Evaluation of Trigonometric Functions and Their Inverses

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Abstract

This paper shows that finding trig functions without a calculator is entirely possible especially when high accuracy is not needed. For educational reasons we focus here on the case where the angle is in degrees rather than radians. Through the use of a simple diagram and a special number it is shown that we can estimate the cosine of an angle with ease and that this process is reversible. Any required accuracy can be obtained for the cosine and other functions can be found from this cosine value. This means that teaching evaluation of these functions in the classroom is entirely possible.

1. INTRODUCTION

1.1 Why Teach Evaluation of Trig Functions?

Trig functions are apparently particularly difficult to calculate, and even many teachers might be hard pressed to say how they may be evaluated. Consequently no-one ever calculates trig functions and their inverses as it is considered too hard – we resort to a calculator, tables or other method. Why should we bother to calculate trig functions when they are so easily available from a calculator?

The same question can be asked about square roots, division, multiplication . . . So where would we stop and say these should be taught with pencil and paper and these not?

The reason for teaching the calculation of products and quotients is so that pupils get to understand better what they mean, and so that in the absence of a calculator, or for quick on-the-spot calculations or estimates, we can get results.

But these same reasons could apply to the evaluation of trig functions and their inverses, which are also of great practical use.

We may also note that in many developing countries, calculators are not available to children.

1.2 Teaching Approach

There are several units in which angles can be measured, the most common being degrees and radians. Degrees are ideal when teaching the idea of ‘angle’ and later on, for more advanced work, radians are found to be more useful.

Although trig functions and their inverses can be more easily evaluated in radian measure however, anyone working in radians is very likely to have a calculator handy and to seek one out for the purpose if it is not.

Therefore evaluating these functions in degree measure is what we will be looking at here as it has educational value.

However since we would not want to burden pupils with unnecessary and tedious calculations our aim is to teach the method using easy examples: whole numbers of degrees, multiples of 5 or small angles (e.g. 1 to 10 degrees).

This way the *method* is assimilated so that more complex calculations *could* be carried out, though we may not require it of them.

We have two stages. The first will give more approximate results but will be very easy. The second will show how the first stage may be extended to a general method.

2. STAGE 1

2.1 Calculation of Cosine

Given a circle of radius 1 and a right-angled triangle with angle *A* degrees, the cosine of *A* is the length of the base of that triangle, and the sine is its height, as shown in Figure 1.

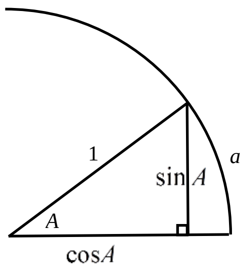


Figure 1

Let the length of the arc subtended by the angle *A* be *a*.

Referring to Figure 2 we draw the tangent at *P* to meet *OR* extended at *T*, and we find that angle *QPT* = *A*.

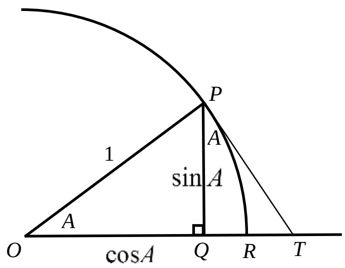


Figure 2

We see that for small angles $\sin A \approx a$, $\tan A \approx a$ and $\cos A \approx$ 1.

Since ΔOQP has sides $\cos A$, sin *A*, 1, and the similar triangle *PQT* has sides sin *A*, *QT*, $\tan A$, therefore *By Proportion QT* = $\sin A \tan A \approx a^2$ for small *A*.

And approximating *QR* as half of *QT*, we find that $\cos A \approx 1 - \frac{a^2}{2}$.

Now, since the circumference of the circle is 2π, the length of the arc *PR* is given by $a = \frac{\pi A}{180}$.

So our approximate expression for $\cos A$ becomes $1 - \left(\frac{\pi}{180}\right)^2 \frac{A^2}{2}$.

At first sight this looks very cumbersome to evaluate for different values of *A*.

However the value of $\frac{1}{3}\left(\frac{\pi}{180}\right)^2$ is 0.0001015 to 7 decimal places and this happens to be a number very easy to work with (see section 3.3).

We give this number a name: $\frac{1}{3}\left(\frac{\pi}{180}\right)^2 = \alpha$.

So we can replace $\left(\frac{\pi}{180}\right)^2$ by 3α.

Then $\cos A \approx 1 - \frac{3A^2\alpha}{2}$. (1)

So to calculate $\cos A$ with formula (1) we:

1) square *A*,
2) increase it by a half of itself,
3) shift the decimal point 4 places to the left (taking α initially as 0.0001)
4) subtract the result from 1 (using *All from 9 and the Last from 10*).

This can all be done mentally.

Example 1 Use equation (1) to find the cosine of 20 degrees.

1) 20^2 = 400,
2) 400 increased by half = 600,
3) Dividing by 10000 gives 0.06,
4) Subtracting from 1 gives 0.94.

∴ Cos 20° = 0.94.

For comparison, the value of cos 20° to 4 decimal places is 0.9397.
The error is 0.03%.

2.2 The Reverse

This procedure can be reversed to give inverse cosines.

Example 2 Find the angle whose cosine is 0.94.

Reversing the above steps:
4) 1 – 0.94 = 0.06
3) 10000 × 0.06 = 600
2) Finding two thirds of 600 gives 400
1) Square root of 400 = 20.

∴ The inverse cosine of 0.94 is 20°.

2.3 Other Functions

A truncated series expansion for $\sec A$ is easily obtained as follows.

Referring to Figure 3, sec $A = OT = OQ + QT \approx 1 - \frac{a^2}{2} + a^2 = 1 + \frac{3A^2\alpha}{2}$.

Sec A is therefore as easy to find as cos A. For example sec 20° = $1 + \frac{3\alpha}{2} \times 20^2 = 1.06$.

However this does not give a good approximation for larger angles and finding $1 \div \cos A$ is therefore a better option.

The other functions are all available through the application of Pythagoras' theorem, reciprocals or similar triangles.

Figure 3 is an extension of the previous diagram and it shows all six trig functions (see table below).
Their values are all obtainable once $\cos A$ is known, as shown below.

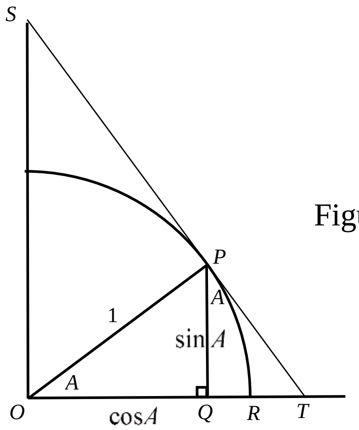


Figure 3

Trig Function	Length in Fig 3	Found from ….
sin A	PQ	ΔOQP (Pythagoras' Theorem)
tan A	PT	ΔOPT (Pythagoras' Theorem)
sec A	OT	$1 \div \cos A$
cosec A	OS	$1 \div \sin A$
cot A	PS	$1 \div \tan A$ (or $\cos A \div \sin A$)

The similar triangles (there are five of them) can also be used to get the various lengths in Figure 3, and, with the other methods indicated finding any one side, given $\cos A$, becomes a fascinating geometrical puzzle for students.

There are of course series expansions for all six trig functions, but with the exception of $\cos A$ and $\sec A$ they all involve odd powers of the angle which, though they can be calculated in a similar way to $\cos A$ above, lead to more complex calculations than we get for $\cos A$.

It is therefore more straightforward and easier to just use the one series (i.e. for $\cos A$) and work with familiar processes like Pythagoras' theorem. Having 3 or 6 series to apply would likely lead to some confusion pedagogically.

2.4 Evaluating the other Functions

Reciprocals are easily found using 'straight division' [1] which we do not need to explain here.

Finding the value of $\sqrt{p^2 \pm q^2}$ for given p and q can be achieved using the Duplexes of Vedic Mathematics and the answer can be found to any accuracy, digit by digit, from the left to right [2]. In our case we are dealing with $\sqrt{1-q^2}$.

The readers knowledge of squaring and square roots using Duplexes [1] is assumed in the following.

The evaluation of $\sqrt{1-q^2}$ will involve:
a) the finding of successive Duplexes of q,
b) the subtraction of q^2 from unity,
c) and finally the square root.

These can all be combined as illustrated in the following example.

Example 3 Find $\sqrt{1-0.74^2}$

1st Iteration
1a) The Duplex of 7 is 49. Look for the carry (if any) from the following Duplex. Since the Duplex of 74 is 56 (2×7×4), we have a carry of 5 and so our 49 is amended to 54.

1b) Apply *All from 9…* to this 54 to get 45.

1c) The square root of 45 is 6 with 9 remainder. So we have $\sqrt{1-0.74^2}$ = $\mathbf{0.6_9}$ initially.
Note, our first answer digit being 6, our divisor will be double this: 12.

2nd Iteration
2a) Go back to the Duplexes: the 2nd Duplex was 56 and only the 6 now remains. Adjust this with the next Duplex: D(4) = 16, so seeing a carry of 1 here we adjust the 6 to a 7.

2b) Take the 7 from 9 to get 2.

2c) This 2 with the 9 (the subscript remainder from the first iteration) gives 92, which we divide by our divisor, 12 to get 7 remainder 8.

So we now have $\mathbf{0.6_9 7_8}$.

At this point we can stop and give 0.67 as the answer since the original number 0.74 is given to only 2 decimal places.

3rd Iteration
To illustrate further we can show another step in the calculation.

3a) Of the Duplexes only the last digit of the last Duplex remains: 6.
3b) This we take from 10 to get 4. And with the 8 remainder we see 84.

3c) Now following the square root procedure we must subtract from this the Duplex of the 2nd digit of the answer: $84 - 7^2 = 35$.

And $35 \div 12 = 3$ remainder $\bar{1}$, giving $\mathbf{0.6_9 7_8 3_{\bar{1}}}$.

From here on, as we have used all the Duplexes in 74, the procedure is exactly as for standard square roots.

Of course we also have the alternative option to find 0.74^2 fully, subtract from 1 and take the square root.

2.5 Range of Values of cos *A* in $\sqrt{1-\cos^2 A}$

We may note that since we can always work with an angle less than or equal to 45° the cosine value will always be between 0.71 and 1.

2.6 Small Angles

We know that the length of the arc in Figure 1 is given by $a = \frac{\pi A}{180}$, and that $\sin A \approx \tan A \approx a$ for small angles. So we may note that $\sin 1° \approx \tan 1° \approx 0.0175$, and use simple proportion find sines and tangents of other small angles.

2.7 Accuracy

The full series for cos *A* is given in the next section but it could be given to students of Stage 1 without proof for the purpose of improving accuracy.

2.8 Inverses

Knowing the value of any one of the basic six trigonometrical functions of an angle, means we know one of the lengths in Figure 3. This means we can find $\cos A$, and hence *A* as indicated in Example 2.

3. STAGE 2

3.1 Increasing Accuracy of $\cos A$

In the Appendix we extend the series expansion for $\cos A$ to get:

$$\cos A = 1 - P + \frac{1}{6}P^2 - \frac{1}{15}PQ + \frac{1}{28}PR \ldots \tag{2}$$

where $P = \frac{3}{2}A^2\alpha$, $Q = \frac{1}{6}P^2$, $R = \frac{1}{15}PQ$ etc.

Note carefully that Q is $|T_3|$, the absolute value of the 3rd term,

R is $|T_4|$ and so on for subsequent terms.

The denominators of the coefficients 6, 15, 28 etc. are given by $n(2n-1)$, $n = 2, 3, 4 \ldots$

The signs alternate, but in actual calculation it may be found preferable to nest the series so that only subtractions are used:

$\cos A = 1 - [P - (\frac{1}{6}P^2 - [\frac{1}{15}PQ - (\frac{1}{28}PR) \ldots])]$

This gives us a simple way to extend the accuracy to any degree. P is used as a constant multiplier, multiplying the previous term and divided by $n(2n-1)$.

3.2 Use of α

It may be considered that accuracy could be increased initially by using a more precise value for α in $\cos A \approx 1 - \frac{3A^2\alpha}{2}$.

That is, to use 0.0001015 instead of 0.0001. However this is not the case.

Illustratively, to 3 decimal places:

$\cos 30^\circ$ = **0.866**

$1 - \frac{3}{2} \times 30^2 \times 0.0001 =$ **0.865**

$1 - \frac{3}{2} \times 30^2 \times 0.0001015 =$ **0.863**

This shows that taking α as 0.0001 gives a more accurate result than taking α as 0.0001015.

The reason for this is that T_3 in (2) is positive and since $\alpha = 0.0001$ will give a greater value for $\cos A$ than $\alpha = 0.0001015$ it will contribute to that 3rd term and hence make the answer more accurate than using a more correct value for α.

However when three or more terms of (2) are considered we do need to use the more accurate value of α.

3.3 Note on multiplication by α.

Since the reader may not be acquainted with the Vedic approach to multiplication of numbers like 0.0001015 note first of all that multiplication by 101 is very simple:
e.g. 34 × 101 = 3434 (the 34 is simply repeated).

And 34 × 1.5 simply means increasing 34 by half of 34, giving 51.

So to find, say, 34 × 0.0001015 we can immediately give: 0.003451.

3.4 Example 4: Use of the first 3 terms of equation (2) to approximate cos 45°

Since we can always work with angles less than 45° finding cos 45° will involve the maximum error.

First we find $P = \frac{3}{2}\alpha A^2 = 1.5 \times 45^2 \times 0.0001015 = 0.3083$ to 4 s.f.

Then $Q = \frac{1}{6}P^2 = \frac{1}{6} \times 0.3083^2 = 0.0158$ (the squaring and division can be combined so that in practice the 0.3803 gets progressively reduced as the parts of Q are obtained).

So cos 45° = 1 – 0.3083 + 0.0158 = 0.7075.

Compared to the true value to 4 s.f. (i.e. 0.7071) we find the error to be 0.06%.

3.5 Notes

1. Although the method given in Section 2.2 for finding inverse cosine is approximate it can be developed to give results to any accuracy [3].
2. The following Appendix includes a number of simple techniques which may be new to the reader i.e. differentiation of trig functions and derivation of series expansions for trig functions [4].

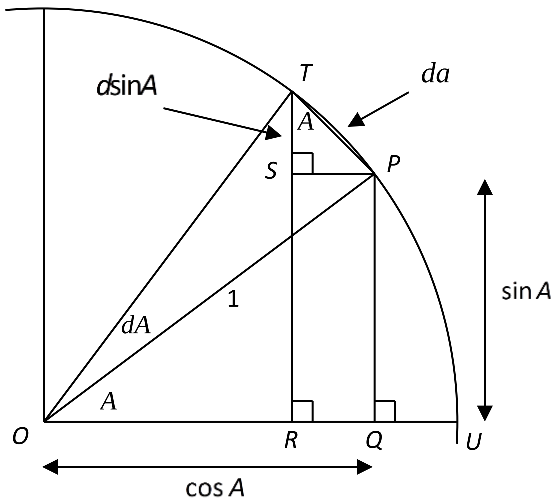
4. Appendix: Derivation of Equation (2)

In this Appendix we first obtain the derivatives of $\cos A$ and $\sin A$, then use these to get a series expansion for $\cos A$.

Next we obtain the series (2) given earlier.

Angles are in degrees throughout.

4.1 Differentiation of Sine and Cosine



In the unit circle above cos A and sin A are as shown and since $\angle OPS = A$, then $\angle STP \approx A$ as $\angle OPT$ is approximately a right angle and approaches a right angle as dA approaches zero.

Since $RT = \sin(A+dA)$ the change in $\sin A$ resulting from the small increase of dA in A is ST. We denote this by $d\sin A$.
If, as previously, a is the length of the arc subtended by angle A then arc $TP = da$.

Now in triangle TSP, $\cos A = \frac{TS}{TP} \approx \frac{d\sin A}{da}$.

And as $dA \to 0$ this approximation tends to equality.

So in the limit $\frac{d\sin A}{da} = \cos A$. (3)

Similarly, from the same diagram, $\sin A = \frac{SP}{TP} = \frac{-d\cos A}{TP} \approx \frac{-d\cos A}{da}$.

Here we have a decrease in $\cos A$ as A increases, hence the minus sign.

Therefore $\frac{d\cos A}{da} = -\sin A$. (4)

It follows from results (3) and (4) that $\frac{d^2\cos A}{da^2} = -\cos A$ (5)

4.2 Series Expansion for Cosine

(5) tells us that if a series expansion for $\cos A$ exists it must be equal to the result of differentiating it twice with respect to a and changing the signs.

We see that this is the case for the two terms of the series already obtained: $\cos A \approx 1 - \frac{a^2}{2}$.

And we can use this result to obtain as many more terms as required by simply integrating a term twice and changing its sign to get the next term:

$$\cos A = 1 - \frac{a^2}{2} + \frac{a^4}{4!} - \frac{a^6}{6!} + \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n a^{2n}}{(2n)!} \qquad (6)$$

From this series we can obtain a series for $\sec A$ by division, using Vedic 'straight division' or *Paravartya* (Transpose and Apply) division.

4.3 Derivation of Equation (2)

Now since $a = \frac{\pi A}{180}$ then $a^2 = \left(\frac{\pi}{180}\right)^2 A^2 = 3\alpha A^2$.

Hence using (6) $\cos A = 1 - \frac{3}{2}\alpha A^2 + \frac{(3\alpha A^2)^2}{4!} - \frac{(3\alpha A^2)^3}{6!} + \ldots$

$$= 1 - \frac{3}{2}\alpha A^2 + \frac{3}{8}\alpha^2 A^4 - \frac{3}{80}\alpha^3 A^6 + \ldots$$

Now suppose we call T_2 (the 2nd term here) P, considering only absolute values.
Then $P^2 = \frac{9}{4}\alpha^2 A^4$, which means T_3 can be written as $\frac{1}{6}P^2$.

Similarly, if T_3 is denoted by Q then T_4 can be written as $\frac{1}{15}PQ$.

That is to say $\cos A = 1 - P + \frac{1}{6}P^2 - \frac{1}{15}PQ + \ldots$ where $P = \frac{3}{2}\alpha A^2$ and $Q = \frac{1}{6}P^2$.

In other words, having found P, we square it and divide by 6 to get T_3.
Then we find the product of T_2 and T_3 and divide by 15 to get T_4.
Thus each term, once obtained, is used to get the next term.

The pattern in the denominators of the coefficients from T_3 on is discernible: 2×3, 3×5, 4×7 etc.

4.4 Proof

From (6): $\cos A = \sum_{n=0}^{\infty}\frac{(-1)^n a^{2n}}{(2n)!} = \sum_{n=0}^{\infty}\frac{(-1)^n (3\alpha)^n A^{2n}}{(2n)!}$, since $a^2 = 3\alpha A^2$. (7)

and $3\alpha = \left(\frac{\pi}{180}\right)^2$ as before.

The 2nd term in this expansion, T_2, is $-\frac{3}{2}\alpha A^2 = -P$ say.

Now from (7): $\frac{T_n}{T_{n-1}} = \frac{\frac{(-1)^n 3^n \alpha^n A^{2n}}{(2n)!}}{\frac{(-1)^{n-1} 3^{n-1}\alpha^{n-1}A^{2n-2}}{(2n-2)!}} = -\frac{3}{2n(2n-1)}\alpha A^2 = -\frac{3}{2}\alpha A^2\frac{1}{n(2n-1)} = -P\frac{1}{n(2n-1)}$, $n \neq 0$.

So $T_n = -P\frac{1}{n(2n-1)}T_{n-1}$, showing that the nth term is the previous term multiplied by $-P$ and the multiplying factor $\frac{1}{n(2n-1)}$.

The multiplying factor $\frac{1}{n(2n-1)}$ for n = 1, 2, 3 . . . gives the magnitudes of the coefficients above: 1, $\frac{1}{6}$, $\frac{1}{15}$ etc.

In a similar way series expansions for other trigonometric functions can be obtained in which a term in A^2 can be used as a multiplying factor: multiplying the previous term and applying a suitable numerical factor. It is also worth noting that all the series for trig functions and their inverses involve an extra factor of A^2 when moving from one term to the next, and therefore the 3α factor, mentioned above, can be utilised to aid their evaluation when A is in degrees.

5. SUMMARY and Concluding Remarks

In Stage 1 we saw that cosine and secant are very easy to obtain estimates for, that the method is easily reversible and that the other trig functions can be found from these. The explanations are also easy to understand. This means that quite young children can be taught these methods. The use of the confusing part-words like ‘cos’, ‘sin’, and the notation for inverse functions could be avoided if thought desirable.

Stage 2 shows that the evaluation of cosine can be extended to any accuracy in a Vedic-type method that uses the previous term and a constant multiplier to repeatedly improve accuracy. This is reminiscent of the ‘osculation’ procedure of Vedic Mathematics [1].

There is another way of evaluating trig functions in the Vedic system: using Pythagorean triples. This is explained in Chapter 11 of [5].

References

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