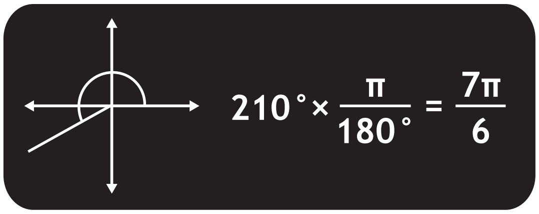


Trigonometry

LESSON ONE - *Degrees and Radians*

Lesson Notes

Example 1: Define each term or phrase and draw a sample angle.

a) Angle in standard position. Draw a standard position angle, θ.
b) Positive and negative angles. Draw θ = 120° and θ = -120°.
c) Reference angle. Find the reference angle of θ = 150°.
d) Co-terminal angles. Draw the first positive co-terminal angle of 60°.
e) Principal angle. Find the principal angle of θ = 420°.
f) General form of co-terminal angles. Find the first four positive and negative co-terminal angles of θ = 45°.

Conversion Multiplier Reference Chart ***(Example 2)***

	degree	radian	revolution
degree	X		
radian		X	
revolution			X

Example 2: Three Angle Types: ***Degrees, Radians, and Revolutions***.

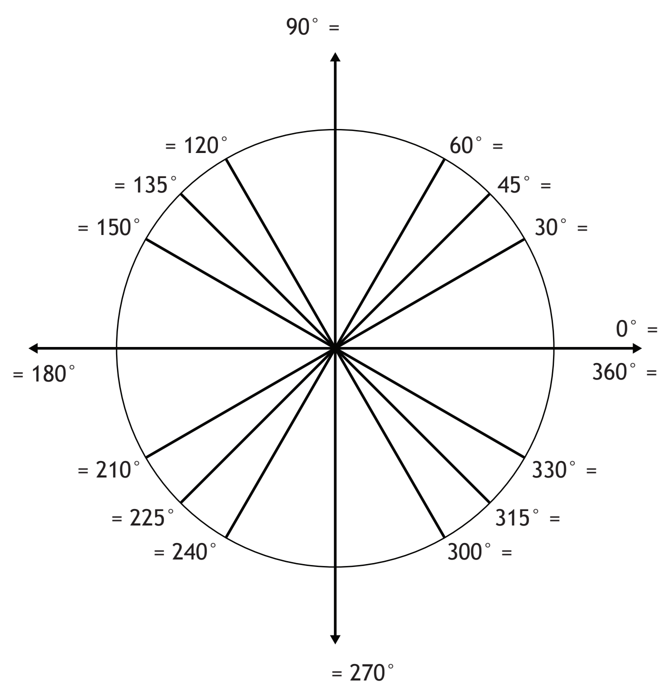
a) i. Define degrees. Draw θ = 1°. **ii.** Define radians. Draw θ = 1 rad. **iii.** Define revolutions. Draw θ = 1 rev.

b) Use conversion multipliers to answer the questions and fill in the reference chart.
i. 23° = __ rad **ii.** 23° = __ rev **iii.** 2.6 = __° **iv.** 2.6 = __ rev **v.** 0.75 rev = __° **vi.** 0.75 rev = __rad

c) Contrast the decimal approximation of a radian with the exact value of a radian.
i. 45° = _____ rad *(decimal approximation)*. **ii.** 45° = _____ rad *(exact value)*.

Example 3: Convert each angle to the requested form. Round all decimals to the nearest hundredth.
a) convert 175° to an approximate radian decimal. **b)** convert 210° to an exact-value radian.
c) convert 120° to an exact-value revolution. **d)** convert 2.5 to degrees. **e)** convert 3π/2 to degrees.
f) write 3π/2 as an approximate radian decimal. **g)** convert π/2 to an exact-value revolution.
h) convert 0.5 rev to degrees. **i)** convert 3 rev to radians.

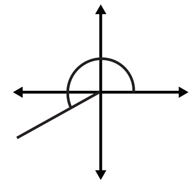
Example 4: The diagram shows commonly used degrees. Find exact-value radians that correspond to each degree. When complete, memorize the diagram.
a) Method One: Find all exact-value radians using a conversion multiplier.
b) Method Two: Use a shortcut *(counting radians)*.



Example 5: Draw each of the following angles in standard position. State the reference angle.
a) 210° **b)** -260° **c)** 5.3 **d)** -5π/4 **e)** 12π/7

Example 6: Draw each of the following angles in standard position. State the principal and reference angles.
a) 930° **b)** -855° **c)** 9 **d)** -10π/3

Trigonometry
LESSON ONE - *Degrees and Radians*
Lesson Notes



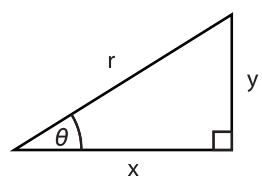
$210^\circ \times \frac{\pi}{180^\circ} = \frac{7\pi}{6}$

Example 7: For each angle, find all co-terminal angles within the stated domain.
a) 60°, Domain: $-360^\circ \leq \theta < 1080^\circ$ **b)** -495°, Domain: $-1080^\circ \leq \theta < 720^\circ$
c) 11.78, Domain: $-2\pi \leq \theta < 4\pi$ **d)** $8\pi/3$, Domain: $-13\pi/2 \leq \theta < 37\pi/5$

Example 8: For each angle, use estimation to find the principal angle.
a) 1893° **b)** -437.24 **c)** $912\pi/15$ **d)** $95\pi/6$

Example 9: Use the general form of co-terminal angles to find the specified angle.
a) $\theta_p = 300^\circ$ (Find θ_c, 3 rotations CC) **b)** $\theta_p = 2\pi/5$ (Find θ_c, 14 rotations C)
c) $\theta_c = -4300^\circ$ (Find *n* and θ_p) **d)** $\theta_c = 32\pi/3$ (Find *n* and θ_p)

Example 10: In addition to the three primary trigonometric ratios (sinθ, cosθ, and tanθ), there are three reciprocal ratios (cscθ, secθ, and cotθ). Given a triangle with side lengths of x and y, and a hypotenuse of length r, the six trigonometric ratios are as follows:



$\sin\theta = \frac{y}{r}$ $\csc\theta = \frac{1}{\sin\theta} = \frac{r}{y}$

$\cos\theta = \frac{x}{r}$ $\sec\theta = \frac{1}{\cos\theta} = \frac{r}{x}$

$\tan\theta = \frac{y}{x}$ $\cot\theta = \frac{1}{\tan\theta} = \frac{x}{y}$

a) If the point P(-5, 12) exists on the terminal arm of an angle θ in standard position, determine the exact values of all six trigonometric ratios. State the reference angle and the standard position angle.

b) If the point P(2, -3) exists on the terminal arm of an angle θ in standard position, determine the exact values of all six trigonometric ratios. State the reference angle and the standard position angle.

Example 11: Determine the sign of each trigonometric ratio in each quadrant.

a) sinθ **b)** cosθ **c)** tanθ **d)** cscθ **e)** secθ **f)** cotθ

g) How do the quadrant signs of the reciprocal trigonometric ratios (cscθ, secθ, and cotθ) compare to the quadrant signs of the primary trigonometric ratios (sinθ, cosθ, and tanθ)?

Example 12: Given the following conditions, find the quadrant(s) where the angle θ could potentially exist.

a) i. $\sin\theta < 0$ **ii.** $\cos\theta > 0$ **iii.** $\tan\theta > 0$

b) i. $\sin\theta > 0$ and $\cos\theta > 0$ **ii.** $\sec\theta > 0$ and $\tan\theta < 0$ **iii.** $\csc\theta < 0$ and $\cot\theta > 0$

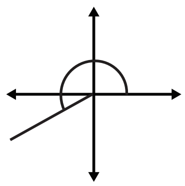
c) i. $\sin\theta < 0$ and $\csc\theta = 1/2$ **ii.** $\cos\theta = -\sqrt{3}/2$ and $\csc\theta < 0$ **iii.** $\sec\theta > 0$ and $\tan\theta = 1$

Example 13: Given one trigonometric ratio, find the exact values of the other five trigonometric ratios. State the reference angle and the standard position angle, to the nearest hundredth of a radian.

a) $\cos\theta = -\frac{12}{13}, \quad \pi \leq \theta < \frac{3\pi}{2}$ **b)** $\csc\theta = \frac{7}{3}, \quad \frac{\pi}{2} \leq \theta < \pi$

Example 14: Given one trigonometric ratio, find the exact values of the other five trigonometric ratios. State the reference angle and the standard position angle, to the nearest hundredth of a degree.

a) $\sec\theta = \frac{5}{4}, \ \sin\theta < 0$ **b)** $\tan\theta = -\frac{2}{3}, \ \sec\theta > 0$



$210^\circ \times \frac{\pi}{180^\circ} = \frac{7\pi}{6}$

Trigonometry
LESSON ONE - *Degrees and Radians*
Lesson Notes

Example 15: Calculating θ with a calculator.

a) When you solve a trigonometric equation in your calculator, the answer you get for θ can seem unexpected. Complete the following chart to learn how the calculator processes your attempt to solve for θ.

If the angle θ could exist in either quadrant ___ or ___ ...	The calculator always picks quadrant
I or II	
I or III	
I or IV	
II or III	
II or IV	
III or IV	

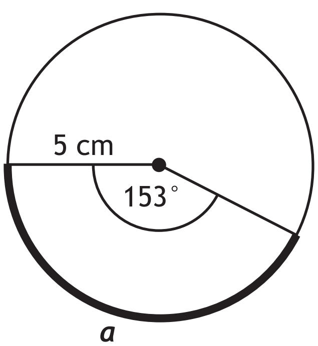
b) Given the point P(-4, 3), Mark tries to find the reference angle using a sine ratio, Jordan tries to find it using a cosine ratio, and Dylan tries to find it using a tangent ratio. Why does each person get a different result from their calculator?

Mark's Calculation of θ (using sine)	Jordan's Calculation of θ (using cosine)	Dylan's Calculation of θ (using tan)
$\sin\theta = \frac{3}{5}$ $\theta = 36.87^\circ$	$\cos\theta = \frac{-4}{5}$ $\theta = 143.13^\circ$	$\tan\theta = \frac{3}{-4}$ $\theta = -36.87^\circ$

Example 16: The formula for arc length is $a = r\theta$, where a is the arc length, θ is the central angle in radians, and r is the radius of the circle. The radius and arc length must have the same units.

a) Derive the formula for arc length, $a = r\theta$. **(b - e)** Solve for the unknown.

b)



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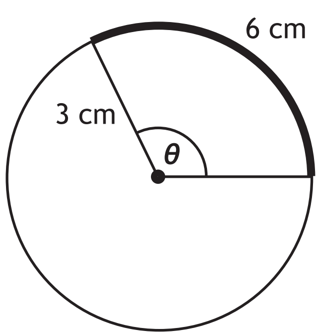
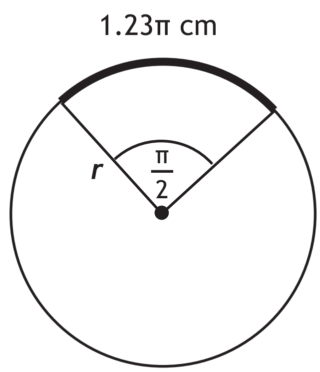
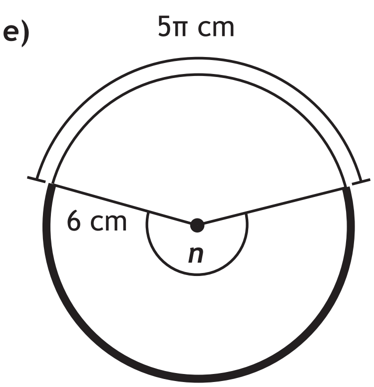


d)



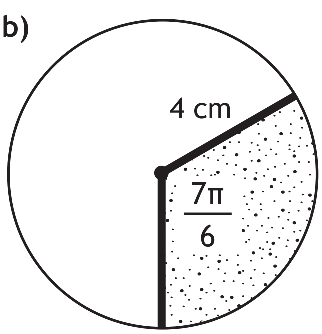
e)



Example 17: Area of a circle sector.

a) Derive the formula for the area of a circle sector, $A = \frac{r^2\theta}{2}$. **(b - e)** Find the area of each shaded region.

b)



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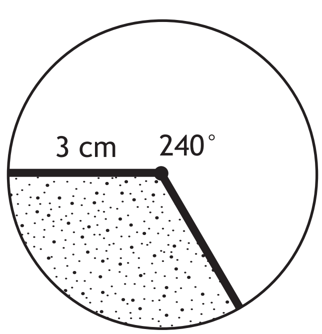
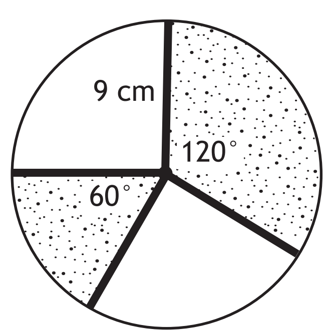
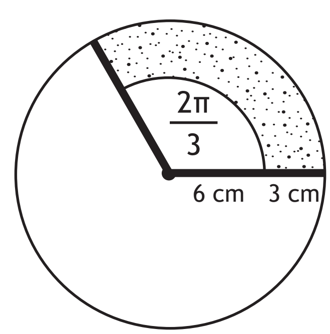


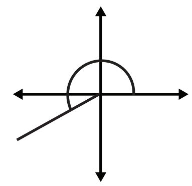
d)



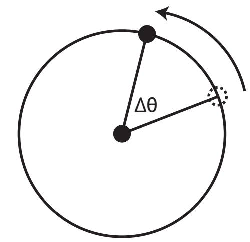
e)



Trigonometry
LESSON ONE - *Degrees and Radians*
Lesson Notes



$$210^\circ \times \frac{\pi}{180^\circ} = \frac{7\pi}{6}$$



Example 18: The formula for angular speed is $\omega = \frac{\Delta\theta}{\Delta T}$, where ω (Greek: Omega) is the angular speed, Δθ is the change in angle, and ΔT is the change in time. Calculate the requested quantity in each scenario. Round all decimals to the nearest hundredth.

a) A bicycle wheel makes 100 complete revolutions in 1 minute. Calculate the angular speed in degrees per second.

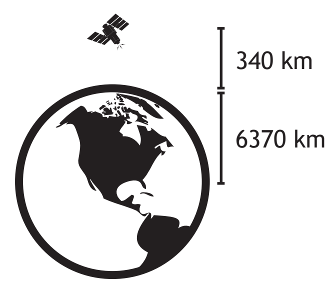
b) A Ferris wheel rotates 1020° in 4.5 minutes. Calculate the angular speed in radians per second.

c) The moon orbits Earth once every 27 days. Calculate the angular speed in revolutions per second. If the average distance from the Earth to the moon is 384 400 km, how far does the moon travel in one second?

d) A cooling fan rotates with an angular speed of 4200 rpm. What is the speed in rps?

e) A bike is ridden at a speed of 20 km/h, and each wheel has a diameter of 68 cm. Calculate the angular speed of one of the bicycle wheels and express the answer using revolutions per second.

Example 19: A satellite orbiting Earth 340 km above the surface makes one complete revolution every 90 minutes. The radius of Earth is approximately 6370 km.



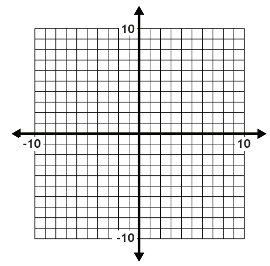
a) Calculate the angular speed of the satellite. Express the answer as an exact value, in radians/second.

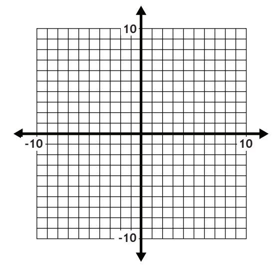
b) How many kilometres does the satellite travel in one minute? Round the answer to the nearest hundredth of a kilometre.

Trigonometry
LESSON TWO - *The Unit Circle*
Lesson Notes

(cosθ, sinθ)

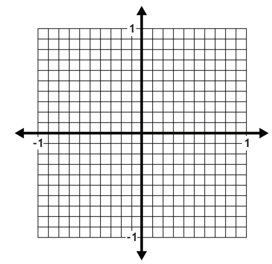
Example 1: Introduction to Circle Equations.





a) A circle centered at the origin can be represented by the relation $x^2 + y^2 = r^2$, where r is the radius of the circle. Draw each circle: **i.** $x^2 + y^2 = 4$ **ii.** $x^2 + y^2 = 49$

b) A circle centered at the origin with a radius of 1 has the equation $x^2 + y^2 = 1$. This special circle is called the *unit circle*. Draw the unit circle and determine if each point exists on the circumference of the unit circle: **i.** (0.6, 0.8) **ii.** (0.5, 0.5)



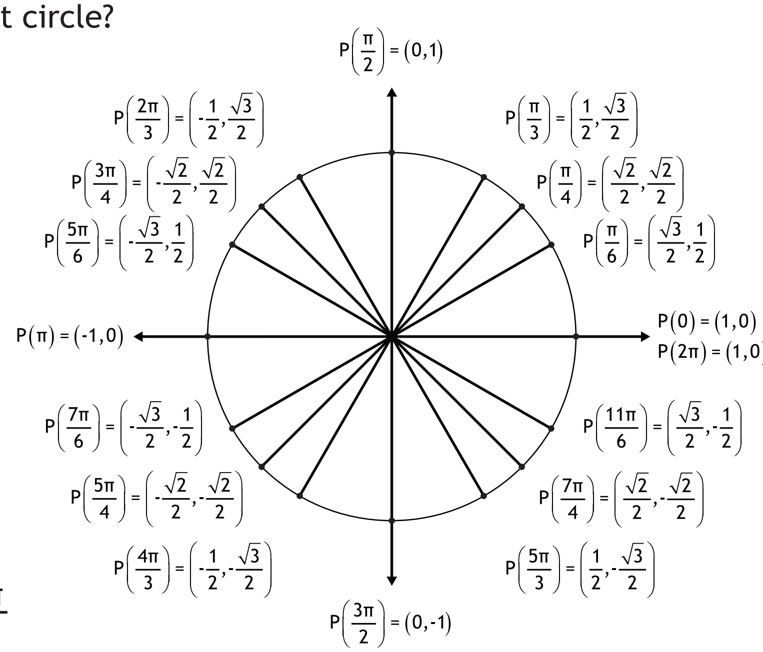
c) Using the equation of the unit circle, $x^2 + y^2 = 1$, find the unknown coordinate of each point. Is there more than one unique answer?

i. $\left(\frac{1}{2}, y\right)$ **ii.** $\left(x, \frac{\sqrt{3}}{2}\right)$, quadrant II. **iii.** (-1, y) **iv.** $\left(x, -\frac{\sqrt{2}}{2}\right)$, cos$\theta$ > 0.

Example 2: The following diagram is called *the unit circle*. Commonly used angles are shown as radians, and their exact-value coordinates are in brackets. Take a few moments to memorize this diagram. When you are done, use the blank unit circle on the next page to practice drawing the unit circle from memory.

a) What are some useful tips to memorize the unit circle?

b) Draw the unit circle from memory.



Example 3: Use the unit circle to find the exact value of each expression.

a) $\sin\frac{\pi}{6}$ **b)** $\cos 180^\circ$ **c)** $\cos\frac{3\pi}{4}$ **d)** $\sin\frac{11\pi}{6}$

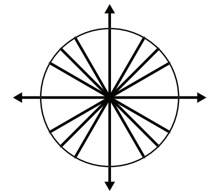
e) $\sin 0$ **f)** $\cos -\frac{\pi}{2}$ **g)** $\sin\frac{4\pi}{3}$ **h)** $\cos -120^\circ$

Example 4: Use the unit circle to find the exact value of each expression.

a) $\cos 420^\circ$ **b)** $-\cos 3\pi$ **c)** $\sin\frac{13\pi}{6}$ **d)** $\cos -\frac{2\pi}{3}$

e) $\sin -\frac{5\pi}{2}$ **f)** $-\sin\frac{9\pi}{4}$ **g)** $\cos^2(-840^\circ)$ **h)** $\cos -\frac{7\pi}{3}$

Trigonometry
LESSON TWO - *The Unit Circle*
Lesson Notes



$(\cos\theta, \sin\theta)$

Example 5: The unit circle contains values for $\cos\theta$ and $\sin\theta$ only. The other four trigonometric ratios can be obtained using the identities on the right. Find the exact values of $\sec\theta$ and $\csc\theta$ in the first quadrant.

$$\sec\theta = \frac{1}{\cos\theta} \qquad \csc\theta = \frac{1}{\sin\theta}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}$$

Example 6: Find the exact values of $\tan\theta$ and $\cot\theta$ in the first quadrant.

Example 7: Use symmetry to fill in quadrants II, III, and IV for $\sec\theta$, $\csc\theta$, $\tan\theta$, and $\cot\theta$.

Example 8: Find the exact value of each expression.

a) $\sec 120^\circ$ **b)** $\sec \frac{3\pi}{2}$ **c)** $\csc \frac{\pi}{3}$ **d)** $\csc -\frac{3\pi}{4}$ **e)** $\tan \frac{\pi}{6}$ **f)** $-\tan \frac{5\pi}{4}$ **g)** $\cot^2(270^\circ)$ **h)** $\cot -\frac{5\pi}{6}$

Example 9: Find the exact value of each expression.

a) $\sin\left(-\frac{\pi}{3}\right) + \cos\left(\frac{5\pi}{4}\right)$ **b)** $\cos^2\frac{\pi}{4} + \sin^2\frac{\pi}{4}$ **c)** $\cot^2\frac{\pi}{3} + 1$ **d)** $\dfrac{\sec\frac{\pi}{6}}{\tan\frac{\pi}{6} + \cot\frac{\pi}{6}}$

Example 10: Find the exact value of each expression.

a) $\sin\frac{\pi}{3}\cos\frac{\pi}{6} + \cos\frac{\pi}{3}\sin\frac{\pi}{6}$ **b)** $\cos\frac{\pi}{4}\cos\frac{\pi}{6} - \sin\frac{\pi}{4}\sin\frac{\pi}{6}$ **c)** $\dfrac{2\tan\frac{\pi}{6}}{1 - \tan^2\frac{\pi}{6}}$ **d)** $\dfrac{\tan\frac{3\pi}{4} - \tan\frac{\pi}{6}}{1 + \tan\frac{3\pi}{4}\tan\frac{\pi}{6}}$

Example 11: Find the exact value of each expression.

a) $\csc\left(-\frac{9\pi}{2}\right)$ **b)** $-\tan^2\left(\frac{617\pi}{6}\right)$ **c)** $\sec\left(\frac{61\pi}{2}\right)$ **d)** $\cot\left(-1980^\circ\right)$

Example 12: Verify each trigonometric statement with a calculator.
Note: Every question in this example has already been seen earlier in the lesson.

a) $\sin\frac{4\pi}{3} = -\frac{\sqrt{3}}{2}$ **b)** $\cos^2\left(-840^\circ\right) = \frac{1}{4}$ **c)** $\sec\frac{3\pi}{2} = \textit{undefined}$ **d)** $\csc\left(-\frac{3\pi}{4}\right) = -\sqrt{2}$

e) $-\tan^2\left(\frac{617\pi}{6}\right) = -\frac{1}{3}$ **f)** $\cot^2\frac{\pi}{3} + 1 = \frac{4}{3}$ **g)** $\dfrac{\sec\frac{\pi}{6}}{\tan\frac{\pi}{6} + \cot\frac{\pi}{6}} = \frac{1}{2}$ **h)** $\dfrac{\tan\frac{3\pi}{4} - \tan\frac{\pi}{6}}{1 + \tan\frac{3\pi}{4}\tan\frac{\pi}{6}} = -2 - \sqrt{3}$

Example 13: Coordinate Relationships on the Unit Circle

a) What is meant when you are asked to find $P\left(\frac{\pi}{3}\right)$ on the unit circle?

b) Find one positive and one negative angle such that $P(\theta) = \left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$

c) How does a half-rotation around the unit circle change the coordinates?
If $\theta = \frac{\pi}{6}$, find the coordinates of the point halfway around the unit circle.

d) How does a quarter-rotation around the unit circle change the coordinates?
If $\theta = \frac{2\pi}{3}$, find the coordinates of the point a quarter-revolution (clockwise) around the unit circle.

e) What are the coordinates of P(3)? Express coordinates to four decimal places.

Example 14: Circumference and Arc Length of the Unit Circle

a) What is the circumference of the unit circle?

b) How is the central angle of the unit circle related to its corresponding arc length?

c) If a point on the terminal arm rotates from $P(\theta) = (1, 0)$ to $P(\theta) = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, what is the arc length?

d) What is the arc length from point A to point B on the unit circle?

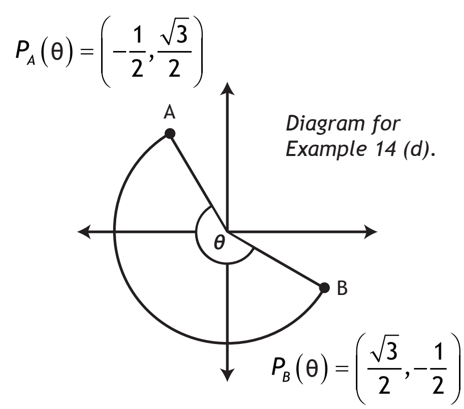


Diagram for Example 14 (d).

Example 15: Domain and Range of the Unit Circle

a) Is $\sin\theta = 2$ possible? Explain, using the unit circle as a reference.

b) Which trigonometric ratios are restricted to a range of $-1 \le y \le 1$? Which trigonometric ratios exist outside that range?

Chart for Example 15 (b).

	Range	Number Line
cosθ & sinθ		
cscθ & secθ		
tanθ & cotθ		

c) If $\sin\theta = -\frac{\sqrt{2}}{2}$ exists on the unit circle, how can the unit circle be used to find cosθ? How many values for cosθ are possible?

d) If $\cos\theta = \frac{3}{5}$ exists on the unit circle, how can the equation of the unit circle be used to find sinθ? How many values for sinθ are possible?

e) If $\cos\theta = 0$, and $0 \le \theta < \pi$, how many values for sinθ are possible?

Example 16: Unit Circle Proofs

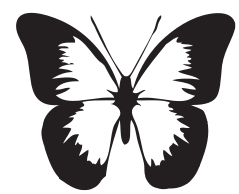
a) Use the Pythagorean Theorem to prove that the equation of the unit circle is $x^2 + y^2 = 1$.

b) Prove that the point where the terminal arm intersects the unit circle, P(θ), has coordinates of (cosθ, sinθ).

c) If the point $P(\theta) = \left(-\frac{40}{41}, \frac{9}{41}\right)$ exists on the terminal arm of a unit circle, find the exact values of the six trigonometric ratios. State the reference angle and standard position angle to the nearest hundredth of a degree.

Example 17: In a video game, the graphic of a butterfly needs to be rotated. To make the butterfly graphic rotate, the programmer uses the equations:

$$x' = x\cos\theta - y\sin\theta$$
$$y' = x\sin\theta + y\cos\theta$$



to transform each pixel of the graphic from its original coordinates, (x, y), to its new coordinates, (x', y'). Pixels may have positive or negative coordinates.

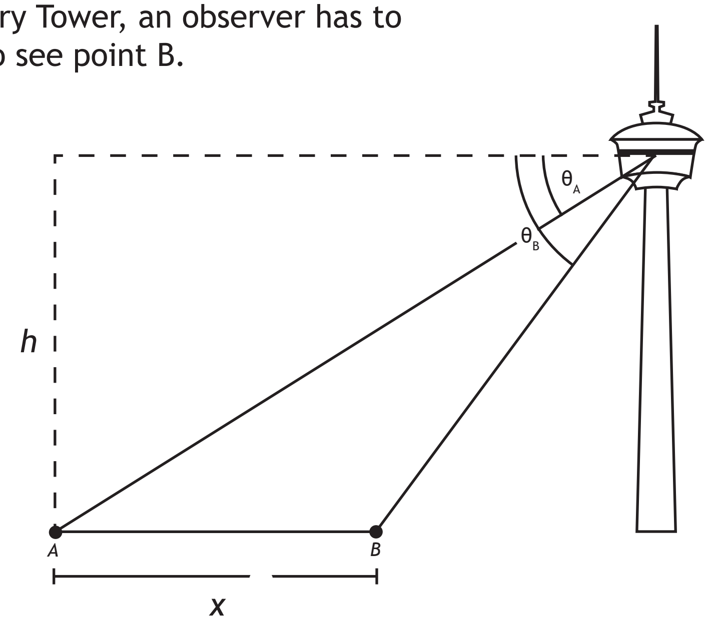
a) If a particular pixel with coordinates of (250, 100) is rotated by $\frac{\pi}{6}$, what are the new coordinates? Round coordinates to the nearest whole pixel.

b) If a particular pixel has the coordinates (640, 480) after a rotation of $\frac{5\pi}{4}$, what were the original coordinates? Round coordinates to the nearest whole pixel.

Example 18: From the observation deck of the Calgary Tower, an observer has to tilt their head θ_A down to see point A, and θ_B down to see point B.

a) Show that the height of the observation deck is $h = \dfrac{x}{\cot\theta_A - \cot\theta_B}$.

b) If $\theta_A = \frac{131}{900}\pi$, $\theta_B = \frac{61}{200}\pi$, and x = 212.92 m, how high is the observation deck above the ground, to the nearest metre?



$y = a\sin b(\theta - c) + d$

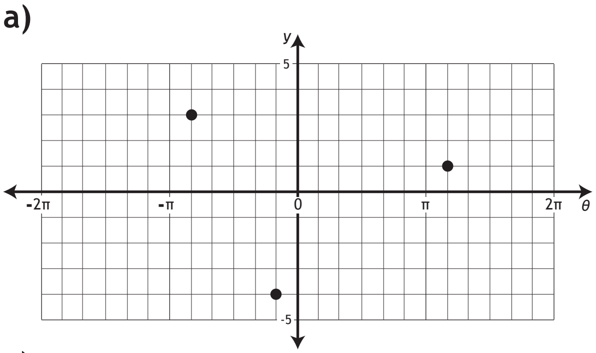
Trigonometry

LESSON THREE - *Trigonometric Functions I*

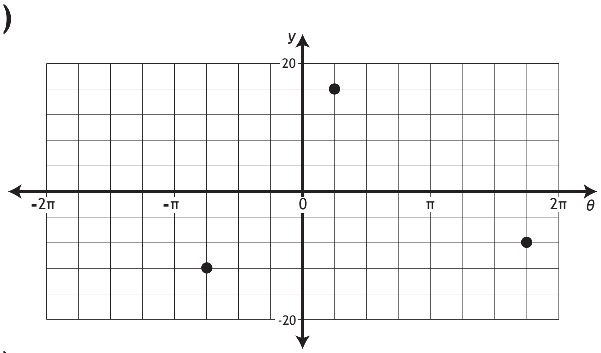
Lesson Notes

Example 1: Label all tick marks in the following grids and state the coordinates of each point.

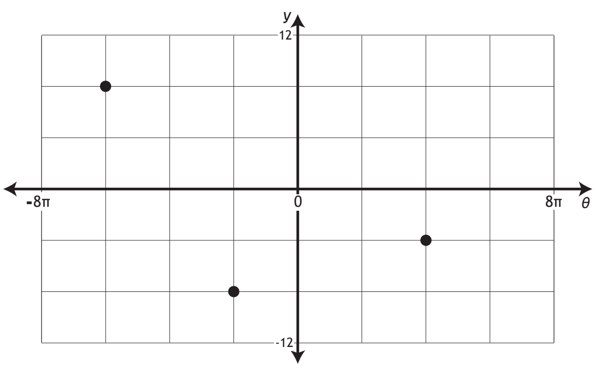
a)



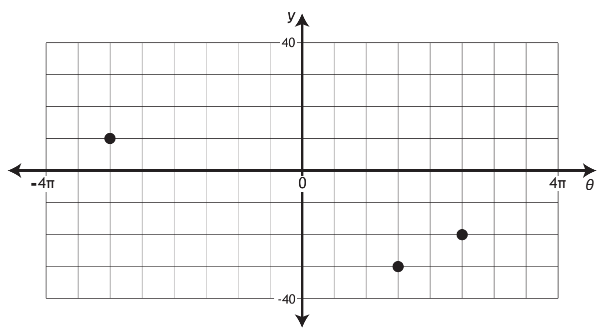
b)



c)



d)



Example 2: Exploring the graph of y = sinθ.

a) Draw y = sinθ. **b)** State the amplitude. **c)** State the period.

d) State the horizontal displacement (phase shift). **e)** State the vertical displacement.

f) State the θ-intercepts. Write your answer using a general form expression.

g) State the y-intercept. **h)** State the domain and range.

Example 3: Exploring the graph of y = cosθ.

a) Draw y = cosθ. **b)** State the amplitude. **c)** State the period.

d) State the horizontal displacement (phase shift). **e)** State the vertical displacement.

f) State the θ-intercepts. Write your answer using a general form expression.

g) State the y-intercept. **h)** State the domain and range.

Example 4: Exploring the graph of y = tanθ.

a) Draw y = tanθ. **b)** Is it correct to say a tangent graph has an amplitude? **c)** State the period.

d) State the horizontal displacement (phase shift). **e)** State the vertical displacement.

f) State the θ-intercepts. Write your answer using a general form expression.

g) State the y-intercept. **h)** State the domain and range.

Trigonometry
LESSON THREE - *Trigonometric Functions I*
Lesson Notes

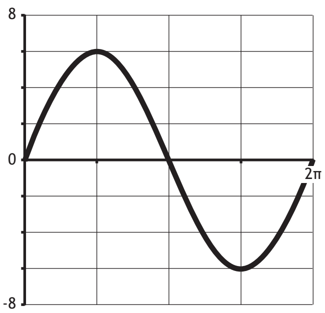
$y = a\sin b(\theta - c) + d$

Example 5: The ***a*** Parameter. Graph each function over the domain $0 \le \theta \le 2\pi$.

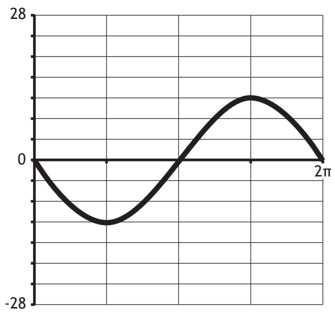
a) $y = 3\sin\theta$ **b)** $y = -2\cos\theta$ **c)** $y = -\frac{1}{2}\sin\theta$ **d)** $y = \frac{5}{2}\cos\theta$

Example 6: The ***a*** Parameter. Determine the trigonometric function corresponding to each graph.

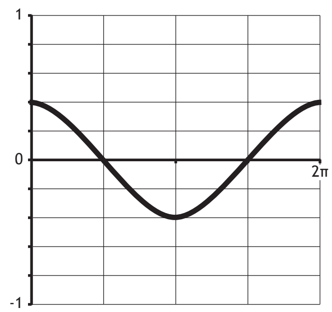
a) write a **sine** function.



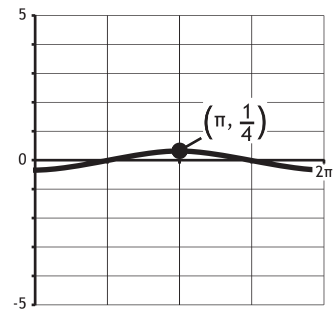
b) write a **sine** function.



c) write a **cosine** function.



d) write a **cosine** function.

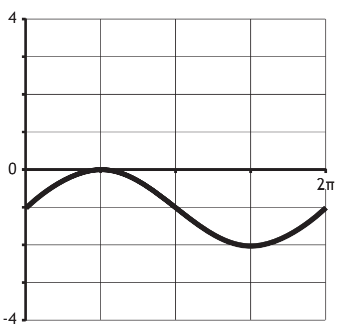


Example 7: The ***d*** Parameter. Graph each function over the domain $0 \le \theta \le 2\pi$.

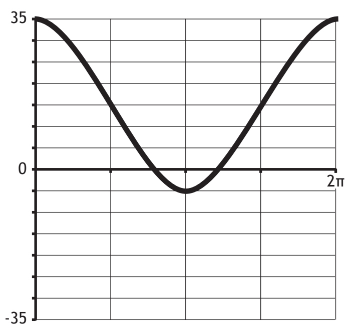
a) $y = \sin\theta - 2$ **b)** $y = \cos\theta + 4$ **c)** $y = -\frac{1}{2}\sin\theta + 2$ **d)** $y = \frac{1}{2}\cos\theta - \frac{1}{2}$

Example 8: The ***d*** Parameter. Determine the trigonometric function corresponding to each graph.

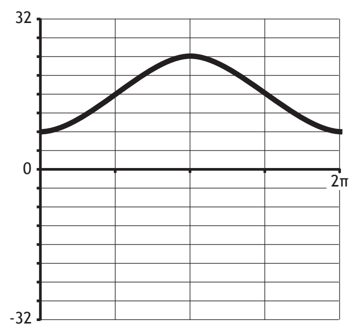
a) write a **sine** function.



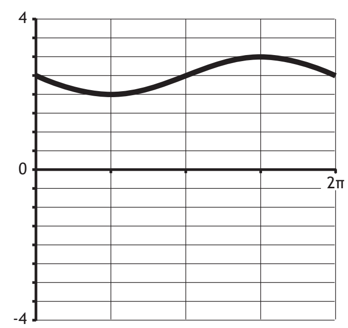
b) write a **cosine** function.



c) write a **cosine** function.



d) write a **sine** function.



Example 9: The ***b*** Parameter. Graph each function over the stated domain.

a) $y = \cos 2\theta$ *($0 \le \theta \le 2\pi$)* **b)** $y = \sin 3\theta$ *($0 \le \theta \le 2\pi$)*

c) $y = \cos \frac{1}{3}\theta$ *($0 \le \theta \le 6\pi$)* **d)** $y = \sin \frac{1}{5}\theta$ *($0 \le \theta \le 10\pi$)*

Example 10: The ***b*** Parameter. Graph each function over the stated domain.

a) $y = -\sin(3\theta)$ *($-2\pi \le \theta \le 2\pi$)* **b)** $y = 4\cos 2\theta + 6$ *($-2\pi \le \theta \le 2\pi$)*

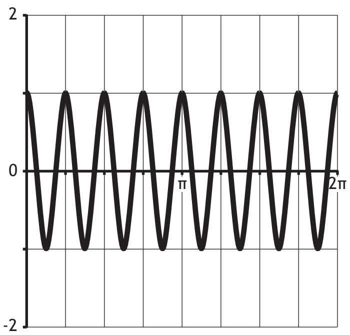
c) $y = 2\cos \frac{1}{2}\theta - 1$ *($-2\pi \le \theta \le 2\pi$)* **d)** $y = \sin \frac{4}{3}\theta$ *($0 \le \theta \le 6\pi$)*

$y = a\sin b(\theta - c) + d$

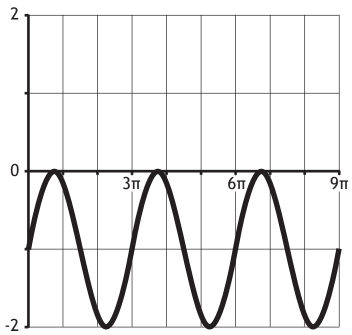
Trigonometry
LESSON THREE - *Trigonometric Functions I*
Lesson Notes

Example 11: The ***b*** Parameter. Determine the trigonometric function corresponding to each graph.

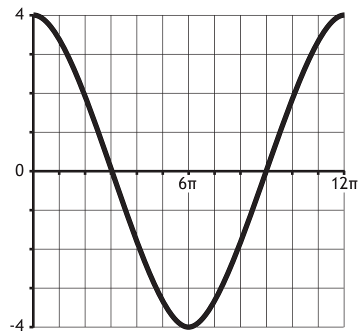
a) write a **cosine** function.



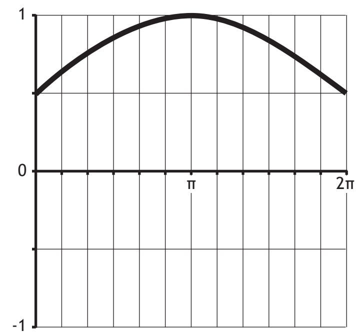
b) write a **sine** function.



c) write a **cosine** function.



d) write a **sine** function.



Example 12: The ***c*** Parameter. Graph each function over the stated domain.

a) $y = \sin\left(\theta - \frac{\pi}{2}\right)$ *(-4π ≤ θ ≤ 4π)*

b) $y = \cos\left(\theta + \pi\right)$ *(-4π ≤ θ ≤ 4π)*

c) $y = \cos\left(\theta - \frac{\pi}{6}\right)$ *(-2π ≤ θ ≤ 2π)*

d) $y = 3\sin\left(\theta + \frac{2\pi}{3}\right)$ *(-2π ≤ θ ≤ 2π)*

Example 13: The ***c*** Parameter. Graph each function over the stated domain.

a) $y = \sin\left(2\theta + \pi\right)$ $\left(-\frac{\pi}{2} \le \theta \le 2\pi\right)$

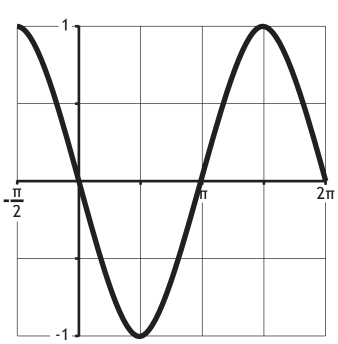
b) $y = \cos\left(\frac{1}{2}\theta + \pi\right)$ *(-2π ≤ θ ≤ 6π)*

c) $y = -\frac{1}{2}\sin\left(2\theta - 3\pi\right) + 1$ *(-π ≤ θ ≤ 4π)*

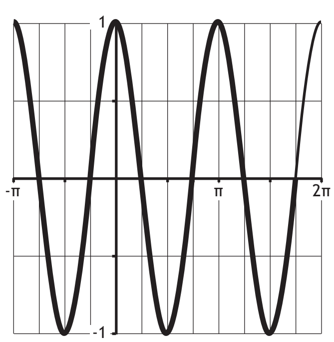
d) $y = -2\cos\left(4\theta + \frac{4\pi}{3}\right) - 2$ *(-2π ≤ θ ≤ 2π)*

Example 14: The ***c*** Parameter. Determine the trigonometric function corresponding to each graph.

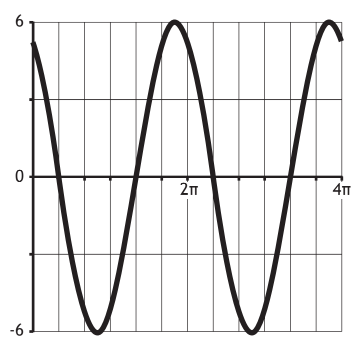
a) write a **cosine** function.



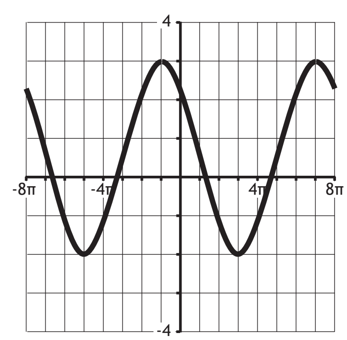
b) write a **sine** function.



c) write a **sine** function.



d) write a **cosine** function.



Example 15: ***a***, ***b***, ***c***, **&** ***d*** Parameters. Graph each function over the stated domain.

a) $y = 2\cos\left(\frac{1}{2}\theta - \frac{\pi}{8}\right) + 3$ *(0 ≤ θ ≤ 6π)*

b) $y = \frac{1}{2}\sin\left(2\theta - \frac{\pi}{2}\right)$ *(0 ≤ θ ≤ 2π)*

c) $y = -2\sin\left(4\theta - \pi\right) - 3$ *(0 ≤ θ ≤ 2π)*

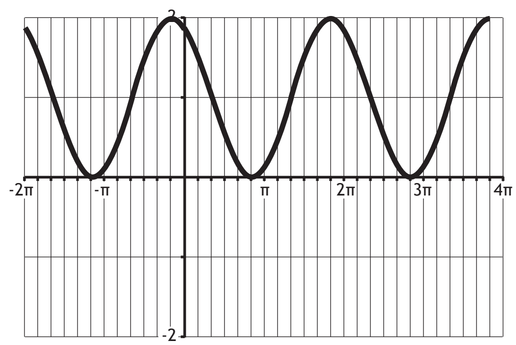
d) $y = -5\cos\left(2\theta - \frac{\pi}{3}\right) + 1$ *(0 ≤ θ ≤ 2π)*

Trigonometry
LESSON THREE - *Trigonometric Functions I*
Lesson Notes

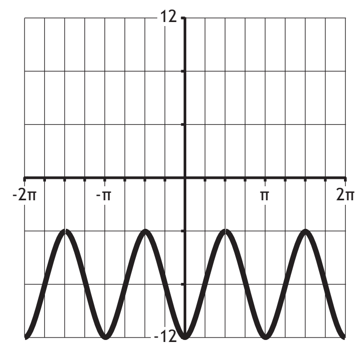
$y = a\sin b(\theta - c) + d$

Example 16: ***a, b, c, & d***. Determine the trigonometric function corresponding to each graph.

a) write a **cosine** function.



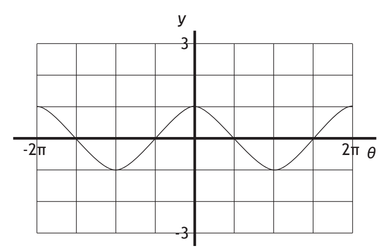
b) write a **cosine** function.



Example 17: Exploring the graph of y = secθ.

a) Draw y = secθ. **b)** State the period. **c)** State the domain and range.
d) Write the general equation of the asymptotes.

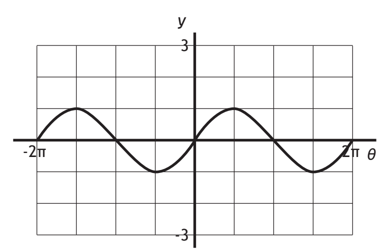
e) Given the graph of f(θ) = cosθ, draw $y = \frac{1}{f(\theta)}$.



Example 18: Exploring the graph of y = cscθ.

a) Draw y = cscθ. **b)** State the period. **c)** State the domain and range.
d) Write the general equation of the asymptotes.

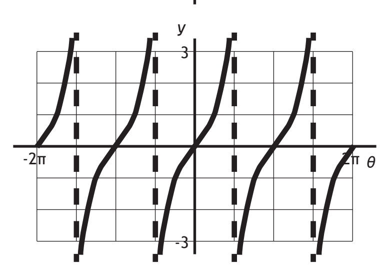
e) Given the graph of f(θ) = sinθ, draw $y = \frac{1}{f(\theta)}$.



Example 19: Exploring the graph of y = cotθ.

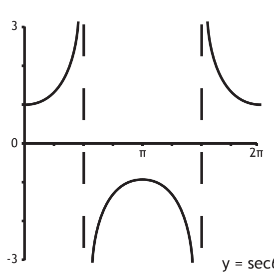
a) Draw y = cotθ. **b)** State the period. **c)** State the domain and range.
d) Write the general equation of the asymptotes.

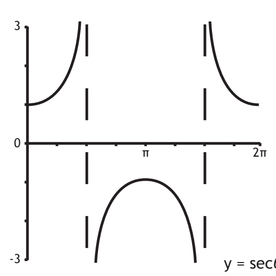
e) Given the graph of f(θ) = tanθ, draw $y = \frac{1}{f(\theta)}$.

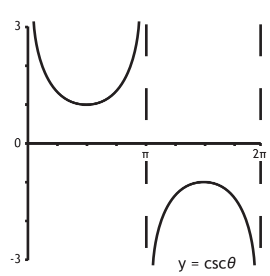


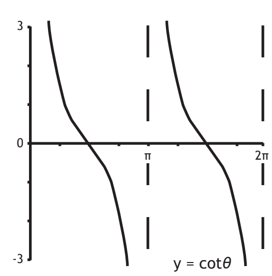
Example 20: Graph each function over the domain 0 ≤ θ ≤ 2π. State the new domain and range.

a) $y = \frac{1}{2}\sec\theta$ **b)** $y = \sec 2\theta$ **c)** $y = \csc\left(\theta - \frac{\pi}{4}\right)$ **d)** $y = \cot\frac{1}{2}\theta$









Trigonometry
LESSON FOUR - *Trigonometric Functions II*
Lesson Notes

Example 1: Trigonometric Functions of Angles

a) i. Graph: $f(\theta) = \cos\left[2\left(\theta - \frac{\pi}{4}\right)\right]$ $(0 \le \theta < 3\pi)$

ii. Graph this function using technology.

b) i. Graph: $f(\theta) = \cos\left[2\left(\theta - 45^\circ\right)\right]$ $(0^\circ \le \theta < 540^\circ)$

ii. Graph this function using technology.

Example 2: Trigonometric Functions of Real Numbers.

a) i. Graph: $h(t) = \cos\left[\frac{\pi}{30}(t - 15)\right]$

ii. Graph this function using technology.

b) i. Graph: $f(x) = \cos\left[\frac{\pi}{8}(x - 4)\right]$

ii. Graph this function using technology.

c) What are three differences between trigonometric functions of angles and trigonometric functions of real numbers?

Example 3: Determine the view window for each function and sketch each graph.

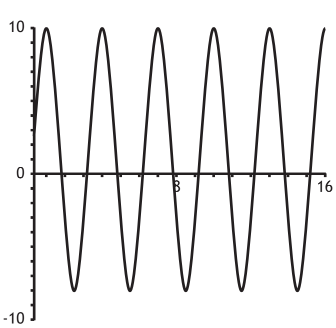
a) $f(x) = 12\sin\left[\frac{\pi}{3}(x - 2)\right] - 14$ **b)** $f(x) = -25\cos\frac{\pi}{250}(x + 225) + 50$

Example 4: Determine the view window for each function and sketch each graph.

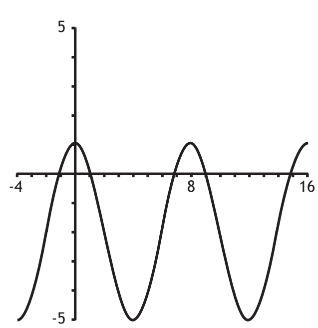
a) $f(x) = 13.5\cos\frac{2\pi}{96}(x - 24) + 6.5$ **b)** $f(x) = 2.5\sin 0.25\pi(x + 3) + 16$

Example 5: Determine the trigonometric function corresponding to each graph.

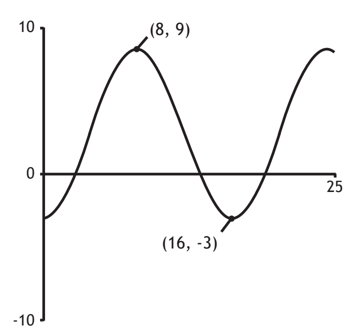
a) write a **cosine** function.



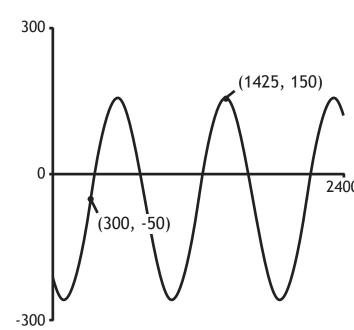
b) write a **sine** function.



c) write a **cosine** function.



d) write a **sine** function.



Example 6: a) If the transformation $g(\theta) - 3 = f(2\theta)$ is applied to the graph of $f(\theta) = \sin\theta$, find the new range.

b) Find the range of $f(\theta) = k\sin\left(\theta - \frac{\pi}{4}\right) - 3$.

c) If the range of $y = 3\cos\theta + d$ is $[-4, k]$, determine the values of d and k.

d) State the range of $f(\theta) - 2 = m\sin(2\theta) + n$.

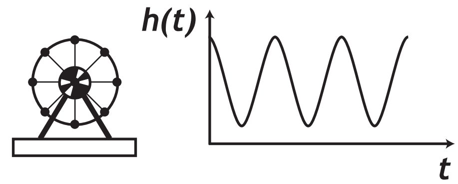
e) The graphs of $f(\theta)$ and $g(\theta)$ intersect at the points $\left(\frac{\pi}{8}, \frac{\sqrt{2}}{2}\right)$ and $\left(\frac{5\pi}{8}, \frac{-\sqrt{2}}{2}\right)$.

If the amplitude of each graph is quadrupled, determine the new points of intersection.

Trigonometry

LESSON FOUR - *Trigonometric Functions II*

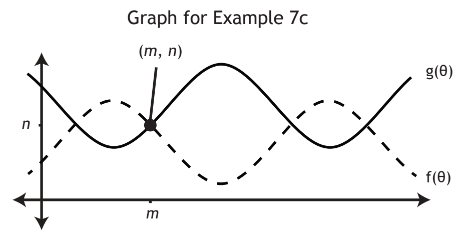
Lesson Notes



Example 7: **a)** If the point $\left(\frac{\pi}{2}, -2\right)$ lies on the graph of $f(\theta) = a\cos\left(\theta - \frac{\pi}{4}\right) - 4$, find the value of a.

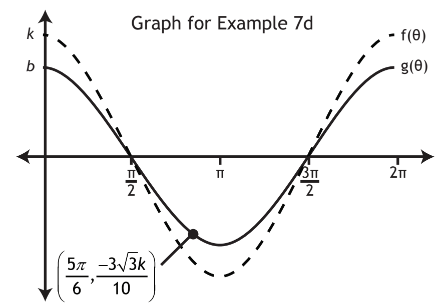
b) Find the y-intercept of $f(\theta) = -3\cos\left(k\theta + \frac{\pi}{2}\right) - b$.

c) The graphs of f(θ) and g(θ) intersect at the point (m, n). Find the value of f(m) + g(m).



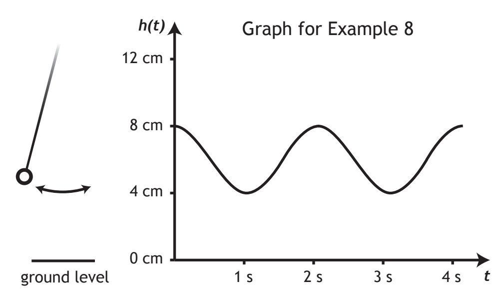
d) The graph of f(θ) = kcosθ is transformed to the graph of g(θ) = bcosθ by a vertical stretch about the x-axis.

If the point $\left(\frac{5\pi}{6}, \frac{-3\sqrt{3}k}{10}\right)$ exists on the graph of g(θ), state the vertical stretch factor.

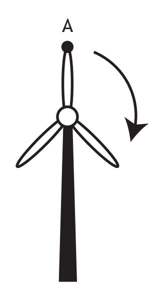


Example 8: The graph shows the height of a pendulum bob as a function of time. One cycle of a pendulum consists of two swings - a right swing and a left swing.

a) Write a function that describes the height of the pendulum bob as a function of time.
b) If the period of the pendulum is halved, how will this change the parameters in the function you wrote in part (a)?
c) If the pendulum is lowered so its lowest point is 2 cm above the ground, how will this change the parameters in the function you wrote in part (a)?



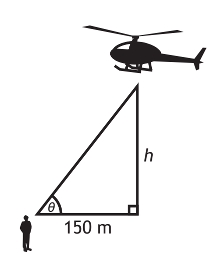
Example 9: A wind turbine has blades that are 30 m long. An observer notes that one blade makes 12 complete rotations (clockwise) every minute. The highest point of the blade during the rotation is 105 m.



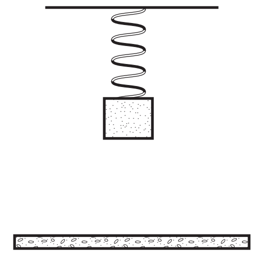
a) Using Point A as the starting point of the graph, draw the height of the blade over two rotations.
b) Write a function that corresponds to the graph.
c) Do we get a different graph if the wind turbine rotates counterclockwise?

Example 10: A person is watching a helicopter ascend from a distance 150 m away from the takeoff point.

a) Write a function, h(*θ*), that expresses the height as a function of the angle of elevation. Assume the height of the person is negligible.
b) Draw the graph, using an appropriate domain.
c) Explain how the shape of the graph relates to the motion of the helicopter.

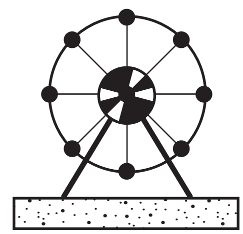


Example 11: A mass is attached to a spring 4 m above the ground and allowed to oscillate from its equilibrium position. The lowest position of the mass is 2.8 m above the ground, and it takes 1 s for one complete oscillation.



a) Draw the graph for two full oscillations of the mass.
b) Write a sine function that gives the height of the mass above the ground as a function of time.
c) Calculate the height of the mass after 1.2 seconds. Round your answer to the nearest hundredth.
d) In one oscillation, how many seconds is the mass lower than 3.2 m? Round your answer to the nearest hundredth.

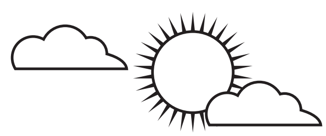
Example 12: A Ferris wheel with a radius of 15 m rotates once every 100 seconds. Riders board the Ferris wheel using a platform 1 m above the ground.



a) Draw the graph for two full rotations of the Ferris wheel.
b) Write a cosine function that gives the height of the rider as a function of time.
c) Calculate the height of the rider after 1.6 rotations of the Ferris wheel. Round your answer to the nearest hundredth.
d) In one rotation, how many seconds is the rider higher than 26 m? Round your answer to the nearest hundredth.

Example 13: The following table shows the number of daylight hours in Grande Prairie.

December 21	March 21	June 21	September 21	December 21
6h, 46m	12h, 17m	17h, 49m	12h, 17m	6h, 46m

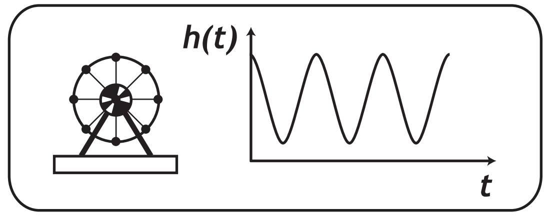


a) Convert each date and time to a number that can be used for graphing.

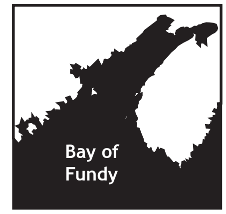
Day Number	December 21 =	March 21 =	June 21 =	September 21 =	December 21 =
Daylight Hours	6h, 46m =	12h, 17m =	17h, 49m =	12h, 17m =	12h, 46m =

b) Draw the graph for one complete cycle *(winter solstice to winter solstice)*.
c) Write a cosine function that relates the number of daylight hours, d, to the day number, n.
d) How many daylight hours are there on May 2? Round your answer to the nearest hundredth.
e) In one year, approximately how many days have more than 17 daylight hours? Round your answer to the nearest day.

Trigonometry
LESSON FOUR - *Trigonometric Functions II*
Lesson Notes



Example 14: The highest tides in the world occur between New Brunswick and Nova Scotia, in the Bay of Fundy. Each day, there are two low tides and two high tides. The chart below contains tidal height data that was collected over a 24-hour period.



Note: Actual tides at the Bay of Fundy are 6 hours and 13 minutes apart due to daily changes in the position of the moon.

In this example, we will use 6 hours for simplicity.

	Time	Decimal Hour	Height of Water (m)
Low Tide	2:12 AM		3.48
High Tide	8:12 AM		13.32
Low Tide	2:12 PM		3.48
High Tide	8:12 PM		13.32

a) Convert each time to a decimal hour.
b) Graph the height of the tide for one full cycle *(low tide to low tide)*.
c) Write a cosine function that relates the height of the water to the elapsed time.
d) What is the height of the water at 6:09 AM? Round your answer to the nearest hundredth.
e) For what percentage of the day is the height of the water greater than 11 m? Round your answer to the nearest tenth.

Example 15: A wooded region has an ecosystem that supports both owls and mice. Owl and mice populations vary over time according to the equations:

Owl population: $O(t) = 50\sin\left[\frac{\pi}{3}(t - 1.5)\right] + 250$ Mouse population: $M(t) = 4000\sin\left(\frac{\pi}{3}t\right) + 12000$

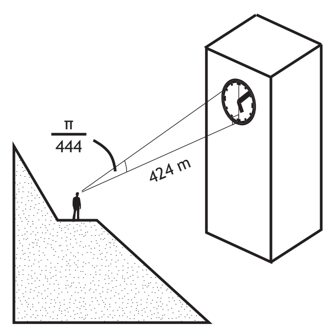
where O is the population of owls, M is the population of mice, and t is the time in years.

a) Graph the population of owls and mice over six years.
b) Describe how the graph shows the relationship between owl and mouse populations.

Example 16: The angle of elevation between the 6:00 position and the 12:00 position of a historical building's clock, as measured from an observer standing on a hill, is $\frac{\pi}{444}$.

The observer also knows that he is standing 424 m away from the clock, and his eyes are at the same height as the base of the clock. The radius of the clock is the same as the length of the minute hand.

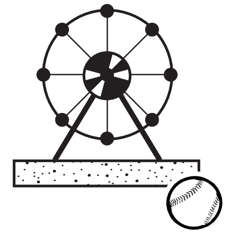
If the height of the minute hand's tip is measured relative to the bottom of the clock, what is the height of the tip at 5:08, to the nearest tenth of a metre?

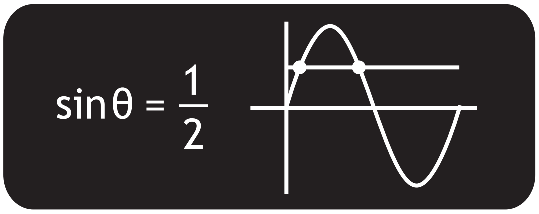


Example 17: Shane is on a Ferris wheel, and his height can be described by the equation $h_{wheel}(t) = -9\cos\frac{\pi}{30}t + 10$.

Tim, a baseball player, can throw a baseball with a speed of 20 m/s. If Tim throws a ball directly upwards, the height can be determined by the equation $h_{ball}(t) = -4.905t^2 + 20t + 1$.

If Tim throws the baseball 15 seconds after the ride begins, when are Shane and the ball at the same height?





Trigonometry

LESSON FIVE - *Trigonometric Equations*

Lesson Notes

Example 1: ***Primary Ratios***. Find all angles in the domain $0 \le \theta \le 2\pi$ that satisfy the given equation. Write the general solution. **Solve equations non-graphically using the unit circle.**

a) $\sin\theta = \frac{\sqrt{3}}{2}$ **b)** $\cos\theta = -\frac{1}{2}$ **c)** $\tan\theta = 0$ **d)** $\tan^2\theta = 1$

Example 2: ***Primary Ratios***. Find all angles in the domain $0 \le \theta \le 2\pi$ that satisfy the given equation. Write the general solution. **Solve equations graphically with intersection points.**

a) $\sin\theta = \frac{1}{2}$ **b)** $\sin\theta = -1$ **c)** $\cos\theta = -\frac{\sqrt{2}}{2}$ **d)** $\cos\theta = 2$ **e)** $\tan\theta = -\sqrt{3}$ **f)** $\tan\theta =$ *undefined*

Example 3: ***Primary Ratios***. Find all angles in the domain $0^\circ \le \theta \le 360^\circ$ that satisfy the given equation. Write the general solution. **Solve equations non-graphically with a calculator (degree mode).**

a) $\sin\theta = \frac{1}{2}$ **b)** $\cos\theta = -\frac{\sqrt{3}}{2}$ **c)** $\tan\theta = 1$

Example 4: ***Primary Ratios***. Find all angles in the domain $0 \le \theta \le 2\pi$ that satisfy the given equation. **Solve equations graphically with θ-intercepts.**

a) $\sin\theta = 1$ **b)** $\cos\theta = \frac{1}{2}$

Example 5: ***Primary Ratios***. Solve $\cos\theta = -\frac{1}{2}$ $\quad 0 \le \theta \le 2\pi$

a) non-graphically, using the $\cos^{-1}$ feature of a calculator. **b)** non-graphically, using the unit circle. **c)** graphically, using point(s) of intersection. **d)** graphically, using θ-intercepts.

Example 6: ***Primary Ratios***. Solve $\sin\theta = -0.30$ $\quad \theta \in R$

a) non-graphically, using the $\cos^{-1}$ feature of a calculator. **b)** non-graphically, using the unit circle. **c)** graphically, using point(s) of intersection. **d)** graphically, using θ-intercepts.

Example 7: ***Reciprocal Ratios***. Find all angles in the domain $0 \le \theta \le 2\pi$ that satisfy the given equation. Write the general solution. **Solve equations non-graphically using the unit circle.**

a) $\sec\theta = -2$ **b)** $\csc\theta =$ undefined **c)** $\cot\theta = -1$

Example 8: ***Reciprocal Ratios***. Find all angles in the domain $0 \le \theta \le 2\pi$ that satisfy the given equation. Write the general solution. **Solve equations graphically with intersection points.**

a) $\csc\theta = \frac{1}{2}$ **b)** $\csc\theta =$ *undefined* **c)** $\sec\theta = 2$ **d)** $\sec\theta = -1$ **e)** $\cot\theta = \frac{\sqrt{3}}{3}$ **f)** $\cot\theta = 0$

Example 9: ***Reciprocal Ratios***. Find all angles in the domain $0^\circ \le \theta \le 360^\circ$ that satisfy the given equation. Write the general solution. **Solve non-graphically with a calculator (degree mode).**

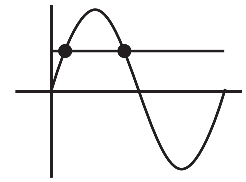
a) $\sec\theta = -2$ **b)** $\csc\theta = \frac{2\sqrt{3}}{3}$ **c)** $\cot\theta = \frac{\sqrt{3}}{3}$

Example 10: ***Reciprocal Ratios***. Find all angles in the domain $0 \le \theta \le 2\pi$ that satisfy the given equation. Write the general solution. **Solve equations graphically with θ-intercepts.**

a) $\csc\theta = -\frac{1}{2}$ **b)** $\sec\theta = 1$

Trigonometry
LESSON FIVE - *Trigonometric Equations*
Lesson Notes

$\sin\theta = \frac{1}{2}$



Example 11: ***Reciprocal Ratios***. Solve $\csc\theta = -2$ $\quad 0 \le \theta \le 2\pi$

a) non-graphically, using the $\cos^{-1}$ feature of a calculator.
b) non-graphically, using the unit circle.
c) graphically, using point(s) of intersection.
d) graphically, using θ-intercepts.

Example 12: ***Reciprocal Ratios***. Solve $\sec\theta = -2.3662$ $\quad 0^\circ \le \theta \le 360^\circ$

a) non-graphically, using the $\cos^{-1}$ feature of a calculator.
b) non-graphically, using the unit circle.
c) graphically, using point(s) of intersection.
d) graphically, using θ-intercepts.

Example 13: ***First-Degree Trigonometric Equations***. Find all angles in the domain $0 \le \theta \le 2\pi$ that satisfy the given equation. Write the general solution.

a) $\cos\theta - 1 = 0$ **b)** $2\sin\theta - \sqrt{3} = 0$ **c)** $3\tan\theta - 5 = 0$ **d)** $4\sec\theta + 3 = 3\sec\theta + 1$

Example 14: ***First-Degree Trigonometric Equations***. Find all angles in the domain $0 \le \theta \le 2\pi$ that satisfy the given equation. Write the general solution.

a) $2\sin\theta\cos\theta = \cos\theta$ **b)** $7\sin\theta = 4\sin\theta$ **c)** $\sin\theta\tan\theta = \sin\theta$ **d)** $\tan\theta + \cos\theta\tan\theta = 0$

Example 15: ***Second-Degree Trigonometric Equations***. Find all angles in the domain $0 \le \theta \le 2\pi$ that satisfy the given equation. Write the general solution.

a) $\sin^2\theta = 1$ **b)** $4\cos^2\theta - 3 = 0$ **c)** $2\cos^2\theta = \cos\theta$ **d)** $\tan^4\theta - \tan^2\theta = 0$

Example 16: ***Second-Degree Trigonometric Equations***. Find all angles in the domain $0 \le \theta \le 2\pi$ that satisfy the given equation. Write the general solution.

a) $2\sin^2\theta - \sin\theta - 1 = 0$ **b)** $\csc^2\theta - 3\csc\theta + 2 = 0$ **c)** $2\sin^3\theta - 5\sin^2\theta + 2\sin\theta = 0$

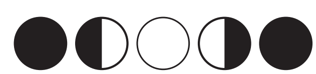
Example 17: ***Double and Triple Angles***. Solve each equation (i) graphically, and (ii) non-graphically.

a) $\sin 2\theta = -\frac{\sqrt{3}}{2}$ $\quad 0 \le \theta \le 2\pi$ **b)** $\cos 3\theta = \frac{\sqrt{2}}{2}$ $\quad 0 \le \theta \le 2\pi$

Example 18: ***Half and Quarter Angles***. Solve each equation (i) graphically, and (ii) non-graphically.

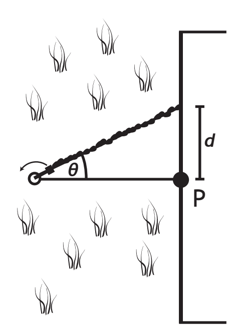
a) $\cos\frac{1}{2}\theta = \frac{1}{2}$ $\quad 0 \le \theta \le 4\pi$ **b)** $\sin\frac{1}{4}\theta = -1$ $\quad 0 \le \theta \le 8\pi$

Example 19: It takes the moon approximately 28 days to go through all of its phases.



a) Write a function, P(t), that expresses the visible percentage of the moon as a function of time. Draw the graph.
b) In one cycle, for how many days is 60% or more of the moon's surface visible?

Example 20: A rotating sprinkler is positioned 4 m away from the wall of a house. The wall is 8 m long. As the sprinkler rotates, the stream of water splashes the house *d* meters from point P.



a) Write a tangent function, d(θ), that expresses the distance where the water splashes the wall as a function of the rotation angle θ.
b) Graph the function for one complete rotation of the sprinkler. Draw only the portion of the graph that actually corresponds to the wall being splashed.
c) If the water splashes the wall 2.0 m north of point P, what is the angle of rotation *(in degrees)*?

$$\begin{array}{r|l} \cos^3 x + \cos x \sin^2 x = \cos x & \\ \cos x\left(\cos^2 x + \sin^2 x\right) & \\ \cos x(1) & \\ \cos x & \cos x \ \checkmark \end{array}$$

Trigonometry

LESSON SIX - *Trigonometric Identities I*

Lesson Notes

Example 1: Understanding Trigonometric Identities.

a) Why are trigonometric identities considered to be a special type of trigonometric equation?

b) Which of the following trigonometric equations are also trigonometric identities?

i. $\sin x = -\frac{1}{2}$ **ii.** $\tan x = 1$ **iii.** $\tan x = \frac{\sin x}{\cos x}$ **iv.** $\csc x = \frac{1}{\sin x}$ **v.** $\sec x = \textit{undefined}$

Example 2: The Pythagorean Identities.

a) Using the definition of the unit circle, derive the identity $\sin^2 x + \cos^2 x = 1$.
Why is $\sin^2 x + \cos^2 x = 1$ called a Pythagorean Identity?

b) Verify that $\sin^2 x + \cos^2 x = 1$ is an identity using (i) $x = \frac{\pi}{6}$ and (ii) $x = \frac{\pi}{2}$.

c) Verify that $\sin^2 x + \cos^2 x = 1$ is an identity using a graphing calculator to draw the graph.

d) Using the identity $\sin^2 x + \cos^2 x = 1$, derive $1 + \cot^2 x = \csc^2 x$ and $\tan^2 x + 1 = \sec^2 x$.

e) Verify that $1 + \cot^2 x = \csc^2 x$ and $\tan^2 x + 1 = \sec^2 x$ are identities for $x = \frac{\pi}{4}$.

f) Verify that $1 + \cot^2 x = \csc^2 x$ and $\tan^2 x + 1 = \sec^2 x$ are identities graphically.

Example 3: ***Reciprocal Identities***. Prove that each trigonometric statement is an identity.
State the non-permissible values of x so the identity is true.

a) $\sin x \sec x = \tan x$ **b)** $\cot x \sin x \sec x = 1$

Example 4: ***Reciprocal Identities***. Prove that each trigonometric statement is an identity.
State the non-permissible values of x so the identity is true.

a) $\frac{\sin x \sec x}{\cot x} = \tan^2 x$ **b)** $\sin 2x \sec 2x = \tan 2x$

Example 5: ***Pythagorean Identities***. Prove that each trigonometric statement is an identity.
State the non-permissible values of x so the identity is true.

a) $\sin^2 x + \frac{1}{\sec^2 x} = 1$ **b)** $\cos x - \cos^3 x = \cos x \sin^2 x$

c) $\sin^3 x - \sin x = -\sin x \cos^2 x$ **d)** $\sin^2 x + \sin^2 x \cos^2 x = \sin^2 x\left(1 + \cos^2 x\right)$

Example 6: ***Pythagorean Identities***. Prove that each trigonometric statement is an identity.
State the non-permissible values of x so the identity is true.

a) $\cos^2 x + \tan^2 x \cos^2 x = 1$ **b)** $\frac{\sec^2 x - 1}{1 + \tan^2 x} = \sin^2 x$

c) $\frac{\sin^2 x}{1 - \cos x} = 1 + \cos x$ **d)** $\left(\frac{\sec^2 x}{\csc^2 x}\right)\left(\csc^2 x - 1\right) = 1$

Trigonometry
LESSON SIX- *Trigonometric Identities I*
Lesson Notes

$$\cos^3 x + \cos x \sin^2 x = \cos x$$
$$\cos x\left(\cos^2 x + \sin^2 x\right)$$
$$\cos x(1)$$
$$\cos x \mid \cos x \checkmark$$

Example 7: ***Common Denominator Proofs***. Prove that each trigonometric statement is an identity. State the non-permissible values of x so the identity is true.

a) $1 + \sec x = \dfrac{\cos x + 1}{\cos x}$ **b)** $\tan^2 x - \sin^2 x = \sin^2 x \tan^2 x$

c) $\cot x + \tan x = \csc x \sec x$ **d)** $\dfrac{1 + \tan x}{1 + \cot x} = \tan x$

Example 8: ***Common Denominator Proofs***. Prove that each trigonometric statement is an identity. State the non-permissible values of x so the identity is true.

a) $\dfrac{\sin x}{\cos x} + \dfrac{\cos x}{1 + \sin x} = \sec x$ **b)** $\dfrac{1 + \tan^2 x}{1 + \cot^2 x} = \tan^2 x$

c) $\dfrac{\cos x}{1 + \sin x} + \dfrac{\cos x}{1 - \sin x} = 2\sec x$ **d)** $\dfrac{\cos x}{1 - \sin x} = \dfrac{1 + \sin x}{\cos x}$

Example 9: ***Assorted Proofs***. Prove each identity. *For simplicity, ignore NPV's and graphs.*

a) $-\dfrac{4\cot x}{1 - \csc^2 x} = 4\tan x$ **b)** $\sin^4 x - \cos^4 x = 2\sin^2 x - 1$

c) $\cot^2 x - \csc^2 x = -1$ **d)** $\csc x - \sin x = \cos x \cot x$

Example 10: ***Assorted Proofs***. Prove each identity. *For simplicity, ignore NPV's and graphs.*

a) $\dfrac{1}{\csc x \sin x \tan x} = \cot x$ **b)** $\dfrac{\csc^2 x \cos x}{\tan x} = \csc^3 x - \csc x$

c) $\dfrac{1}{5}\sin^2 x + \dfrac{1}{5}\cos^2 x = \dfrac{1}{5}$ **d)** $\dfrac{\sec x - \cos x}{\sin x} = \tan x$

Example 11: ***Assorted Proofs***. Prove each identity. *For simplicity, ignore NPV's and graphs.*

a) $\dfrac{\sin x}{1 - \cos x} = \dfrac{1 + \cos x}{\sin x}$ **b)** $\dfrac{1 - \cos x}{\sin x} - \dfrac{\sin x}{1 + \cos x} = 0$

c) $(\tan x - 1)^2 = \dfrac{1 - 2\sin x \cos x}{\cos^2 x}$ **d)** $\dfrac{1 + \cos x}{1 - \cos x} = \left(\dfrac{1 + \cos x}{\sin x}\right)^2$

Example 12: Exploring the proof of $\sin x = \tan x \cos x$

a) Prove algebraically that $\sin x = \tan x \cos x$.

b) Verify that $\sin x = \tan x \cos x$ for $\frac{\pi}{3}$.

c) State the non-permissible values for $\sin x = \tan x \cos x$.

d) Show graphically that $\sin x = \tan x \cos x$. Are the graphs *exactly* the same?

$$\begin{array}{r|l} \cos^3 x + \cos x \sin^2 x & = \cos x \\ \cos x\left(\cos^2 x + \sin^2 x\right) & \\ \cos x(1) & \\ \cos x & \cos x \ \checkmark \end{array}$$

Example 13: Exploring the proof of $\csc x + \cot x = \dfrac{1+\cos x}{\sin x}$.

a) Prove algebraically that $\csc x + \cot x = \dfrac{1+\cos x}{\sin x}$.

b) Verify that $\csc x + \cot x = \dfrac{1+\cos x}{\sin x}$ for $\dfrac{\pi}{3}$.

c) State the non-permissible values for $\csc x + \cot x = \dfrac{1+\cos x}{\sin x}$.

d) Show graphically that $\csc x + \cot x = \dfrac{1+\cos x}{\sin x}$. Are the graphs *exactly* the same?

Example 14: Exploring the proof of $\dfrac{1}{1-\cos x} + \dfrac{1}{1+\cos x} = 2\csc^2 x$

a) Prove algebraically that $\dfrac{1}{1-\cos x} + \dfrac{1}{1+\cos x} = 2\csc^2 x$

b) Verify that $\dfrac{1}{1-\cos x} + \dfrac{1}{1+\cos x} = 2\csc^2 x$ for $\dfrac{\pi}{2}$.

c) State the the non-permissible values for $\dfrac{1}{1-\cos x} + \dfrac{1}{1+\cos x} = 2\csc^2 x$.

d) Show graphically that $\dfrac{1}{1-\cos x} + \dfrac{1}{1+\cos x} = 2\csc^2 x$. Are the graphs *exactly* the same?

Example 15: ***Equations with Identites***. Solve each trigonometric equation over the domain $0 \le x \le 2\pi$.

a) $2\sin^2 x - \cos x - 1 = 0$ **b)** $\sin x = \sec x \cot x$ **c)** $2\tan^2 x = -3\sec x$ **d)** $\cos^2 x = \sin^2 x$

Example 16: ***Equations with Identites***. Solve each trigonometric equation over the domain $0 \le x \le 2\pi$.

a) $3 - 3\csc x + \cot^2 x = 0$ **b)** $3\sin^2 x + 3\cos x - 4 = \sin^2 x - 2\cos x$

c) $\sin^3 x = \sin x$ **d)** $2\sin^3 x - 2\cos^2 x - \sin x + 1 = 0$

Example 17: ***Equations with Identites***. Solve each trigonometric equation over the domain $0 \le x \le 2\pi$.

a) $2\sec^2 x - \tan^4 x = -1$ **b)** $2\cos^3 x + 3\cos x = 7\cos^2 x$

c) $\tan^2 x + 2\sec^2 x - 3 = 0$ **d)** $4\sin^2 x + 2\sqrt{2}\sin x + 2\sqrt{3}\sin x + \sqrt{6} = 0$

Trigonometry

LESSON SIX- *Trigonometric Identities I*

Lesson Notes

$$\cos^3 x + \cos x \sin^2 x = \cos x$$
$$\cos x\left(\cos^2 x + \sin^2 x\right)$$
$$\cos x(1)$$
$$\cos x \quad \Big| \quad \cos x \quad \checkmark$$

Example 18: Use the Pythagorean identities to find the indicated value and draw the corresponding triangle.

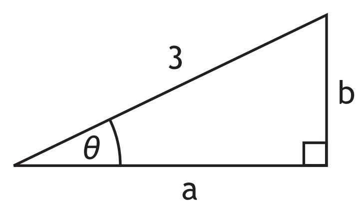
a) If the value of $\sin x = \frac{4}{7}$, $0 \le x \le \frac{\pi}{2}$, find the value of cos*x* within the same domain.

b) If the value of $\tan A = \frac{3}{2}$, $\pi < A < \frac{3\pi}{2}$, find the value of sec*A* within the same domain.

c) If $\cos\theta = \frac{\sqrt{7}}{7}$, and $\cot\theta < 0$, find the exact value of $\sin\theta$.

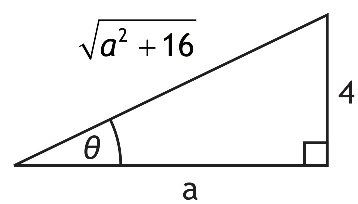
Example 19: Trigonometric Substitution.

a) Using the triangle to the right, show that $\frac{\sqrt{9-b^2}}{b^2}$ can be expressed as $\frac{\cos\theta}{3\sin^2\theta}$.



Hint: *Use the triangle to find a trigonometric expression equivalent to b.*

b) Using the triangle to the right, show that $\frac{a^2}{\sqrt{a^2+16}}$ can be expressed as $4\cot\theta\cos\theta$.



Hint: *Use the triangle to find a trigonometric expression equivalent to a.*

$\cos 15^\circ = \dfrac{\sqrt{6}+\sqrt{2}}{4}$

$\sin 15^\circ = \dfrac{\sqrt{6}-\sqrt{2}}{4}$

Trigonometry

LESSON SEVEN - *Trigonometric Identities II*

Lesson Notes

Example 1: Evaluate each trigonometric sum or difference.

a) $\sin\left(\frac{\pi}{3}+\frac{\pi}{4}\right)$ **b)** $\sin\left(\frac{\pi}{2}-\frac{\pi}{6}\right)$ **c)** $\cos\left(45^\circ-60^\circ\right)$ **d)** $\cos\left(\frac{\pi}{3}+\frac{\pi}{6}\right)$ **e)** $\tan\left(\frac{\pi}{4}+\frac{\pi}{3}\right)$ **f)** $\tan\left(\frac{\pi}{6}-\frac{\pi}{3}\right)$

Example 2: Write each expression as a single trigonometric ratio.

a) $\sin\frac{\pi}{6}\cos\frac{\pi}{2}+\cos\frac{\pi}{6}\sin\frac{\pi}{2}$ **b)** $\dfrac{\tan\frac{\pi}{4}-\tan\frac{\pi}{6}}{1+\tan\frac{\pi}{4}\tan\frac{\pi}{6}}$ **c)** $\cos\frac{\pi}{3}\cos\frac{\pi}{6}+\sin\frac{\pi}{3}\sin\frac{\pi}{6}$

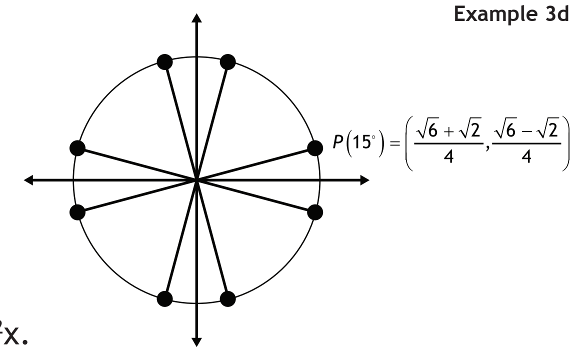
Example 3: Find the exact value of each expression.

a) $\cos 15^\circ$ **b)** $\sin\frac{5\pi}{12}$ **c)** $\tan 195^\circ$ **d)** Given the exact values of cosine and sine for 15°, fill in the blanks for the other angles.

Example 4: Find the exact value of each expression.

a) $\csc\left(\frac{\pi}{3}+\frac{\pi}{4}\right)$ **b)** $\sec\left(\frac{\pi}{12}\right)$ **c)** $\cot\left(\frac{\pi}{2}-\frac{\pi}{4}\right)$

Example 3d



Example 5: Double-angle identities.

a) Prove the double-angle sine identity, $\sin 2x = 2\sin x\cos x$.

b) Prove the double-angle cosine identity, $\cos 2x = \cos^2 x - \sin^2 x$.

c) The double-angle cosine identity, $\cos 2x = \cos^2 x - \sin^2 x$, can be expressed as $\cos 2x = 1 - 2\sin^2 x$ or $\cos 2x = 2\cos^2 x - 1$. Derive each identity.

d) Derive the double-angle tan identity, $\tan 2x = \dfrac{2\tan x}{1-\tan^2 x}$.

Example 6: Double-angle identities.

a) Evaluate each of the following expressions using a double-angle identity.

i. $\sin 60^\circ$ **ii.** $\cos\frac{\pi}{2}$ **iii.** $\tan 90^\circ$

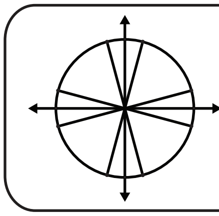
b) Express each of the following expressions using a double-angle identity.

i. $\sin 8x$ **ii.** $\cos 4x$ **iii.** $\sin x$ **iv.** $\cos\frac{1}{2}x$

c) Write each of the following expression as a single trigonometric ratio using a double-angle identity.

i. $\cos^2 30^\circ - \sin^2 30^\circ$ **ii.** $\sin\frac{\pi}{8}\cos\frac{\pi}{8}$ **iii.** $1-\sin^2\frac{1}{2}x$ **iv.** $\dfrac{2\tan\frac{x}{8}}{1-\tan^2\frac{x}{8}}$

Trigonometry
LESSON SEVEN- *Trigonometric Identities II*
Lesson Notes



$\cos 15° = \dfrac{\sqrt{6}+\sqrt{2}}{4}$

$\sin 15° = \dfrac{\sqrt{6}-\sqrt{2}}{4}$

Example 7: Prove each trigonometric identity.

Note: Variable restrictions may be ignored for the proofs in this lesson.

a) $\cos\left(x+\dfrac{5\pi}{6}\right)=-\dfrac{\sqrt{3}\cos x+\sin x}{2}$ **b)** $\sin\left(\dfrac{3\pi}{2}-x\right)=-\cos x$

c) $\tan\left(x-\dfrac{3\pi}{4}\right)=\dfrac{\tan x+1}{1-\tan x}$ **d)** $\cos(x+y)+\cos(x-y)=2\cos x\cos y$

Example 8: Prove each trigonometric identity.

a) $\cos\left(x+\dfrac{\pi}{6}\right)-\sin\left(x+\dfrac{2\pi}{3}\right)=0$ **b)** $\dfrac{\sin(x-y)}{\cos x\cos y}=\tan x-\tan y$

c) $\cos(x+y)\cos(x-y)=(\cos x\cos y)^2-(\sin x\sin y)^2$ **d)** $\cos 2x=\cos^2 x-\sin^2 x$

Example 9: Prove each trigonometric identity.

a) $\cos 2x+2\sin^2 x=1$ **b)** $\dfrac{2}{1+\cos 2x}=\sec^2 x$

c) $\dfrac{\sin 2x}{\cos 2x+\sin^2 x}=2\tan x$ **d)** $\dfrac{2\sin x\cos x}{\cos^2 x-\sin^2 x}=\tan 2x$

Example 10: Prove each trigonometric identity.

a) $\cos^4 x-\sin^4 x=\cos 2x$ **b)** $1-(\sin x+\cos x)^2=-\sin 2x$

c) $\dfrac{2(\tan x-\cot x)}{\tan^2 x-\cot^2 x}=\sin 2x$ **d)** $\dfrac{1}{1-\tan x}-\dfrac{1}{1+\tan x}=\tan 2x$

Example 11: Prove each trigonometric identity.

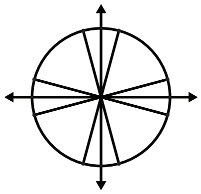
a) $2\csc 2x=\csc x\sec x$ **b)** $\dfrac{\sin(x+y)}{\cos x\sin y}=\tan x\cot y+1$

c) $\sin 88^\circ\cos 58^\circ-\cos 88^\circ\sin 58^\circ=\dfrac{1}{2}$ **d)** $\tan\left(x+\dfrac{\pi}{4}\right)=\dfrac{\tan x+1}{1-\tan x}$

Example 12: Prove each trigonometric identity.

a) $(\sin x+\cos x)^2-1=\sin 2x$ **b)** $\dfrac{1}{2}\sin\dfrac{2x}{5}=\sin\dfrac{x}{5}\cos\dfrac{x}{5}$

c) $\cos^2\left(x-\dfrac{\pi}{2}\right)=\sin^2 x$ **d)** $\sin 3x=3\sin x-4\sin^3 x$



$\cos 15^\circ = \dfrac{\sqrt{6}+\sqrt{2}}{4}$

$\sin 15^\circ = \dfrac{\sqrt{6}-\sqrt{2}}{4}$

Trigonometry

LESSON SEVEN - *Trigonometric Identities II*

Lesson Notes

Example 13: Prove each trigonometric identity.

a) $\dfrac{5\sin x - \cos 2x - 11}{2\sin x - 3} = \sin x + 4$ **b)** $\cos 3x = 4\cos^3 x - 3\cos x$

c) $\cos 34^\circ \cos 41^\circ - \sin 34^\circ \sin 41^\circ = \dfrac{\sqrt{6}-\sqrt{2}}{4}$ **d)** $\dfrac{\tan x + \tan y}{\sec x \sec y} = \sin(x+y)$

Example 14: Solve each trigonometric equation over the domain 0 ≤ x ≤ 2π.

a) $\cos 2x = \cos^2 x$ **b)** $\cos\left(x+\dfrac{\pi}{4}\right) + \cos\left(x-\dfrac{\pi}{4}\right) = -1$

c) $4\sin^2 x + 4\cos 2x - 1 = 0$ **d)** $2\cos^2 \dfrac{1}{2}x - 2\sin^2 \dfrac{1}{2}x = 1$

Example 15: Solve each trigonometric equation over the domain 0 ≤ x ≤ 2π.

a) $\cos 2x + 7\sin x - 4 = 0$ **b)** $\sin 2x - \cos x = 0$

c) $\sin\left(\dfrac{\pi}{3}+x\right) - \sin\left(\dfrac{\pi}{3}-x\right) = 1$ **d)** $\sin x \cos x = \dfrac{1}{4}$

Example 16: Solve each trigonometric equation over the domain 0 ≤ x ≤ 2π.

a) $\cos 2x - \cos x = 0$ **b)** $\csc\left(x+\dfrac{\pi}{2}\right) - \csc\left(x-\dfrac{\pi}{2}\right) = 4$

c) $\dfrac{1}{2}\sin 2x + \sin x = 0$ **d)** $2\cot^2 x - 3\csc x = 0$

Example 17: Solve each trigonometric equation over the domain 0 ≤ x ≤ 2π.

a) $8\sin x \cos x = 2$ **b)** $(\cos x - \sin x)^2 = \sin 2x + 1$

c) $\tan(x-\pi) + \sec x = 0$ **d)** $\cos(x+\pi) - \cos^2 x = 0$