed arc length s is the distance P travels along the circle from A to B.

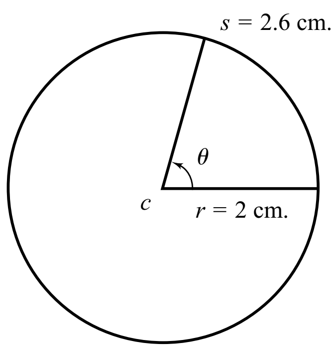


FIGURE 5

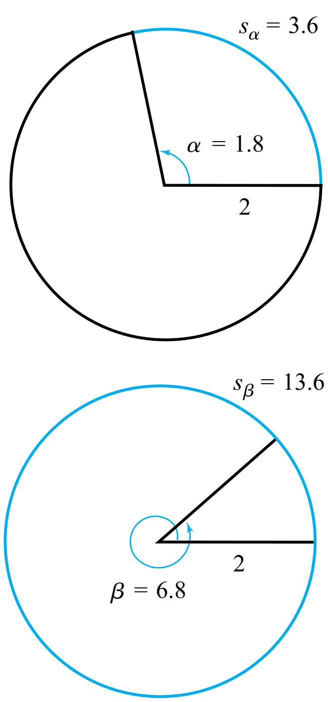


FIGURE 6

Definition: radian measure of an angle

Suppose θ is any angle and C is a circle of radius r with its center at the vertex of θ. If s is the directed arc length associated with θ, then the radian measure of θ is $\frac{s}{r}$; that is,

$$\theta = \frac{s}{r}.$$

In taking the ratio of two lengths, units cancel; thus *radian measure is simply a real number* (no units). If, for example, θ is an angle such that in a circle of radius 2 centimeters, the associated arc length is 2.6 centimeters (see Figure 5). Then the measure of θ in radians is given by

$$\theta = \frac{s}{r} = \frac{2.6 \text{ cm}}{2 \text{ cm}} = 1.3.$$

The units cancel, and we write simply $\theta = 1.3$. We could emphasize that the radian measure of θ is 1.3 by writing $\theta = 1.3$ radians, but our normal convention is that radians need not be written. *When the measure of an angle is given as a real number, it is understood that the measure is radians.*

▶EXAMPLE 3 ***Radian measure of central angles*** In Figure 6, α and β are central angles of a circle of radius 2. The lengths of the subtended arcs are $s_\alpha = 3.6$ for α and $s_\beta = 13.6$ for β. Determine the measures of α and β in radians.

Solution

$$\alpha = \frac{s_\alpha}{r} = \frac{3.6}{2} = 1.8$$

The measure of α is 1.8 radians.

$$\beta = \frac{s_\beta}{r} = \frac{13.6}{2} = 6.8$$

The measure of β is 6.8 radians. The measure of β is greater than one revolution ($2\pi \approx 6.28$), as the arrow in the figure indicates. ◀

Degree–Radian Relationships

In many cases we may have the measure of an angle in degrees when we need the radian measure, or vice versa. This requires a technique for conversion. Since one complete rotation is measured by either 360° or 2π radians, we have the necessary equivalence. The basic relationship $360° = 2\pi$ radians connects degree and radian measures.

Degree–radian conversions

$$180° = \pi \text{ radians.} \qquad \textbf{(1)}$$

$$1° = \frac{\pi}{180} \text{ radians} \quad \text{or} \quad 1° \approx 0.017453 \text{ radians},$$

$$1 \text{ radian} = \left(\frac{180}{\pi}\right)^\circ \quad \text{or} \quad 1 \text{ radian} \approx 57.296°.$$

See Figure 7 for equivalent measures of selected angles.

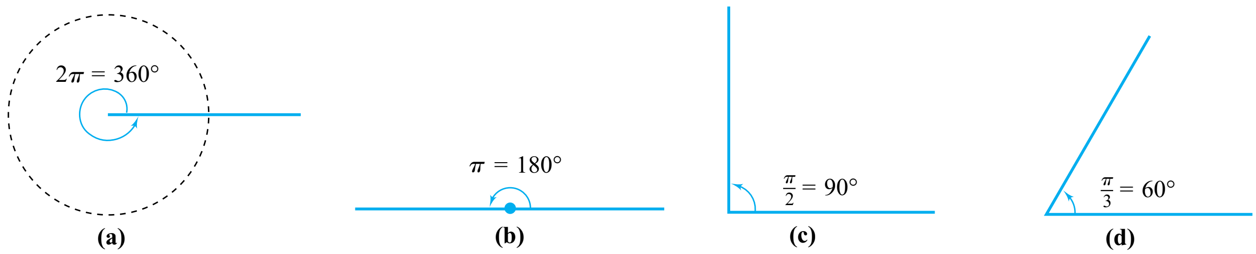


FIGURE 7
Degree–radian measure for some familiar angles.

Most of us are familiar with degree measure, and because radian measure is so important in calculus, we need to learn some convenient equivalences. There are a few angles that are used so often, we need to recognize them immediately in both degree and radian measure:

$$\pi = 180°, \frac{\pi}{2} = 90°, \frac{\pi}{3} = 60°, \frac{\pi}{4} = 45°, \frac{\pi}{6} = 30°.$$

For most of us, a visual reminder of these relations is helpful, as in Figure 8.

In addition to thinking in terms of fractions of π radians, you need to develop a feeling for radian measure expressed simply as numbers. For example, 1 radian is about 57.3°, almost 60°. Similarly, an angle of 3 radians is very nearly a straight angle (remember that $3.14 \approx \pi = 180°$). See Figure 9. Right angles are so common that the decimal approximation of $\pi/2$ as 1.57 will become very familiar.

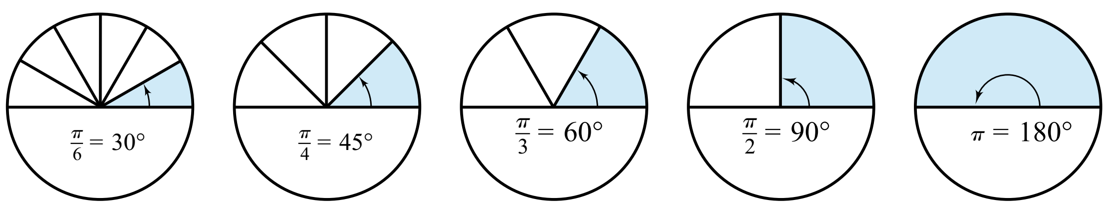


FIGURE 8
Degree and radian measure for some familiar angles.

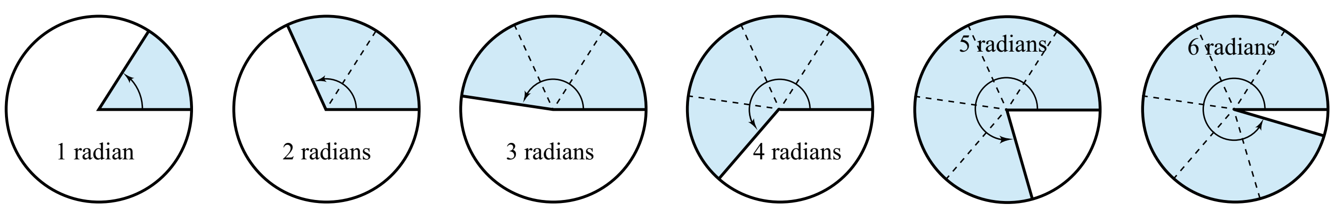


FIGURE 9
Integer multiples of 1 radian.

▶EXAMPLE 4 ***Degree–radian conversion*** Draw a diagram that shows the angle and then find the corresponding radian measure. Give the result both in exact form and as a decimal approximation rounded off to two decimal places.

(a) $\alpha = 210°$ **(b)** $\beta = 585°$ **(c)** $\gamma = -150°$

Solution

Diagrams in Figure 10 show the angles. To convert from degree measure to radian measure, multiply by $\frac{\pi}{180}$.

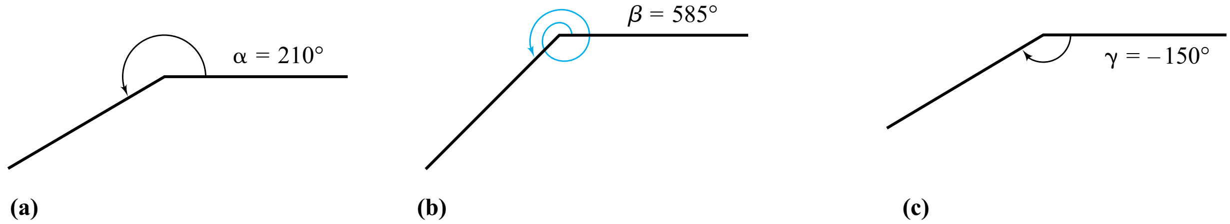


FIGURE 10

(a) $\alpha = 210° = 210\left(\frac{\pi}{180}\right) = \frac{7\pi}{6} \approx 3.67$

(b) $\beta = 585° = 585\left(\frac{\pi}{180}\right) = \left(\frac{13\pi}{4}\right) \approx 10.21$

(c) $\gamma = -150° = -150\left(\frac{\pi}{180}\right) = -\frac{5\pi}{6} \approx -2.62$ ◀

▶EXAMPLE 5 ***Radian–degree conversion*** If the radian measure of θ is 2.47, find its degree measure rounded off to one decimal place and then to the nearest minute.

Strategy: To get the degree measure, multiply by $\frac{180}{\pi}$. Convert the decimal part of a degree to minutes by multiplying by 60.

Solution

Follow the strategy.

$$2.47 \text{ radians} = 2.47\left(\frac{180}{\pi}\right)^{\circ} \approx 141.5205754°.$$

Rounded off to one decimal place, 2.47 radians $\approx 141.5°$; to the nearest minute, 2.47 radians $\approx 141°31'$. Hence, θ is approximately $141°31'$. ◀

Applications of Radian Measure: Arc Length and Area

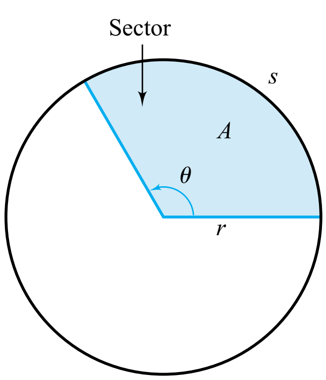


FIGURE 11

In a circle a given central angle between 0 and 2π determines a portion of the circle called a **sector,** as indicated by the shaded region shown in Figure 11. For the sector shown in the figure with central angle θ, suppose the length of the subtended arc is s and the area of the sector is A. The ratios of θ, s, and A to the respective measures 2π, $2\pi r$, and πr^2 for the entire circle are equal, that is,

$$\frac{\theta}{2\pi} = \frac{s}{2\pi r}, \qquad \frac{\theta}{2\pi} = \frac{A}{\pi r^2}, \qquad \frac{s}{2\pi r} = \frac{A}{\pi r^2}.$$

Solving for s and A ,

$$s = r\theta, \qquad A = \frac{1}{2}r^2\theta, \qquad A = \frac{1}{2}rs.$$

Arc length and the area of a circular sector

Suppose θ is a central angle of a circle of radius r. Let s denote the length of the subtended arc and let A denote the area of the sector. If θ is measured in radians, then s and A are given by

$$s = r\theta \qquad \textbf{(2)}$$

$$A = \frac{1}{2}r^2\theta \qquad A = \frac{1}{2}rs \qquad \textbf{(3)}$$

▶EXAMPLE 6 ***Arc length/area of circular sector*** The radius of a circle is 3.6 centimeters and the central angle of a circular sector is 150°. Draw a diagram to show the sector and find the arc length and the area of the sector.

Strategy: Equations (2) and (3) require θ to be in radians. First convert 150° to radians and then use Equations (2) and (3).

Solution

The sector is the shaded region in Figure 12. Following the strategy,

$$\theta = 150° = 150\left(\frac{\pi}{180}\right) = \frac{5\pi}{6}.$$

Substitute 3.6 for r and $\frac{5\pi}{6}$ for θ in Equations (2) and (3)

$$s = 3.6\left(\frac{5\pi}{6}\right) = 3\pi \approx 9.42,$$

$$A = \frac{1}{2}(3.6)^2\left(\frac{5\pi}{6}\right) = 5.4\pi \approx 16.96.$$

Round off to two significant digits to get an arc length of 9.4 cm and an area of 17 cm^2. ◀

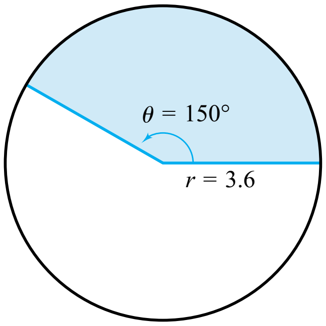


FIGURE 12

▶EXAMPLE 7 ***Area as a function*** Given a circular sector with central angle of 30°. **(a)** Give a formula for the area A of the sector as a function of the radius r. Draw a graph of $A(r)$ in the window $[0, 20] \times [0, 50]$. **(b)** Evaluate A when $r = 12.7$ (one decimal place). **(c)** Find r when $A = 25.6$ (one decimal place).

Solution

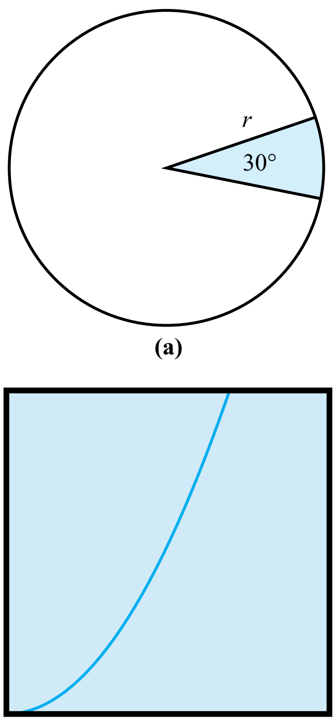
(a) Since the formula for area (Equation (3)) requires radian measure for the central angle, we use $30° = \pi/6$, and the area (see Figure 13a) is given by

$$A(r) = \frac{\pi}{12}r^2$$

Graphing `Y = πX²/12` in the specified window gives us a calculator graph as shown in Figure 13b, clearly part of a parabola.

(b) Tracing on the calculator graph does not allow us to read with sufficient accuracy the value of A when $r = 12.7$. We can zoom in for more accuracy, or we can return to the home screen and evaluate `π(12.7)²/12`. Either way, we get an area of approximately 42.2.

(c) If $\pi r^2/12 = 25.6$, then $r = \sqrt{(12)(25.6)/\pi} \approx 9.9$. ◀



(a)

[0, 20] by [0, 50]

(b)

$A = \frac{\pi x^2}{12}$

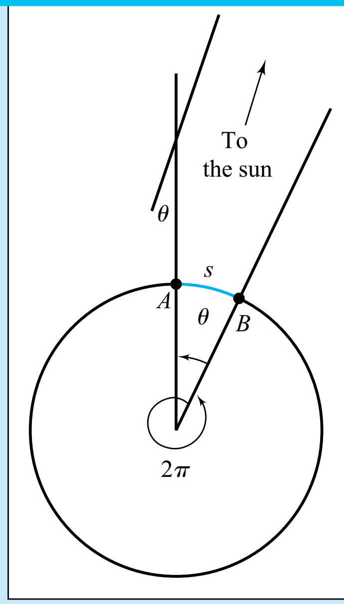
FIGURE 13

▶EXAMPLE 8 ***Circular motion*** Assume that the moon travels around the earth in a circular path of radius 239,000 miles and that it makes one complete revolution every 28 days.

(a) Find a formula for the distance $D(x)$ (in thousands of miles) that the moon travels in x days. Draw a calculator graph. How far does the moon travel **(b)** in 10 days? in 21 days and 6 hours? **(c)** How many days does it take for the moon to travel a million miles? A billion miles?

HISTORICAL NOTE MEASUREMENT OF THE CIRCUMFERENCE OF THE EARTH

One of the earliest and most dramatic applications of trigonometry was made by Eratosthenes a little before 200 B.C. As his name suggests, Eratosthenes was of Greek descent, but he spent his life in Egypt during the reign of Ptolemy II and later became the head of the greatest scientific library in the ancient world at Alexandria.



Travelers reported that at Syene (the modern city Aswan), the sun cast no shadow at noon on the summer solstice (the longest day of the year). Eratosthenes reasoned, then, that at Syene on that date, the sun's rays were coming directly toward the center of the Earth. Alexandria was supposed to be directly north of Syene. By measuring the angle of the sun's rays at Alexandria at noon on the same day, Eratosthenes realized that he could use geometric relationships to find the circumference of the Earth.

In the diagram, A represents Alexandria and B, Syene. The ratio of the distance from A to B, the arc length s, to the entire circumference C must equal the ratio of angle θ to the entire central angle (a complete revolution).

Symbolically,

$$\frac{s}{C} = \frac{\theta}{\text{One revolution}}$$

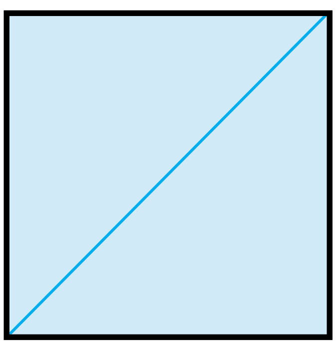
By the best estimates of the day, the distance from Alexandria to Syene was 5000 stadia, a distance estimated from travel by camel caravans by surveyors trained to count paces of constant length. Eratosthenes measured angle θ at Alexandria to be $\frac{1}{50}$ of a complete revolution, leading to the equation,

$$\frac{5000}{C} = \frac{1}{50},$$

from which he calculated the circum-circumference of the earth to be

$$C = 50 \cdot 5000 = 250{,}000 \text{ stadia}.$$

Comparison with modern measurements is difficult because the measuring unit, the stadium, varied in size, but Eratosthenes' estimate would compare to something near 24,000 miles. Several compensating errors probably contributed to the truly remarkable accuracy of his figure, but the real genius Eratosthenes lay in his analysis of the problem and his recognition that geometric figures can tell us something about the nature of the world that we can learn in no other way.



[0, 30] by [0, 1600]

$D(x) = \frac{239\pi x}{14}$

FIGURE 14

Solution

(a) The distance traveled in one revolution is $2\pi r$, or $478{,}000\pi$ miles ($\approx$ 1.50 million), every 28 days. One day's distance is $2\pi r/28$, where the radius is 239 thousand miles, from which the number of thousands of miles is given by

$$D(x) = \frac{239\pi}{14}x.$$

A calculator graph in $[0, 30] \times [0, 1600]$ is shown in Figure 14.

(b) Either from the graph or from the formula, $D(10) \approx 536{,}000$ miles. Six hours is a quarter of a day; in 21 days, 6 hours, $D(21.25) \approx 1{,}140{,}000$ miles.

(c) A million miles is a thousand thousand ($10^6 = 10^3 10^3$), so we want to solve for x when $D(x) = 1000$: $x = 14 \cdot 1000/(239\pi) \approx 18.6$ days, less than three weeks. A billion miles is $10^9/10^3 = 10^6$ thousands. When $D(x) = 10^6$, $x = 14{,}000{,}000/(239\pi) \approx 18{,}600$ days. Assuming 365 days in a year, it will take more than 51 *years* for the moon to travel a billion miles in circling the earth. ◀

►EXAMPLE 9 ***Making a paper cup*** Draw a circle of radius 4 on a piece of paper. Cut from a point A on the circle to the center O. You can make a conical cup by sliding OA to any other radius OB, effectively cutting out a sector with central angle $x = \angle AOB$. See Figure 15.

(a) Using the circumference of the cone given by $C(x) = 4(2\pi - x)$, as in the diagram, find a formula for the radius $r(x)$ at the top of the cone and the height $h(x)$.
(b) Express the volume $V(x)$ as a function of x and draw a calculator graph.
(c) Find the approximate value of x giving the maximum volume. What is the maximum volume?

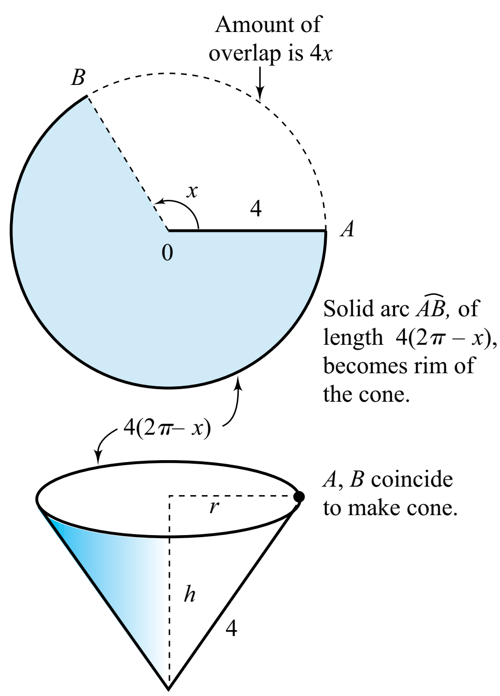


FIGURE 15

Solution

(a) For the top of the cup, $C = 2\pi r$, so

$$r(x) = C(x)/2\pi = \frac{4(2\pi - x)}{2\pi} = \frac{2}{\pi}(2\pi - x).$$

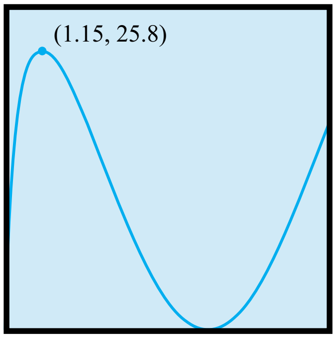
Looking at a cross-section of the cup, r and h are legs of a right triangle with hypotenuse 4 (the radius of the original circle), so $r^2 + h^2 = 4^2$.

$$h^2 = 16 - \frac{4}{\pi^2}(2\pi - x)^2 = 16 - \frac{4}{\pi^2}(4\pi^2 - 4\pi x + x^2),$$

or

$$h^2 = \frac{4(4\pi x - x^2)}{\pi^2}.$$

Thus, $h(x) = \dfrac{2}{\pi}\sqrt{4\pi x - x^2}$.



[0, 10] by [0, 30]

FIGURE 16

(b) The volume of a cone is a third of the volume of the cylinder with the same base, or $V = \frac{1}{3}\pi r^2 h$, so we have

$$V(x) = \frac{\pi}{3}(r(x))^2 h(x) = \frac{\pi}{3}\frac{4}{\pi^2}(2\pi - x)^2 \frac{2}{\pi}\sqrt{4\pi x - x^2}.$$

Thus,

$$V(x) = \frac{8}{3\pi^2}(2\pi - x)^2\sqrt{4\pi x - x^2},\ 0 < x < 2\pi.$$

(c) Graphing V in $[0, 10] \times [0, 30]$ gives the graph shown in Figure 16. Tracing and zooming to find the high point, we find that the maximum volume is about 25.8 cubic inches when x is about 1.15 radians, or about 66°. A volume of 25.8 cubic inches is just over 14 ounces. We suggest that you make a cone with $x \approx 66°$ and observe that your cone is not standard. Since that is the shape with the maximum volume and the least waste (overlap), why do you suppose paper cups are not made to hold the maximum volume? ◀

Linear and Angular Speed

There are two kinds of speeds associated with rotational motion. To introduce the basic ideas, consider an example. Suppose a bicycle wheel is rotating at a constant rate of 40 revolutions per minute (40 rpm). One measure of speed, **angular speed,** gives the rate of rotation, frequently denoted by the Greek letter ω. The bicycle wheel's angular speed is 40 rpm by one measure. Since one revolution is equivalent to 2π radians, the angular speed can also be expressed as $40(2\pi)$ or 80π radians per minute, or 4800π radians per hour.

Suppose the diameter of the bicycle wheel is 26.4 inches, so that its radius r is 13.2 inches or 1.1 feet. In 1 minute a point on the circumference turns through an angle θ equal to 80π radians, and the distance s traveled by point on the circumference in 1 minute is

$$s = r\theta = (1.1)(80\pi) = 88\pi.$$

Hence, the point travels 88π feet in 1 minute. The **linear speed** of a point on the circumference is $88\pi \frac{\text{ft}}{\text{min}} \approx 276$ ft/min.

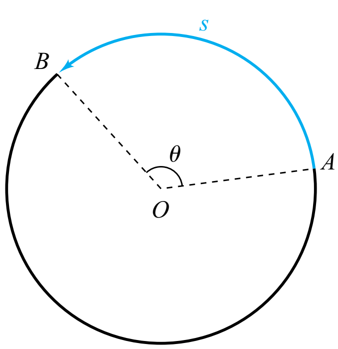


FIGURE 17

Relationship between linear and angular speeds. For a particle moving in a circular path at a uniform rate, the linear and angular speeds are obviously related. Suppose that such a particle P moves from point A to point B along a circle of radius r, as indicated in Figure 17. If the central angle AOB is θ, the distance (arc length) from A to B is s, and P moves from A to B in time t, then the linear and angular speeds are

$$v = \frac{s}{t} \quad \text{and} \quad \omega = \frac{\theta}{t}.$$

Since $s = r\theta$ (where θ is measured in radians),

$$v = \frac{s}{t} = \frac{r\theta}{t} = r\left(\frac{\theta}{t}\right) = r\omega$$

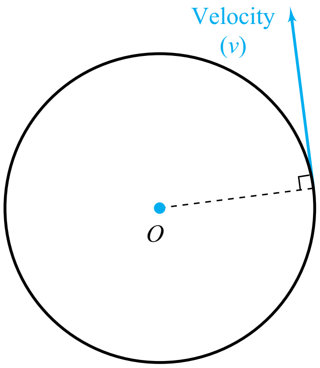


FIGURE 18

Linear and angular speed relationship

For a particle moving in a circular path of radius r at a uniform rate, the linear speed v and angular speed ω are related by the equation

$$v = r\omega. \qquad \textbf{(4)}$$

Linear speed and angular speed are sometimes called linear velocity and angular velocity, but we reserve the term *velocity* for directed speeds. Velocity is a vector quantity, meaning that it has both a magnitude and a direction (see Section 7.5). In uniform circular motion, the magnitude of the linear velocity vector is the linear speed defined above, in a direction tangent to the circular path of motion, as indicated by the arrow in Figure 18.

Strategy: First convert r and ω to units that are consistent with feet per second.

▶EXAMPLE 10 ***Linear and angular speed*** The wheel of a grindstone with a radius of 8 inches is rotating at 300 rpm.

(a) Find the angular speed in radians per second.
(b) Find the linear speed of a point on the circumference of the wheel in feet per second.
(c) For a certain job, it is desirable to have the linear speed of the grinding edge of the wheel at 30 ft/sec. What change in angular speed (in revolutions per minute) is required?

Solution

Given that r is 8 inches and ω is 300 rpm, follow the strategy and express r and ω in units of feet and seconds.

$$r = 8 \text{ in} \times \left(\frac{1 \text{ ft}}{12 \text{ in}}\right) = \frac{2}{3} \text{ ft}$$

and

$$\omega = \frac{300 \text{ rev}}{\text{min}} \times \left(\frac{2\pi \text{ rad}}{\text{rev}}\right) \times \left(\frac{1 \text{ min}}{60 \text{ sec}}\right) = 10\pi \text{ rad/sec}$$

(a) The angular speed is $10\pi \frac{\text{rad}}{\text{sec}}$.
(b) Using Equation (4),

$$v = r\omega = \left(\frac{2}{3}\right)(10\pi) = \frac{20\pi}{3} \approx 20.9.$$

Hence, the linear speed of a point on the grinding edge of the wheel is 20.9 ft/sec.

(c) For v to be $30 \frac{\text{ft}}{\text{sec}}$, specify $r\omega = 30$, or

$$\omega = \frac{30}{r} = \frac{30}{\frac{2}{3}} = 45 \text{ rad/sec}.$$

To express ω in revolutions per minute, convert radians per second.

$$\omega = 45\frac{\text{rad}}{\text{sec}} \times \left(\frac{1 \text{ rev}}{2\pi \text{ rad}}\right) \times \left(\frac{60 \text{ sec}}{\text{min}}\right) = \frac{1350 \text{ rev}}{\pi \text{ min}} \approx 430 \text{ rpm}$$

For a linear speed of 30 ft/sec, the wheel speed must increase to about 430 rpm, almost half again as fast as the present angular speed. ◀

Does this answer make sense? Always ask yourself if a solution is reasonable and make an independent check or a simple estimate, if possible. In Example 10, for instance, we found in part **(b)** that when ω is 300 rpm, the speed is near 20 ft/sec. Therefore, in part **(c)** an angular speed of 30 ft/sec should correspond to about $\frac{3}{2}$ of 300 rpm, giving an estimate of about 450 rpm. The result of 430 rpm in part **(c)** is entirely reasonable.

EXERCISES 5.1

Check Your Understanding

Exercises 1–5 True or False. Give reasons.

1. An angle of $\frac{22}{7}$ radians is equal to an angle of 180°.

2. An angle of 180° is greater than an angle of 3.16 radians.

3. The sector of a circle with a central angle of 1 radian and radius r cm has an area of $\frac{1}{2}r^2$ cm².

4. If the central angle θ of a circle measures 1 radian, then the length of the arc subtended by θ is equal to the length of the radius.

5. Assume that the planets travel in circular orbits about the sun. If the planet Venus takes 225 days for one revolution about the sun and the Earth takes 365 days, then the angular speed of Venus is less than the angular speed of the Earth.

Exercises 6–10 Fill in the blank so that the resulting statement is true.

6. An angle of $\frac{5\pi}{4}$ is equal to an angle of ______ degrees.

7. An angle of 210° is equal to an angle of ______ radians.

8. In a circular sector, if $s = 48$ and $r = 24$, then $\theta =$ ______ radians.

9. In a circular sector, if $s = 12$ and $\theta = 60°$, then $r =$ ______ .

10. An angular speed of 5 rpm is equal to ______ $\frac{\text{rad}}{\text{min}}$.

Develop Mastery

Unless otherwise specified, results given as decimal approximations should be rounded off to the number of significant digits consistent with the given data.

Exercises 1–4 Draw a diagram to show the angle. Include a curved arrow to indicate the amount and direction of rotation from the initial side to the terminal side.

1. **(a)** $A = 240°$ **(b)** $B = 720°$ **(c)** $C = -210°$

2. **(a)** $A = 540°$ **(b)** $B = -135°$ **(c)** $C = 67°30'$

3. **(a)** $A = \frac{2\pi}{3}$ **(b)** $B = -\frac{7\pi}{4}$ **(c)** $C = 1.8$

4. **(a)** $A = \frac{5\pi}{3}$ **(b)** $B = -3\pi$ **(c)** $C = -2.36$

Exercises 5–6 Sketch an angle θ that satisfies the inequality. Include a curved arrow and also illustrate the range of position for the terminal side with dashed rays.

5. **(a)** $\frac{\pi}{2} < \theta < \pi$ **(b)** $-\pi < \theta < -\frac{\pi}{2}$
(c) $1.7 < \theta < 2.5$

6. **(a)** $\frac{3\pi}{4} < \theta < \pi$ **(b)** $-\frac{5\pi}{5} < \theta < -\pi$
(c) $0.79 < \theta < 1.05$

Exercises 7–8 ***DMS to Decimal*** *Express the angle as a decimal number of degrees rounded off to three decimal places.*

7. **(a)** 23°38′ **(b)** 143°16′23″ **(c)** −95°31′

8. **(a)** 57°34′ **(b)** 241°15′51″ **(c)** −73°43′

Exercises 9–10 Find the radian measure of the angle and give the result in both exact form (involving the number π) and decimal form rounded off to two decimal places.

9. **(a)** 60° **(b)** 330° **(c)** 22°30′ **(d)** 105°

10. **(a)** 90° **(b)** 450° **(c)** 67°30′ **(d)** −165°

Exercises 11–12 ***Radians to Degrees*** *Express the angle in decimal degree form, rounded off, if necessary, to one decimal place.*

11. **(a)** $\frac{2\pi}{3}$ **(b)** $\frac{5\pi}{12}$ **(c)** 4π **(d)** 3.6

12. **(a)** $\frac{7\pi}{4}$ **(b)** $\frac{11\pi}{12}$ **(c)** -5π **(d)** 5.4

Exercises 13–16 ***Ordering Angles*** *Order angles α, β, and γ from smallest to largest (as, for example, $\alpha < \gamma < \beta$).*

13. $\alpha = 47°24'$, $\beta = 47.48°$, $\gamma = 0.824$

14. $\alpha = 154°35'$, $\beta = 154.32°$, $\gamma = 2.705$

15. $\alpha = \frac{22}{7}$, $\beta = \frac{355}{113}$, $\gamma = \pi$

16. $\alpha = 120°36'$, $\beta = 120.53°$, $\gamma = \frac{21}{10}$

Exercises 17–20 ***Triangle Angles*** *Two of the three angles A, B, and C of a triangle are given. Find the third angle. Remember that the sum of the three angles of any triangle is equal to 180° (or π).*

17. $A = 58°$, $B = 73°$

18. $B = 37°41'$, $C = 84°37'$

19. $A = \frac{\pi}{4}$, $C = \frac{5\pi}{12}$

20. $A = \frac{2\pi}{3}$, $B = \frac{\pi}{15}$

Exercises 21–26 ***Arc Length, Area*** *The radius r and the central angle θ of a circular sector are given. Draw a diagram that shows the sector and determine **(a)** the arc length s and **(b)** the area A for the sector.*

21. $r = 24$, $\theta = 30°$

22. $r = 32.1$, $\theta = 96.3°$

23. $r = 164$, $\theta = 256°$

24. $r = 47$, $\theta = \frac{3\pi}{5}$

25. $r = 36$, $\theta = 4.3$

26. $r = 16.2$, $\theta = \frac{7\pi}{8}$

27. The radius of a circular sector is 12.5 centimeters and its area is 182 square centimeters. Find the central angle **(a)** in radians and **(b)** in degrees.

28. What is the radius of a circular sector with central angle 37.5° and area 6.80 square feet?

29. Assume that the Earth travels a circular orbit of radius 93 million miles about the sun and that it takes 365 days to complete an orbit.
(a) Through what angle (in radians) will the radial line from the sun to the Earth sweep in 73 days?
(b) How far does the Earth travel in its orbit about the sun in 73 days?

30. The diameter of a bicycle wheel is 26 inches. Through what angle does a spoke of the wheel rotate when the bicycle moves forward 24 feet? Give your result in radians to two significant digits.

31. What is the measure in degrees of the smaller angle between the hour and minute hands of a clock **(a)** At 2:30? **(b)** At 2:45?

32. At what times to the nearest tenth of a minute between 1:00 and 2:00 is the smaller angle between the hour and minute hands 15°?

33. The minute hand of a clock is 6 inches long.
(a) How far does the tip of the hand travel in 15 minutes?
(b) How far does the tip of the hand travel between 8:00 A.M. and 4:15 P.M. of the same day?

34. **(a)** What is the linear speed (in inches per hour) of the tip of the minute hand in Exercise 33?
(b) What is the linear speed of a point 1 inch from the tip of the minute hand?

35. What is the angular speed in radians per minute of
(a) the hour hand of a clock?
(b) the minute hand?

36. ***Nautical Mile*** A nautical mile is the length of an arc of a great circle subtended on the surface of the Earth by an angle of one minute (1′) at the center of the Earth. Assuming that the Earth is a sphere of radius 3960 miles, a nautical mile is equal to how many ordinary miles (5280 ft)?

37. ***Speed in Knots*** A ship is traveling along a great circle route at a speed of 20 knots.
(a) How fast is it moving in miles per hour?
(b) How far does it travel in 4 hours in nautical miles? In ordinary miles?
(c) Through what angle does a line from the center of the Earth to the ship revolve in 4 hours? (*Hint:* If you are not familiar with the word "knots," look it up in the dictionary.)

38. A circular sector with central angle 90° is cut out of a circular piece of tin of radius 15 inches. The remaining piece is formed into a cone (see Example 9). Find the volume of the cone.

39. Repeat Exercise 38 reducing the central angle of the sector cut out of the piece of tin to 60°.

40. A circular piece of tin of radius 12 inches is cut into three equal sectors, each of which is then formed into a cone.
(a) What is the height of each cone?
(b) What is the volume of each cone?

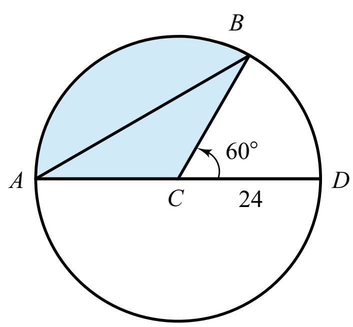
41. In Example 9 find the value of x (2 decimal places) for which $V = 20.7$ cubic inches.

42. ***Looking Ahead to Calculus*** Using calculus, we can show that the x-value in Example 9 giving a maximum volume is a root of the equation

$$3x^2 - 12\pi x + 4\pi^2 = 0.$$

Solve for x. Does this agree with the value of x found in Example 9?

43. In the diagram C is the center and AD is a diameter of the circle with radius 24 cm. $\angle BCD$ measures 60°. Find in exact form the area of **(a)** $\triangle ABC$, **(b)** circular sector BCD, **(c)** the shaded region.



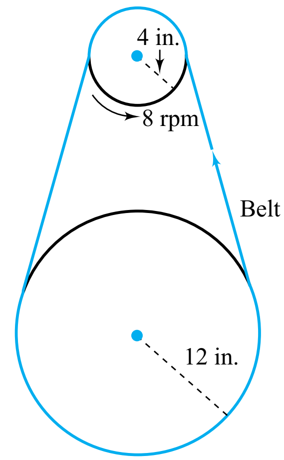
44. A satellite travels in a circular orbit 140 miles above the surface of the Earth. It makes one complete revolution every 150 minutes.
(a) What is its angular speed in revolutions per hour and in radians per hour?
(b) What is its linear speed? Assume that the radius of the Earth is 3960 miles.

45. The face of a windmill is 4.0 meters in diameter and a wind is causing it to rotate at 30 rev/min. What is the linear speed of the tip of one of the blades (in meters per minute)?

46. Assume that the moon follows a circular orbit about the Earth with a radius of 239,000 miles and that one revolution takes 27.3 days. Find the linear speed (in miles per hour) of the moon in its orbit about the Earth.

47. Assume that the Earth travels about the sun in a circular orbit with a radius of 93 million miles and that one revolution takes 365 days. Find the linear speed (in miles per hour) of the Earth in its orbit about the sun.

48. Two pulleys, one with a radius of 4 inches and the other with a radius of 12 inches, are connected by a belt (see the diagram). If the smaller pulley is being driven by a motor at 8 rev/min,
(a) determine the angular speed of the larger pulley (in revolutions per minute).
(b) What is the linear speed of a point on the belt?



49. The diameter of a bicycle wheel is 26 inches. When the bicycle moves at a speed of 30 mph, determine the angular speed of the wheel in revolutions per minute.

50. To measure the approximate speed of the current of a river, a circular paddle wheel with a radius of 3 feet is lowered into the water just far enough to cause it to rotate. If the wheel rotates at a speed of 12 rev/min, what is the speed of the current in miles per hour?

51. The blade of a rotary lawnmower is 34 cm long and rotates at 31 rad/sec.
(a) What is the blade's angular speed in revolutions per minute?
(b) What is the linear speed (in kilometers per hour) of the tip of the blade?

52. A record was set in rope turning with 49,299 turns in 5 hours and 33 minutes.
(a) What is the average angular speed of the rope in revolutions per minute?
(b) Assuming that the rope forms an arc so that its midpoint travels in a circular path of radius 3.5 feet, what is the average linear speed (in feet per minute) of the midpoint of the rope?
(c) How far (in miles) did the midpoint travel during the record-setting turning session?