Trigonometric Identities and Equations

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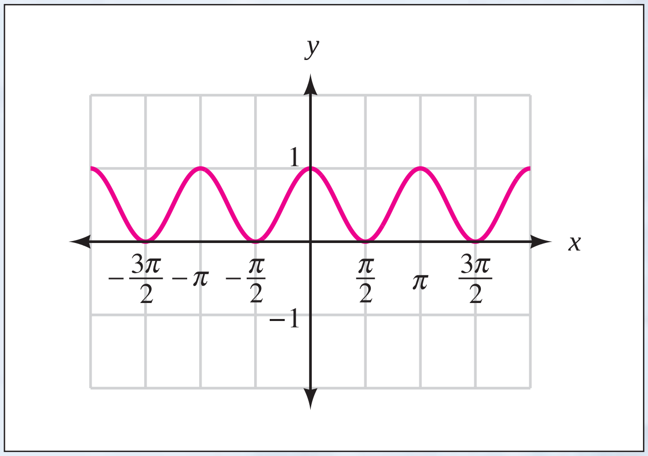


Although it doesn't look like it, Figure 1 above shows the graphs of two functions, namely

$$y = \cos^2 x \qquad \text{and} \qquad y = \frac{1 - \sin^4 x}{1 + \sin^2 x}$$

Although these two functions look quite different from one another, they are in fact the same function. This means that, for all values of x,

$$\cos^2 x = \frac{1 - \sin^4 x}{1 + \sin^2 x}$$

This last expression is an *identity*, and identities are one of the topics we will study in this chapter.

CHAPTER OUTLINE

11.1 Introduction to Identities

In this section, we will turn our attention to identities. In algebra, statements such as $2x = x + x$, $x^3 = x \cdot x \cdot x$, and $x/(4x) = 1/4$ are called **identities.** They are identities because they are true for all replacements of the variable for which they are defined.

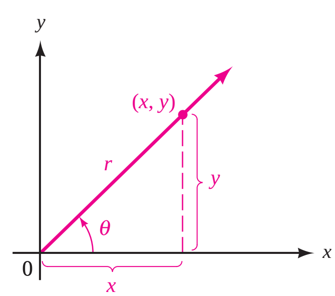
The eight basic **trigonometric identities** are listed in Table 1. As we will see, they are all derived from the definition of the trigonometric functions. Since many of the trigonometric identities have more than one form, we list the basic identity first and then give the most common equivalent forms.

TABLE 1

	Basic Identities	Common Equivalent Forms
Reciprocal	$\csc\theta = \dfrac{1}{\sin\theta}$	$\sin\theta = \dfrac{1}{\csc\theta}$
	$\sec\theta = \dfrac{1}{\cos\theta}$	$\cos\theta = \dfrac{1}{\sec\theta}$
	$\cot\theta = \dfrac{1}{\tan\theta}$	$\tan\theta = \dfrac{1}{\cot\theta}$
Ratio	$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$	
	$\cot\theta = \dfrac{\cos\theta}{\sin\theta}$	
Pythagorean	$\cos^2\theta + \sin^2\theta = 1$	$\sin^2\theta = 1 - \cos^2\theta$
	$1 + \tan^2\theta = \sec^2\theta$	$\sin\theta = \pm\sqrt{1 - \cos^2\theta}$
	$1 + \cot^2\theta = \csc^2\theta$	$\cos^2\theta = 1 - \sin^2\theta$
		$\cos\theta = \pm\sqrt{1 - \sin^2\theta}$

Reciprocal Identities

Note that, in Table 1, the eight basic identities are grouped in categories. For example, since $\csc\theta = 1/(\sin\theta)$, cosecant and sine must be reciprocals. It is for this reason that we call the identities in this category **reciprocal identities.**



As we mentioned above, the eight basic identities are all derived from the definition of the six trigonometric functions. To derive the first reciprocal identity, we use the definition of $\sin\theta$ to write

$$\frac{1}{\sin\theta} = \frac{1}{y/r} = \frac{r}{y} = \csc\theta$$

Note that we can write this same relationship between sin θ and csc θ as

$$\sin\theta = \frac{1}{\csc\theta}$$

because

$$\frac{1}{\csc\theta} = \frac{1}{r/y} = \frac{y}{r} = \sin\theta$$

The first identity we wrote, $\csc\theta = 1/(\sin\theta)$, is the basic identity. The second one, $\sin\theta = 1/(\csc\theta)$, is an equivalent form of the first one.

The other reciprocal identities and their common equivalent forms are derived in a similar manner.

Examples 1–6 show how we use the reciprocal identities to find the value of one trigonometric function, given the value of its reciprocal.

Examples

1. If $\sin\theta = \frac{3}{5}$, then $\csc\theta = \frac{5}{3}$, because

$$\csc\theta = \frac{1}{\sin\theta} = \frac{1}{\frac{3}{5}} = \frac{5}{3}$$

2. If $\cos\theta = -\frac{\sqrt{3}}{2}$, then $\sec\theta = -\frac{2}{\sqrt{3}}$.

(*Remember:* Reciprocals always have the same algebraic sign.)

3. If $\tan\theta = 2$, then $\cot\theta = \frac{1}{2}$.

4. If $\csc\theta = a$, then $\sin\theta = \frac{1}{a}$.

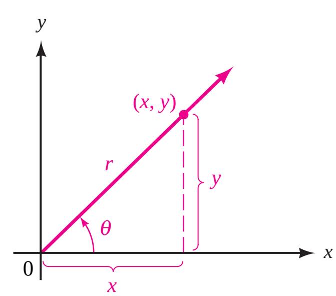
5. If $\sec\theta = 1$, then $\cos\theta = 1$.

6. If $\cot\theta = -1$, then $\tan\theta = -1$.

Ratio Identities

Unlike the reciprocal identities, the **ratio identities** do not have any common equivalent forms. Here is how we derive the ratio identity for tan θ:

$$\frac{\sin\theta}{\cos\theta} = \frac{y/r}{x/r} = \frac{y}{x} = \tan\theta$$



Example 7 If $\sin\theta = -\frac{3}{5}$ and $\cos\theta = \frac{4}{5}$, find $\tan\theta$ and $\cot\theta$.

Solution Using the ratio identities we have

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{-\frac{3}{5}}{\frac{4}{5}} = -\frac{3}{4}$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{\frac{4}{5}}{-\frac{3}{5}} = -\frac{4}{3}$$

Note that, once we found $\tan\theta$, we could have used a reciprocal identity to find $\cot\theta$:

$$\cot\theta = \frac{1}{\tan\theta} = \frac{1}{-\frac{3}{4}} = -\frac{4}{3}$$

Pythagorean Identities

The identity $\cos^2\theta + \sin^2\theta = 1$ is called a **Pythagorean identity** because it is derived from the Pythagorean Theorem. Recall from the definition of $\sin\theta$ and $\cos\theta$ that if (x, y) is a point on the terminal side of θ and r is the distance to (x, y) from the origin, the relationship between x, y, and r is $x^2 + y^2 = r^2$. This relationship comes from the Pythagorean Theorem. Here is how we use it to derive the first Pythagorean identity.

$$x^2 + y^2 = r^2$$

$$\frac{x^2}{r^2} + \frac{y^2}{r^2} = 1 \qquad \text{Divide each side by } r^2.$$

$$\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1 \qquad \text{Property of exponents.}$$

$$(\cos\theta)^2 + (\sin\theta)^2 = 1 \qquad \text{Definition of } \sin\theta \text{ and } \cos\theta$$

$$\cos^2\theta + \sin^2\theta = 1 \qquad \text{Notation}$$

There are four very useful equivalent forms of the first Pythagorean identity. Two of the forms occur when we solve $\cos^2\theta + \sin^2\theta = 1$ for $\cos\theta$, while the other two forms are the result of solving for $\sin\theta$.

Solving $\cos^2\theta + \sin^2\theta = 1$ for $\cos\theta$, we have

$$\cos^2\theta + \sin^2\theta = 1$$

$$\cos^2\theta = 1 - \sin^2\theta \qquad \text{Add } -\sin^2\theta \text{ to each side.}$$

$$\cos\theta = \pm\sqrt{1 - \sin^2\theta} \qquad \text{Take the square root of each side.}$$

Similarly, solving for $\sin\theta$ gives us

$$\sin^2\theta = 1 - \cos^2\theta$$

and

$$\sin\theta = \pm\sqrt{1 - \cos^2\theta}$$

Example 8 If $\sin\theta = \dfrac{3}{5}$ and θ terminates in quadrant II, find $\cos\theta$.

Solution We can obtain $\cos\theta$ from $\sin\theta$ by using the identity

$$\cos\theta = \pm\sqrt{1 - \sin^2\theta}$$

If $\sin\theta = \frac{3}{5}$, the identity becomes

$$\cos\theta = \pm\sqrt{1 - \left(\frac{3}{5}\right)^2} \quad \text{Substitute } \tfrac{3}{5} \text{ for } \sin\theta.$$

$$= \pm\sqrt{1 - \frac{9}{25}} \quad \text{Square } \tfrac{3}{5} \text{ to get } \tfrac{9}{25}$$

$$= \pm\sqrt{\frac{16}{25}} \quad \text{Subtract.}$$

$$= \pm\frac{4}{5} \quad \text{Take the square root of the numerator and denominator separately.}$$

Now we know that $\cos\theta$ is either $+\frac{4}{5}$ or $-\frac{4}{5}$. Looking back to the original statement of the problem, however, we see that θ terminates in quadrant II; therefore, $\cos\theta$ must be negative.

$$\cos\theta = -\frac{4}{5}$$

Example 9 If $\cos\theta = \frac{1}{2}$ and θ terminates in quadrant IV, find the remaining trigonometric ratios for θ.

Solution The first, and easiest, ratio to find is $\sec\theta$, because it is the reciprocal of $\cos\theta$.

$$\sec\theta = \frac{1}{\cos\theta} = \frac{1}{\frac{1}{2}} = 2$$

Next, we find $\sin\theta$. Since θ terminates in QIV, $\sin\theta$ will be negative. Using one of the equivalent forms of the Pythagorean identity, we have

$\sin\theta = -\sqrt{1-\cos^2\theta}$	Negative sign because θ is in QIV.
$= -\sqrt{1-\left(\frac{1}{2}\right)^2}$	Substitute $\frac{1}{2}$ for $\cos\theta$.
$= -\sqrt{1-\frac{1}{4}}$	Square $\frac{1}{2}$ to get $\frac{1}{4}$
$= -\sqrt{\frac{3}{4}}$	Subtract.
$= -\frac{\sqrt{3}}{2}$	Take the square root of the numerator and denominator separately.

Now that we have $\sin\theta$ and $\cos\theta$, we can find $\tan\theta$ by using a ratio identity.

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{-\sqrt{3}/2}{1/2} = -\sqrt{3}$$

Cot θ and csc θ are the reciprocals of $\tan\theta$ and $\sin\theta$, respectively. Therefore,

$$\cot\theta = \frac{1}{\tan\theta} = -\frac{1}{\sqrt{3}} \qquad \csc\theta = \frac{1}{\sin\theta} = -\frac{2}{\sqrt{3}}$$

Here are all six ratios together:

$$\sin\theta = -\frac{\sqrt{3}}{2} \qquad \csc\theta = -\frac{2}{\sqrt{3}}$$

$$\cos\theta = \frac{1}{2} \qquad \sec\theta = 2$$

$$\tan\theta = -\sqrt{3} \qquad \cot\theta = -\frac{1}{\sqrt{3}}$$

The basic identities allow us to write any of the trigonometric functions in terms of sine and cosine. The next examples illustrate this.

Example 10 Write $\tan\theta$ in terms of $\sin\theta$.

Solution When we say we want $\tan\theta$ written in terms of $\sin\theta$, we mean that we want to write an expression that is equivalent to $\tan\theta$ but involves no trigonometric function other than $\sin\theta$. Let's begin by using a ratio identity to write $\tan\theta$ in terms of $\sin\theta$ and $\cos\theta$:

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

Now we need to replace $\cos\theta$ with an expression involving only $\sin\theta$. Since $\cos\theta = \pm\sqrt{1-\sin^2\theta}$, we have

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$= \frac{\sin\theta}{\pm\sqrt{1-\sin^2\theta}}$$

$$= \pm\frac{\sin\theta}{\sqrt{1-\sin^2\theta}}$$

This last expression is equivalent to $\tan\theta$ and is written in terms of $\sin\theta$ only. (In a problem like this it is okay to include numbers and algebraic symbols with $\sin\theta$—just no other trigonometric functions.)

Here is another example. This one involves simplification of the product of two trigonometric functions.

Example 11 Write $\sec\theta\tan\theta$ in terms of $\sin\theta$ and $\cos\theta$, and then simplify.

Note The notation $\sec\theta\tan\theta$ means $\sec\theta\cdot\tan\theta$.

Solution Since $\sec\theta = 1/(\cos\theta)$ and $\tan\theta = (\sin\theta)/(\cos\theta)$, we have

$$\sec\theta\tan\theta = \frac{1}{\cos\theta}\cdot\frac{\sin\theta}{\cos\theta}$$

$$= \frac{\sin\theta}{\cos^2\theta}$$

The next examples show how we manipulate trigonometric expressions using algebraic techniques.

Example 12 Add $\dfrac{1}{\sin\theta} + \dfrac{1}{\cos\theta}$.

Solution We can add these two expressions in the same way we would add $\frac{1}{3}$ and $\frac{1}{4}$, by first finding a least common denominator, and then writing each expression again with the LCD for its denominator.

$$\begin{aligned}\frac{1}{\sin\theta}+\frac{1}{\cos\theta}&=\frac{1}{\sin\theta}\cdot\frac{\mathbf{cos}\,\boldsymbol{\theta}}{\mathbf{cos}\,\boldsymbol{\theta}}+\frac{1}{\cos\theta}\cdot\frac{\mathbf{sin}\,\boldsymbol{\theta}}{\mathbf{sin}\,\boldsymbol{\theta}} && \text{The LCD is } \sin\theta\cos\theta.\\&=\frac{\cos\theta}{\sin\theta\cos\theta}+\frac{\sin\theta}{\cos\theta\sin\theta}\\&=\frac{\cos\theta+\sin\theta}{\sin\theta\cos\theta}\end{aligned}$$

Example 13 Multiply $(\sin\theta + 2)(\sin\theta - 5)$.

Solution We multiply these two expressions in the same way we would multiply $(x + 2)(x - 5)$.

$$\begin{aligned}(\sin\theta+2)(\sin\theta-5)&=\overset{\text{F}}{\sin\theta\sin\theta}\overset{\text{O}}{-5\sin\theta}\overset{\text{I}}{+2\sin\theta}\overset{\text{L}}{-10}\\&=\sin^2\theta-3\sin\theta-10\end{aligned}$$

Getting Ready for Class

After reading through the preceding section, respond in your own words and in complete sentences.

A. State the reciprocal identities for $\csc\theta$, $\sec\theta$, and $\cot\theta$.

B. State the ratio identities for $\tan\theta$ and $\cot\theta$.

C. State the three Pythagorean identities.

D. Write $\tan\theta$ in terms of $\sin\theta$.

PROBLEM SET 11.1

Use the reciprocal identities in the following problems.

1. If $\sin\theta = \dfrac{4}{5}$, find $\csc\theta$.

2. If $\cos\theta = \sqrt{3}/2$, find $\sec\theta$.

3. If $\sec\theta = -2$, find $\cos\theta$.

4. If $\csc\theta = -\dfrac{13}{12}$, find $\sin\theta$.

5. If $\tan\theta = a\ (a \neq 0)$, find $\cot\theta$.

6. If $\cot\theta = -b\ (b \neq 0)$, find $\tan\theta$.

Use a ratio identity to find $\tan\theta$ if:

7. $\sin\theta = \dfrac{3}{5}$ and $\cos\theta = -\dfrac{4}{5}$

8. $\sin\theta = 2/\sqrt{5}$ and $\cos\theta = 1/\sqrt{5}$

Use a ratio identity to find $\cot\theta$ if:

9. $\sin\theta = -\dfrac{5}{13}$ and $\cos\theta = -\dfrac{12}{13}$

10. $\sin\theta = 2/\sqrt{13}$ and $\cos\theta = 3/\sqrt{13}$

Use the equivalent forms of the Pythagorean identity on Problems 11–20.

11. Find $\sin\theta$ if $\cos\theta = \dfrac{3}{5}$ and θ terminates in QI.

12. Find $\sin\theta$ if $\cos\theta = \dfrac{5}{13}$ and θ terminates in QI.

13. Find $\cos\theta$ if $\sin\theta = \dfrac{1}{3}$ and θ terminates in QII.

14. Find $\cos\theta$ if $\sin\theta = \sqrt{3}/2$ and θ terminates in QII.

15. If $\sin\theta = -\dfrac{4}{5}$ and θ terminates in QIII, find $\cos\theta$.

16. If $\sin\theta = -\dfrac{4}{5}$ and θ terminates in QIV, find $\cos\theta$.

17. If $\cos\theta = \sqrt{3}/2$ and θ terminates in QI, find $\sin\theta$.

18. If $\cos\theta = -\dfrac{1}{2}$ and θ terminates in QII, find $\sin\theta$.

19. If $\sin\theta = 1/\sqrt{5}$ and $\theta \in$ QII, find $\cos\theta$.

20. If $\cos\theta = -1/\sqrt{10}$ and $\theta \in$ QIII, find $\sin\theta$.

Find the remaining trigonometric ratios of θ if:

21. $\cos\theta = \dfrac{12}{13}$ and θ terminates in QI

22. $\sin\theta = \dfrac{12}{13}$ and θ terminates in QI

23. $\sin\theta = -\dfrac{1}{2}$ and θ terminates in QIV

24. $\cos\theta = -\dfrac{1}{3}$ and θ terminates in QIII

25. $\cos\theta = 2/\sqrt{13}$ and $\theta \in$ QIV

26. $\sin\theta = 3/\sqrt{10}$ and $\theta \in$ QII

27. $\sec\theta = -3$ and $\theta \in$ QIII

28. $\sec\theta = -4$ and $\theta \in$ QII

Write each of the following in terms of $\sin\theta$ and $\cos\theta$, and then simplify if possible:

29. $\csc\theta\cot\theta$

30. $\sec\theta\cot\theta$

31. $\csc\theta\tan\theta$

32. $\sec\theta\tan\theta\csc\theta$

33. $\dfrac{\sec\theta}{\csc\theta}$

34. $\dfrac{\csc\theta}{\sec\theta}$

35. $\dfrac{\sin\theta}{\csc\theta}$

36. $\dfrac{\cos\theta}{\sec\theta}$

37. $\tan\theta + \sec\theta$

38. $\cot\theta - \csc\theta$

39. $\sin\theta\cot\theta + \cos\theta$

40. $\cos\theta\tan\theta + \sin\theta$

Add and subtract as indicated. Then simplify your answers if possible. Leave all answers in terms of $\sin\theta$ and/or $\cos\theta$.

41. $\dfrac{\sin\theta}{\cos\theta} + \dfrac{1}{\sin\theta}$

42. $\dfrac{\cos\theta}{\sin\theta} + \dfrac{\sin\theta}{\cos\theta}$

43. $\dfrac{1}{\sin\theta} - \dfrac{1}{\cos\theta}$

44. $\dfrac{1}{\cos\theta} - \dfrac{1}{\sin\theta}$

45. $\sin\theta + \dfrac{1}{\cos\theta}$

46. $\cos\theta + \dfrac{1}{\sin\theta}$

47. $\dfrac{1}{\sin\theta} - \sin\theta$

48. $\dfrac{1}{\cos\theta} - \cos\theta$

Multiply.

49. $(\sin\theta + 4)(\sin\theta + 3)$

50. $(\cos\theta + 2)(\cos\theta - 5)$

51. $(2\cos\theta + 3)(4\cos\theta - 5)$

52. $(3\sin\theta - 2)(5\sin\theta - 4)$

53. $(1 - \sin\theta)(1 + \sin\theta)$

54. $(1 - \cos\theta)(1 + \cos\theta)$

55. $(1 - \tan\theta)(1 + \tan\theta)$

56. $(1 - \cot\theta)(1 + \cot\theta)$

57. $(\sin\theta - \cos\theta)^2$

58. $(\cos\theta + \sin\theta)^2$

59. $(\sin\theta - 4)^2$

60. $(\cos\theta - 2)^2$

Review Problems

The problems that follow review material we covered in Section 10.1.

Convert to radian measure.

61. 120°

62. 330°

63. 135°

64. 270°

Convert to degree measure.

65. $\dfrac{\pi}{6}$

66. $\dfrac{5\pi}{6}$

67. $\dfrac{5\pi}{4}$

68. $\dfrac{4\pi}{3}$

Extending the Concepts

Recall from algebra that the slope of the line through (x_1, y_1) and (x_2, y_2) is

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

It is the change in the y-coordinates divided by the change in the x-coordinates.

69. The line $y = 3x$ passes through the points (0, 0) and (1, 3). Find its slope.

70. Suppose the angle formed by the line $y = 3x$ and the positive x-axis is θ. Find the tangent of θ. (See Figure 1.)

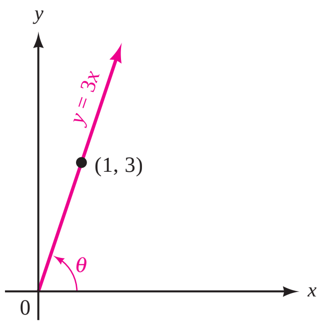


FIGURE 1

71. Find the slope of the line $y = mx$.

72. Find $\tan \theta$ if θ is the angle formed by the line $y = mx$ and the positive x-axis. (See Figure 2.)

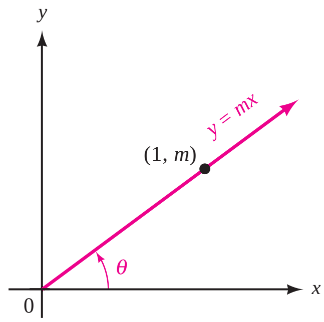


FIGURE 2

11.2 Proving Identities

Next we want to use the eight basic identities and their equivalent forms to verify other trigonometric identities. To prove (or verify) that a trigonometric identity is true, we use trigonometric substitutions and algebraic manipulations to either:

1. Transform the right side into the left side.

Or:

2. Transform the left side into the right side.

The main thing to remember in proving identities is to work on each side of the identity separately. We do not want to use properties from algebra that involve both sides of the identity—such as the addition property of equality. We prove identities in order to develop the ability to transform one trigonometric expression into another. When we encounter problems in other courses that require the use of the techniques used to verify identities, we usually find that the solution to these problems hinges upon transforming an expression containing trigonometric functions into a less complicated expression. In these cases, we do not usually have an equal sign to work with.

Example 1 Verify the identity: $\sin \theta \cot \theta = \cos \theta$.

Proof To prove this identity we transform the left side into the right side:

$$\begin{aligned} \sin\theta\cot\theta &= \sin\theta\cdot\frac{\cos\theta}{\sin\theta} && \text{Ratio identity} \\ &= \frac{\sin\theta\cos\theta}{\sin\theta} && \text{Multiply.} \\ &= \cos\theta && \text{Divide out common factor } \sin\theta. \end{aligned}$$

Example 2 Prove: $\tan x + \cos x = \sin x(\sec x + \cot x)$.

Proof We begin by applying the distributive property to the right side of the identity. Then we change each expression on the right side to an equivalent expression involving only $\sin x$ and $\cos x$.

$$\begin{aligned} \sin x(\sec x + \cot x) &= \sin x\sec x + \sin x\cot x && \text{Multiply.} \\ &= \sin x\cdot\frac{1}{\cos x} + \sin x\cdot\frac{\cos x}{\sin x} && \text{Reciprocal and ratio identities} \\ &= \frac{\sin x}{\cos x} + \cos x && \text{Multiply and divide out common factor } \sin x. \\ &= \tan x + \cos x && \text{Ratio identity} \end{aligned}$$

In this case, we transformed the right side into the left side.

Before we go on to the next example, let's list some guidelines that may be useful in learning how to prove identities.

Probably the best advice is to remember that these are simply guidelines. The best way to become proficient at proving trigonometric identities is to practice. The more identities you prove, the more you will be able to prove and the more confident you will become. *Don't be afraid to stop and start over if you don't seem to be getting anywhere.* With most identities, there are a number of different proofs that will lead to the same result. Some of the proofs will be longer than others.

Guidelines for Proving Identities

1. It is usually best to work on the more complicated side first.
2. Look for trigonometric substitutions involving the basic identities that may help simplify things.
3. Look for algebraic operations, such as adding fractions, the distributive property, or factoring, that may simplify the side you are working with or that will at least lead to an expression that will be easier to simplify.
4. If you cannot think of anything else to do, change everything to sines and cosines and see if that helps.
5. Always keep and eye on the side you are not working with to be sure you are working toward it. There is a certain sense of direction that accompanies a successful proof.

Example 3 Prove: $\dfrac{\cos^4 t - \sin^4 t}{\cos^2 t} = 1 - \tan^2 t$.

Proof In this example, factoring the numerator on the left side will reduce the exponents there from 4 to 2.

$$\begin{aligned}
\frac{\cos^4 t - \sin^4 t}{\cos^2 t} &= \frac{(\cos^2 t + \sin^2 t)(\cos^2 t - \sin^2 t)}{\cos^2 t} && \text{Factor.}\\
&= \frac{1\,(\cos^2 t - \sin^2 t)}{\cos^2 t} && \text{Pythagorean identity}\\
&= \frac{\cos^2 t}{\cos^2 t} - \frac{\sin^2 t}{\cos^2 t} && \text{Separate into two fractions.}\\
&= 1 - \tan^2 t && \text{Ratio identity}
\end{aligned}$$

Example 4 Prove: $1 + \cos\theta = \dfrac{\sin^2\theta}{1 - \cos\theta}$.

Proof We begin by applying an alternative form of the Pythagorean identity to the right side to write $\sin^2\theta$ as $1 - \cos^2\theta$. Then we factor $1 - \cos^2\theta$ and reduce to lowest terms.

$$\begin{aligned}
\frac{\sin^2\theta}{1 - \cos\theta} &= \frac{1 - \cos^2\theta}{1 - \cos\theta} && \text{Pythagorean identity}\\
&= \frac{(1 - \cos\theta)(1 + \cos\theta)}{1 - \cos\theta} && \text{Factor.}\\
&= 1 + \cos\theta && \text{Reduce.}
\end{aligned}$$

Example 5 Prove: $\tan x + \cot x = \sec x \csc x$.

Proof We begin by rewriting the left side in terms of $\sin x$ and $\cos x$. Then we simplify by finding a common denominator, changing to equivalent fractions, and adding, as we did when we combined rational expressions in Chapter 4.

$$\begin{aligned}
\tan x + \cot x &= \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} && \text{Change to expressions in } \sin x \text{ and } \cos x.\\
&= \frac{\sin x}{\cos x}\cdot\frac{\mathbf{\sin x}}{\mathbf{\sin x}} + \frac{\cos x}{\sin x}\cdot\frac{\mathbf{\cos x}}{\mathbf{\cos x}} && \text{LCD}\\
&= \frac{\sin^2 x + \cos^2 x}{\cos x \sin x} && \text{Add fractions.}\\
&= \frac{1}{\cos x \sin x} && \text{Pythagorean identity}\\
&= \frac{1}{\cos x}\cdot\frac{1}{\sin x} && \text{Write as separate fractions.}\\
&= \sec x \csc x && \text{Reciprocal identities}
\end{aligned}$$

Example 6 Prove: $\dfrac{\sin A}{1+\cos A}+\dfrac{1+\cos A}{\sin A}=2\csc A$.

Proof The LCD for the left side is $\sin A(1+\cos A)$.

$$\frac{\sin A}{1+\cos A}+\frac{1+\cos A}{\sin A}=\frac{\mathbf{\sin A}}{\mathbf{\sin A}}\cdot\frac{\sin A}{1+\cos A}+\frac{1+\cos A}{\sin A}\cdot\frac{\mathbf{1+\cos A}}{\mathbf{1+\cos A}} \quad \text{LCD}$$

$$=\frac{\sin^2 A+(1+\cos A)^2}{\sin A(1+\cos A)} \quad \text{Add fractions.}$$

$$=\frac{\sin^2 A+1+2\cos A+\cos^2 A}{\sin A(1+\cos A)} \quad \text{Expand } (1+\cos A)^2.$$

$$=\frac{2+2\cos A}{\sin A(1+\cos A)} \quad \text{Pythagorean identity}$$

$$=\frac{2(1+\cos A)}{\sin A(1+\cos A)} \quad \text{Factor out 2.}$$

$$=\frac{2}{\sin A} \quad \text{Reduce.}$$

$$=2\csc A \quad \text{Reciprocal identity}$$

Example 7 Prove: $\dfrac{1+\sin t}{\cos t}=\dfrac{\cos t}{1-\sin t}$.

Proof The trick to proving this identity is to multiply the numerator and denominator on the right side by $1+\sin t$.

$$\frac{\cos t}{1-\sin t}=\frac{\cos t}{1-\sin t}\cdot\frac{\mathbf{1+\sin t}}{\mathbf{1+\sin t}} \quad \text{Multiply numerator and denominator by } 1+\sin t.$$

$$=\frac{\cos t(1+\sin t)}{1-\sin^2 t} \quad \text{Multiply out the denominator.}$$

$$=\frac{\cos t(1+\sin t)}{\cos^2 t} \quad \text{Pythagorean identity}$$

$$=\frac{1+\sin t}{\cos t} \quad \text{Reduce.}$$

Note that it would have been just as easy for us to verify this identity by multiplying the numerator and denominator on the left side by $1-\sin t$.

Getting Ready for Class

After reading through the preceding section, respond in your own words and in complete sentences.

A. What is an identity?

B. In trigonometry, how do we prove an identity?

C. What is a first step in simplifying the expression, $\dfrac{\cos^4 t - \sin^4 t}{\cos^2 t}$?

D. What is a first step in simplifying the expression, $\dfrac{\sin A}{1 + \cos A} + \dfrac{1 + \cos A}{\sin A}$?

PROBLEM SET 11.2

Prove that each of the following identities is true:

1. $\cos\theta \tan\theta = \sin\theta$

2. $\sec\theta \cot\theta = \csc\theta$

3. $\csc\theta \tan\theta = \sec\theta$

4. $\tan\theta \cot\theta = 1$

5. $\dfrac{\tan A}{\sec A} = \sin A$

6. $\dfrac{\cot A}{\csc A} = \cos A$

7. $\sec\theta \cot\theta \sin\theta = 1$

8. $\tan\theta \csc\theta \cos\theta = 1$

9. $\cos x(\csc x + \tan x) = \cot x + \sin x$

10. $\sin x(\sec x + \csc x) = \tan x + 1$

11. $\cot x - 1 = \cos x(\csc x - \sec x)$

12. $\tan x(\cos x + \cot x) = \sin x + 1$

13. $\cos^2 x(1 + \tan^2 x) = 1$

14. $\sin^2 x(\cot^2 x + 1) = 1$

15. $(1 - \sin x)(1 + \sin x) = \cos^2 x$

16. $(1 - \cos x)(1 + \cos x) = \sin^2 x$

17. $\dfrac{\cos^4 t - \sin^4 t}{\sin^2 t} = \cot^2 t - 1$

18. $\dfrac{\sin^4 t - \cos^4 t}{\sin^2 t \cos^2 t} = \sec^2 t - \csc^2 t$

19. $1 + \sin\theta = \dfrac{\cos^2\theta}{1 - \sin\theta}$

20. $1 - \sin\theta = \dfrac{\cos^2\theta}{1 + \sin\theta}$

21. $\dfrac{1 - \sin^4\theta}{1 + \sin^2\theta} = \cos^2\theta$

22. $\dfrac{1 - \cos^4\theta}{1 + \cos^2\theta} = \sin^2\theta$

23. $\sec^2\theta - \tan^2\theta = 1$

24. $\csc^2\theta - \cot^2\theta = 1$

25. $\sec^4\theta - \tan^4\theta = \dfrac{1 + \sin^2\theta}{\cos^2\theta}$

26. $\csc^4\theta - \cot^4\theta = \dfrac{1 + \cos^2\theta}{\sin^2\theta}$

27. $\tan\theta - \cot\theta = \dfrac{\sin^2\theta - \cos^2\theta}{\sin\theta\cos\theta}$

28. $\sec\theta - \csc\theta = \dfrac{\sin\theta - \cos\theta}{\sin\theta\cos\theta}$

29. $\csc B - \sin B = \cot B \cos B$

30. $\sec B - \cos B = \tan B \sin B$

31. $\cot\theta\cos\theta + \sin\theta = \csc\theta$

32. $\tan\theta\sin\theta + \cos\theta = \sec\theta$

33. $\dfrac{\cos x}{1 + \sin x} + \dfrac{1 + \sin x}{\cos x} = 2\sec x$

34. $\dfrac{\cos x}{1 + \sin x} - \dfrac{1 - \sin x}{\cos x} = 0$

35. $\dfrac{1}{1 + \cos x} + \dfrac{1}{1 - \cos x} = 2\csc^2 x$

36. $\dfrac{1}{1 - \sin x} + \dfrac{1}{1 + \sin x} = 2 \sec^2 x$

37. $\dfrac{1 - \sec x}{1 + \sec x} = \dfrac{\cos x - 1}{\cos x + 1}$

38. $\dfrac{\csc x - 1}{\csc x + 1} = \dfrac{1 - \sin x}{1 + \sin x}$

39. $\dfrac{\cos t}{1 + \sin t} = \dfrac{1 - \sin t}{\cos t}$

40. $\dfrac{\sin t}{1 + \cos t} = \dfrac{1 - \cos t}{\sin t}$

41. $\dfrac{(1 - \sin t)^2}{\cos^2 t} = \dfrac{1 - \sin t}{1 + \sin t}$

42. $\dfrac{\sin^2 t}{(1 - \cos t)^2} = \dfrac{1 + \cos t}{1 - \cos t}$

43. $\dfrac{\sec \theta + 1}{\tan \theta} = \dfrac{\tan \theta}{\sec \theta - 1}$

44. $\dfrac{\csc \theta - 1}{\cot \theta} = \dfrac{\cot \theta}{\csc \theta + 1}$

45. Show that $\sin(A + B)$ is, in general, not equal to $\sin A + \sin B$ by substituting 30° for A and 60° for B in both expressions and simplifying.

46. Show that $\sin 2x \neq 2 \sin x$ by substituting 30° for x and then simplifying both sides.

Review Problems

The problems that follow review material we covered in Section 10.2. Reviewing these problems will help you with some of the material in the next section.

Give the exact value of each of the following:

47. $\sin \dfrac{\pi}{3}$

48. $\cos \dfrac{\pi}{3}$

49. $\cos \dfrac{\pi}{6}$

50. $\sin \dfrac{\pi}{6}$

51. $\tan 45°$

52. $\cot 45°$

53. $\sin 90°$

54. $\cos 90°$

Extending the Concepts

Prove each identity.

55. $\dfrac{\sec^4 y - \tan^4 y}{\sec^2 y + \tan^2 y} = 1$

56. $\dfrac{\csc^2 y + \cot^2 y}{\csc^4 y - \cot^4 y} = 1$

57. $\dfrac{\sin^3 A - 8}{\sin A - 2} = \sin^2 A + 2 \sin A + 4$

58. $\dfrac{1 - \cos^3 A}{1 - \cos A} = \cos^2 A + \cos A + 1$

59. $\dfrac{1 - \tan^3 t}{1 - \tan t} = \sec^2 t + \tan t$

60. $\dfrac{1 + \cot^3 t}{1 + \cot t} = \csc^2 t - \cot t$

61. $\dfrac{\sec B}{\sin B + 1} = \dfrac{1 - \sin B}{\cos^3 B}$

62. $\dfrac{1 - \cos B}{\csc B} = \dfrac{\sin^3 B}{1 + \cos B}$

11.3 Sum and Difference Formulas

The expressions $\sin(A + B)$ and $\cos(A + B)$ occur frequently enough in mathematics that it is necessary to find expressions equivalent to them that involve sines and cosines of single angles. The most obvious question to begin with is

$$\sin(A + B) = \sin A + \sin B?$$

Note A counterexample is an example that shows that a statement is not, in general, true.

The answer is no. Substituting almost any pair of numbers for A and B in the formula will yield a false statement. As a counterexample, we can let $A = 30°$ and

$B = 60°$ in the formula above and then simplify each side.

$$\sin(30° + 60°) \stackrel{?}{=} \sin 30° + \sin 60°$$

$$\sin 90° \stackrel{?}{=} \frac{1}{2} + \frac{\sqrt{3}}{2}$$

$$1 \neq \frac{1 + \sqrt{3}}{2}$$

The formula just doesn't work. The next question is, what are the formulas for $\sin(A + B)$ and $\cos(A + B)$? The answer to that question is what this section is all about. Let's start by deriving the formula for $\cos(A + B)$.

We begin by drawing A in standard position and then adding B and $-B$ to it. These angles are shown in Figure 1 in relation to the unit circle. The **unit circle** is the circle with its center at the origin and with a radius of 1. Since the radius of the unit circle is 1, the point through which the terminal side of A passes will have coordinates $(\cos A, \sin A)$. [If P_2 in Figure 1 has coordinates (x, y), then by the definition of $\sin A$, $\cos A$, and the unit circle, $\cos A = x/r = x/1 = x$ and $\sin A = y/r = y/1 = y$. Therefore, $(x, y) = (\cos A, \sin A)$.] The points on the unit circle through which the terminal sides of the other angles in Figure 1 pass are found in the same manner.

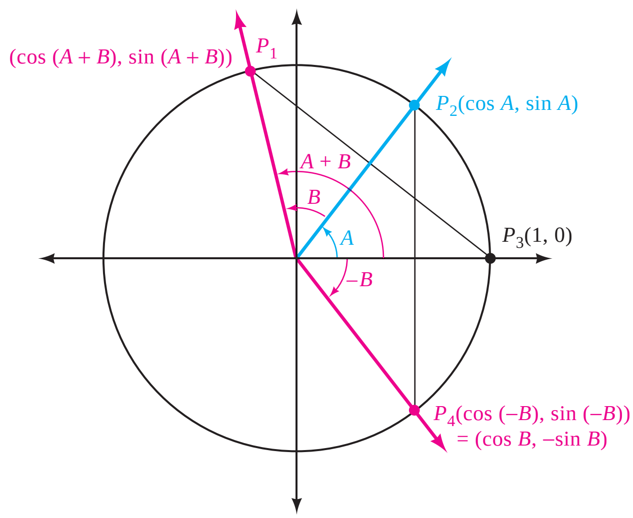


FIGURE 1

To derive the formula for $\cos(A + B)$, we simply have to see that line segment P_1P_3 is equal to line segment P_2P_4. (From geometry, they are chords cut off by equal central angles.)

$$\overline{P_1P_3} = \overline{P_2P_4}$$

Squaring both sides gives us

$$(\overline{P_1 P_3})^2 = (\overline{P_2 P_4})^2$$

Now, applying the distance formula, we have

$$[\cos(A + B) - 1]^2 + [\sin(A + B) - 0]^2 = (\cos A - \cos B)^2 + (\sin A + \sin B)^2$$

Let's call this Equation 1. Taking the left side of Equation 1, expanding it, and then simplifying by using the Pythagorean identity, gives us

Left side of Equation 1

$$\begin{aligned} \cos^2(A + B) - 2\cos(A + B) + 1 + \sin^2(A + B) & \quad \text{Expand squares.} \\ = -2\cos(A + B) + 2 & \quad \text{Pythagorean identity} \end{aligned}$$

Applying the same two steps to the right side of Equation 1 looks like this:

Right side of Equation 1

$$\begin{aligned} &\cos^2 A - 2\cos A\cos B + \cos^2 B + \sin^2 A + 2\sin A\sin B + \sin^2 B \\ &\qquad = -2\cos A\cos B + 2\sin A\sin B + 2 \end{aligned}$$

Equating the simplified versions of the left and right sides of Equation 1, we have

$$-2\cos(A + B) + 2 = -2\cos A\cos B + 2\sin A\sin B + 2$$

Adding -2 to both sides and then dividing both sides by -2 gives us the formula we are after.

$$\cos(A + B) = \cos A\cos B - \sin A\sin B$$

This is the first formula in a series of formulas for trigonometric functions of the sum or difference of two angles. It must be memorized. Before we derive the others, let's look at some of the ways we can use our first formula.

Example 1 Find the exact value for cos 75°.

Solution We write 75° as 45° + 30° and then apply the formula for $\cos(A + B)$.

$$\begin{aligned} \cos 75° &= \cos(45° + 30°) \\ &= \cos 45°\cos 30° - \sin 45°\sin 30° \\ &= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\cdot\frac{1}{2} \\ &= \frac{\sqrt{6} - \sqrt{2}}{4} \end{aligned}$$