TRIGONOMETRIC RATIOS AND FUNCTIONS

▶ *How can you find the width of the opening between two halves of a bridge?*

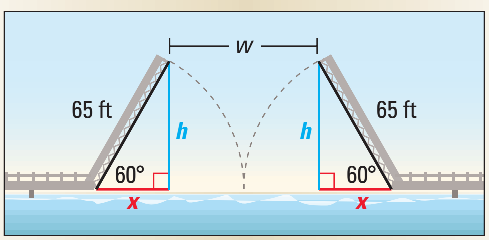
CHAPTER 13

APPLICATION: Drawbridges

The Chicago River System has 52 movable bridges, more than any other city in the world. One type of moveable bridge is a double-leaf drawbridge, a series of which are shown open in the photo at the left and closed in the photo at the right.

Think & Discuss

The diagram below gives the dimensions for a double-leaf drawbridge. Each leaf of the bridge is 65 feet long and has a maximum opening angle of 60°. For a 30°-60°-90° triangle, the ratio of the lengths of the sides, from shortest to longest, is $1 : \sqrt{3} : 2$.

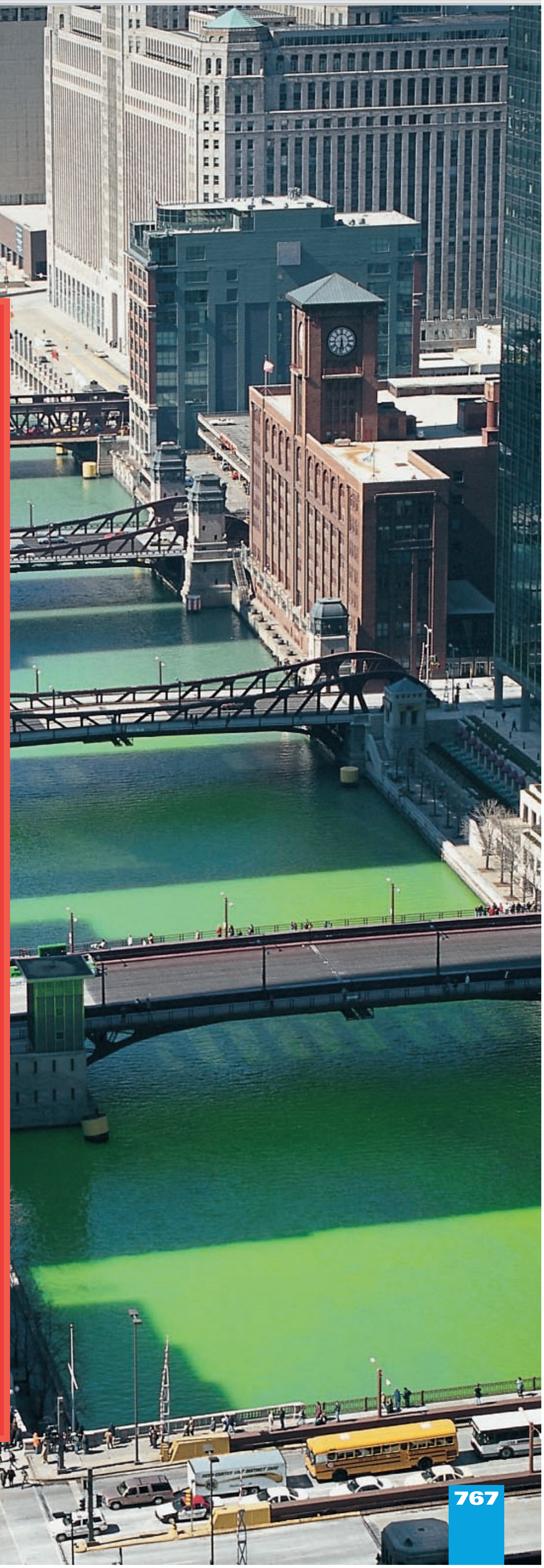


1. Find the height h to which the end of each leaf is lifted.
2. What is the maximum width w that a boat could have at the height of the opening and still fit through the opening? (*Hint*: Use the fact that $2x + w = 130$.)

Learn More About It

You will find the angle at which a drawbridge must open to allow a ship to pass in Ex. 56 on p. 796.

INTERNET **APPLICATION LINK** Visit www.mcdougallittell.com for more information on drawbridges.



CHAPTER 13

Study Guide

PREVIEW

What's the chapter about?

Chapter 13 is about **trigonometry**. In Chapter 13 you'll learn

- how to evaluate trigonometric functions and inverse trigonometric functions.
- how to find side lengths, angle measures, and areas of triangles.
- how to use parametric equations.

KEY VOCABULARY

▶ Review
- reciprocal, p. 5
- inverse functions, p. 422

▶ New
- sine, p. 769
- cosine, p. 769
- tangent, p. 769
- cosecant, p. 769
- secant, p. 769
- cotangent, p. 769
- radian, p. 777
- sector, p. 779
- inverse sine, p. 792
- inverse cosine, p. 792
- inverse tangent, p. 792
- law of sines, p. 799
- law of cosines, p. 807
- parametric equations, p. 813

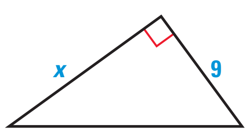
PREPARE

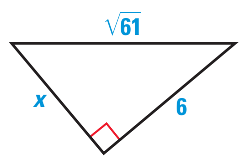
Are you ready for the chapter?

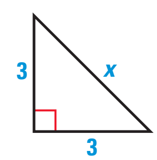
SKILL REVIEW Do these exercises to review key skills that you'll apply in this chapter. See the given **reference page** if there is something you don't understand.

> **STUDENT HELP**
>
> **Study Tip**
> "Student Help" boxes throughout the chapter give you study tips and tell you where to look for extra help in this book and on the Internet.

Find the missing side length. (Skills Review, p. 917)

1. 

2. 

3. 

Simplify the expression. (Review Example 1, p. 264)

4. $\sqrt{96}$ **5.** $\sqrt{18}$ **6.** $\sqrt{200}$ **7.** $\frac{2}{\sqrt{2}}$ **8.** $\frac{2}{\sqrt{3}}$ **9.** $\sqrt{\frac{3}{4}}$

Solve the equation. (Review Example 4, p. 569)

10. $\frac{3}{x} = \frac{6}{x-1}$ **11.** $\frac{4x}{5} = \frac{5}{x}$ **12.** $\frac{7}{4} = \frac{x}{8}$ **13.** $\frac{x+3}{x} = \frac{7}{10}$

STUDY STRATEGY

Here's a study strategy!

Draw Diagrams

Drawing a diagram can help you figure out how to solve a problem from the given information. A diagram can help you with almost every problem (not just word problems) in this chapter. Whenever you are given lengths, angle measures, points, or ratios, try drawing and labeling a diagram.

13.1 Right Triangle Trigonometry

GOAL 1 Evaluating Trigonometric Functions

What you should learn

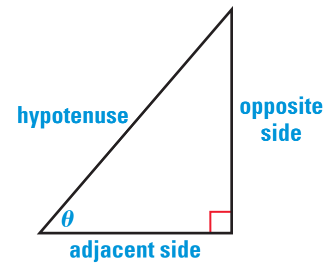
GOAL 1 Use trigonometric relationships to evaluate trigonometric functions of acute angles.

GOAL 2 Use trigonometric functions to solve **real-life** problems, such as finding the altitude of a kite in **Example 4**.

Why you should learn it

▼ To solve **real-life** problems, such as finding the length of a zip-line at a ropes course in **Ex. 50**.

Consider a right triangle, one of whose acute angles is θ (the Greek letter *theta*). The three sides of the triangle are the *hypotenuse,* the side *opposite* θ, and the side *adjacent* to θ.



Ratios of a right triangle's three sides are used to define the six trigonometric functions: **sine**, **cosine**, **tangent**, **cosecant**, **secant**, and **cotangent**. These six functions are abbreviated sin, cos, tan, csc, sec, and cot, respectively.

RIGHT TRIANGLE DEFINITION OF TRIGONOMETRIC FUNCTIONS

Let θ be an acute angle of a right triangle. The six trigonometric functions of θ are defined as follows.

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \qquad \cos\theta = \frac{\text{adj}}{\text{hyp}} \qquad \tan\theta = \frac{\text{opp}}{\text{adj}}$$

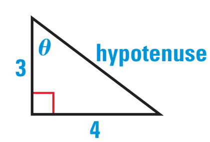
$$\csc\theta = \frac{\text{hyp}}{\text{opp}} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}}$$

The abbreviations *opp, adj,* and *hyp* represent the lengths of the three sides of the right triangle. Note that the ratios in the second row are the reciprocals of the ratios in the first row. That is:

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

EXAMPLE 1 *Evaluating Trigonometric Functions*

Evaluate the six trigonometric functions of the angle θ shown in the right triangle.



SOLUTION

From the Pythagorean theorem, the length of the hypotenuse is:

$$\sqrt{3^2 + 4^2} = \sqrt{25} = 5$$

Using adj = 3, opp = 4, and hyp = 5, you can write the following.

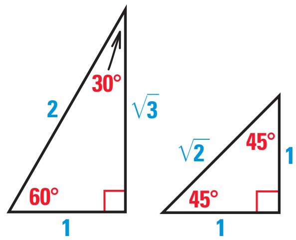
$$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{4}{5} \qquad \cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{3}{5} \qquad \tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{4}{3}$$

$$\csc\theta = \frac{\text{hyp}}{\text{opp}} = \frac{5}{4} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}} = \frac{5}{3} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}} = \frac{3}{4}$$

STUDENT HELP

Skills Review
For help with the Pythagorean theorem, see p. 917.

The angles 30°, 45°, and 60° occur frequently in trigonometry. The table below gives the values of the six trigonometric functions for these angles. To remember these values, you may find it easier to draw the triangles shown, rather than memorize the table.



θ	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\csc\theta$	$\sec\theta$	$\cot\theta$
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$	$\sqrt{3}$
45°	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\sqrt{2}$	$\sqrt{2}$	1
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2	$\frac{\sqrt{3}}{3}$

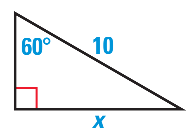
Trigonometric functions can be used to find a missing side length or angle measure of a right triangle. Finding *all* missing side lengths and angle measures is called **solving a right triangle**.

EXAMPLE 2 *Finding a Missing Side Length of a Right Triangle*

STUDENT HELP

INTERNET HOMEWORK HELP Visit our Web site www.mcdougallittell.com for extra examples.

Find the value of x for the right triangle shown.



SOLUTION

Write an equation using a trigonometric function that involves the ratio of x and 10. Solve the equation for x.

$$\sin 60° = \frac{\text{opp}}{\text{hyp}}$$ **Write trigonometric equation.**

$$\frac{\sqrt{3}}{2} = \frac{x}{10}$$ **Substitute.**

$$5\sqrt{3} = x$$ **Multiply each side by 10.**

▶ The length of the side is $x = 5\sqrt{3} \approx 8.66$.

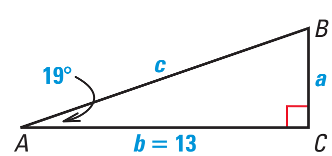
You can use a calculator to evaluate trigonometric functions of *any* angle, not just 30°, 45°, and 60°. Use the keys SIN, COS, and TAN for sine, cosine, and tangent. Use these keys and the reciprocal key for cosecant, secant, and cotangent. Before using the calculator be sure it is set in degree mode.

STUDENT HELP

Study Tip
In Example 3, B is used to represent both the angle and its measure. Throughout this chapter, a capital letter is used to denote a vertex of a triangle and the same letter in lowercase is used to denote the side opposite that angle.

EXAMPLE 3 *Using a Calculator to Solve a Right Triangle*

Solve $\triangle ABC$.



SOLUTION

Because the triangle is a right triangle, A and B are complementary angles, so $B = 90° - 19° = 71°$.

$$\frac{a}{13} = \tan 19° \approx 0.3443 \qquad \frac{c}{13} = \sec 19° = \frac{1}{\cos 19°} \approx 1.058$$

$$a \approx 4.48 \qquad c \approx 13.8$$

FOCUS ON APPLICATIONS

KITE FLYING In the late 1800s and early 1900s, kites were used to lift weather instruments. In 1919 the German Weather Bureau set a kite-flying record. Eight kites on a single line, like those pictured above, were flown at an altitude of 9740 meters.

GOAL 2 USING TRIGONOMETRY IN REAL LIFE

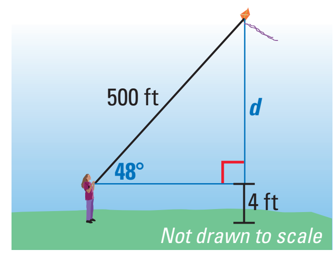
EXAMPLE 4 *Finding the Altitude of a Kite*

KITE FLYING Wind speed affects the angle at which a kite flies. The table at the right shows the angle the kite line makes with a line parallel to the ground for several different wind speeds. You are flying a kite 4 feet above the ground and are using 500 feet of line. At what altitude is the kite flying if the wind speed is 35 miles per hour?

Wind speed (miles per hour)	Angle of kite line (degrees)
25	70
30	60
35	48
40	29
45	0

SOLUTION

At a wind speed of 35 miles per hour, the angle the kite line makes with a line parallel to the ground is 48°. Write an equation using a trigonometric function that involves the ratio of the distance d and 500.



$\sin 48° = \frac{d}{500}$ **Write trigonometric equation.**

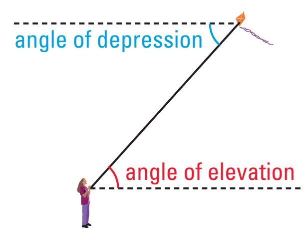
$0.7431 \approx \frac{d}{500}$ **Simplify.**

$372 \approx d$ **Solve for *d*.**

▶ When you add 4 feet for the height at which you are holding the kite line, the kite's altitude is about 376 feet.

. . . .

In Example 4 the angle the kite line makes with a line parallel to the ground is the **angle of elevation**. At the height of the kite, the angle from a line parallel to the ground to the kite line is the **angle of depression**. These two angles have the same measure.



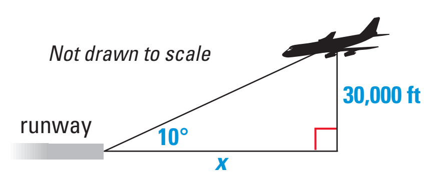
REAL LIFE Aviation

EXAMPLE 5 *Finding the Distance to an Airport*

An airplane flying at an altitude of 30,000 feet is headed toward an airport. To guide the airplane to a safe landing, the airport's landing system sends radar signals from the runway to the airplane at a 10° angle of elevation. How far is the airplane (measured along the ground) from the airport runway?

SOLUTION

Begin by drawing a diagram.



$$\frac{x}{30{,}000} = \cot 10° = \frac{1}{\tan 10°} \approx 5.671$$

$$x \approx 170{,}100$$

▶ The plane is about 170,100 feet (or 32.2 miles) from the airport.

GUIDED PRACTICE

Vocabulary Check ✓

1. Explain what it means to solve a right triangle.

Concept Check ✓

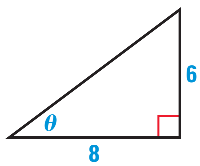
2. Given a 30°-60°-90° triangle with only the measures of the angles labeled, can you find the lengths of any of the sides? Explain.

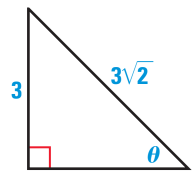
3. If you are given a right triangle with an acute angle θ, what two trigonometric functions of θ can you calculate using the lengths of the hypotenuse and the side opposite θ?

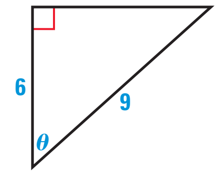
4. For which acute angle θ is $\cos \theta = \frac{\sqrt{3}}{2}$?

Skill Check ✓

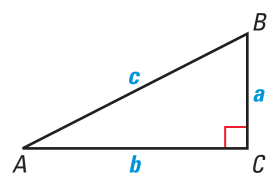
Evaluate the six trigonometric functions of the angle θ.

5.


6.


7.


Solve △*ABC* using the diagram at the right and the given measurements.



8. $A = 20°, a = 12$ **9.** $A = 75°, c = 20$

10. $B = 40°, c = 5$ **11.** $A = 62°, b = 30$

12. $B = 63°, a = 15$ **13.** $B = 15°, b = 42$

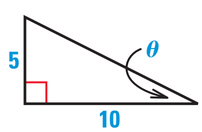
14. **KITE FLYING** Look back at Example 4 on page 771. Suppose you are flying a kite 4 feet above the ground on a line that is 300 feet long. If the wind speed is 30 miles per hour, what is the altitude of the kite?

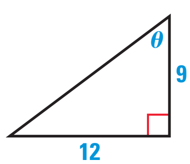
PRACTICE AND APPLICATIONS

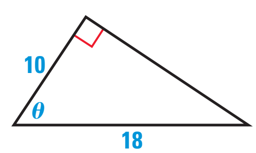
STUDENT HELP

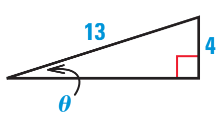
Extra Practice to help you master skills is on p. 957.

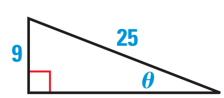
EVALUATING FUNCTIONS **Evaluate the six trigonometric functions of the angle θ.**

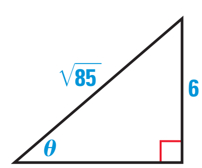
15.


16.


17.


18.


19.


20.


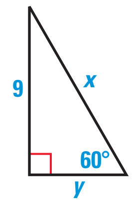
STUDENT HELP

HOMEWORK HELP
Example 1: Exs. 15–21
Example 2: Exs. 22–24
Example 3: Exs. 25–40
Examples 4, 5: Exs. 43–50

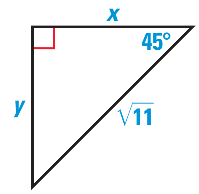
21. **VISUAL THINKING** The lengths of the sides of a right triangle are 5 centimeters, 12 centimeters, and 13 centimeters. Sketch the triangle. Let θ represent the angle that is opposite the side whose length is 5 centimeters. Evaluate the six trigonometric functions of θ.

FINDING SIDE LENGTHS **Find the missing side lengths x and y.**

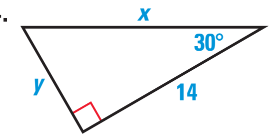
22.



23.



24.



EVALUATING FUNCTIONS **Use a calculator to evaluate the trigonometric function. Round the result to four decimal places.**

25. $\sin 14°$ **26.** $\cos 31°$ **27.** $\tan 59°$ **28.** $\sec 23°$

29. $\csc 80°$ **30.** $\cot 36°$ **31.** $\csc 6°$ **32.** $\cot 11°$

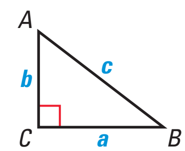
SOLVING TRIANGLES **Solve $\triangle ABC$ using the diagram and the given measurements.**

33. $B = 24°, a = 8$ **34.** $A = 37°, c = 22$

35. $A = 19°, b = 4$ **36.** $B = 41°, c = 18$

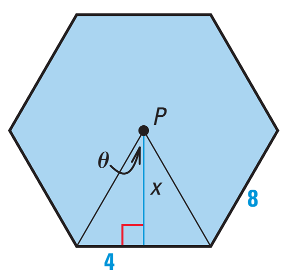
37. $A = 29°, b = 21$ **38.** $B = 56°, a = 6.8$

39. $B = 65°, c = 12$ **40.** $A = 70°, c = 30$

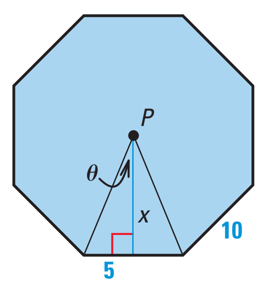


GEOMETRY CONNECTION **Find the area of the regular polygon with point P at its center.**

41.



42.



FOCUS ON APPLICATIONS

REAL LIFE **DUQUESNE INCLINE** Built in Pittsburgh in 1877, the Duquesne Incline transports people up and down the side of a mountain in cable cars. In 1877 the cost of a one-way trip was \$.05. Today the cost is \$1.

DUQUESNE INCLINE **In Exercises 43 and 44, use the following information.** The track of the Duquesne Incline is about 800 feet long and the angle of elevation is 30°. The average speed of the cable cars is about 320 feet per minute.

43. How high does the Duquesne Incline rise?

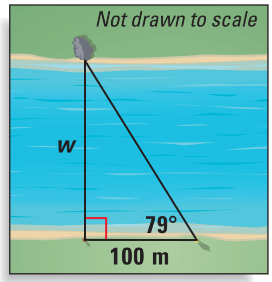
44. What is the vertical speed of the cable cars (in feet per minute)?

45. **SKI SLOPE** A ski slope at a mountain has an angle of elevation of 25.2°. The vertical height of the slope is 1808 feet. How long is the ski slope?

46. **BOARDING A SHIP** A gangplank is a narrow ramp used for boarding or leaving a ship. The maximum safe angle of elevation for a gangplank is 20°. Suppose a gangplank is 10 feet long. What is the closest a ship can come to the dock for the gangplank to be used?

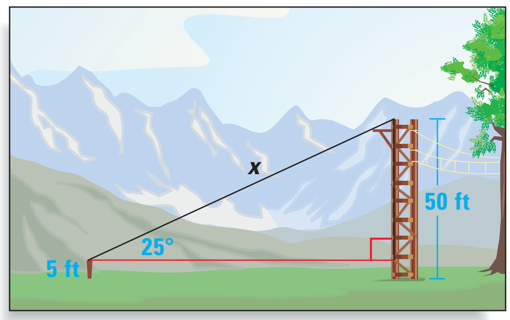
47. **JIN MAO BUILDING** You are standing 75 meters from the base of the Jin Mao Building in Shanghai, China. You estimate that the angle of elevation to the top of the building is 80°. What is the approximate height of the building? Suppose one of your friends is at the top of the building. What is the distance between you and your friend?

48. **MEASURING RIVER WIDTH** To measure the width of a river you plant a stake on one side of the river, directly across from a boulder. You then walk 100 meters to the right of the stake and measure a 79° angle between the stake and the boulder. What is the width *w* of the river?



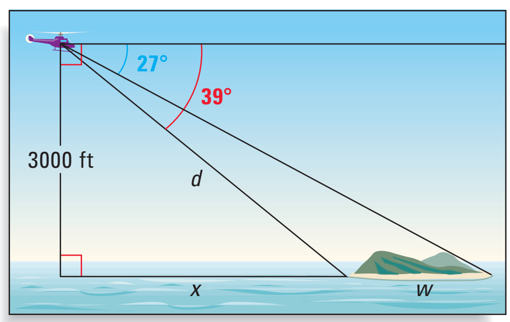
49. **MOUNT COOK** You are climbing Mount Cook in New Zealand. You are below the mountain's peak at an altitude of 8580 feet. Using surveying instruments, you measure the angle of elevation to the peak to be 30.5°. The distance (along the face of the mountain) between you and the peak is 7426 feet. What is the altitude of the peak?

50. **ROPES COURSE** You are designing a zip-line for a ropes course at a summer camp. A zip-line is a cable to which people can attach their safety harnesses and slide down to the ground. You want to attach one end of the cable to a pole 50 feet high and the other end to a pole 5 feet high. The maximum safe angle of elevation for the zip-line is 25°. Calculate the minimum length *x* of cable needed.



Test Preparation

51. **MULTI-STEP PROBLEM** You are a surveyor in a helicopter and are trying to determine the width of an island, as illustrated at the right.



a. What is the shortest distance *d* the helicopter would have to travel to land on the island?

b. What is the horizontal distance *x* that the helicopter has to travel before it is directly over the nearer end of the island?

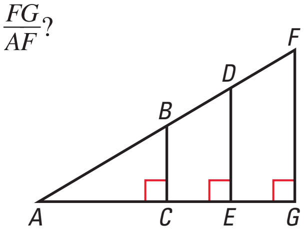
c. *Writing* Find the width *w* of the island. Explain the process you used to find your answer.

★ **Challenge**

ANALYZING SIMILAR TRIANGLES **In Exercises 52–54, use the diagram below.**

52. Explain why $\triangle ABC$, $\triangle ADE$, and $\triangle AFG$ are similar triangles.

53. What does similarity imply about the ratios $\frac{BC}{AB}$, $\frac{DE}{AD}$, and $\frac{FG}{AF}$? Does the value of sin *A* depend on which triangle from Exercise 52 is used to calculate it? Would the value of sin *A* change if it were found using a different right triangle that is similar to the three given triangles?



54. Do your observations about sin *A* also apply to the other five trigonometric functions? Explain.

STUDENT HELP

Skills Review
For help with similar triangles, see p. 923.

MIXED REVIEW

UNIT ANALYSIS Find the product. Give the answer with the appropriate unit of measure. (Review 1.1 for 13.2)

55. $(3.5 \text{ hours}) \cdot \frac{45 \text{ miles}}{1 \text{ hour}}$

56. $(500 \text{ dollars}) \cdot \frac{12.2 \text{ schillings}}{1 \text{ dollar}}$

57. $\frac{3 \text{ dollars}}{1 \text{ square foot}} \cdot (1222 \text{ square feet})$

58. $(12 \text{ seconds}) \cdot \frac{254 \text{ feet}}{1 \text{ second}}$

CLASSIFYING Classify the conic section. (Review 10.6)

59. $y^2 - 16x - 14y + 17 = 0$

60. $25x^2 + y^2 - 100x - 2y + 76 = 0$

61. $x^2 + y^2 = 25$

62. $x^2 - y^2 = 100$

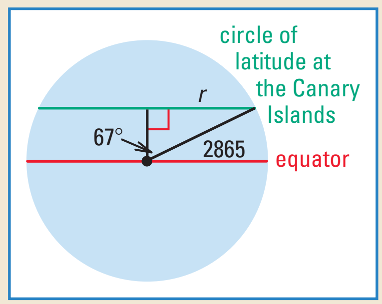
63. **ESSAY TOPICS** For a homework assignment you have to choose from 15 possible topics on which to write an essay. If all of the topics are equally interesting, what is the probability that you and your five friends will all choose different topics? (Review 12.5)

MATH & History: Columbus's Voyage

APPLICATION LINK www.mcdougallittell.com

THEN

IN 1492 Christopher Columbus set sail west from the Canary Islands intending to reach Japan. Due to miscalculations of Earth's circumference and the relative location of Japan, he instead sailed to the New World.



Columbus believed the distance west from the Canary Islands to Japan to be $\frac{1}{6}$ the circumference of Earth at that latitude. He supposed Earth's radius at the equator to be about 2865 miles.

1. Use the diagram at the right to calculate what Columbus believed to be the radius r of Earth at the latitude of the Canary Islands.
2. Use your answer to Exercise 1 to calculate the distance west from the Canary Islands that Columbus believed he would find Japan.
3. Use reference materials to find the true distance west from the Canary Islands to Japan. How far off were Columbus's calculations?

NOW

TODAY aerial photography and computers are used to make maps. Accurate maps in combination with satellite-based navigation make travel a more exact science.

The oldest existing map was made on a clay tablet in Babylonia.

Influential map maker Claudius Ptolemy wrote his eight-volume *Geography*.

Columbus sails to the Bahama Islands and Cuba, intending to reach Japan.

The Landsat 7 satellite was launched.

13.2 General Angles and Radian Measure

What you should learn

GOAL 1 Measure angles in standard position using degree measure and radian measure.

GOAL 2 Calculate arc lengths and areas of sectors, as applied in **Example 6**.

Why you should learn it

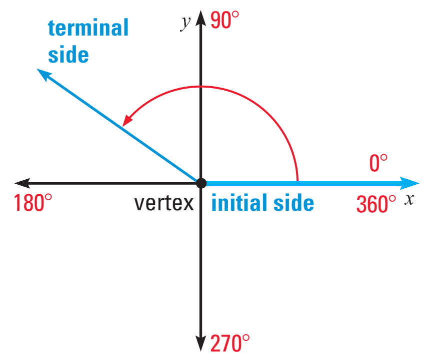
▼ To solve **real-life** problems, such as finding the angle generated by a rotating figure skater in **Exs. 77–79**.

REAL LIFE

GOAL 1 ANGLES IN STANDARD POSITION

In Lesson 13.1 you worked only with acute angles (angles measuring between 0° and 90°). In this lesson you will study angles whose measures can be any real numbers.

Recall that an angle is formed by two rays that have a common endpoint, called the vertex. You can generate any angle by fixing one ray, called the **initial side**, and rotating the other ray, called the **terminal side**, about the vertex. In a coordinate plane, an angle whose vertex is at the origin and whose initial side is the positive *x*-axis is in **standard position**.



The measure of an angle is determined by the amount and direction of rotation from the initial side to the terminal side. The angle measure is positive if the rotation is counterclockwise, and negative if the rotation is clockwise. The terminal side of an angle can make more than one complete rotation.

EXAMPLE 1 *Drawing Angles in Standard Position*

Draw an angle with the given measure in standard position. Then tell in which quadrant the terminal side lies.

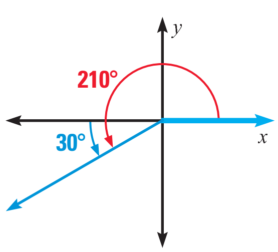
a. 210° **b.** −45° **c.** 510°

SOLUTION

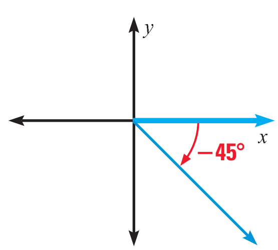
a. Use the fact that 210° = 180° + 30°. So, the terminal side is 30° counterclockwise past the negative *x*-axis.

b. Because −45° is negative, the terminal side is 45° clockwise from the positive *x*-axis.

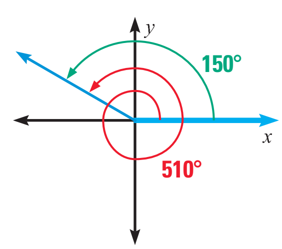
c. Use the fact that 510° = 360° + 150°. So, the terminal side makes one complete revolution counterclockwise and continues another 150°.



Terminal side in Quadrant III



Terminal side in Quadrant IV



Terminal side in Quadrant II

In Example 1 the angles 510° and 150° are *coterminal*. Two angles in standard position are **coterminal** if their terminal sides coincide. An angle coterminal with a given angle can be found by adding or subtracting multiples of 360°.

EXAMPLE 2 *Finding Coterminal Angles*

Find one positive angle and one negative angle that are coterminal with (**a**) −60° and (**b**) 495°.

STUDENT HELP

INTERNET HOMEWORK HELP Visit our Web site www.mcdougallittell.com for extra examples.

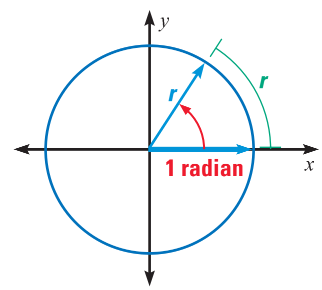
SOLUTION

There are many such angles, depending on what multiple of 360° is added or subtracted.

a. Positive coterminal angle: $-60° + 360° = 300°$
Negative coterminal angle: $-60° - 360° = -420°$

b. Positive coterminal angle: $495° - 360° = 135°$
Negative coterminal angle: $495° - 2(360°) = -225°$

So far, all the angles you have worked with have been measured in degrees. You can also measure angles in *radians*. To define a radian, consider a circle with radius r centered at the origin. One **radian** is the measure of an angle in standard position whose terminal side intercepts an arc of length r.

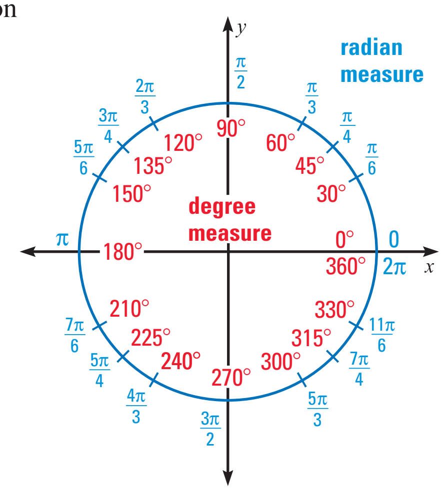


Because the circumference of a circle is $2\pi r$, there are 2π radians in a full circle. Degree measure and radian measure are therefore related by the equation $360° = 2\pi$ radians, or $180° = \pi$ radians.

The diagram shows equivalent radian and degree measures for special angles from 0° to 360° (0 radians to 2π radians).

You may find it helpful to memorize the equivalent degree and radian measures of special angles in the first quadrant and for $90° = \frac{\pi}{2}$ radians. All other special angles are just multiples of these angles.

You can use the following rules to convert degrees to radians and radians to degrees.



STUDENT HELP

Study Tip
When no units of angle measure are specified, radian measure is implied. For instance, $\theta = 2$ means that $\theta = 2$ radians.

CONVERSIONS BETWEEN DEGREES AND RADIANS

- To rewrite a degree measure in radians, multiply by $\frac{\pi \text{ radians}}{180°}$.
- To rewrite a radian measure in degrees, multiply by $\frac{180°}{\pi \text{ radians}}$.

EXAMPLE 3 Converting Between Degrees and Radians

a. Convert 110° to radians. **b.** Convert $-\frac{\pi}{9}$ radians to degrees.

SOLUTION

a. $110° = 110°\left(\frac{\pi \text{ radians}}{180°}\right)$

$= \frac{11\pi}{18}$ radians

✓**CHECK** Check that your answer is reasonable:

The angle 110° is between the special angles 90° and 120°. The angle $\frac{11\pi}{18}$ is between the same special angles: $\frac{\pi}{2} = \frac{9\pi}{18}$ and $\frac{2\pi}{3} = \frac{12\pi}{18}$.

b. $-\frac{\pi}{9} = \left(-\frac{\pi}{9} \text{ radians}\right)\left(\frac{180°}{\pi \text{ radians}}\right)$

$= -20°$

✓**CHECK** Check that your answer is reasonable:

The angle $-\frac{\pi}{9}$ is between the special angles 0 and $-\frac{\pi}{6}$. The angle $-20°$ is between the same special angles: 0° and $-30°$.

REAL LIFE **Bicycles**

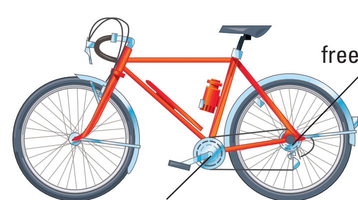
EXAMPLE 4 Measuring an Angle for a Bicycle

A bicycle's *gear ratio* is the number of times the freewheel turns for every one turn of the chainwheel. The table shows the number of teeth in the freewheel and chainwheel for the first 5 gears on an 18-speed touring bicycle. In fourth gear, if the chainwheel completes 3 rotations, through what angle does the freewheel turn? Give your answer in both degrees and radians. ▶ Source: *The All New Complete Book of Bicycling*

STUDENT HELP

INTERNET **APPLICATION LINK** Visit our Web site www.mcdougallittell.com for more information about bicycle gears in Example 4.

Gear number	Number of teeth in freewheel	Number of teeth in chainwheel
1	32	24
2	26	24
3	22	24
4	32	40
5	19	24



SOLUTION

In fourth gear, the gear ratio is $\frac{40}{32}$. For every one turn of the chainwheel in this gear, the freewheel makes 1.25 rotations. The measure of the angle θ through which the freewheel turns when the chainwheel completes 3 rotations is:

$$\theta = (3.75 \text{ rotations})\left(\frac{360°}{1 \text{ rotation}}\right) = 1350°$$

To find the angle measure in radians, multiply by $\frac{\pi \text{ radians}}{180°}$:

$$\theta = 1350°\left(\frac{\pi \text{ radians}}{180°}\right) = \frac{15\pi}{2} \text{ radians}$$

GOAL 2 ARC LENGTHS AND AREAS OF SECTORS

A **sector** is a region of a circle that is bounded by two radii and an arc of the circle. The **central angle** θ of a sector is the angle formed by the two radii. There are simple formulas for the arc length and area of a sector when the central angle is measured in radians.

STUDENT HELP

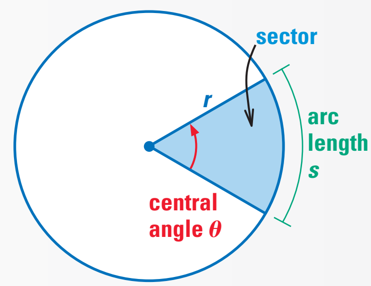
Derivations
For a derivation of the formula for arc length, see p. 900.

ARC LENGTH AND AREA OF A SECTOR

The arc length s and area A of a sector with radius r and central angle θ (measured in radians) are as follows.

Arc length: $s = r\theta$

Area: $A = \frac{1}{2}r^2\theta$



EXAMPLE 5 *Finding Arc Length and Area*

Find the arc length and area of a sector with a radius of 9 cm and a central angle of 60°.

SOLUTION

First convert the angle measure to radians.

$$\theta = 60°\left(\frac{\pi \text{ radians}}{180°}\right) = \frac{\pi}{3} \text{ radians}$$

Then use the formulas for arc length and area.

$$\text{Arc length: } s = r\theta = 9\left(\frac{\pi}{3}\right) = 3\pi \text{ centimeters}$$

$$\text{Area: } A = \frac{1}{2}r^2\theta = \frac{1}{2}(9^2)\left(\frac{\pi}{3}\right) = \frac{27\pi}{2} \text{ square centimeters}$$

FOCUS ON APPLICATIONS

REAL LIFE **SPACE NEEDLE** The restaurant at the top of the Space Needle in Seattle, Washington, is circular and has a radius of 47.25 feet. The dining part of the restaurant (by the windows) revolves, making about one complete revolution per hour.

EXAMPLE 6 *Finding an Angle and Arc Length*

SPACE NEEDLE Read the photo caption at the left. You go to dinner at the Space Needle and sit at a window table at 6:42 P.M. Your dinner ends at 8:18 P.M. Through what angle do you rotate during your stay? How many feet do you revolve?

SOLUTION

You spend 96 minutes at dinner. Because the Space Needle makes one complete revolution every 60 minutes, your angle of rotation is:

$$\theta = \frac{96}{60}(2\pi) = \frac{16\pi}{5} \text{ radians}$$

Because the radius is 47.25 feet, you move through an arc length of:

$$s = r\theta = 47.25\left(\frac{16\pi}{5}\right) \approx 475 \text{ feet}$$

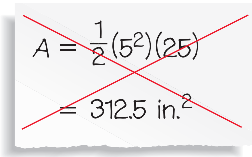
GUIDED PRACTICE

Vocabulary Check ✓

1. In your own words, describe what a radian is.

Concept Check ✓

2. **ERROR ANALYSIS** An error has been made in finding the area of a sector with a radius of 5 inches and a central angle of 25°. Find and correct the error.



3. How does the sign of an angle's measure determine its direction of rotation?

4. A circle has radius r. What is the length of the arc corresponding to a central angle of π radians?

Skill Check ✓

Draw an angle with the given measure in standard position. Then find one positive and one negative coterminal angle.

5. 60°   6. −45°   7. $\frac{7\pi}{4}$ 8. 300°

9. $-\frac{3\pi}{2}$ 10. $\frac{7\pi}{8}$ 11. 150°   12. $-\frac{5\pi}{4}$

Rewrite each degree measure in radians and each radian measure in degrees.

13. 30°   14. 100°   15. 260°   16. −320°

17. $\frac{7\pi}{4}$ 18. $\frac{18\pi}{4}$ 19. $\frac{\pi}{12}$ 20. $-\frac{5\pi}{2}$

Find the arc length and area of a sector with the given radius r and central angle θ.

21. $r = 4$ in., $\theta = 55°$ 22. $r = 5$ m, $\theta = 135°$ 23. $r = 2$ cm, $\theta = 85°$

24. **SPACE NEEDLE** Recall from Example 6 on page 779 that the circular restaurant at the Space Needle has a radius of 47.25 feet and rotates about once per hour. If you are seated at a window table from 6:00 P.M. to 8:10 P.M., through what angle do you rotate? How many feet do you revolve?

PRACTICE AND APPLICATIONS

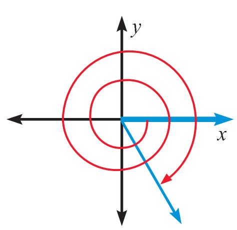
STUDENT HELP

Extra Practice to help you master skills is on p. 958.

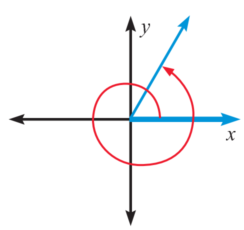
VISUAL THINKING **Match the angle measure with the angle.**

25. −210°   26. 420°   27. $-\frac{13\pi}{3}$

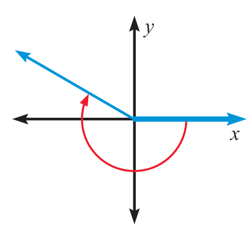
A.



B.



C.



DRAWING ANGLES **Draw an angle with the given measure in standard position.**

28. 144°   29. $\frac{2\pi}{9}$ 30. −15°   31. $-\frac{7\pi}{6}$

32. $\frac{19\pi}{12}$ 33. 1620°   34. -5π 35. $-\frac{13\pi}{4}$

STUDENT HELP

HOMEWORK HELP
Example 1: Exs. 25–35
Example 2: Exs. 36–43
Example 3: Exs. 44–59
Example 4: Exs. 77–81
Example 5: Exs. 60–68
Example 6: Exs. 82–88

FINDING COTERMINAL ANGLES **Find one positive angle and one negative angle coterminal with the given angle.**

36. $55°$ **37.** $210°$ **38.** $420°$ **39.** $780°$

40. $\frac{13\pi}{2}$ **41.** $\frac{17\pi}{4}$ **42.** $\frac{24\pi}{7}$ **43.** $\frac{16\pi}{3}$

CONVERTING MEASURES **Rewrite each degree measure in radians and each radian measure in degrees.**

44. $25°$ **45.** $225°$ **46.** $160°$ **47.** $45°$

48. $-110°$ **49.** $325°$ **50.** $400°$ **51.** $-290°$

52. $\frac{7\pi}{3}$ **53.** $-\frac{9\pi}{2}$ **54.** $\frac{\pi}{10}$ **55.** $-\frac{5\pi}{12}$

56. $\frac{7\pi}{15}$ **57.** $-\frac{15\pi}{4}$ **58.** $-\frac{5\pi}{6}$ **59.** $\frac{8\pi}{5}$

FINDING ARC LENGTH AND AREA **Find the arc length and area of a sector with the given radius *r* and central angle *θ*.**

60. $r = 3$ in., $\theta = \frac{\pi}{4}$ **61.** $r = 3$ ft, $\theta = \frac{\pi}{18}$ **62.** $r = 2$ cm, $\theta = \frac{9\pi}{20}$

63. $r = 12$ in., $\theta = 90°$ **64.** $r = 5$ m, $\theta = 120°$ **65.** $r = 15$ mm, $\theta = 175°$

66. $r = 4$ ft, $\theta = 200°$ **67.** $r = 16$ cm, $\theta = 50°$ **68.** $r = 20$ ft, $\theta = 270°$

EVALUATING FUNCTIONS **Evaluate the trigonometric function using a calculator if necessary. If possible, give an exact answer.**

69. $\sin \frac{\pi}{6}$ **70.** $\cos \frac{\pi}{4}$ **71.** $\tan \frac{\pi}{3}$ **72.** $\cos \frac{4\pi}{11}$

73. $\cot \frac{\pi}{5}$ **74.** $\sec \frac{\pi}{8}$ **75.** $\sin \frac{2\pi}{9}$ **76.** $\csc \frac{3\pi}{10}$

FIGURE SKATING **In Exercises 77–79, use the following information.** The number of revolutions made by a figure skater for each type of Axel jump is given. Determine the measure of the angle generated as the skater performs the jump. Give the answer in both degrees and radians.

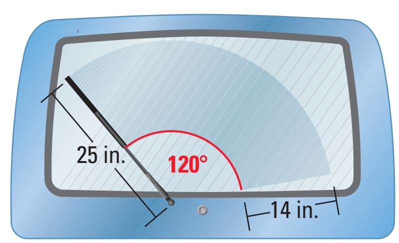
77. Single Axel: $1\frac{1}{2}$ **78.** Double Axel: $2\frac{1}{2}$ **79.** Triple Axel: $3\frac{1}{2}$

80. **TIME IN SCHOOL** You are in school from 8:00 A.M. to 3:00 P.M. Draw a diagram that shows the number of rotations completed by the minute hand of a clock during this time. Find the measure of the angle generated by the minute hand. Give the answer in both degrees and radians.

81. **BICYCLE GEARS** Look back at Example 4 on page 778. In fifth gear, if the bicycle's chainwheel completes 4 rotations, through what angle does the freewheel turn? Give your answer in both degrees and radians.

82. **FARMING TECHNOLOGY** A sprinkler system on a farm rotates 140° and sprays water up to 35 meters. Draw a diagram that shows the region that can be irrigated with the sprinkler. Then find the area of the region.

83. **WINDSHIELD WIPERS** A car's rear windshield wiper rotates 120° as shown. Find the area covered by the wiper.



FOCUS ON PEOPLE

AXEL PAULSEN, a Norwegian speed skating champion, invented the Axel jump in 1882. It is the only jump in figure skating that requires taking off from a forward position.

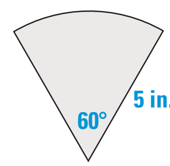
SPIRAL STAIRS **In Exercises 84–86, use the following information.**
A spiral staircase has 13 steps. Each step is a sector with a radius of 36 inches and a central angle of $\frac{\pi}{7}$.

84. What is the length of the arc formed by the outer edge of each step?

85. Through what angle would you rotate by climbing the stairs? Include a fourteenth turn for stepping up on the landing.

86. How many square inches of carpeting would you need to cover the 13 steps?

SNOW CONES **In Exercises 87 and 88, use the following information.**
You are starting a business selling homemade snow cones in paper cups. You cut out a paper cup in the shape of a sector.



87. The sector has a central angle of 60° and a radius of 5 inches. When you shape the sector into a cone without overlapping edges, what will the cone's diameter be?

88. Suppose you want to make a cone that has a diameter of 4 inches and a slant height of 6 inches. What should the radius and central angle of the sector be?

Test Preparation

QUANTITATIVE COMPARISON **In Exercises 89 and 90, choose the statement that is true about the given quantities.**

(A) The quantity in column A is greater.

(B) The quantity in column B is greater.

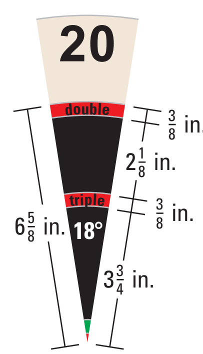
(C) The two quantities are equal.

(D) The relationship cannot be determined from the given information.

	Column A	Column B
89.	Arc length of a sector with $r = 2$ inches and $\theta = 45°$	Arc length of a sector with $r = 2.5$ inches and $\theta = \frac{\pi}{5}$
90.	Area of a sector with $r = 2$ inches and $\theta = 45°$	Area of a sector with $r = 2.5$ inches and $\theta = \frac{\pi}{5}$

★ Challenge

DARTS **In Exercises 91 and 92, use the following information.**
A dart board is divided into 20 sectors. Each sector is worth a point value from 1 to 20 and has shaded regions that double or triple this value. The 20 point sector is shown at the right.



91. Find the area of the sector. Then find the areas of the double region and the triple region in the sector.

92. If you throw a dart and it randomly lands somewhere inside the sector, what is the probability that it lands within the double region? within the triple region?

EXTRA CHALLENGE
www.mcdougallittell.com

MIXED REVIEW

PROPERTIES OF SQUARE ROOTS **Simplify the expression.** (Review 5.3 for 13.3)

93. $\sqrt{275}$ **94.** $\sqrt{1216}$ **95.** $\sqrt{8} \cdot \sqrt{32}$ **96.** $\sqrt{18} \cdot \sqrt{24}$

97. $\sqrt{\frac{7}{16}}$ **98.** $\sqrt{\frac{11}{36}}$ **99.** $\frac{\sqrt{8}}{\sqrt{7}}$ **100.** $\frac{\sqrt{12}}{\sqrt{5}}$

EVALUATING EXPRESSIONS **Evaluate the expression $\frac{x^2}{2y + 5}$ for the given values of x and y.** (Review 1.2 for 13.3)

101. $x = 6, y = 11$ **102.** $x = 3, y = -3$ **103.** $x = 12, y = 15$

104. $x = -1, y = -5$ **105.** $x = -10, y = 16$ **106.** $x = -20, y = -25$

WRITING EQUATIONS **Write the standard form of the equation of the parabola with the given focus and vertex at (0, 0).** (Review 10.2)

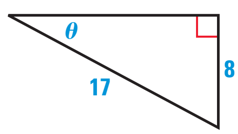
107. $(5, 0)$ **108.** $(-3, 0)$ **109.** $(6, 0)$

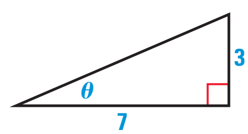
110. $(0, -12)$ **111.** $(0, -4.4)$ **112.** $(0, 15)$

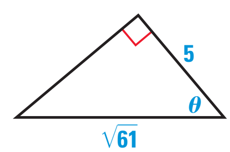
QUIZ 1

Self-Test for Lessons 13.1 and 13.2

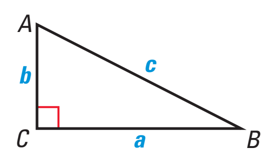
Evaluate the six trigonometric functions of θ. (Lesson 13.1)

1.


2.


3.


Solve △*ABC* using the diagram at the right and the given measurements. (Lesson 13.1)



4. $B = 50°, a = 18$ **5.** $A = 33°, c = 12$

6. $A = 10°, a = 3$ **7.** $B = 71°, c = 2.3$

Find one positive angle and one negative angle coterminal with the given angle. (Lesson 13.2)

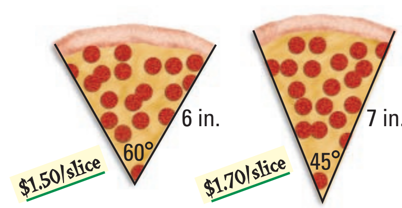
8. $25°$ **9.** $-\frac{14\pi}{3}$ **10.** $\frac{33\pi}{4}$ **11.** $-6200°$

Find the arc length and area of a sector with the given radius r and central angle θ. (Lesson 13.2)

12. $r = 6$ m, $\theta = \frac{\pi}{3}$ **13.** $r = 2$ ft, $\theta = \frac{5\pi}{6}$ **14.** $r = 8$ cm, $\theta = 20°$

15. $r = 22$ in., $\theta = 220°$ **16.** $r = 5$ ft, $\theta = 75°$ **17.** $r = 12$ mm, $\theta = 160°$

18. **THE BEST DEAL** Decide which of the two pizza slices shown is the best deal. Explain your reasoning. (Lesson 13.2)



13.3 Trigonometric Functions of Any Angle

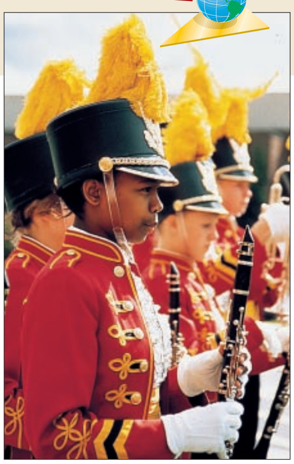
What *you should learn*

GOAL 1 Evaluate trigonometric functions of any angle.

GOAL 2 Use trigonometric functions to solve **real-life** problems, such as finding the distance a soccer ball is kicked in **Ex. 71**.

Why *you should learn it*

▼ To solve **real-life** problems, such as finding distances for a marching band on a football field in **Example 6**.

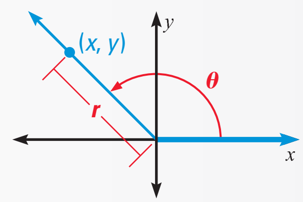


GOAL 1 EVALUATING TRIGONOMETRIC FUNCTIONS

In Lesson 13.1 you learned how to evaluate trigonometric functions of an acute angle. In this lesson you will learn to evaluate trigonometric functions of *any* angle.

GENERAL DEFINITION OF TRIGONOMETRIC FUNCTIONS

Let θ be an angle in standard position and (x, y) be any point (except the origin) on the terminal side of θ. The six trigonometric functions of θ are defined as follows.



$\sin\theta = \frac{y}{r}$ $\qquad$ $\csc\theta = \frac{r}{y},\ y \neq 0$

$\cos\theta = \frac{x}{r}$ $\qquad$ $\sec\theta = \frac{r}{x},\ x \neq 0$

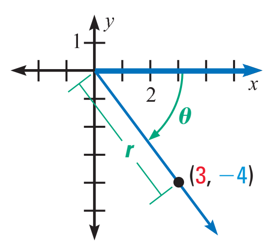
$\tan\theta = \frac{y}{x},\ x \neq 0$ $\qquad$ $\cot\theta = \frac{x}{y},\ y \neq 0$

Pythagorean theorem gives $r = \sqrt{x^2 + y^2}$.

For acute angles, these definitions give the same values as those given by the definitions in Lesson 13.1.

EXAMPLE 1 *Evaluating Trigonometric Functions Given a Point*

Let $(3, -4)$ be a point on the terminal side of an angle θ in standard position. Evaluate the six trigonometric functions of θ.



SOLUTION

Use the Pythagorean theorem to find the value of r.

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ &= \sqrt{3^2 + (-4)^2} \\ &= \sqrt{25} \\ &= 5 \end{aligned}$$

Using $x = 3$, $y = -4$, and $r = 5$, you can write the following.

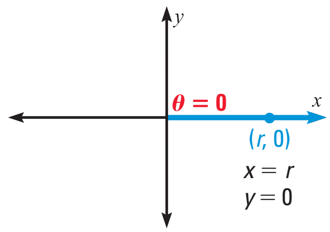
$\sin\theta = \frac{y}{r} = -\frac{4}{5}$ $\qquad$ $\csc\theta = \frac{r}{y} = -\frac{5}{4}$

$\cos\theta = \frac{x}{r} = \frac{3}{5}$ $\qquad$ $\sec\theta = \frac{r}{x} = \frac{5}{3}$

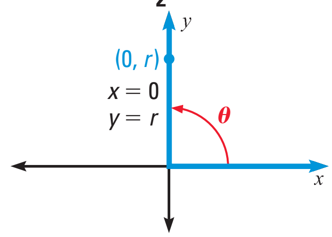
$\tan\theta = \frac{y}{x} = -\frac{4}{3}$ $\qquad$ $\cot\theta = \frac{x}{y} = -\frac{3}{4}$

If the terminal side of θ lies on an axis, then θ is a **quadrantal angle**. The diagrams below show the values of x and y for the quadrantal angles 0°, 90°, 180°, and 270°.

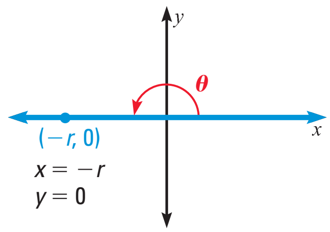
0° or 0 radians



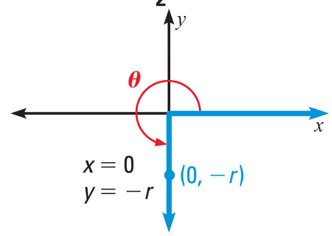
90° or $\frac{\pi}{2}$ radians



180° or π radians



270° or $\frac{3\pi}{2}$ radians



EXAMPLE 2 *Trigonometric Functions of a Quadrantal Angle*

Evaluate the six trigonometric functions of $\theta = 180°$.

Solution

When $\theta = 180°$, $x = -r$ and $y = 0$. The six trigonometric functions of θ are as follows.

$$\sin\theta = \frac{y}{r} = \frac{0}{r} = 0 \qquad \csc\theta = \frac{r}{y} = \frac{r}{0} = \text{undefined}$$

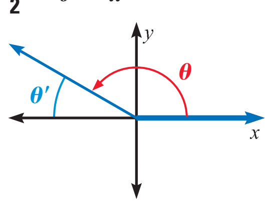
$$\cos\theta = \frac{x}{r} = \frac{-r}{r} = -1 \qquad \sec\theta = \frac{r}{x} = \frac{r}{-r} = -1$$

$$\tan\theta = \frac{y}{x} = \frac{0}{-r} = 0 \qquad \cot\theta = \frac{x}{y} = \frac{-r}{0} = \text{undefined}$$

. . . .

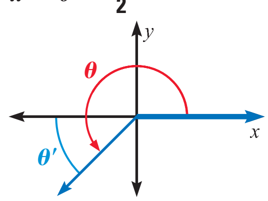
The values of trigonometric functions of angles greater than 90° (or less than 0°) can be found using corresponding acute angles called *reference angles*. Let θ be an angle in standard position. Its **reference angle** is the acute angle θ' (read *theta prime*) formed by the terminal side of θ and the x-axis. The relationship between θ and θ' is given below for nonquadrantal angles θ such that $90° < \theta < 360°$ $\left(\frac{\pi}{2} < \theta < 2\pi\right)$.

$90° < \theta < 180°$;
$\frac{\pi}{2} < \theta < \pi$



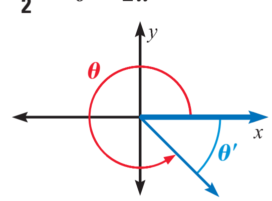
Degrees: $\theta' = 180° - \theta$
Radians: $\theta' = \pi - \theta$

$180° < \theta < 270°$;
$\pi < \theta < \frac{3\pi}{2}$



Degrees: $\theta' = \theta - 180°$
Radians: $\theta' = \theta - \pi$

$270° < \theta < 360°$;
$\frac{3\pi}{2} < \theta < 2\pi$



Degrees: $\theta' = 360° - \theta$
Radians: $\theta' = 2\pi - \theta$

EXAMPLE 3 *Finding Reference Angles*

Find the reference angle θ' for each angle θ.

a. $\theta = 320°$ **b.** $\theta = -\frac{5\pi}{6}$

SOLUTION

a. Because $270° < \theta < 360°$, the reference angle is $\theta' = 360° - 320° = 40°$.

b. Because θ is coterminal with $\frac{7\pi}{6}$ and $\pi < \frac{7\pi}{6} < \frac{3\pi}{2}$, the reference angle is $\theta' = \frac{7\pi}{6} - \pi = \frac{\pi}{6}$.

....

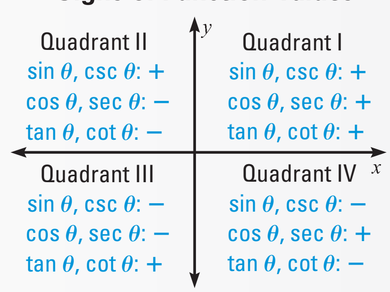
The signs of the trigonometric function values in the four quadrants can be determined from the function definitions. For instance, because $\cos\theta = \frac{x}{r}$ and r is always positive, it follows that $\cos\theta$ is positive wherever $x > 0$, which is in Quadrants I and IV.

CONCEPT SUMMARY — EVALUATING TRIGONOMETRIC FUNCTIONS

Use these steps to evaluate a trigonometric function of any angle θ.

1. Find the reference angle θ'.
2. Evaluate the trigonometric function for the angle θ'.
3. Use the quadrant in which θ lies to determine the sign of the trigonometric function value of θ. (See the diagram at the right.)

Signs of Function Values



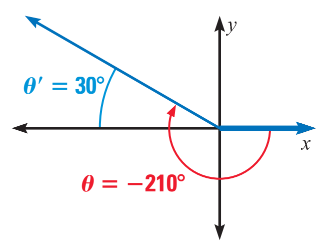
EXAMPLE 4 *Using Reference Angles to Evaluate Trigonometric Functions*

STUDENT HELP

INTERNET HOMEWORK HELP
Visit our Web site www.mcdougallittell.com for extra examples.

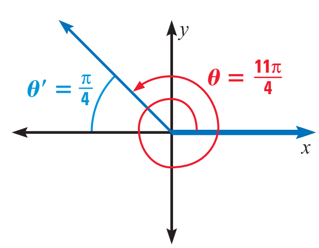
Evaluate (**a**) $\tan(-210°)$ and (**b**) $\csc\frac{11\pi}{4}$.

SOLUTION



a. The angle $-210°$ is coterminal with $150°$.
The reference angle is $\theta' = 180° - 150° = 30°$.
The tangent function is negative in Quadrant II, so you can write:

$$\tan(-210°) = -\tan 30° = -\frac{\sqrt{3}}{3}$$



b. The angle $\frac{11\pi}{4}$ is coterminal with $\frac{3\pi}{4}$. The reference angle is $\theta' = \pi - \frac{3\pi}{4} = \frac{\pi}{4}$. The cosecant function is positive in Quadrant II, so you can write:

$$\csc\frac{11\pi}{4} = \csc\frac{\pi}{4} = \sqrt{2}$$

FOCUS ON APPLICATIONS

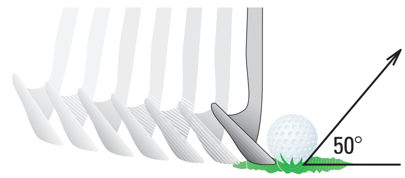
GOLF BALLS The dimples on a golf ball create pockets of air turbulence that keep the ball in the air for a longer period of time than if the ball were smooth. The longest drive of a golf ball on record is 473 yards, 2 feet, 6 inches.

INTERNET **DATA UPDATE**
www.mcdougallittell.com

GOAL 2 USING TRIGONOMETRIC FUNCTIONS IN REAL LIFE

EXAMPLE 5 *Calculating Projectile Distance*

GOLF The horizontal distance d (in feet) traveled by a projectile with an initial speed v (in feet per second) is given by



$$d = \frac{v^2}{32}\sin 2\theta$$

where θ is the angle at which the projectile is launched. Estimate the horizontal distance traveled by a golf ball that is hit at an angle of 50° with an initial speed of 105 feet per second. (This model neglects air resistance and wind conditions. It also assumes that the projectile's starting and ending heights are the same.)

SOLUTION

The horizontal distance given by the model is:

$$d = \frac{v^2}{32}\sin 2\theta \qquad \text{Write distance model.}$$

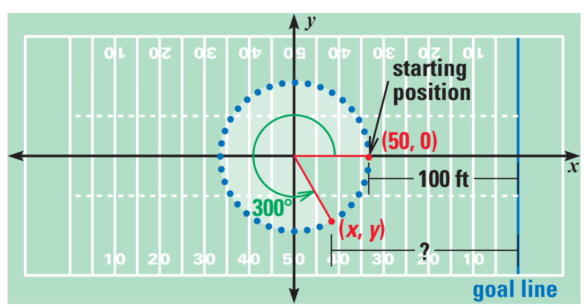
$$= \frac{105^2}{32}\sin(2 \cdot 50°) \approx 339 \text{ feet} \qquad \text{Substitute and use a calculator.}$$

▶ The golf ball travels a horizontal distance of about 339 feet.

Marching Band

EXAMPLE 6 *Modeling with Trigonometric Functions*

Your school's marching band is performing at halftime during a football game. In the last formation, the band members form a circle 100 feet wide in the center of the field. Your starting position is 100 feet from the goal line, where you will exit the field. How far from the goal line will you be after you have marched 300° around the circle?



SOLUTION

The radius of the circle is $r = 50$. So, you can write:

$$\cos 300° = \frac{x}{r} \qquad \text{Use definition of cosine.}$$

$$\frac{1}{2} = \frac{x}{50} \qquad \text{Substitute.}$$

$$25 = x \qquad \text{Solve for } x.$$

▶ You will be $100 + (50 - 25) = 125$ feet from the goal line.

GUIDED PRACTICE

Vocabulary Check ✓

1. Define the terms quadrantal angle and reference angle.

Concept Check ✓

2. Given an angle θ in Quadrant III, explain how you can use a reference angle to find $\sin \theta$.

3. Explain why $\tan 270°$ is undefined.

4. In which quadrant(s) must θ lie for $\cos \theta$ to be positive?

Skill Check ✓

5. Let $(-4, -5)$ be a point on the terminal side of an angle θ in standard position. Evaluate the six trigonometric functions of θ.

Sketch the angle. Then find its reference angle.

6. $\frac{7\pi}{4}$ **7.** $-120°$ **8.** $\frac{7\pi}{8}$ **9.** $390°$

10. $-\frac{2\pi}{3}$ **11.** $-370°$ **12.** $\frac{2\pi}{3}$ **13.** $230°$

Evaluate the function without using a calculator.

14. $\cos\left(-\frac{4\pi}{3}\right)$ **15.** $\tan 240°$ **16.** $\sin \frac{7\pi}{4}$ **17.** $\csc(-225°)$

18. $\cot\left(-\frac{3\pi}{4}\right)$ **19.** $\cos 240°$ **20.** $\sec \frac{11\pi}{6}$ **21.** $\tan \frac{5\pi}{6}$

22. **MARCHING BAND** Look back at Example 6 on page 787. Suppose you marched 135° around the circle from the same starting position. How far from the goal line would you be?

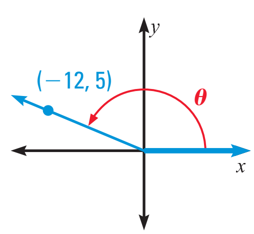
PRACTICE AND APPLICATIONS

STUDENT HELP

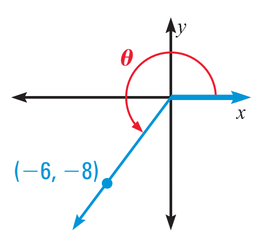
Extra Practice to help you master skills is on p. 958.

USING A POINT **Use the given point on the terminal side of an angle θ in standard position. Evaluate the six trigonometric functions of θ.**

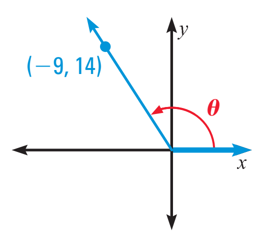
23.



24.



25.



26. $(-12, -15)$ **27.** $(-1, 1)$ **28.** $(15, -8)$ **29.** $(6, -9)$

30. $(7, 10)$ **31.** $(1, -\sqrt{3})$ **32.** $(-3, -4)$ **33.** $(-15, 5\sqrt{7})$

STUDENT HELP

HOMEWORK HELP
Example 1: Exs. 23–33
Example 2: Exs. 34–36
Example 3: Exs. 37–44
Example 4: Exs. 45–60
Example 5: Exs. 69–71
Example 6: Exs. 72–76

QUADRANTAL ANGLES **Evaluate the six trigonometric functions of θ.**

34. $\theta = 90°$ **35.** $\theta = 270°$ **36.** $\theta = 0°$

FINDING REFERENCE ANGLES **Sketch the angle. Then find its reference angle.**

37. $240°$ **38.** $-515°$ **39.** $-170°$ **40.** $315°$

41. $-440°$ **42.** $-\frac{3\pi}{4}$ **43.** $\frac{25\pi}{4}$ **44.** $-\frac{11\pi}{3}$

EVALUATING FUNCTIONS **Evaluate the function without using a calculator.**

45. $\cos 315°$ **46.** $\cos(-210°)$ **47.** $\csc(-240°)$ **48.** $\tan 210°$

49. $\sec 780°$ **50.** $\sin 225°$ **51.** $\cos(-225°)$ **52.** $\tan(-120°)$

53. $\cot \frac{11\pi}{6}$ **54.** $\sec \frac{9\pi}{4}$ **55.** $\sin\left(-\frac{5\pi}{6}\right)$ **56.** $\cos \frac{5\pi}{3}$

57. $\sin\left(-\frac{17\pi}{6}\right)$ **58.** $\sec \frac{23\pi}{6}$ **59.** $\csc \frac{17\pi}{3}$ **60.** $\cot\left(-\frac{13\pi}{4}\right)$

STUDENT HELP

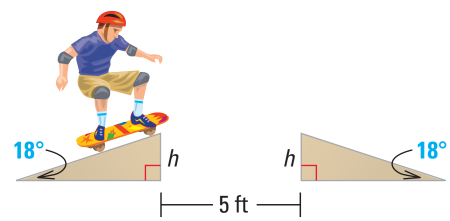
Study Tip
Make sure your calculator is in radian mode when finding trigonometric functions of angles measured in radians.

USING A CALCULATOR **Use a calculator to evaluate the function. Round the result to four decimal places.**

61. $\sec 137°$ **62.** $\cot 400°$ **63.** $\sin(-10°)$ **64.** $\csc 540°$

65. $\cot\left(-\frac{4\pi}{5}\right)$ **66.** $\sec \frac{11\pi}{2}$ **67.** $\cos \frac{6\pi}{5}$ **68.** $\csc \frac{23\pi}{8}$

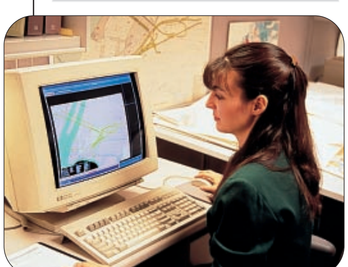
69. **SKATEBOARDING** A skateboarder is setting up two ramps for a jump as shown. He wants to jump off one ramp and land on the other. If the ramps are placed 5 feet apart, at what speed must the skateboarder launch off the first ramp to land on the second ramp?



70. **VOLLEYBALL** While playing a game of volleyball, you set the ball to your teammate. You hit the ball with an initial speed of 24 feet per second at an angle of 70°. About how far away should your teammate be to receive your set?

71. **SOCCER** You and a friend are playing soccer. Both of you kick the ball with an initial speed of 42 feet per second. Your kick was projected at an angle of 45° and your friend's kick was projected at an angle of 60°. About how much farther will your soccer ball go than your friend's soccer ball?

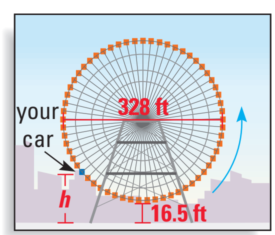
FOCUS ON CAREERS



REAL LIFE **CARTOGRAPHER**
Cartographers compile information from aerial photographs and satellite data to map Earth's surface. A map's circles of latitude and longitude, as discussed in Exs. 73 and 74, are used to describe location.

INTERNET **CAREER LINK**
www.mcdougallittell.com

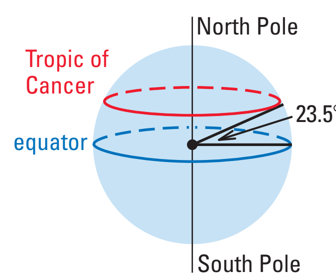
72. **FERRIS WHEEL** The largest Ferris wheel in operation is the Cosmolock 21 at Yokohama City, Japan. It has a diameter of 328 feet. Passengers board the cars at the bottom of the wheel, about 16.5 feet above the ground. Imagine that you have boarded the Cosmolock 21. The wheel rotates 312° and then stops. How high above the ground are you?



SOCIAL STUDIES CONNECTION **In Exercises 73 and 74, use the information below.**
The Tropic of Cancer is the circle of latitude farthest north of the equator where the sun can appear directly overhead. It lies 23.5° north of the equator, as shown below.

73. Find the circumference of the Tropic of Cancer using 3960 miles as Earth's approximate radius.

74. What is the distance between two points that lie directly across from each other (through the axis) on the Tropic of Cancer?



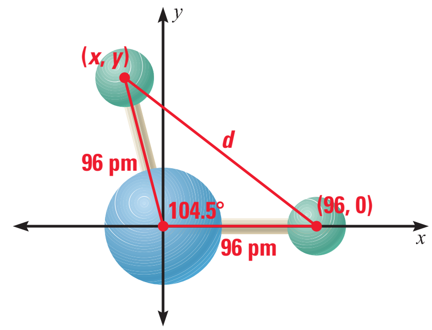
STUDENT HELP

Look Back
For help with the distance formula, see p. 589.

SCIENCE CONNECTION **In Exercises 75 and 76, use the following information.** When two atoms in a molecule are bonded to a common atom, chemists are interested in both the bond angle and the bond length. A water molecule (H_2O) is made up of two hydrogen atoms bonded to an oxygen atom. The diagram below shows a coordinate plane superimposed on a cross section of a water molecule.

75. In the diagram, coordinates are given in picometers (pm). (Note: 1 pm $= 10^{-12}$ m). If the center of one hydrogen atom has coordinates (96, 0), find the coordinates (x, y) of the center of the other hydrogen atom.

76. Use your answer to Exercise 75 and the distance formula to find the distance d (in picometers) between the centers of the two hydrogen atoms.



Test Preparation

77. MULTIPLE CHOICE What is the value of $\sec\left(\frac{40\pi}{3}\right)$?

Ⓐ -2 Ⓑ $-\sqrt{2}$ Ⓒ $-\frac{\sqrt{2}}{2}$ Ⓓ $-\frac{1}{2}$ Ⓔ $\sqrt{2}$

78. MULTIPLE CHOICE What is the approximate horizontal distance traveled by a football that is kicked at an angle of 40° with an initial speed of 70 feet per second?

Ⓐ 98 feet Ⓑ 142 feet Ⓒ 151 feet Ⓓ 157 feet Ⓔ 280 feet

★ Challenge

79. CRITICAL THINKING If θ is an angle in Quadrant II and $\tan\theta = -2$, find the values of the other five trigonometric functions of θ.

80. CRITICAL THINKING If θ is an angle in Quadrant III and $\cos\theta = -0.64$, find the values of the other five trigonometric functions of θ.

MIXED REVIEW

HORIZONTAL LINE TEST **Graph the function. Then use the graph to determine whether the inverse of f is a function.** (Review 7.4 for 13.4)

81. $f(x) = x - 3$ **82.** $f(x) = 4x + 5$ **83.** $f(x) = 5x^2$

84. $f(x) = 5x^3$ **85.** $f(x) = 3x^2 - 7$ **86.** $f(x) = -|x + 2|$

CHOOSING CARDS **A card is randomly drawn from a standard 52-card deck. Find the probability of the given event. (A face card is a king, queen, or jack.)** (Review 12.4)

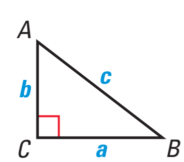
87. a king and a diamond **88.** a jack or a club **89.** a ten or a face card

SOLVING TRIANGLES **Solve $\triangle ABC$ using the diagram and the given measurements.** (Review 13.1)

90. $A = 62°, b = 5$ **91.** $B = 20°, c = 22$

92. $B = 31°, a = 17$ **93.** $A = 50°, c = 3$

94. $B = 75°, b = 34$ **95.** $A = 83°, a = 50$



ACTIVITY 13.4

Developing Concepts

Group Activity for use with Lesson 13.4

Investigating Inverse Trigonometric Functions

GROUP ACTIVITY
Work in a small group.

MATERIALS
- paper
- pencil

QUESTION **Do the trigonometric functions sine, cosine, and tangent have inverses that are also functions?**

EXPLORING THE CONCEPT

1 Copy and complete the table to find the value of $f(x) = x^2$ for each of the given x-values.

x	−4	−3	−2	−1	0	1	2	3	4
$f(x) = x^2$	?	?	?	?	?	?	?	?	?

2 For a function to have an inverse, it must be true that no two values of x are paired with the same value of y. Use your completed table to explain why the function $f(x) = x^2$ does not have an inverse on the domain $-4 \le x \le 4$.

3 Restrict the domain of $f(x) = x^2$ so that it does have an inverse. Explain why you chose the domain you did.

4 Copy and complete the table to find the values of $f(\theta) = \sin\theta$, $g(\theta) = \cos\theta$, and $h(\theta) = \tan\theta$ for each of the given values of θ.

θ	$-\pi$	$-\frac{3\pi}{4}$	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
$f(\theta) = \sin\theta$	?	?	?	?	?	?	?	?	?
$g(\theta) = \cos\theta$	?	?	?	?	?	?	?	?	?
$h(\theta) = \tan\theta$	?	?	?	?	?	?	?	?	?

5 Use the table to explain why $f(\theta) = \sin\theta$ does not have an inverse on the domain $-\pi \le \theta \le \pi$.

6 Does $g(\theta) = \cos\theta$ have an inverse on the domain $-\pi \le \theta \le \pi$? Explain why or why not.

7 Does $h(\theta) = \tan\theta$ have an inverse on the domain $-\pi \le \theta \le \pi$? Explain why or why not.

DRAWING CONCLUSIONS

1. Use the table you completed in **Step 4** to choose a restricted domain for which $f(\theta) = \sin\theta$ does have an inverse. Explain how you made your choice.
2. Write a restricted domain for which $g(\theta) = \cos\theta$ has an inverse. Explain how you chose the domain.
3. Write a restricted domain for which $h(\theta) = \tan\theta$ has an inverse. Explain how you chose the domain.
4. Are the domains that you wrote in Exercises 1–3 the *only* domains for which the trigonometric functions have inverses? Explain.

13.4 Inverse Trigonometric Functions

What *you should learn*

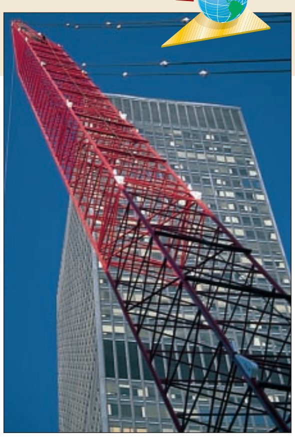
GOAL 1 Evaluate inverse trigonometric functions.

GOAL 2 Use inverse trigonometric functions to solve **real-life** problems, such as finding an angle of repose in **Example 4**.

Why *you should learn it*

▼ To solve **real-life** problems, such as finding the angle at which to set the arm of a crane in **Example 5**.

REAL LIFE



GOAL 1 EVALUATING AN INVERSE TRIGONOMETRIC FUNCTION

In the first three lessons of this chapter, you learned to evaluate trigonometric functions of a given angle. In this lesson you will study the reverse problem—finding angles that correspond to a given value of a trigonometric function.

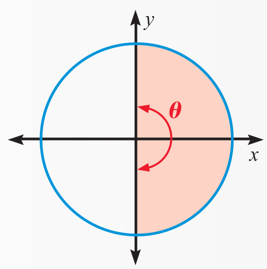
Suppose you were asked to find an angle θ whose sine is 0.5. After thinking about the problem for a while, you would probably realize that there are *many* such angles. For instance, the angles

$$\frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}, \text{ and } -\frac{7\pi}{6}$$

all have a sine value of 0.5. (Try checking this with a calculator.) Of these, the value of the *inverse sine function* at 0.5 is defined to be $\frac{\pi}{6}$. General definitions of inverse sine, inverse cosine, and inverse tangent are given below.

INVERSE TRIGONOMETRIC FUNCTIONS

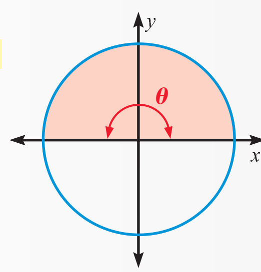
- If $-1 \le a \le 1$, then the **inverse sine** of a is $\sin^{-1} a = \theta$ where $\sin \theta = a$ and $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ (or $-90° \le \theta \le 90°$).



$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$-1 \le \sin \theta \le 1$

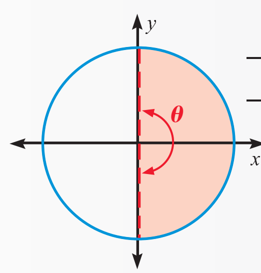
- If $-1 \le a \le 1$, then the **inverse cosine** of a is $\cos^{-1} a = \theta$ where $\cos \theta = a$ and $0 \le \theta \le \pi$ (or $0° \le \theta \le 180°$).



$0 \le \theta \le \pi$

$-1 \le \cos \theta \le 1$

- If a is any real number, then the **inverse tangent** of a is $\tan^{-1} a = \theta$ where $\tan \theta = a$ and $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ (or $-90° < \theta < 90°$).



$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$-\infty < \tan \theta < +\infty$

EXAMPLE 1 *Evaluating Inverse Trigonometric Functions*

Evaluate the expression in both radians and degrees.

a. $\sin^{-1}\frac{\sqrt{3}}{2}$ **b.** $\cos^{-1} 2$ **c.** $\tan^{-1}(-1)$

SOLUTION

a. When $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, or $-90° \le \theta \le 90°$, the angle whose sine is $\frac{\sqrt{3}}{2}$ is:

$$\theta = \sin^{-1}\frac{\sqrt{3}}{2} = \frac{\pi}{3} \quad \text{or} \quad \theta = \sin^{-1}\frac{\sqrt{3}}{2} = 60°$$

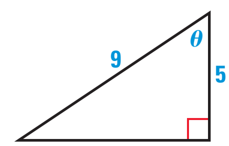
b. There is no angle whose cosine is 2. So, $\cos^{-1} 2$ is undefined.

c. When $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, or $-90° < \theta < 90°$, the angle whose tangent is -1 is:

$$\theta = \tan^{-1}(-1) = -\frac{\pi}{4} \quad \text{or} \quad \theta = \tan^{-1}(-1) = -45°$$

EXAMPLE 2 *Finding an Angle Measure*

Find the measure of the angle θ for the triangle shown.



SOLUTION

In the right triangle, you are given the adjacent side and the hypotenuse. You can write:

$$\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{5}{9}$$

This equation is asking you to find the acute angle whose cosine is $\frac{5}{9}$. Use a calculator to find the measure of θ.

$$\theta = \cos^{-1}\frac{5}{9} \approx 0.982 \text{ radians} \quad \text{or} \quad \theta = \cos^{-1}\frac{5}{9} \approx 56.3°$$

STUDENT HELP

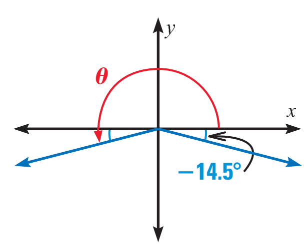
Study Tip
When approximating the value of an angle, make sure your calculator is set to radian mode if you want your answer in radians, or to degree mode if you want your answer in degrees.

EXAMPLE 3 *Solving a Trigonometric Equation*

Solve the equation $\sin\theta = -\frac{1}{4}$ where $180° < \theta < 270°$.

SOLUTION

In the interval $-90° < \theta < 90°$, the angle whose sine is $-\frac{1}{4}$ is $\sin^{-1}\left(-\frac{1}{4}\right) \approx -14.5°$. This angle is in Quadrant IV as shown. In Quadrant III (where $180° < \theta < 270°$), the angle that has the same sine value is:



$$\theta \approx 180° + 14.5° = 194.5°$$

✓ **CHECK** Use a calculator to check the answer.

$$\sin 194.5° \approx -0.25 \checkmark$$

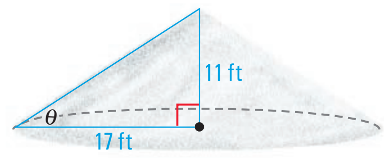
FOCUS ON APPLICATIONS

REAL LIFE **ROCK SALT** Each year about 9 million tons of rock salt are poured on highways in North America to melt ice. Although rock salt is the best deicing material, it also eats away at cars and road surfaces.

GOAL 2 USING INVERSE TRIGONOMETRIC FUNCTIONS IN REAL LIFE

EXAMPLE 4 *Writing and Solving a Trigonometric Equation*

ROCK SALT Different types of granular substances naturally settle at different angles when stored in cone-shaped piles. This angle θ is called the *angle of repose*. When rock salt is stored in a cone-shaped pile 11 feet high, the diameter of the pile's base is about 34 feet. ▶ Source: Bulk-Store Structures, Inc.



a. Find the angle of repose for rock salt.

b. How tall is a pile of rock salt that has a base diameter of 50 feet?

SOLUTION

a. In the right triangle shown inside the cone, you are given the opposite side and the adjacent side. You can write:

$$\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{11}{17}$$

This equation is asking you to find the acute angle whose tangent is $\frac{11}{17}$.

$$\theta = \tan^{-1}\frac{11}{17} \approx 32.9°$$

▶ The angle of repose for rock salt is about 32.9°.

b. The pile of rock salt has a base radius of 25 feet. From part (a) you know that the angle of repose for rock salt is about 32.9°. To find the height h (in feet) of the pile you can write:

$$\tan 32.9° = \frac{h}{25}$$

$$h = 25\tan 32.9°$$

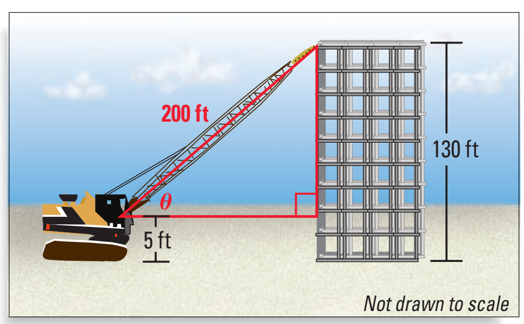
$$\approx 16.2$$

▶ The pile of rock salt is about 16.2 feet tall.

REAL LIFE *Construction*

EXAMPLE 5 *Writing and Solving a Trigonometric Equation*

A crane has a 200 foot arm whose lower end is 5 feet off the ground. The arm has to reach the top of a building 130 feet high. At what angle θ should the arm be set?



SOLUTION

In the right triangle in the diagram, you know the opposite side and the hypotenuse. You can write:

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{130 - 5}{200} = \frac{5}{8}$$

This equation is asking you to find the acute angle whose sine is $\frac{5}{8}$.

$$\theta = \sin^{-1}\frac{5}{8} \approx 38.7°$$

▶ The crane's arm should be set at an acute angle of about 38.7°.

GUIDED PRACTICE

Vocabulary Check ✓

1. Complete this statement: The __?__ sine of 1 equals $\frac{\pi}{2}$, or 90°.

Concept Check ✓

2. Explain why the domain of $y = \cos\theta$ cannot be restricted to $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ if the inverse is to be a function.

3. Explain why $\tan^{-1} 3$ is defined, but $\cos^{-1} 3$ is undefined.

4. **ERROR ANALYSIS** A student needed to find an angle θ in Quadrant III such that $\sin\theta = -0.3221$. She used a calculator to find that $\sin^{-1}(-0.3221) \approx -18.8°$. Then she added this result to 180° to get an answer of $\theta = 161.2°$. What did she do wrong?

Skill Check ✓

Evaluate the expression without using a calculator.

5. $\tan^{-1}\sqrt{3}$ **6.** $\cos^{-1}\frac{\sqrt{2}}{2}$ **7.** $\sin^{-1}\frac{1}{2}$ **8.** $\cos^{-1}\left(-\frac{1}{2}\right)$

Use a calculator to evaluate the expression in both radians and degrees. Round to three significant digits.

9. $\tan^{-1} 3.9$ **10.** $\cos^{-1}(-0.94)$ **11.** $\cos^{-1} 0.34$ **12.** $\sin^{-1}(-0.4)$

Solve the equation for θ. Round to three significant digits.

13. $\sin\theta = -0.35;\ 180° < \theta < 270°$ **14.** $\tan\theta = 2.4;\ 180° < \theta < 270°$

15. $\cos\theta = 0.43;\ 270° < \theta < 360°$ **16.** $\sin\theta = 0.8;\ 90° < \theta < 180°$

17. **CONSTRUCTION** A crane has a 150 foot arm whose lower end is 4 feet off the ground. The arm has to reach the top of a building 105 feet high. At what angle should the crane's arm be set?

PRACTICE AND APPLICATIONS

STUDENT HELP

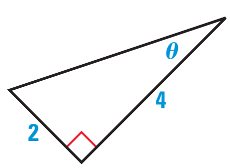
Extra Practice to help you master skills is on p. 958.

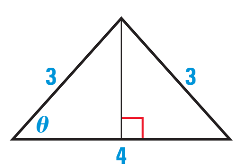
EVALUATING EXPRESSIONS **Evaluate the expression without using a calculator. Give your answer in both radians and degrees.**

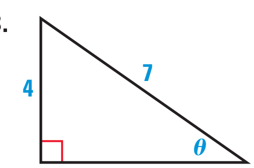
18. $\sin^{-1}\frac{\sqrt{2}}{2}$ **19.** $\cos^{-1}\frac{1}{2}$ **20.** $\tan^{-1} 1$ **21.** $\sin^{-1} 0$

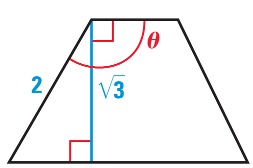
22. $\cos^{-1}(-1)$ **23.** $\sin^{-1}(-1)$ **24.** $\tan^{-1}\left(-\frac{\sqrt{3}}{3}\right)$ **25.** $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

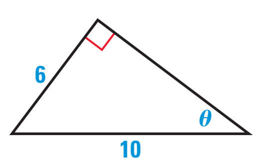
FINDING ANGLES **Find the measure of the angle θ. Round to three significant digits.**

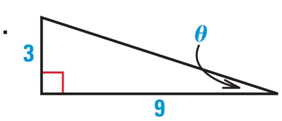
26.


27.


28.


29.


30.


31.


STUDENT HELP

HOMEWORK HELP
Example 1: Exs. 18–25, 32–43
Example 2: Exs. 26–43
Example 3: Exs. 44–51
Examples 4, 5: Exs. 52–57

EVALUATING EXPRESSIONS **Use a calculator to evaluate the expression in both radians and degrees. Round to three significant digits.**

32. $\tan^{-1} 3.9$ **33.** $\cos^{-1} 0.24$ **34.** $\cos^{-1} 0.34$ **35.** $\sin^{-1} 0.75$

36. $\sin^{-1} (-0.4)$ **37.** $\cos^{-1} (-0.6)$ **38.** $\tan^{-1} (-0.2)$ **39.** $\tan^{-1} 2.25$

40. $\cos^{-1} (-0.8)$ **41.** $\sin^{-1} 0.99$ **42.** $\tan^{-1} 12$ **43.** $\cos^{-1} 0.55$

STUDENT HELP

INTERNET **HOMEWORK HELP** Visit our Web site www.mcdougallittell.com for help with Exs. 44–51.

SOLVING EQUATIONS **Solve the equation for θ. Round to three significant digits.**

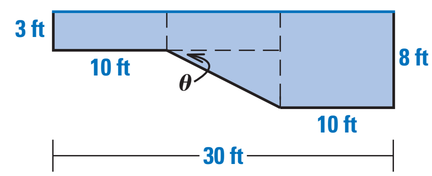
44. $\sin \theta = -0.35;\ 180° < \theta < 270°$ **45.** $\tan \theta = 2.4;\ 180° < \theta < 270°$

46. $\cos \theta = 0.43;\ 270° < \theta < 360°$ **47.** $\sin \theta = 0.8;\ 90° < \theta < 180°$

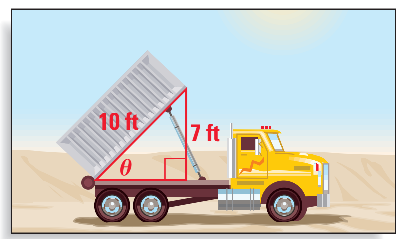
48. $\tan \theta = -2.1;\ 90° < \theta < 180°$ **49.** $\cos \theta = -0.72;\ 180° < \theta < 270°$

50. $\sin \theta = 0.2;\ 90° < \theta < 180°$ **51.** $\tan \theta = 0.9;\ 180° < \theta < 270°$

52. **SWIMMING POOL** The swimming pool shown in cross section at the right ranges in depth from 3 feet at the shallow end to 8 feet at the deep end. Find the angle of depression θ between the shallow end and the deep end.

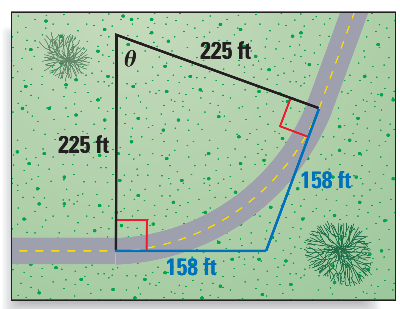


53. **DUMP TRUCK** The dump truck shown has a 10 foot bed. When tilted at its maximum angle, the bed reaches a height of 7 feet above its original position. What is the maximum angle θ that the truck bed can tilt?

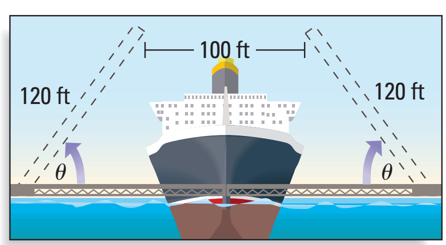


54. **GRANULAR ANGLE OF REPOSE** Look back at Example 4 on page 794. When whole corn is stored in a cone-shaped pile 20 feet high, the diameter of the pile's base is about 82 feet. Find the angle of repose for whole corn.

55. **ROAD DESIGN** Curves that connect two straight sections of a road are often constructed as arcs of circles. In the diagram, θ is the central angle of a circular arc that has a radius of 225 feet. Each radius line shown is perpendicular to one of the straight sections. The straight sections are therefore tangent to the arc. The extension of each straight section to their point of intersection is 158 feet in length. Find the degree measure of θ.

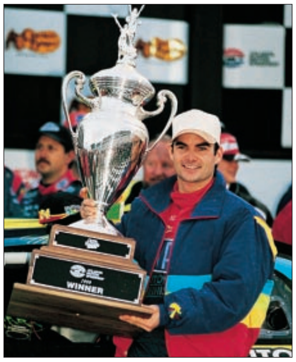


56. **DRAWBRIDGE** The Park Street Bridge in Alameda County, California, is a double-leaf drawbridge. Each leaf of the bridge is 120 feet long. A ship that is 100 feet wide needs to pass through the bridge. What is the minimum angle θ that each leaf of the bridge should be opened to in order to ensure that the ship will fit?



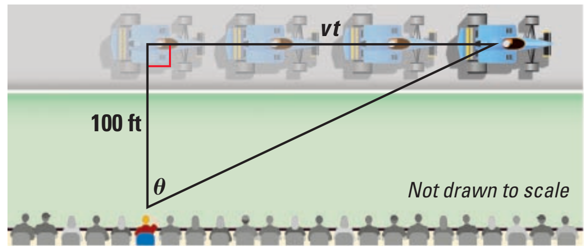
▶ Source: Alameda County Drawbridges

FOCUS ON PEOPLE



JEFF GORDON began racing in the top division of the National Association for Stock Car Auto Racing (NASCAR) in 1993. He has won three NASCAR division championships in the past four years.

57. **RACEWAY** Suppose you are at a raceway and are sitting on the straightaway, 100 feet from the center of the track. If a car traveling 145 miles per hour passes directly in front of you, at what angle do you have to turn your head to see the car t seconds later? Assume that the car is still on the straightaway and is traveling at a constant speed. (*Hint:* First convert 145 miles per hour to a speed v in feet per second. The expression vt represents the distance in feet traveled by the car.)



GEOMETRY CONNECTION **In Exercises 58–62, use the following information.** Consider a line with positive slope m that makes an angle θ with the x-axis (measuring counterclockwise from the x-axis).

58. Find the slope m of the line $y = 3x - 2$.

59. Find θ for the line $y = 3x - 2$.

60. **CRITICAL THINKING** How could you have found θ for the line $y = 3x - 2$ by using the slope m of the line? Write an equation relating θ and m.

61. Find an equation of the line that makes an angle of 58° with the x-axis and whose y-intercept is 3.

62. Find an equation of the line that makes an angle of 35° with the x-axis and whose x-intercept is 4.

Test Preparation

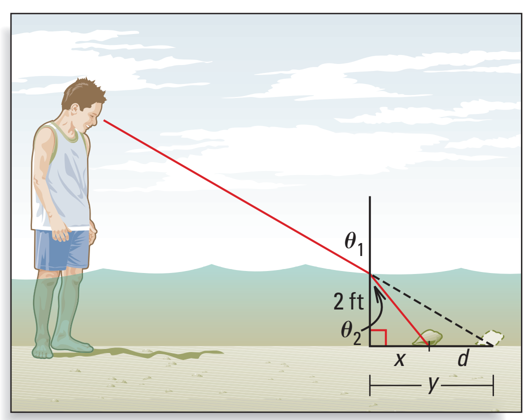
63. **MULTI-STEP PROBLEM** If you stand in shallow water and look at an object below the surface of the water, the object will look farther away from you than it really is. This is because when light rays pass between air and water, the water *refracts*, or bends, the light rays. The *index of refraction* for seawater is 1.341. This is the ratio of the sine of θ_1 to the sine of θ_2 for angles θ_1 and θ_2 below.

a. You are standing in seawater that is 2 feet deep and are looking at a shell at angle $\theta_1 = 60°$ (measured from a line perpendicular to the surface of the water). Find θ_2.

b. Find the distances x and y.

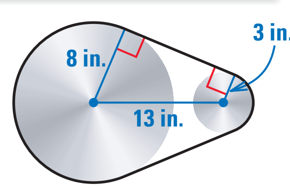
c. Find the distance d between where the shell is and where it appears to be.

d. *Writing* What happens to d as you move closer to the shell? Explain your reasoning.



★ Challenge

64. **LENGTH OF A PULLEY BELT** Find the length of the pulley belt shown at the right. (*Hint:* Partition the belt into four parts: the two straight segments, the arc around the small wheel, and the arc around the large wheel.)



Mixed Review

Solving Equations **Solve the rational equation. Check for extraneous solutions.** (Review 9.6 for 13.5)

65. $\frac{6}{x} = \frac{7}{x+3}$

66. $\frac{3}{x-3} = \frac{7}{x}$

67. $\frac{-1}{4+x} = \frac{6}{2x}$

68. $\frac{3}{x+3} + 7 = \frac{-4}{x+3}$

69. $\frac{1}{x+2} = \frac{x}{2x+9}$

70. $\frac{3x}{x-2} = 2 + \frac{6}{x-2}$

Choosing Numbers **You have an equally likely chance of choosing any number 1 through 30. Find the probability of the given event.** (Review 12.3)

71. A multiple of 5 is chosen.

72. A prime number is chosen.

73. An even number is chosen.

74. A factor of 90 is chosen.

75. A number less than 12 is chosen.

76. A number greater than 23 is chosen.

Evaluating Functions **Use a calculator to evaluate the function. Round to four decimal places.** (Review 13.3 for 13.5)

77. $\sin 27°$

78. $\sin \frac{23\pi}{8}$

79. $\cos 67°$

80. $\sec \frac{53\pi}{9}$

81. $\tan 192°$

82. $\csc 219°$

Quiz 2

Self-Test for Lessons 13.3 and 13.4

Use the given point on the terminal side of an angle θ in standard position. Evaluate the six trigonometric functions of θ. (Lesson 13.3)

1. $(-9, -16)$

2. $(7, -2)$

3. $(-1, 5)$

4. $(6, -11)$

5. $(3, 6)$

6. $(-12, 3)$

7. $(9, -5)$

8. $(-7, -8)$

Evaluate the function without using a calculator. (Lesson 13.3)

9. $\sin(-135°)$

10. $\tan \frac{8\pi}{3}$

11. $\cos(-420°)$

12. $\tan\left(-\frac{2\pi}{3}\right)$

13. $\sin \frac{5\pi}{3}$

14. $\cos 870°$

15. $\tan(-30°)$

16. $\sin \frac{23\pi}{6}$

Use a calculator to evaluate the expression in both radians and degrees. Round to three significant digits. (Lesson 13.4)

17. $\tan^{-1} 2.3$

18. $\sin^{-1}(-0.6)$

19. $\cos^{-1} 0.95$

20. $\sin^{-1} 0.23$

21. $\tan^{-1}(-4)$

22. $\cos^{-1}(-0.8)$

23. $\sin^{-1} 0.1$

24. $\tan^{-1} 10$

Solve the equation for θ. Round to three significant digits. (Lesson 13.4)

25. $\sin\theta = 0.25;\ 90° < \theta < 180°$

26. $\cos\theta = 0.21;\ 270° < \theta < 360°$

27. $\tan\theta = 7;\ 180° < \theta < 270°$

28. $\sin\theta = -0.44;\ 180° < \theta < 270°$

29. $\cos\theta = -0.3;\ 180° < \theta < 270°$

30. $\tan\theta = -4.5;\ 90° < \theta < 180°$

31. **Lacrosse** A lacrosse player throws a ball at an angle of 55° and at an initial speed of 40 feet per second. How far away should her teammate be to catch the ball at the same height from which it was thrown? (Lesson 13.3)

13.5 The Law of Sines

What *you should learn*

GOAL 1 Use the law of sines to find the sides and angles of a triangle.

GOAL 2 Find the area of any triangle, as applied in **Example 6**.

Why *you should learn it*

▼ To solve **real-life** problems, such as finding the distance between the Empire State Building and the Statue of Liberty in **Ex. 60**.

REAL LIFE

GOAL 1 Using the Law of Sines

In Lesson 13.1 you learned how to solve right triangles. To solve a triangle with no right angle, you need to know the measure of at least one side and any two other parts of the triangle. This breaks down into four possible cases.

1. Two angles and any side (AAS or ASA)
2. Two sides and an angle opposite one of them (SSA)
3. Three sides (SSS)
4. Two sides and their included angle (SAS)

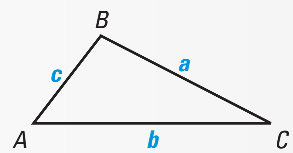
The first two cases can be solved using the **law of sines**. The last two cases require the law of cosines, which you will study in Lesson 13.6.

LAW OF SINES

If $\triangle ABC$ has sides of length a, b, and c as shown, then:

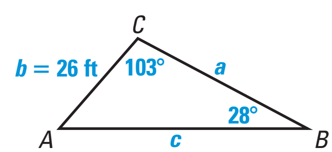
$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

An equivalent form is $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$.



EXAMPLE 1 *The AAS or ASA Case*

Solve $\triangle ABC$ with $C = 103°$, $B = 28°$, and $b = 26$ feet.



Solution

You can find the third angle of $\triangle ABC$ as follows.

$$A = 180° - 103° - 28° = 49°$$

By the law of sines you can write:

$$\frac{a}{\sin 49°} = \frac{26}{\sin 28°} = \frac{c}{\sin 103°}$$

You can then solve for a and c as follows.

$\frac{a}{\sin 49°} = \frac{26}{\sin 28°}$ **Write two equations, each with one variable.** $\frac{c}{\sin 103°} = \frac{26}{\sin 28°}$

$a = \frac{26 \sin 49°}{\sin 28°}$ **Solve for the variable.** $c = \frac{26 \sin 103°}{\sin 28°}$

$a \approx 41.8$ feet **Use a calculator.** $c \approx 54.0$ feet

STUDENT HELP

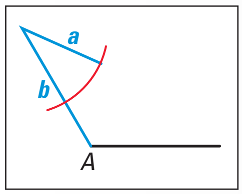
Derivations
For a derivation of the law of sines, see p. 900.

Two angles and one side (AAS or ASA) determine exactly one triangle. Two sides and an angle opposite one of those sides (SSA) may determine no triangle, one triangle, or two triangles. The SSA case is called the *ambiguous case*.

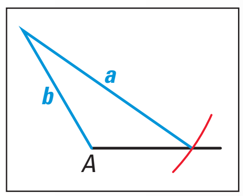
POSSIBLE TRIANGLES IN THE SSA CASE

Consider a triangle in which you are given a, b, and A.

A IS OBTUSE.

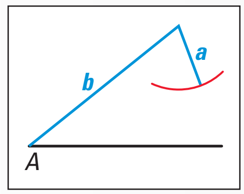


$a \le b$
No triangle

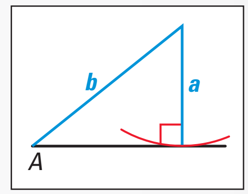


$a > b$
One triangle

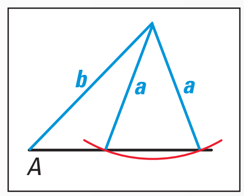
A IS ACUTE.



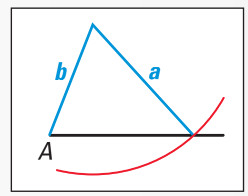
$b \sin A > a$
No triangle



$b \sin A = a$
One triangle



$b \sin A < a < b$
Two triangles



$a > b$
One triangle

EXAMPLE 2 *The SSA Case—One Triangle*

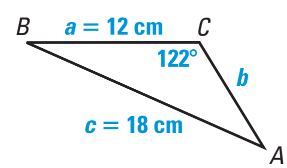
Solve $\triangle ABC$ with $C = 122°$, $a = 12$ cm, and $c = 18$ cm.

STUDENT HELP

Look Back
For help with solving rational equations, see p. 569.

SOLUTION

First make a sketch. Because C is obtuse and the side opposite C is longer than the given adjacent side, you know that only one triangle can be formed. Use the law of sines to find A.



$$\frac{\sin A}{12} = \frac{\sin 122°}{18}$$ **Law of sines**

$$\sin A = \frac{12 \sin 122°}{18}$$ **Multiply each side by 12.**

$$\sin A \approx 0.5654$$ **Use a calculator.**

$$A \approx 34.4°$$ **Use inverse sine function.**

You then know that $B \approx 180° - 122° - 34.4° = 23.6°$. Use the law of sines again to find the remaining side length b of the triangle.

$$\frac{b}{\sin 23.6°} = \frac{18}{\sin 122°}$$

$$b = \frac{18 \sin 23.6°}{\sin 122°} \approx 8.5 \text{ centimeters}$$

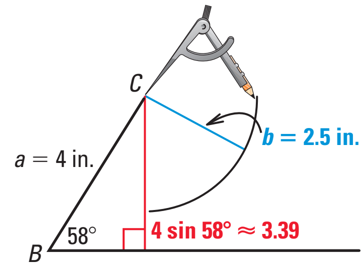
EXAMPLE 3 *The SSA Case—No Triangle*

Solve △ABC with $a = 4$ inches, $b = 2.5$ inches, and $B = 58°$.

SOLUTION

Begin by drawing a horizontal line. On one end form a 58° angle (B) and draw a segment ($\overline{BC}$) 4 inches long. At vertex C, use a compass to draw an arc of radius 2.5 inches. This arc does not intersect the horizontal line, so it is not possible to draw the indicated triangle.

You can see that b needs to be at least $4 \sin 58° \approx 3.39$ inches long to reach the horizontal side and form a triangle.



EXAMPLE 4 *The SSA Case—Two Triangles*

ASTRONOMY At certain times during the year, you can see Venus in the morning sky. The distance between Venus and the sun is approximately 67 million miles. The distance between Earth and the sun is approximately 93 million miles. Estimate the distance between Venus and Earth if the observed angle between the sun and Venus is 34°.

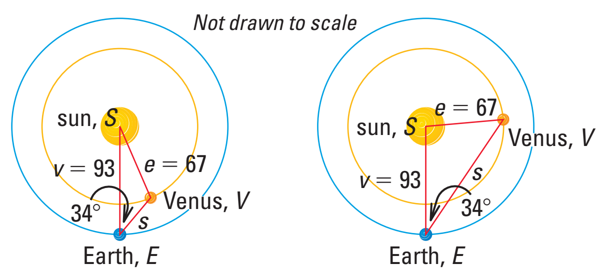
SOLUTION

Venus's distance from the sun, $e = 67$, is greater than $v \sin E = 93 \sin 34° \approx 52$ and less than Earth's distance from the sun, $v = 93$. Therefore, two possible triangles can be formed. Draw diagrams as shown. Use the law of sines to find the possible measures of V.

$$\frac{\sin 34°}{67} = \frac{\sin V}{93}$$

$$\sin V = \frac{93 \sin 34°}{67}$$

$$\sin V \approx 0.7762$$



There are two angles between 0° and 180° for which $\sin V \approx 0.7762$. Use your calculator to find the angle between 0° and 90°: $\sin^{-1} 0.7762 \approx 50.9°$. To find the second angle, subtract the angle given by your calculator from 180°: $180° - 50.9° = 129.1°$. So, $V \approx 50.9°$ or $V \approx 129.1°$.

Because the sum of the angle measures in a triangle equals 180°, you know that $S \approx 95.1°$ when $V \approx 50.9°$ or $S \approx 16.9°$ when $V \approx 129.1°$. Finally, use the law of sines again to find the side length s.

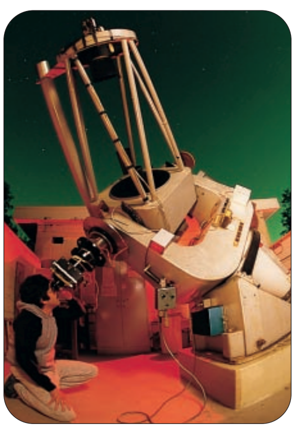
$$\frac{s}{\sin 95.1°} = \frac{67}{\sin 34°} \qquad \frac{s}{\sin 16.9°} = \frac{67}{\sin 34°}$$

$$s = \frac{67 \sin 95.1°}{\sin 34°} \qquad s = \frac{67 \sin 16.9°}{\sin 34°}$$

$$\approx 119 \qquad \approx 34.8$$

▶ The approximate distance between Venus and Earth is either 119 million miles or 34.8 million miles.

FOCUS ON CAREERS



REAL LIFE

ASTRONOMER Astronomers study energy, matter, and natural processes throughout the universe. A doctoral degree and an aptitude for physics and mathematics are needed to become an astronomer.

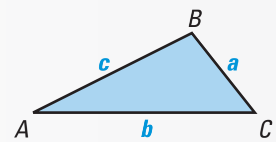
INTERNET CAREER LINK www.mcdougallittell.com

GOAL 2 FINDING THE AREA OF A TRIANGLE

You can find the area of any triangle if you know the lengths of two sides and the measure of the included angle.

AREA OF A TRIANGLE

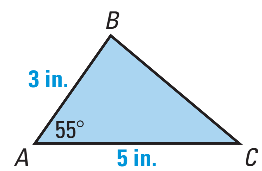
The area of any triangle is given by one half the product of the lengths of two sides times the sine of their included angle. For $\triangle ABC$ shown, there are three ways to calculate the area:



$\text{Area} = \frac{1}{2}bc \sin A$ $\qquad$ $\text{Area} = \frac{1}{2}ac \sin B$ $\qquad$ $\text{Area} = \frac{1}{2}ab \sin C$

EXAMPLE 5 *Finding a Triangle's Area*

Find the area of $\triangle ABC$.



SOLUTION

Use the appropriate formula for the area of a triangle.

$$\text{Area} = \frac{1}{2}bc \sin A$$

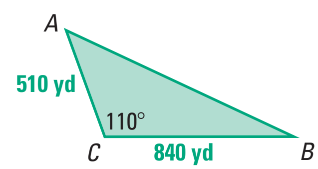
$$= \frac{1}{2}(5)(3) \sin 55^\circ$$

$$\approx 6.14 \text{ square inches}$$

REAL LIFE *Real Estate*

EXAMPLE 6 *Calculating the Price of Land*

You are buying the triangular piece of land shown. The price of the land is \$2000 per acre (1 acre = 4840 square yards). How much does the land cost?



SOLUTION

The area of the land is:

$$\text{Area} = \frac{1}{2}ab \sin C$$

$$= \frac{1}{2}(840)(510) \sin 110^\circ$$

$$\approx 201{,}000 \text{ square yards}$$

▶ The property contains $201{,}000 \div 4840 \approx 41.5$ acres. At \$2000 per acre, the price of the land is about $(2000)(41.5) = \$83{,}000$.

GUIDED PRACTICE

Vocabulary Check ✓

1. What is the SSA case called? Why is it called this?

Concept Check ✓

2. Which two of the following cases can be solved using the law of sines?

   A. SSS **B.** SSA **C.** AAS or ASA **D.** SAS

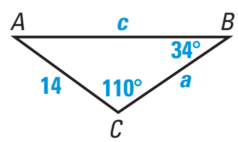
3. Suppose a, b, and A are given for $\triangle ABC$ where $A < 90°$. Under what conditions would you have no triangle? one triangle? two triangles?

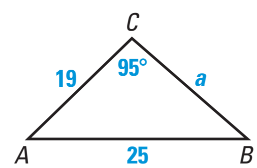
Skill Check ✓

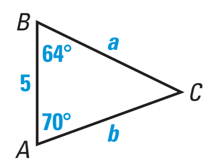
Decide whether the given measurements can form exactly *one triangle*, exactly *two triangles*, or *no triangle*. (You do not need to solve the triangle.)

4. $C = 65°, c = 44, b = 32$
5. $A = 140°, a = 5, c = 7$
6. $A = 18°, a = 16, c = 10$
7. $A = 70°, a = 155, c = 160$

Solve $\triangle ABC$.

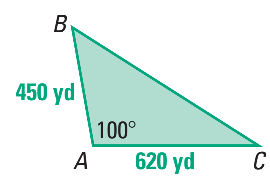
8. 

9. 

10. 

Find the area of the triangle with the given side lengths and included angle.

11. $b = 2, c = 3, A = 47°$
12. $a = 23, b = 15, C = 51°$
13. $a = 13, c = 24, B = 127°$
14. $b = 12, c = 17, A = 103°$
15. **REAL ESTATE** Suppose you are buying the triangular piece of land shown. The price of the land is \$2200 per acre (1 acre = 4840 square yards). How much does the land cost?



PRACTICE AND APPLICATIONS

STUDENT HELP

Extra Practice to help you master skills is on p. 958.

NUMBER OF SOLUTIONS **Decide whether the given measurements can form exactly *one triangle*, exactly *two triangles*, or *no triangle*.**

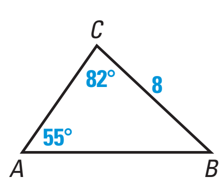
16. $C = 65°, c = 44, b = 32$
17. $A = 140°, a = 5, c = 7$
18. $A = 18°, a = 16, c = 10$
19. $A = 70°, a = 155, c = 160$
20. $C = 160°, c = 12, b = 15$
21. $B = 105°, b = 11, a = 5$
22. $B = 56°, b = 13, a = 14$
23. $C = 25°, c = 6, b = 20$

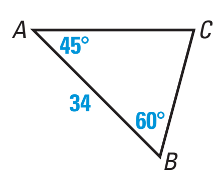
STUDENT HELP

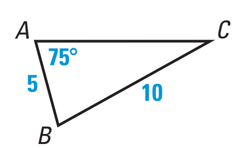
HOMEWORK HELP

Examples 1–3: Exs. 16–36
Example 4: Exs. 16–36, 56–62
Example 5: Exs. 37–52
Example 6: Exs. 63–67

SOLVING TRIANGLES **Solve $\triangle ABC$.**

24. 

25. 

26. 

STUDENT HELP

HOMEWORK HELP Visit our Web site www.mcdougallittell.com for help with Exs. 27–36.

SOLVING TRIANGLES **Solve $\triangle ABC$. (*Hint:* Some of the "triangles" have no solution and some have two solutions.)**

27. $B = 60°, b = 30, c = 20$

28. $B = 110°, C = 30°, a = 15$

29. $B = 130°, a = 10, b = 8$

30. $A = 20°, a = 10, c = 11$

31. $C = 95°, a = 8, c = 9$

32. $A = 70°, B = 60°, c = 25$

33. $C = 16°, b = 92, c = 32$

34. $A = 10°, C = 130°, b = 5$

35. $B = 35°, a = 12, b = 26$

36. $C = 145°, b = 5, c = 9$

FINDING AREA **Find the area of the triangle with the given side lengths and included angle.**

37. $B = 25°, a = 17, c = 33$

38. $C = 130°, a = 21, b = 17$

39. $C = 120°, a = 8, b = 5$

40. $A = 85°, b = 11, c = 18$

41. $A = 75°, b = 16, c = 21$

42. $B = 110°, a = 11, c = 24$

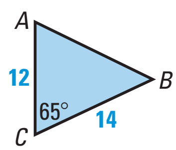
43. $C = 125°, a = 3, b = 8$

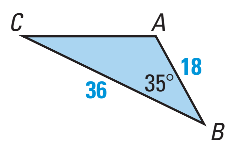
44. $B = 29°, a = 13, c = 13$

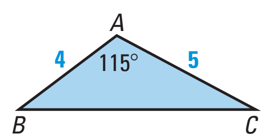
45. $B = 96°, a = 15, c = 9$

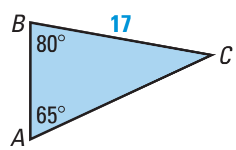
46. $A = 32°, b = 10, c = 12$

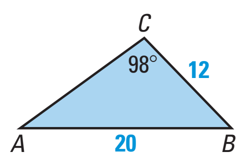
FINDING AREA **Find the area of $\triangle ABC$.**

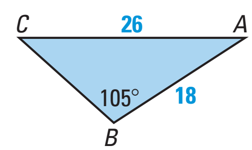
47.


48.


49.


50.


51.


52.


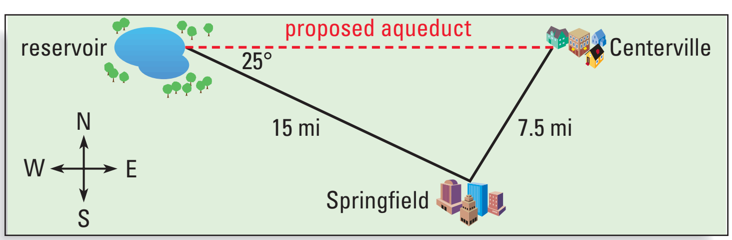
FINDING A PATTERN **In Exercises 53–55, use a graphing calculator to explore how the angle measure between two sides of a triangle affects the area of the triangle.**

53. Choose a fixed length for each of two sides of a triangle. Let x represent the measure of the included angle. Enter an equation for the area of this triangle into the calculator.

54. Use the *Table* feature to look at the y-values for $0° < x < 180°$. Does area always increase for increasing values of x? Explain.

55. What value of x maximizes area?

56. **AQUEDUCT** A reservoir supplies water through an aqueduct to Springfield, which is 15 miles from the reservoir at 25° south of east. A pumping station at Springfield pumps water 7.5 miles to Centerville, which is due east from the reservoir. Plans have been made to build an aqueduct directly from the reservoir to Centerville. How long will the aqueduct be?

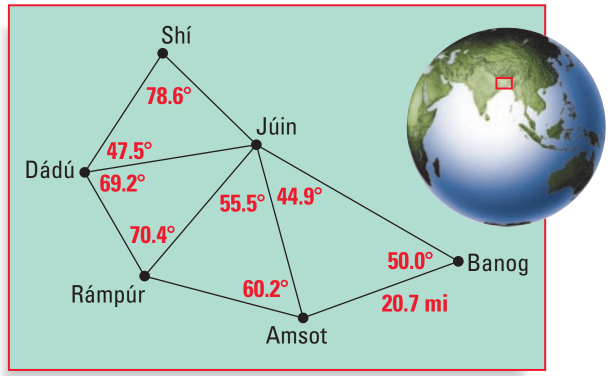


HISTORY CONNECTION **In Exercises 57–59, use the following information.**
In 1802 Captain William Lambton began what is known as the Great Trigonometrical Survey of India. Lambton and his company systematically divided India into triangles. They used trigonometry to find unknown distances from a known distance they measured, called a *baseline*. The map below shows a section of the Great Trigonometrical Survey of India.

57. Use the given measurements to find the distance between Júin and Amsot.

58. Find the distance between Júin and Rámpúr.

59. *Writing* How could you find the distance from Shí to Dádú? Explain.

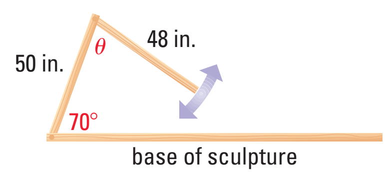


60. **NEW YORK CITY** You are on the observation deck of the Empire State Building looking at the Chrysler Building. When you turn about 145° you see the Statue of Liberty. You know that the Chrysler Building and the Empire State Building are about 0.6 mile apart and that the Chrysler Building is about 5.7 miles from the Statue of Liberty. Find the approximate distance between the Empire State Building and the Statue of Liberty.

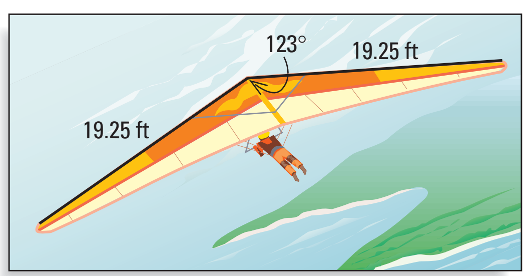
ART CONNECTION **In Exercises 61 and 62, use the following information.**
You are creating a sculpture for an art show at your school. One 50 inch wooden beam makes an angle of 70° with the base of your sculpture. You have another wooden beam 48 inches long that you would like to attach to the top of the 50 inch beam and to the base of the sculpture, as shown below.

61. Find all possible angles θ that the 48 inch beam can make with the 50 inch beam.

62. Find all possible distances d that the bottom of the 48 inch beam can be from the left end of the base.



63. **HANG GLIDER** A hang glider is shown at the right. Use the given nose angle and wing measurements to approximate the area of the sail.



COURTYARD **In Exercises 64 and 65, use the following information.**
You are seeding a triangular courtyard. One side of the courtyard is 52 feet long and another side is 46 feet long. The angle opposite the 52 foot side is 65°.

64. How long is the third side of the courtyard?

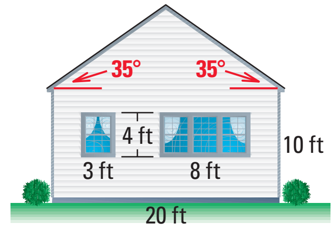
65. One bag of grass seed covers an area of 50 square feet. How many bags of grass seed will you need to cover the courtyard?

FOCUS ON CAREERS

REAL LIFE **SCULPTOR**
Many sculptors create large geometric pieces that require precise calculation of angle measures and side lengths. Shown above is *Center Peace* by sculptor Linda Howard.

INTERNET **CAREER LINK**
www.mcdougallittell.com

BUYING PAINT In Exercises 66 and 67, use the following information. You plan to paint the side of the house shown below. One gallon of paint will cover an area of 400 square feet.

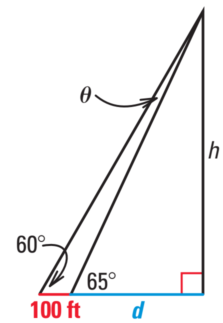


66. Find the area to be painted. Do not include the window area.

67. How many gallons of paint do you need?

Test Preparation

68. MULTI-STEP PROBLEM You are at an unknown distance d from a mountain, as shown below. The angle of elevation to the top of the mountain is 65°. You step back 100 feet and measure the angle of elevation to be 60°.



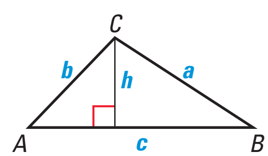
a. Find the height h of the mountain using the law of sines and right triangle trigonometry. (*Hint*: First find θ.)

b. Find the height h of the mountain using a system of equations. Set up one tangent equation involving the ratio of d and h, and another tangent equation involving the ratio of $100 + d$ and h, and then solve the system.

c. *Writing* Which method was easier for you to use? Explain.

★ **Challenge**

69. DERIVING FORMULAS Using the triangle shown at the right as a reference, derive the formulas for the area of a triangle given in the property box on page 802. Then show how to derive the law of sines using the area formulas.



EXTRA CHALLENGE
www.mcdougallittell.com

MIXED REVIEW

COMBINING EXPRESSIONS Perform the indicated operations. (Review 7.2 for 13.6)

70. $5\sqrt{11} + \sqrt{11} - 9\sqrt{11}$

71. $2\sqrt{12} + 5\sqrt{12} + 3\sqrt{27}$

72. $\sqrt{125} - 7\sqrt{45} + 10\sqrt{40}$

73. $\sqrt{7} + 5\sqrt{63} - 2\sqrt{112}$

74. $2\sqrt{486} - 5\sqrt{54} - 2\sqrt{150}$

75. $\sqrt{72} + 6\sqrt{98} - 10\sqrt{8}$

FINDING COSINE VALUES Use a calculator to evaluate the trigonometric function. Round the result to four decimal places. (Review 13.1, 13.3 for 13.6)

76. $\cos 52°$

77. $\cos \frac{12\pi}{5}$

78. $\cos \frac{9\pi}{5}$

79. $\cos \frac{10\pi}{7}$

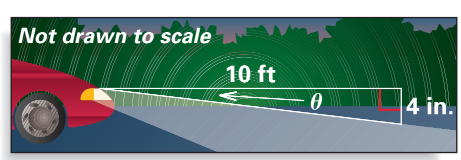
80. $\cos 20°$

81. $\cos 305°$

82. $\cos(-200°)$

83. $\cos 5°$

84. CAR HEADLIGHTS In Massachusetts the low-beam headlights of cars are set to focus down 4 inches at a distance of 10 feet. At what angle θ are the beams directed? (Review 13.4)



13.6 The Law of Cosines

What you should learn

GOAL 1 Use the law of cosines to find the sides and angles of a triangle.

GOAL 2 Use Heron's formula to find the area of a triangle, as applied in **Example 5**.

Why you should learn it

▼ To solve **real-life** problems, such as finding the angle at which two swinging trapeze artists meet in **Ex. 50**.

REAL LIFE

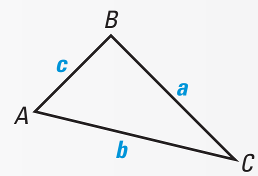
GOAL 1 USING THE LAW OF COSINES

You have not yet solved triangles for which two sides and the included angle (SAS) or three sides (SSS) are given. You can solve both of these cases using the **law of cosines**.

LAW OF COSINES

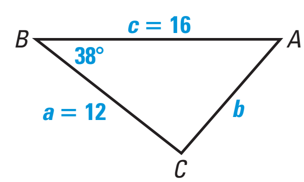
If $\triangle ABC$ has sides of length a, b, and c as shown, then:

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$b^2 = a^2 + c^2 - 2ac\cos B$$
$$c^2 = a^2 + b^2 - 2ab\cos C$$



EXAMPLE 1 *The SAS Case*

Solve $\triangle ABC$ with $a = 12$, $c = 16$, and $B = 38°$.



SOLUTION

Begin by using the law of cosines to find the length b of the third side.

$b^2 = a^2 + c^2 - 2ac\cos B$ **Write law of cosines.**

$b^2 = 12^2 + 16^2 - 2(12)(16)\cos 38°$ **Substitute for *a*, *c*, and *B*.**

$b^2 \approx 97.4$ **Simplify.**

$b \approx \sqrt{97.4} \approx 9.87$ **Take square root.**

Now that you know all three sides and one angle, you can use the law of cosines *or* the law of sines to find a second angle.

$\frac{\sin A}{a} = \frac{\sin B}{b}$ **Write law of sines.**

$\frac{\sin A}{12} = \frac{\sin 38°}{9.87}$ **Substitute for *a*, *b*, and *B*.**

$\sin A = \frac{12 \sin 38°}{9.87}$ **Multiply each side by 12.**

$\sin A \approx 0.7485$ **Simplify.**

$A \approx \sin^{-1} 0.7485 \approx 48.5°$ **Use inverse sine.**

You can find the third angle as follows.

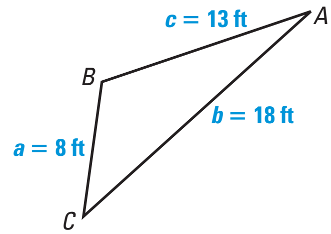
$C \approx 180° - 38° - 48.5° = 93.5°$

STUDENT HELP

Derivations
For a derivation of the law of cosines, see p. 901.

EXAMPLE 2 *The SSS Case*

Solve $\triangle ABC$ with $a = 8$ feet, $b = 18$ feet, and $c = 13$ feet.



SOLUTION

First find the angle opposite the longest side, $\overline{AC}$. Using the law of cosines, you can write:

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac} = \frac{8^2 + 13^2 - 18^2}{2(8)(13)} = -0.4375$$

Using the inverse cosine function, you can find the measure of obtuse angle B:

$$B = \cos^{-1}(-0.4375) \approx 115.9°$$

Now use the law of sines to find A.

$\frac{\sin A}{a} = \frac{\sin B}{b}$ — **Write law of sines.**

$\frac{\sin A}{8} = \frac{\sin 115.9°}{18}$ — **Substitute.**

$\sin A = \frac{8 \sin 115.9°}{18}$ — **Multiply each side by 8.**

$\sin A \approx 0.3998$ — **Simplify.**

$A \approx \sin^{-1} 0.3998 \approx 23.6°$ — **Use inverse sine.**

Finally, you can find the measure of angle C:

$$C \approx 180° - 23.6° - 115.9° = 40.5°$$

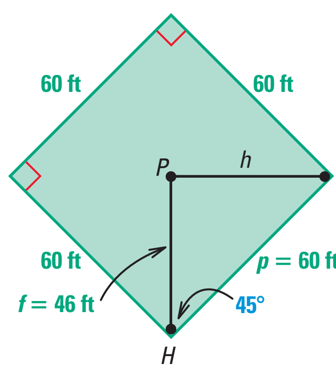
STUDENT HELP

Study Tip
In Example 2 the largest angle is found first to make sure that the other two angles are acute. This way, when you use the law of sines to find another angle measure, you will know that it is between 0° and 90°.

REAL LIFE
Softball

EXAMPLE 3 *The SAS Case*

The pitcher's mound on a softball field is 46 feet from home plate. The distance between the bases is 60 feet. How far is the pitcher's mound from first base?



SOLUTION

Begin by forming $\triangle HPF$. In this triangle you know that $H = 45°$ because the line HP bisects the right angle at home plate. From the given information you know that $f = 46$ and $p = 60$. Using the law of cosines, you can solve for h.

$h^2 = f^2 + p^2 - 2fp \cos H$ — **Write law of cosines.**

$h^2 = 46^2 + 60^2 - 2(46)(60) \cos 45°$ — **Substitute for *f*, *p*, and *H*.**

$h^2 \approx 1812.8$ — **Simplify.**

$h \approx \sqrt{1812.8}$ — **Take square root.**

≈ 42.6 feet — **Simplify.**

▶ The distance between the pitcher's mound and first base is about 42.6 feet.

GOAL 2 USING HERON'S FORMULA

The law of cosines can be used to establish the following formula for the area of a triangle. This formula is credited to the Greek mathematician Heron (circa A.D. 100).

HERON'S AREA FORMULA

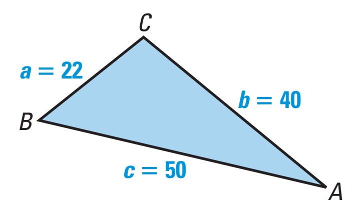
The area of the triangle with sides of length *a, b,* and *c* is

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}$$

where $s = \frac{1}{2}(a + b + c)$. The variable *s* is called the *semiperimeter*, or half-perimeter, of the triangle.

EXAMPLE 4 *Finding the Area of a Triangle*

Find the area of $\triangle ABC$.



STUDENT HELP

Look Back
For help with simplifying radical expressions, see p. 264.

SOLUTION

Begin by finding the semiperimeter.

$$s = \frac{1}{2}(a + b + c) = \frac{1}{2}(22 + 40 + 50) = 56$$

Now use Heron's formula to find the area of $\triangle ABC$:

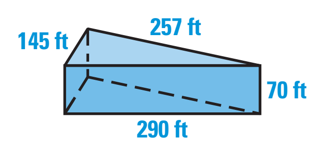
$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}$$

$$= \sqrt{56(56-22)(56-40)(56-50)}$$

$$= \sqrt{182{,}784} \approx 428 \text{ square units}$$

EXAMPLE 5 *Finding the Volume of a Building*

LANDAU BUILDING The dimensions of the Landau Building are given at the right. Find the volume of the building.



FOCUS ON APPLICATIONS

REAL LIFE LANDAU BUILDING
The Landau Building, located in Cambridge, Massachusetts, was designed by architect I.M. Pei. Pei's work is known to have a sharp, geometric look.

SOLUTION

Begin by finding the area of the base. The semiperimeter of the base is:

$$s = \frac{1}{2}(a + b + c) = \frac{1}{2}(145 + 257 + 290) = 346$$

So, the area of the base is:

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}$$

$$= \sqrt{346(346-145)(346-257)(346-290)}$$

$$\approx 18{,}600 \text{ square feet}$$

To find the volume, multiply this area by the building's height:

$$\text{Volume} = (\text{Area of base})(\text{Height}) \approx (18{,}600)(70) = 1{,}302{,}000 \text{ cubic feet}$$

GUIDED PRACTICE

Vocabulary Check ✓

1. Complete this statement: In a triangle with sides of length a, b, and c, $\frac{1}{2}(a + b + c)$ is called the _?_.

Concept Check ✓

2. For each case, tell whether you would use the *law of sines* or the *law of cosines* to solve the triangle.

a. SSS **b.** SSA **c.** SAS **d.** ASA **e.** AAS

3. If when using the law of cosines to find angle A in $\triangle ABC$, you get $\cos A < 0$, what type of angle is A?

4. Express Heron's formula in words.

Skill Check ✓

Solve $\triangle ABC$.

5. $B = 20°, a = 120, c = 100$

6. $C = 95°, a = 10, b = 12$

7. $a = 25, b = 11, c = 24$

8. $a = 2, b = 4, c = 5$

Find the area of $\triangle ABC$ having the given side lengths.

9. $a = 25, b = 60, c = 45$

10. $a = 9, b = 4, c = 11$

11. $a = 100, b = 55, c = 61$

12. $a = 5, b = 27, c = 29$

BASEBALL **In Exercises 13 and 14, use the following information.** The pitcher's mound on a baseball field is 60.5 feet from home plate. The distance between the bases is 90 feet.

13. How far is the pitcher's mound from first base?

14. Using Heron's formula, find the area of the triangle formed by the pitcher's mound, home plate, and first base.

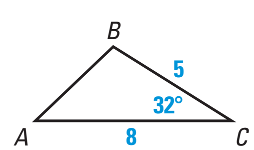
PRACTICE AND APPLICATIONS

STUDENT HELP

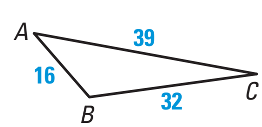
Extra Practice to help you master skills is on p. 959.

SOLVING TRIANGLES **Solve $\triangle ABC$.**

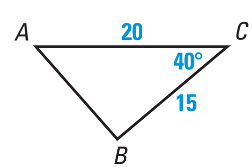
15.



16.



17.



SOLVING TRIANGLES **Solve $\triangle ABC$.**

18. $B = 20°, a = 120, c = 100$

19. $C = 95°, a = 10, b = 12$

20. $a = 25, b = 11, c = 24$

21. $a = 2, b = 4, c = 5$

22. $A = 78°, b = 2, c = 4$

23. $A = 60°, b = 30, c = 28$

24. $B = 45°, a = 11, c = 22$

25. $C = 30°, a = 20, b = 20$

26. $a = 9, b = 3, c = 11$

27. $B = 15°, a = 12, c = 6$

28. $a = 25, b = 26, c = 5$

29. $a = 47, b = 30, c = 62$

STUDENT HELP

HOMEWORK HELP
Examples 1, 2: Exs. 15–37
Example 3: Exs. 50, 51
Example 4: Exs. 38–48
Example 5: Exs. 52–54

CHOOSING A METHOD **Use the law of sines, the law of cosines, or the Pythagorean theorem to solve $\triangle ABC$.**

30. $A = 96°, B = 39°, b = 13$

31. $B = 80°, C = 30°, b = 34$

32. $A = 34°, b = 17, c = 48$

33. $C = 104°, b = 11, c = 32$

34. $A = 48°, B = 51°, c = 36$

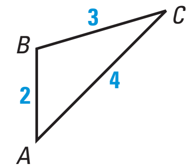
35. $a = 48, b = 51, c = 36$

36. $B = 10°, b = 5, c = 25$

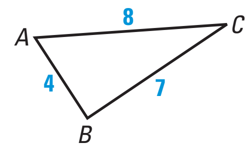
37. $C = 90°, a = 4, b = 11$

FINDING AREA **Find the area of $\triangle ABC$.**

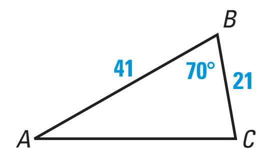
38.



39.



40.



FINDING AREA **Find the area of $\triangle ABC$ having the given side lengths.**

41. $a = 15, b = 20, c = 25$

42. $a = 13, b = 10, c = 4$

43. $a = 75, b = 68, c = 72$

44. $a = 3, b = 19, c = 21$

45. $a = 4, b = 2, c = 4$

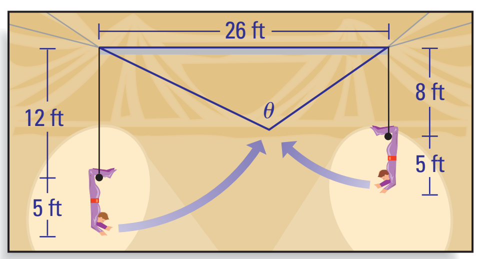
46. $a = 20, b = 21, c = 37$

47. $a = 8, b = 8, c = 8$

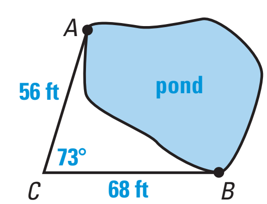
48. $a = 18, b = 15, c = 10$

49. **CRITICAL THINKING** Explain why the Pythagorean theorem is a special case of the law of cosines.

50. **TRAPEZE ARTISTS** The diagram shows the path of two trapeze artists who are both 5 feet long when hanging by their knees. The "flyer" on the left bar is preparing to make hand-to-hand contact with the "catcher" on the right bar. At what angle θ will the two meet? ▶ Source: Trapeze Arts, Inc.



51. **SURVEYING** You are a surveyor measuring the width of a pond from point A to point B, as shown. You set up your transit at point C and measure an angle of 73°. You also measure the distance from point C to points A and B, getting 56 feet and 68 feet, respectively. What is the width of the pond?



52. **GIBSON BLOCK** Built in 1913, the Gibson Block in Alberta, Canada, is shaped like a flat clothing iron of that time period. The approximately triangular base of the building has sides of length 18.3 meters, 37.1 meters, and 41.0 meters. The height of the Gibson Block is about 13.2 meters. Find the volume of the Gibson Block. ▶ Source: Stantec Architecture Ltd.

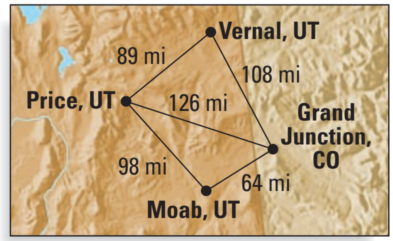
FOCUS ON CAREERS

SURVEYOR
A surveyor takes precise measurements to establish official land airspace, and water boundaries. A surveyor often uses an instrument called a *transit*, as pictured above, to measure angles.

CAREER LINK
www.mcdougallittell.com

53. **DINOSAUR DIAMOND** In Utah and Colorado, an area called the Dinosaur Diamond is known for containing many dinosaur fossils. The map at the right shows the towns at the four vertices of the diamond. Use the given distances to find the area of the Dinosaur Diamond.

▶ Source: Dinomation



54. **FERTILIZER** A farmer has a triangular field with sides that are 240 feet, 300 feet, and 360 feet long. He wants to apply fall fertilizer to the field. If it takes one 40 pound bag of fertilizer to cover 6000 square feet, how many bags does he need to cover the field?

Test Preparation

55. **MULTIPLE CHOICE** Two airplanes leave an airport at the same time, the first headed due north and the second headed 37° east of north. At 2:00 P.M. the first airplane is 250 miles from the airport and the second airplane is 316 miles from the airport. How far apart are the two airplanes?

Ⓐ about 190 miles Ⓑ about 210 miles Ⓒ about 200 miles

Ⓓ about 310 miles Ⓔ about 165 miles

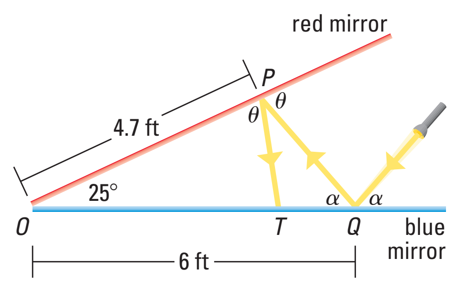
56. **MULTIPLE CHOICE** Find the area of a triangle with sides of length 37 feet, 23 feet, and 42 feet.

Ⓐ about 189 ft^2 Ⓑ about 134 ft^2 Ⓒ about 477 ft^2

Ⓓ about 424 ft^2 Ⓔ about 777 ft^2

★ Challenge

57. **MIRRORS** In the diagram, a beam of light is directed at the blue mirror, reflected to the red mirror, and then reflected back to the blue mirror. Find the distance PT that the light travels from the red mirror back to the blue mirror given that $OQ = 6$ feet and $OP = 4.7$ feet. (*Hint:* You will need to find θ. To do this, find $\angle OPQ$ and use the fact that $2\theta + m\angle TPQ = 180°$.)



EXTRA CHALLENGE
www.mcdougallittell.com

MIXED REVIEW

WRITING EQUATIONS **Write an equation of the hyperbola with the given foci and vertices. (Review 10.5)**

58. Foci: $(-7, 0), (7, 0)$
Vertices: $(-2, 0), (2, 0)$

59. Foci: $(0, -11), (0, 11)$
Vertices: $(0, -3), (0, 3)$

60. Foci: $(-9, 0), (9, 0)$
Vertices: $(-5, 0), (5, 0)$

61. Foci: $(0, -2\sqrt{5}), (0, 2\sqrt{5})$
Vertices: $(0, -1), (0, 1)$

CALCULATING PROBABILITIES **Calculate the probability of rolling a die 30 times and getting the given number of 4's. (Review 12.6)**

62. 1 **63.** 3 **64.** 5 **65.** 6 **66.** 8 **67.** 10

68. **VERTICAL MOTION** From a height of 120 feet, how long does it take a ball thrown downward at 20 feet per second to hit the ground? **(Review 5.6 for 13.7)**

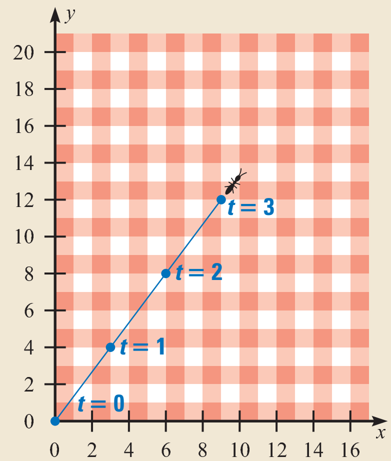
EXPLORING DATA AND STATISTICS

13.7 Parametric Equations and Projectile Motion

GOAL 1 USING PARAMETRIC EQUATIONS

ACTIVITY Developing Concepts: Investigating Linear Motion

Suppose an ant starts at one corner of a picnic tablecloth and moves in a straight line, as shown. The ant's position (x, y) relative to the edges of the tablecloth is given for different times t (in seconds).



1. Write two equations: one that gives the ant's horizontal position x as a function of t, and one that gives the ant's vertical position y as a function of t.
2. What is the ant's position after 5 seconds?
3. How long will it take the ant to reach an edge of the tablecloth?

What you should learn

GOAL 1 Use parametric equations to represent motion in a plane.

GOAL 2 Use parametric equations to represent projectile motion, as applied in **Example 4**.

Why you should learn it

▼ To solve **real-life** problems, such as modeling the path of a leaping dolphin in **Exs. 36–38**.

REAL LIFE

In the investigation you wrote a pair of equations that expressed x and y in terms of a third variable t. These equations, $x = f(t)$ and $y = g(t)$, are called **parametric equations**, and t is called the **parameter**.

EXAMPLE 1 *Graphing a Set of Parametric Equations*

Graph $x = 3t - 12$ and $y = -2t + 3$ for $0 \le t \le 5$.

SOLUTION

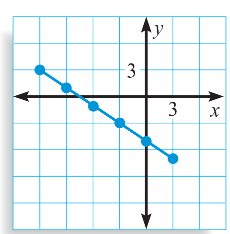
Begin by making a table of values.

t	0	1	2	3	4	5
x	−12	−9	−6	−3	0	3
y	3	1	−1	−3	−5	−7

Plot the points (x, y) given in the table:

$(-12, 3), (-9, 1), (-6, -1),$
$(-3, -3), (0, -5), (3, -7)$

Then connect the points with a line segment as shown.



EXAMPLE 2 *Eliminating the Parameter*

Write an xy-equation for the parametric equations in Example 1: $x = 3t - 12$ and $y = -2t + 3$ for $0 \le t \le 5$. State the domain for the equation.

SOLUTION

First solve one of the parametric equations for t.

$$x = 3t - 12 \quad \text{Write original equation.}$$

$$x + 12 = 3t \quad \text{Add 12 to each side.}$$

$$\frac{1}{3}x + 4 = t \quad \text{Multiply each side by } \frac{1}{3}.$$

Then substitute for t in the other parametric equation.

$$y = -2t + 3 \quad \text{Write original equation.}$$

$$y = -2\left(\frac{1}{3}x + 4\right) + 3 \quad \text{Substitute for } t.$$

$$y = -\frac{2}{3}x - 5 \quad \text{Simplify.}$$

This process is called *eliminating the parameter* because the parameter t is not in the final equation. When $t = 0$, $x = -12$ and when $t = 5$, $x = 3$. So, the domain of the xy-equation is $-12 \le x \le 3$.

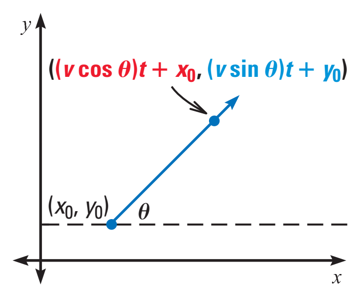
. . . .

Consider an object that is moving with constant speed v along a straight line that makes an angle θ measured counterclockwise from a line parallel to the x-axis. The position of the object at any time t can be represented by the parametric equations

$$x = (v \cos \theta)t + x_0$$

$$y = (v \sin \theta)t + y_0$$

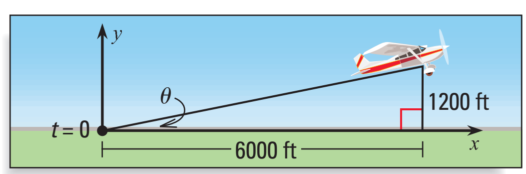
where (x_0, y_0) is the object's location when $t = 0$.



REAL LIFE Aviation

EXAMPLE 3 *Modeling Linear Motion*

Write a set of parametric equations for the airplane shown, given that its speed is 306 feet per second.



SOLUTION

The angle of elevation is $\theta = \tan^{-1}\left(\frac{1200}{6000}\right) \approx 11.3°$.

Using $v = 306$, $\theta = 11.3°$, and $(x_0, y_0) = (0, 0)$, you can write the following.

$$x = (v \cos \theta)t + x_0 \qquad \text{and} \qquad y = (v \sin \theta)t + y_0$$

$$x \approx (306 \cos 11.3°)t + 0 \qquad\qquad y \approx (306 \sin 11.3°)t + 0$$

$$\approx 300t \qquad\qquad \approx 60t$$

FOCUS ON APPLICATIONS

REAL LIFE **PUMPKIN TOSSING**
In the annual Morton, Illinois, pumpkin tossing contest, contestants use machines they built to throw pumpkins. In 1998 an air cannon entrant set a record by throwing a pumpkin 4491 feet.

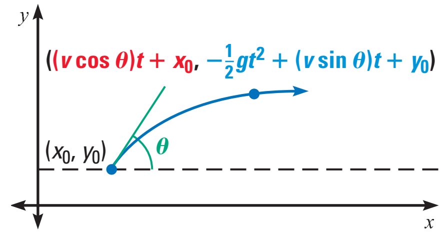
INTERNET DATA UPDATE
www.mcdougallittell.com

GOAL 2 MODELING PROJECTILE MOTION

Parametric equations can also be used to model nonlinear motion in a plane. For instance, consider an object that is projected into the air at an angle θ with an initial speed v. The object's parabolic path can be modeled with the parametric equations

$$x = (v\cos\theta)t + x_0$$

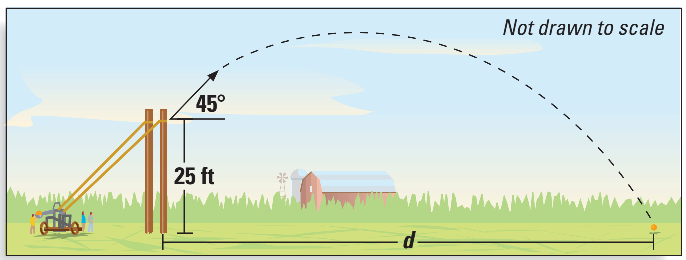
$$y = -\frac{1}{2}gt^2 + (v\sin\theta)t + y_0$$



where (x_0, y_0) is the object's location when $t = 0$. The constant g is the acceleration due to gravity. Its value is 32 ft/sec^2 or 9.8 m/sec^2. (Note that this model neglects air resistance.)

EXAMPLE 4 *Modeling Projectile Motion*

PUMPKIN TOSSING In a pumpkin tossing contest in Morton, Illinois, a contestant won the catapult competition by using two telephone poles, huge rubber bands, and a power winch. Suppose the pumpkin was launched with an initial speed of 125 feet per second, at an angle of 45°, and from an initial height of 25 feet.



a. Write a set of parametric equations for the motion of the pumpkin.

b. Use the equations to find how far the pumpkin traveled.

SOLUTION

a. Using $v = 125$ ft/sec, $\theta = 45°$, and $(x_0, y_0) = (0, 25)$, you can write the following.

$$x = (v\cos\theta)t + x_0 \qquad \text{and} \qquad y = -\frac{1}{2}gt^2 + (v\sin\theta)t + y_0$$

$$\approx 88.4t \qquad\qquad \approx -16t^2 + 88.4t + 25$$

b. The pumpkin hits the ground when $y = 0$.

$$-16t^2 + 88.4t + 25 = y \qquad \text{Write parametric equation for } y.$$

$$-16t^2 + 88.4t + 25 = 0 \qquad \text{Substitute 0 for } y.$$

$$t = \frac{-88.4 \pm \sqrt{(88.4)^2 - 4(-16)(25)}}{2(-16)} \qquad \text{Use the quadratic formula to find } t.$$

$$t \approx 5.8 \text{ seconds} \qquad \text{Simplify and choose positive } t\text{-value.}$$

When $t = 5.8$ seconds, the pumpkin's location will have an x-value of $x = (88.4)(5.8) \approx 513$ feet. So, the pumpkin traveled about 513 feet.

STUDENT HELP

Look Back
For help with the quadratic formula, see p. 291.

GUIDED PRACTICE

Vocabulary Check ✓

1. Complete this statement: Parametric equations express variables like x and y in terms of another variable such as t. In this case, t is called the __?__.

Concept Check ✓

2. For an object moving in a straight line at a constant speed v, what do you need to know in order to write parametric equations describing the object's motion?

3. In this lesson you studied two parametric models for describing motion:

$$x = (v\cos\theta)t + x_0 \qquad\qquad x = (v\cos\theta)t + x_0$$
$$\text{and}$$
$$y = (v\sin\theta)t + y_0 \qquad\qquad y = -\frac{1}{2}gt^2 + (v\sin\theta)t + y_0$$

Under what circumstances would you use each model?

Skill Check ✓

Graph the parametric equations.

4. $x = 2t$ and $y = t$ for $0 \le t \le 4$

5. $x = 3t + 4$ and $y = t - 3$ for $0 \le t \le 5$

6. $x = (20\cos 60°)t$ and $y = (20\sin 60°)t$ for $2 \le t \le 6$

Write an *xy*-equation for the parametric equations. State the domain.

7. $x = 7t$ and $y = 3t - 2$ for $0 \le t \le 5$

8. $x = -4t + 2$ and $y = 5t - 4$ for $0 \le t \le 6$

9. $x = (11.5\cos 72.1°)t$ and $y = (11.5\sin 72.1°)t + 3$ for $0 \le t \le 10$

10. **SOFTBALL CONTEST** At a softball throwing contest, you throw a softball with an initial speed of 60 feet per second, at an angle of 50°, and from an initial height of 5.5 feet. Write parametric equations for the softball's motion.

PRACTICE AND APPLICATIONS

STUDENT HELP

Extra Practice to help you master skills is on p. 959.

GRAPHING **Graph the parametric equations.**

11. $x = 2t - 2$ and $y = -t + 3$ for $0 \le t \le 5$

12. $x = 5 - 5t$ and $y = 3t - 2$ for $0 \le t \le 5$

13. $x = 2t - 6$ and $y = t - 3$ for $3 \le t \le 8$

14. $x = 30t + 10$ and $y = 60t - 20$ for $0 \le t \le 4$

15. $x = (80.6\cos 7.1°)t$ and $y = (80.6\sin 7.1°)t$ for $0 \le t \le 5$

ELIMINATING THE PARAMETER **Write an *xy*-equation for the parametric equations. State the domain.**

STUDENT HELP

HOMEWORK HELP
Example 1: Exs. 11–15
Example 2: Exs. 16–20
Example 3: Exs. 24–32
Example 4: Exs. 33–40

16. $x = 2t$ and $y = -4t$ for $0 \le t \le 5$

17. $x = t + 1$ and $y = 2t - 3$ for $0 \le t \le 5$

18. $x = 3t + 6$ and $y = 5t - 1$ for $0 \le t \le 20$

19. $x = (14.14\cos 45°)t$ and $y = (14.14\sin 45°)t$ for $0 \le t \le 10$

20. $x = (111.8\cos 63.43°)t$ and $y = (111.8\sin 63.43°)t$ for $0 \le t \le 10$

STUDENT HELP

HOMEWORK HELP
Visit our Web site www.mcdougallittell.com for help with Exs. 21–23.

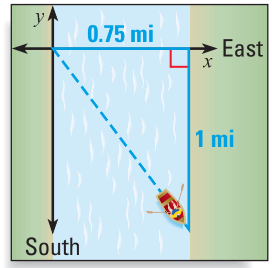
DESCRIBING LINEAR MOTION Use the given information to write parametric equations describing the linear motion.

21. An object is at (0, 0) at time $t = 0$ and then at (19, 57) at time $t = 3$.

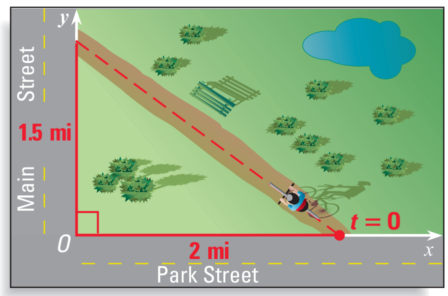
22. An object is at (18, 8) at time $t = 4$ and then at (40.8, 19.0) at time $t = 9$.

23. An object is at (3, 2) at time $t = 0$ and then at (14.3, 66.1) at time $t = 5$.

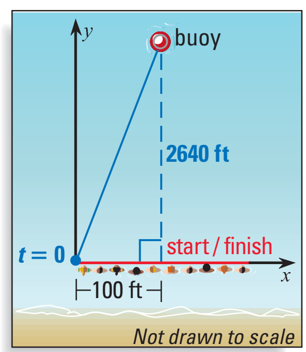
24. ROWBOAT You are trying to row a boat due east across a river that is 0.75 mile wide and flows due south. You reach the other side in 15 minutes, but the current has pulled you 1 mile downstream. Write a set of parametric equations to describe the path you traveled. Then write an *xy*-equation for the parametric equations. State the domain of the *xy*-equation.



25. BIKE PATH A bike trail connects Park Street and Main Street as shown. You enter the trail 2 miles from the intersection of the streets and bike at a speed of 10 miles per hour. You reach Main Street 1.5 miles from the intersection. Write a set of parametric equations to describe your path.



SWIMMING In Exercises 26–28, use the following information.
You are swimming in a race across a lake and back. Swimmers must swim to, and then back from, a buoy placed 2640 feet from the center of the start/finish line. You start the race 100 feet from the center of the start/finish line as shown.



26. You swim to the buoy at a steady rate of 0.7 foot per second. Write a set of parametric equations for your path.

27. Use the equations to determine how long it takes you to reach the buoy.

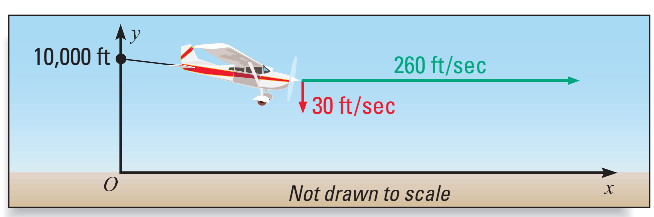
28. If you continue to swim at a steady rate of 0.7 foot per second straight back to the center of the start/finish line, how long will it take you to complete the race?

FOCUS ON APPLICATIONS

INSTRUMENT PANEL In addition to gauges that give speed and altitude readings, the instrument panel of an airplane also contains an *attitude indicator*. This instrument tells how the airplane is tilted in relation to Earth's horizon.

APPLICATION LINK
www.mcdougallittell.com

LANDING A PLANE In Exercises 29–32, use the following information.
You are flying in a small airplane at an altitude of 10,000 feet. When you descend to land the plane, your horizontal air speed will be 260 feet per second (177 miles per hour) and your rate of descent will be 30 feet per second.



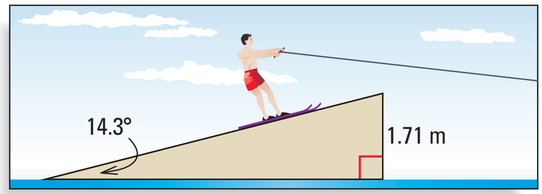
29. Write a set of parametric equations for the plane's descent.

30. What is the angle of descent?

31. How long will it take for the plane to land?

32. How far from the airport should you begin the descent?

WATER SKIING **In Exercises 33–35, use the following information.**
A water skier jumps off a ramp at a speed of 17.9 meters per second. The ramp's angle of elevation is 14.3°, and the height of the end of the ramp above the surface of the water is 1.71 meters. ▶ Source: American Water Ski Association



33. Write a set of parametric equations for the water skier's jump.

34. For how many seconds is the water skier in the air?

35. How far from the ramp does the water skier land?

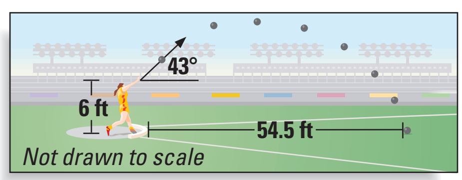
LEAPING DOLPHIN **In Exercises 36–38, use the following information.**
A dolphin is performing in a show at an oceanic park and makes a leap out of the water. The dolphin leaves the water traveling at a speed of 32 feet per second and at an angle of 48° with the surface of the water.

36. Write a set of parametric equations for the dolphin's motion.

37. For how many seconds is the dolphin in the air?

38. How far across the water does the dolphin travel in the air?

SHOT PUT **In Exercises 39 and 40, use the following information.**
A shot put is thrown a distance of 54.5 feet at a high school track and field meet. The shot put was released from a height of 6 feet and at an angle of 43°.



39. Write a set of parametric equations for the path of the shot put.

40. Use the equations to determine the speed of the shot put at the time of release.

Test Preparation

41. **MULTIPLE CHOICE** Which equation is an xy-equation for the parametric equations $x = 3t + 12$ and $y = 12t - 8$ where $0 \le t \le 20$?

Ⓐ $y = 9x + 4;\ 0 \le x \le 20$ Ⓑ $y = 36x + 132;\ -8 \le x \le 132$

Ⓒ $y = 4x - 56;\ 12 \le x \le 72$ Ⓓ $y = 15x + 4;\ 4 \le x \le 304$

42. **MULTIPLE CHOICE** An airplane takes off at an angle of 10.6° with the ground and travels at a constant speed of 324 miles per hour. Which set of parametric equations describes the airplane's ascent?

Ⓐ $x = 4670t,\ y = 874t$ Ⓑ $x = 318t,\ y = 60t$

Ⓒ $x = 5000t,\ y = 80t$ Ⓓ $x = 800t,\ y = 150t$

★ Challenge

43. **CRITICAL THINKING** Write the following pairs of equations in the form $y = f(x)$.

$$x = (v \cos \theta)t + x_0 \qquad x = (v \cos \theta)t + x_0$$
$$y = (v \sin \theta)t + y_0 \qquad y = -\frac{1}{2}gt^2 + (v \sin \theta)t + y_0$$

In which case is the path of the moving object *not* affected by changing the speed v? Explain why this makes sense.

EXTRA CHALLENGE
www.mcdougallittell.com

MIXED REVIEW

GRAPHING **Graph the function.** (Review 5.1, 7.5, 8.1 for 14.1)

44. $y = 9x^2$

45. $y = -10x^2$

46. $y = 7\sqrt{x}$

47. $y = -6\sqrt{x}$

48. $y = 7 \cdot 2^x$

49. $y = -\frac{3}{4} \cdot 3^x$

FINDING SUMS **Find the sum of the series.** (Review 11.1, 11.4)

50. $\sum_{i=1}^{10} -3i$

51. $\sum_{i=1}^{27} i^2$

52. $\sum_{n=1}^{\infty} 20\left(\frac{4}{5}\right)^{n-1}$

53. $\sum_{n=1}^{\infty} -\frac{1}{6}\left(-\frac{1}{2}\right)^{n-1}$

NORMAL DISTRIBUTIONS **Find the probability that a randomly selected *x*-value is in the given interval.** (Review 12.7)

54. to the left of the mean

55. between the mean and 1 standard deviation to the left of the mean

56. between 2 and 3 standard deviations from the mean

57. more than 3 standard deviations to the right of the mean

QUIZ 3

Self-Test for Lessons 13.5–13.7

Solve △*ABC*. (Lessons 13.5 and 13.6)

1. $B = 70°, b = 30, c = 25$

2. $B = 10°, C = 100°, a = 15$

3. $A = 40°, B = 110°, b = 30$

4. $A = 122°, a = 9, c = 13$

5. $a = 45, b = 32, c = 24$

6. $A = 107°, b = 15, c = 28$

Find the area of △*ABC*. (Lessons 13.5 and 13.6)

7. $B = 95°, a = 12, c = 30$

8. $C = 103°, a = 41, b = 25$

9. $A = 117°, b = 16, c = 8$

10. $a = 7, b = 7, c = 5$

11. $a = 89, b = 55, c = 71$

12. $a = 40, b = 21, c = 32$

Graph the parametric equations. (Lesson 13.7)

13. $x = 4 - 2t$ and $y = 3t + 1$ for $0 \le t \le 5$

14. $x = 2t - 5$ and $y = 4t - 3$ for $3 \le t \le 7$

15. $x = (10.5 \cos 45°)t$ and $y = (10.5 \sin 45°)t + 4$ for $0 \le t \le 5$

Write an *xy*-equation for the parametric equations. State the domain. (Lesson 13.7)

16. $x = -5t + 3$ and $y = t - 6$ for $0 \le t \le 5$

17. $x = (10 \cos 35°)t$ and $y = (10 \sin 35°)t$ for $0 \le t \le 30$

18. **SOCCER** You are a goalie in a soccer game. You save the ball and then drop kick it as far as you can down the field. Your kick has an initial speed of 26 feet per second and starts at a height of 2 feet. If you kick the ball at an angle of 45°, how far down the field does the ball hit the ground? (Lesson 13.7)

ACTIVITY 13.7

Using Technology

Graphing Calculator Activity for use with Lesson 13.7

Graphing Parametric Equations

You can use a graphing calculator to graph a set of parametric equations.

EXAMPLE

At a driving range, you hit a golf ball at ground level with an initial speed of 120 feet per second and at an angle of 45°. Use a graphing calculator to graph a set of parametric equations that describe the path of the ball. Then use the graphing calculator to estimate how far the ball travels.

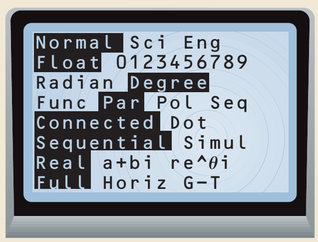
SOLUTION

Assume that $x_0 = 0$ and $y_0 = 0$. The parametric equations that describe the path of the golf ball are as follows.

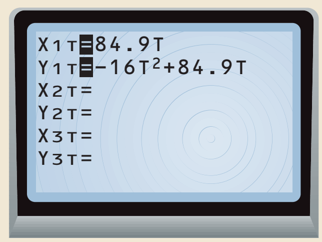
$$x = (v \cos \theta)t + x_0 \qquad y = -\frac{1}{2}gt^2 + (v \sin \theta)t + y_0$$

$$\approx 84.9t \qquad \approx -16t^2 + 84.9t$$

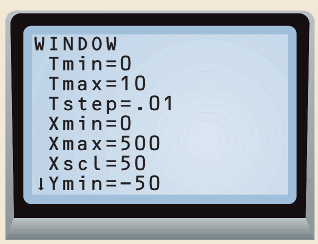
1 Put your calculator in *Parametric* mode.



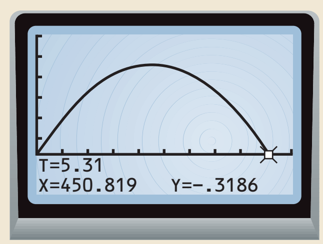
2 Enter the parametric equations.



3 Set the viewing window so that $0 \le t \le 10$, $0 \le x \le 500$, and $-50 \le y \le 150$.



4 Graph the parametric equations. Use the graphing calculator's *Trace* feature to find the value of x when $y = 0$.



► An estimate of the distance the ball travels is about 450 feet, or 150 yards.

EXERCISES

1. Graph a set of parametric equations that describe the path of the golf ball in the example when it is hit at different angles. Copy and complete the table.

Angle, θ	30°	35°	40°	50°	55°	60°
Horizontal distance (ft), x	?	?	?	?	?	?

2. At what angle should you hit the ball so that it travels the maximum horizontal distance? Explain.

STUDENT HELP

INTERNET

KEYSTROKE HELP

See keystrokes for several models of calculators at www.mcdougallittell.com

CHAPTER 13

Chapter Summary

WHAT did you learn?	*WHY did you learn it?*
Evaluate trigonometric functions.	
• of acute angles **(13.1)**	Find the altitude of a kite. **(p. 771)**
• of any angle **(13.3)**	Find the horizontal distance traveled by a golf ball. **(p. 787)**
Find the sides and angles of a triangle.	
• solve right triangles **(13.1)**	Find the length of a zip-line at a ropes course. **(p. 774)**
• use the law of sines **(13.5)**	Find the distance between two buildings. **(p. 805)**
• use the law of cosines **(13.6)**	Find the angle at which two trapeze artists meet. **(p. 811)**
Measure angles using degree measure and radian measure. **(13.2)**	Find the angle generated by a figure skater performing a jump. **(p. 781)**
Find arc lengths and areas of sectors. **(13.2)**	Find the area irrigated by a rotating sprinkler. **(p. 781)**
Evaluate inverse trigonometric functions. **(13.4)**	Find the angle at which to set the arm of a crane. **(p. 794)**
Find the area of a triangle.	
• using two sides and the included angle **(13.5)**	Find the amount of paint needed for the side of a house. **(p. 806)**
• using Heron's formula **(13.6)**	Find the area of the Dinosaur Diamond. **(p. 812)**
Use parametric equations to model linear or projectile motion. **(13.7)**	Model the path of a leaping dolphin. **(p. 818)**
Use trigonometric and inverse trigonometric functions to solve real-life problems. **(13.1, 13.3–13.7)**	Find distances for a marching band on a football field. **(p. 787)**

How does Chapter 13 fit into the BIGGER PICTURE of algebra?

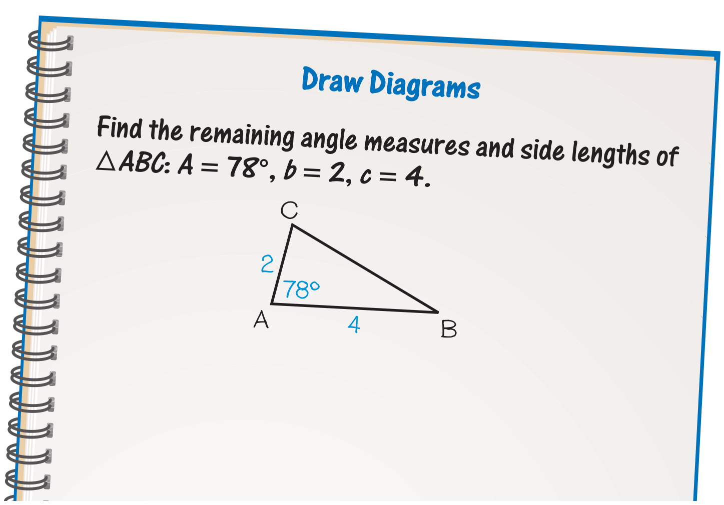
Trigonometry is closely tied to both algebra and geometry. In this chapter you studied trigonometric functions of *angles*, defined by ratios of side lengths of right triangles.

In the next chapter you will study trigonometric functions of *real numbers*, used to model periodic behavior. You will see even more connections between trigonometry and algebra as you graph trigonometric functions in a coordinate plane.

STUDY STRATEGY

How did you draw diagrams?

Here is an example of a diagram drawn for Exercise 22 on page 810, following the **Study Strategy** on page 768.



CHAPTER 13

Chapter Review

VOCABULARY

- sine, p. 769
- cosine, p. 769
- tangent, p. 769
- cosecant, p. 769
- secant, p. 769
- cotangent, p. 769
- solving a right triangle, p. 770
- angle of elevation, p. 771
- angle of depression, p. 771
- initial side of an angle, p. 776
- terminal side of an angle, p. 776
- standard position, p. 776
- coterminal angles, p. 777
- radian, p. 777
- sector, p. 779
- central angle, p. 779
- quadrantal angle, p. 785
- reference angle, p. 785
- inverse sine, p. 792
- inverse cosine, p. 792
- inverse tangent, p. 792
- law of sines, p. 799
- law of cosines, p. 807
- parametric equations, p. 813
- parameter, p. 813

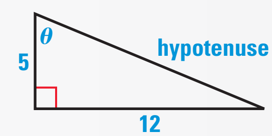
13.1 RIGHT TRIANGLE TRIGONOMETRY

Examples on pp. 769–771

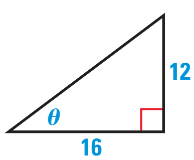
EXAMPLE You can evaluate the six trigonometric functions of θ for the triangle shown. First find the hypotenuse length: $\sqrt{5^2 + 12^2} = \sqrt{169} = 13$.

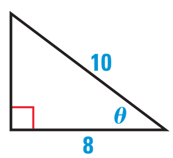
$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{12}{13}$ $\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{5}{13}$ $\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{12}{5}$

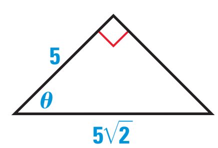
$\csc\theta = \frac{\text{hyp}}{\text{opp}} = \frac{13}{12}$ $\sec\theta = \frac{\text{hyp}}{\text{adj}} = \frac{13}{5}$ $\cot\theta = \frac{\text{adj}}{\text{opp}} = \frac{5}{12}$

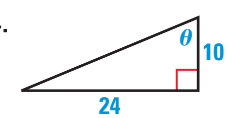


Evaluate the six trigonometric functions of θ.

1.


2.


3.


4.


13.2 GENERAL ANGLES AND RADIAN MEASURE

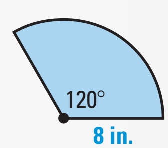
Examples on pp. 776–779

EXAMPLES You can measure angles using degree measure or radian measure.

$20° = 20°\left(\frac{\pi \text{ radians}}{180°}\right) = \frac{\pi}{9}$ radians $\frac{7\pi}{6}$ radians $= \left(\frac{7\pi}{6} \text{ radians}\right)\left(\frac{180°}{\pi \text{ radians}}\right) = 210°$

Arc length of the sector at the right: $s = r\theta = 8\left(\frac{2\pi}{3}\right) = \frac{16\pi}{3}$ inches

Area of the sector at the right: $A = \frac{1}{2}r^2\theta = \frac{1}{2}(8^2)\left(\frac{2\pi}{3}\right) = \frac{64\pi}{3}$ square inches



Rewrite each degree measure in radians and each radian measure in degrees.

5. $30°$ **6.** $225°$ **7.** $-15°$ **8.** $\frac{3\pi}{4}$ **9.** $\frac{5\pi}{3}$ **10.** $\frac{\pi}{3}$

Find the arc length and area of a sector with the given radius r and central angle θ.

11. $r = 5$ ft, $\theta = \frac{\pi}{2}$ **12.** $r = 12$ in., $\theta = 25°$ **13.** $r = 16$ cm, $\theta = 210°$

13.3 Trigonometric Functions of Any Angle

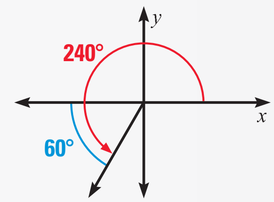
Examples on pp. 784–787

EXAMPLE You can evaluate the six trigonometric functions of $\theta = 240°$ using a reference angle: $\theta' = \theta - 180° = 240° - 180° = 60°$.

$\sin 240° = -\sin 60° = -\frac{\sqrt{3}}{2}$ $\csc 240° = -\csc 60° = -\frac{2\sqrt{3}}{3}$

$\cos 240° = -\cos 60° = -\frac{1}{2}$ $\sec 240° = -\sec 60° = -2$

$\tan 240° = +\tan 60° = \sqrt{3}$ $\cot 240° = +\cot 60° = \frac{\sqrt{3}}{3}$



Evaluate the function without using a calculator.

14. $\tan \frac{11\pi}{4}$ **15.** $\cos \frac{11\pi}{6}$ **16.** $\sec 225°$ **17.** $\sin 390°$ **18.** $\csc(-120°)$

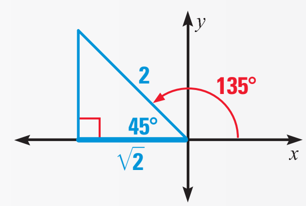
13.4 Inverse Trigonometric Functions

Examples on pp. 792–794

EXAMPLE You can find an angle within a certain range that corresponds to a given value of a trigonometric function.

To find $\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$, find θ so that $\cos \theta = -\frac{\sqrt{2}}{2}$ and $0° \le \theta \le 180°$.

So, $\theta = \cos^{-1}\left(-\frac{\sqrt{2}}{2}\right) = 135°$ $\left(\text{or } \frac{3\pi}{4} \text{ radians}\right)$.



Evaluate the expression without using a calculator. Give your answer in both radians and degrees.

19. $\sin^{-1} \frac{\sqrt{2}}{2}$ **20.** $\tan^{-1} \frac{\sqrt{3}}{3}$ **21.** $\cos^{-1} 0$ **22.** $\tan^{-1}(-1)$ **23.** $\cos^{-1}\left(-\frac{1}{2}\right)$

13.5 The Law of Sines

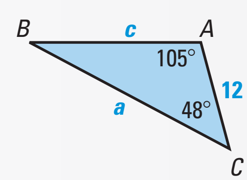
Examples on pp. 799–802

EXAMPLE You can solve the triangle shown using the law of sines.

The measure of the third angle is: $B = 180° - 105° - 48° = 27°$.

$\frac{a}{\sin 105°} = \frac{12}{\sin 27°}$ $\frac{c}{\sin 48°} = \frac{12}{\sin 27°}$

$a = \frac{12 \sin 105°}{\sin 27°} \approx 25.5$ $c = \frac{12 \sin 48°}{\sin 27°} \approx 19.6$



Area of this triangle $= \frac{1}{2}bc \sin A = \frac{1}{2}(12)(19.6) \sin 105° \approx 114$ square units

13.5 continued

Solve △*ABC*. (*Hint:* Some of the "triangles" may have no solution and some may have two.)

24. $A = 45°, B = 60°, c = 44$ **25.** $B = 18°, b = 12, a = 19$ **26.** $C = 140°, c = 40, b = 20$

Find the area of the triangle with the given side lengths and included angle.

27. $C = 35°, b = 10, a = 22$ **28.** $A = 110°, b = 8, c = 7$ **29.** $B = 25°, a = 15, c = 31$

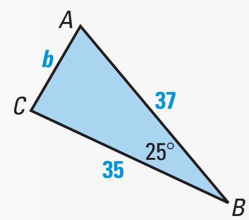
13.6 The Law of Cosines

Examples on pp. 807–809

EXAMPLE You can solve the triangle below using the law of cosines.

Law of cosines: $b^2 = 35^2 + 37^2 - 2(35)(37)\cos 25° \approx 247$

$b \approx 15.7$



Law of sines: $\frac{\sin A}{35} \approx \frac{\sin 25°}{15.7}$, $\sin A \approx \frac{35 \sin 25°}{15.7}$, $A \approx 70.4°$

$C \approx 180° - 25° - 70.4° = 84.6°$

You can use Heron's formula to find the area of this triangle:

$s \approx \frac{1}{2}(35 + 15.7 + 37) \approx 44$, so area $\approx \sqrt{44(44 - 35)(44 - 15.7)(44 - 37)} \approx 280$ square units

Solve △*ABC*.

30. $a = 25, b = 18, c = 28$ **31.** $a = 6, b = 11, c = 14$ **32.** $B = 30°, a = 80, c = 70$

Find the area of △*ABC* having the given side lengths.

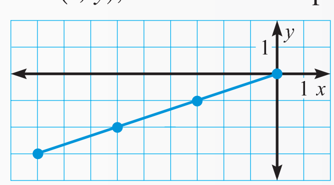
33. $a = 11, b = 2, c = 12$ **34.** $a = 4, b = 24, c = 26$ **35.** $a = 15, b = 8, c = 21$

13.7 Parametric Equations and Projectile Motion

Examples on pp. 813–815

EXAMPLE You can graph the parametric equations $x = -3t$ and $y = -t$ for $0 \le t \le 3$. Make a table of values, plot the points (x, y), and connect the points.

t	0	1	2	3
x	0	−3	−6	−9
y	0	−1	−2	−3



To write an *xy*-equation for these parametric equations, solve the first equation for *t*: $t = -\frac{1}{3}x$. Substitute into the second equation: $y = \frac{1}{3}x$. The domain is $-9 \le x \le 0$.

Graph the parametric equations.

36. $x = 3t + 1$ and $y = 3t + 6$ for $0 \le t \le 5$ **37.** $x = 2t + 4$ and $y = -4t + 2$ for $2 \le t \le 5$

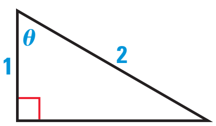
Write an *xy*-equation for the parametric equations. State the domain.

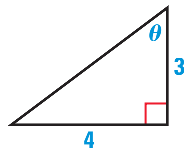
38. $x = 5t$ and $y = t + 7$ for $0 \le t \le 20$ **39.** $x = 2t - 3$ and $y = -4t + 5$ for $0 \le t \le 8$

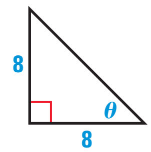
CHAPTER 13

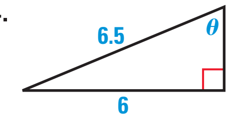
Chapter Test

Evaluate the six trigonometric functions of θ.

1.


2.


3.


4.


Rewrite each degree measure in radians and each radian measure in degrees.

5. $120°$ **6.** $360°$ **7.** $-60°$ **8.** $\frac{\pi}{9}$ **9.** 5π **10.** $-\frac{5\pi}{4}$

Find the arc length and area of a sector with the given radius *r* and central angle θ.

11. $r = 4$ ft, $\theta = 240°$ **12.** $r = 20$ cm, $\theta = 45°$ **13.** $r = 12$ in., $\theta = 150°$

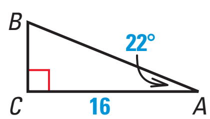
Evaluate the function without using a calculator.

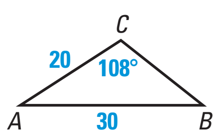
14. $\cos 180°$ **15.** $\sec(-30°)$ **16.** $\cot 495°$ **17.** $\sin \frac{7\pi}{6}$ **18.** $\tan\left(-\frac{\pi}{4}\right)$ **19.** $\csc\left(-\frac{7\pi}{4}\right)$

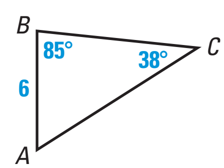
Evaluate the expression without using a calculator. Give your answer in both radians and degrees.

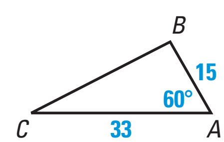
20. $\sin^{-1} 1$ **21.** $\tan^{-1} \sqrt{3}$ **22.** $\cos^{-1} \frac{\sqrt{3}}{2}$ **23.** $\tan^{-1} 0$ **24.** $\cos^{-1} 1$ **25.** $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$

Solve $\triangle ABC$.

26.


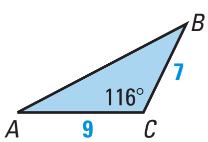
27.


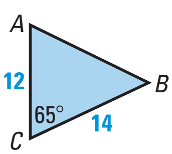
28.


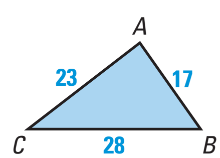
29.


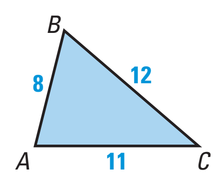
30. $A = 120°, a = 14, b = 10$ **31.** $B = 40°, a = 7, c = 10$ **32.** $C = 105°, a = 4, b = 3$

Find the area of $\triangle ABC$.

33.


34.


35.


36.


Graph the parametric equations. Then write an *xy*-equation and state the domain.

37. $x = 2t - 3$ and $y = -5t + 6$ for $1 \le t \le 4$ **38.** $x = t - 4$ and $y = -t + 6$ for $0 \le t \le 6$

39. **BOAT RIDE** A boat travels 50 miles due west before adjusting its course 25° north of west and traveling an additional 35 miles. How far is the boat from its point of departure?

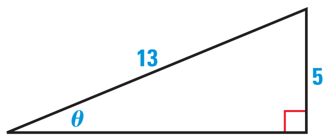
40. **PROJECTILE MOTION** You throw a ball at an angle of 50°, from a height of 6 feet, and with an initial speed of 25 feet per second. Write a set of parametric equations for the path of the ball. How far from you does the ball land?

CHAPTER 13

Chapter Standardized Test

TEST-TAKING STRATEGY When taking a test, first tackle the questions that you know are easy for you to answer. Then go back and answer questions that you suspect will take you extra time and effort.

1. **MULTIPLE CHOICE** Given the diagram, which equation is correct?



Ⓐ $\sec\theta = \frac{12}{13}$ Ⓑ $\cot\theta = \frac{12}{5}$

Ⓒ $\cos\theta = \frac{5}{13}$ Ⓓ $\csc\theta = \frac{12}{5}$

Ⓔ $\sin\theta = \frac{13}{5}$

2. **MULTIPLE CHOICE** Suppose $(8, -15)$ is a point on the terminal side of an angle θ in standard position. Which equation is *not* true?

Ⓐ $\csc\theta = -\frac{15}{17}$ Ⓑ $\cos\theta = \frac{8}{17}$

Ⓒ $\tan\theta = -\frac{15}{8}$ Ⓓ $\cot\theta = -\frac{8}{15}$

Ⓔ $\sec\theta = \frac{17}{8}$

3. **MULTIPLE CHOICE** What is the value of $\cos 765°$?

Ⓐ $\frac{\sqrt{2}}{2}$ Ⓑ $\frac{2\sqrt{3}}{3}$ Ⓒ $\sqrt{2}$

Ⓓ 2 Ⓔ undefined

4. **MULTIPLE CHOICE** What is the value of $\sin\left(-\frac{13\pi}{6}\right)$?

Ⓐ $-\frac{\sqrt{3}}{2}$ Ⓑ $-\frac{\sqrt{2}}{2}$ Ⓒ $-\frac{1}{2}$

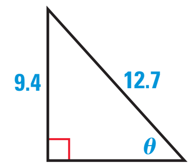
Ⓓ $\frac{1}{2}$ Ⓔ 1

5. **MULTIPLE CHOICE** What is the solution of the equation $\sin\theta = -\frac{3}{8}$, where $180° < \theta < 270°$?

Ⓐ $-202.02°$ Ⓑ $-22.02°$ Ⓒ $22.02°$

Ⓓ $202.02°$ Ⓔ $222.02°$

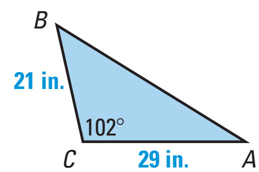
6. **MULTIPLE CHOICE** What is the approximate value of θ in the triangle shown?



Ⓐ 36.5° Ⓑ 42.3° Ⓒ 47.7°

Ⓓ 48.9° Ⓔ 52.6°

7. **MULTIPLE CHOICE** What is the area of the triangle shown?

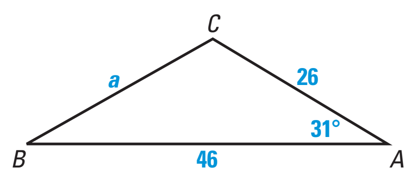


Ⓐ about 182.4 in.^{2} Ⓑ about 297.8 in.^{2}

Ⓒ about 300.8 in.^{2} Ⓓ about 304.5 in.^{2}

Ⓔ about 595.7 in.^{2}

8. **MULTIPLE CHOICE** What is the value of a in the triangle shown?



Ⓐ about 24.6 Ⓑ about 27.2

Ⓒ about 28.9 Ⓓ about 29.2

Ⓔ about 30.5

9. **MULTIPLE CHOICE** Which equation is an xy-equation for the parametric equations $x = 5t - 9$ and $y = -3t + 11$?

Ⓐ $y = \frac{3}{5}x + \frac{28}{5}$ Ⓑ $y = -\frac{3}{5}x + \frac{82}{5}$

Ⓒ $y = -\frac{3}{5}x - \frac{82}{5}$ Ⓓ $y = -\frac{3}{5}x - \frac{28}{5}$

Ⓔ $y = -\frac{3}{5}x + \frac{28}{5}$

QUANTITATIVE COMPARISON **In Exercises 10–12, choose the statement that is true about the given quantities.**

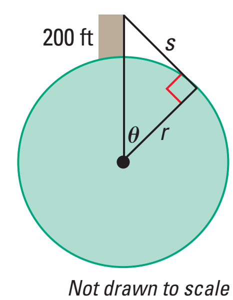
Ⓐ The quantity in column A is greater.

Ⓑ The quantity in column B is greater.

Ⓒ The two quantities are equal.

Ⓓ The relationship cannot be determined from the given information.

	Column A	Column B
10.	Solution of $\tan\theta = -\sqrt{3}$, where $270° \le \theta \le 360°$	Solution of $\sin\theta = -\frac{\sqrt{2}}{2}$, where $270° \le \theta \le 360°$
11.	Area of a sector with $r = 5$ in. and $\theta = 60°$	Area of a sector with $r = 6$ in. and $\theta = 45°$
12.	Area of a triangle with side lengths 5 cm, 8 cm, and 11 cm	Area of a triangle with side lengths 7 cm, 7 cm, and 12 cm



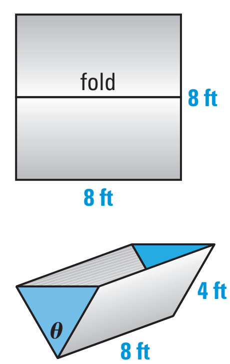
13. MULTI-STEP PROBLEM You are enjoying the view at the top of a 200 foot tall building on a clear day. To find the distance you can see to the horizon, you draw the diagram at the right.

a. Find the value of θ in the diagram. Use 3960 miles for the value of r.

b. Use your answer to part (a) to find the distance s you can see to the horizon.

c. How much farther could you see to the horizon if the building were 400 feet tall?

d. **CRITICAL THINKING** Write a general formula for finding the distance s you can see to the horizon from the top of a building that is h feet tall.



14. MULTI-STEP PROBLEM A trough can be made by folding a rectangular piece of metal in half and then enclosing the ends. The volume of water the trough can hold depends on how far you bend the metal.

a. Predict the value of θ that will maximize the volume of the trough shown.

b. Find the volume of the trough as a function of θ. (*Hint:* You will need to find the area of one of the triangular faces.)

c. **CRITICAL THINKING** Find the value of θ that maximizes the volume. What is the maximum volume? How close was your prediction?

15. MULTI-STEP PROBLEM Two motorized toy boats are released in a pool at time $t = 0$. Boat 1 travels 65° north of east at a rate of 0.75 meter per second. Boat 2 travels due east at a rate of 0.5 meter per second.

a. Write a set of parametric equations to describe the path of each boat.

b. At what point will each boat hit the east edge of the pool?

c. At what point do the paths of the two boats cross?

d. *Writing* If the two boats are released at the same time, will they collide? If so, how many seconds after the boats are released? If not, explain why not.