UNIVERSITY OF CAMBRIDGE

NST2AS NATURAL SCIENCES TRIPOS Part II

Tuesday 2 June 2020

ASTROPHYSICS - PAPER 1

Candidates may attempt not more than six questions.

Each question is divided into Part **(i)** *and Part* **(ii)**, *which may or may not be related. Candidates may attempt either or both Parts.*

The number of marks for each question is the same, with Part **(ii)** *of each question carrying twice as many marks as Part* **(i)**. *The approximate number of marks allocated to each component of a question is indicated in the right margin. Additional credit will be given for a substantially complete answer to either Part.*

Write on **one side** *of the paper only and* **begin each answer on a separate sheet.** *Please ensure that your candidate number is written at the top of each sheet and number the pages within each answer.*

Answers must be uploaded to Moodle as separate PDF files, named with your candidate number followed by an underscore and **X, Y, Z**, *according to the letter associated with each question. (For example,* **3X and 6X** *should be in one file named 7850T_X.pdf and* **2Y, 4Y and 7Y** *in another file.) The first page of each such file should be a title page bearing your candidate number, the appropriate letter* **X, Y, Z**, *and a list of the questions (including whether Part* **(i)**, *Part* **(ii)** *or both) attempted.*

A master cover sheet bearing your candidate number and listing all Parts of all questions attempted must also be completed and uploaded as a separate PDF file.

STATIONERY REQUIREMENTS	SPECIAL REQUIREMENTS
Script Paper	*Astrophysics Formulae Booklet*
Master Cover Sheet	*Approved Calculators Allowed*

You should spend three hours working on this paper (plus any pre-agreed individual adjustment). Downloading and uploading times should not be included in the allocated exam time.

Question 1Z - Relativity

(i) Explain what is meant by a *Lorentz boost.* [4]

Show that the composition of two co-linear Lorentz boosts, with speeds $\beta_1 c$ and $\beta_2 c$, is also a Lorentz boost, and find the speed βc of the resulting boost in terms of β_1 and β_2. [*Hint: You may wish to work in terms of the rapidity* ψ*, defined by* $\beta = \tanh\psi$.] [6]

(ii) In Minkowski spacetime in inertial coordinates, a particle transports a 4-vector $\boldsymbol{V}$ along its worldline according to

$$\frac{dV^\mu}{d\tau} = \frac{1}{c^2}\left(V^\nu u_\nu a^\mu - V^\nu a_\nu u^\mu\right) ,$$

where τ is the particle's proper time, and u^μ and a^μ are the components of the particle's velocity and acceleration 4-vectors, respectively. Show that $V^\mu u_\mu$ and $V^\mu V_\mu$ are constant along the particle's path. [5]

Consider the case where $V^\mu u_\mu = 0$ and a particle that, in the laboratory frame, moves in a circle in the x-y plane of radius r with constant angular speed ω, so that its path is

$$x^\mu(\tau) = [c\gamma\tau, r\cos(\gamma\omega\tau), r\sin(\gamma\omega\tau), 0] ,$$

where γ is the Lorentz factor for speed ωr. Show that, in the laboratory frame,

$$\begin{aligned} \frac{dV^1}{d\xi} &= \nu^2 \sin\xi \left(V^1\cos\xi + V^2\sin\xi\right) , \\ \frac{dV^2}{d\xi} &= -\nu^2 \cos\xi \left(V^1\cos\xi + V^2\sin\xi\right) , \end{aligned} \qquad (*)$$

where $\xi = \gamma\omega\tau$ and ν is a dimensionless constant that you should specify. [6]

Show further that

$$\gamma V^0 = -\nu\left(V^1\sin\xi - V^2\cos\xi\right) .$$ [2]

Suppose that the instantaneous rest frame of the particle at $\tau = 0$, denoted $S'(0)$, is obtained from the laboratory frame by a Lorentz boost along the y-axis, and at $\tau = 0$ the vector $\boldsymbol{V}$ lies in the x'-y' plane of $S'(0)$. Given that the general solution to $(*)$ is

$$\begin{aligned} V^1 &= A\left[\cos\xi\cos(\gamma\xi - \alpha) + \gamma\sin\xi\sin(\gamma\xi - \alpha)\right] , \\ V^2 &= A\left[\sin\xi\cos(\gamma\xi - \alpha) - \gamma\cos\xi\sin(\gamma\xi - \alpha)\right] , \end{aligned}$$

where A and α are constants, show that when the particle returns to its starting point, $\boldsymbol{V}$ has rotated in $S'(0)$ by an angle $2\pi(\gamma - 1)$. [7]

Question 2Y - Astrophysical Fluid Dynamics

(i) Consider an axisymmetric accretion disk with surface density $\Sigma(R, t)$ orbiting around a central body with angular velocity $\Omega(R)$. By considering the disk as a set of interacting rings, or otherwise, show that

$$R\frac{\partial \Sigma}{\partial t} + \frac{\partial}{\partial R}(R u_R \Sigma) = 0\,,$$
$$R\frac{\partial}{\partial t}\left(\Sigma R^2 \Omega\right) + \frac{\partial}{\partial R}\left(R\Sigma u_R R^2 \Omega\right) = \frac{1}{2\pi}\frac{\partial G_{\rm tot}}{\partial R}\,,$$

where $u_R(R, t)$ is the radial velocity of the matter and $G_{\rm tot}(R, t)$ is the total torque exerted by the disk outside of radius R on the disk inside of that radius. [10]

(ii) Suppose that the accretion disk of Part (i) is in orbit around a central object of mass M, radial pressure gradients are negligible, and that the mass of the accretion flow itself is negligible in comparison to M. Further, suppose that this accretion disk is subject to the sum of an internal viscous torque, $G_\nu(R, t) = 2\pi R^3 \nu \Sigma \Omega'$ (where ν is the effective kinematic viscosity and primes denotes differentiation with respect to R) and an external torque, $G_{\rm m}(t)$, due to large-scale magnetic fields that connect it with the central object. By first finding $u_R\Sigma$ in terms of $\partial G_{\rm tot}/\partial R$, or otherwise, show that the evolution of the disk is governed by

$$\frac{\partial \Sigma}{\partial t} = \frac{3}{R}\frac{\partial}{\partial R}\left(R^{1/2}\frac{\partial}{\partial R}\left(\nu \Sigma R^{1/2}\right)\right) - \frac{1}{\pi R (GM)^{1/2}}\frac{\partial}{\partial R}\left(R^{1/2}\frac{\partial G_{\rm m}}{\partial R}\right)\,.$$ [8]

Consider the case where $G'_{\rm m} = \beta\delta(R - R_{\rm m})$, corresponding to all of the external torque being applied at a single radius $R = R_{\rm m} > R_*$. Further, assume that the viscous torque vanishes at some innermost radius R_*. Show that, in steady state, the local rate of viscous dissipation per unit surface area of the disk is given by

$$D_{\rm ss}(R) = \frac{3GM\dot{M}}{8\pi R^3}\left(1 - \sqrt{\frac{R_*}{R}}\right) + \frac{3(GM)^{1/2}\beta}{8\pi R^{7/2}}\Theta(R - R_{\rm m})\,,$$

where $\Theta(x)$ is the Heaviside step function and $\dot{M}$ is the mass accretion rate onto the central object. [8]

Sketch $D_{\rm ss}(R)$. Comment on its behaviour around $R = R_{\rm m}$ and at $R \gg R_{\rm m}$. [4]

[*You may assume without proof that the viscous dissipation rate per unit surface area of the disk is* $D(R) = \frac{1}{2}\nu\Sigma R^2 \left(\partial\Omega/\partial R\right)^2$.]

TURN OVER...

Question 3X - Introduction to Cosmology

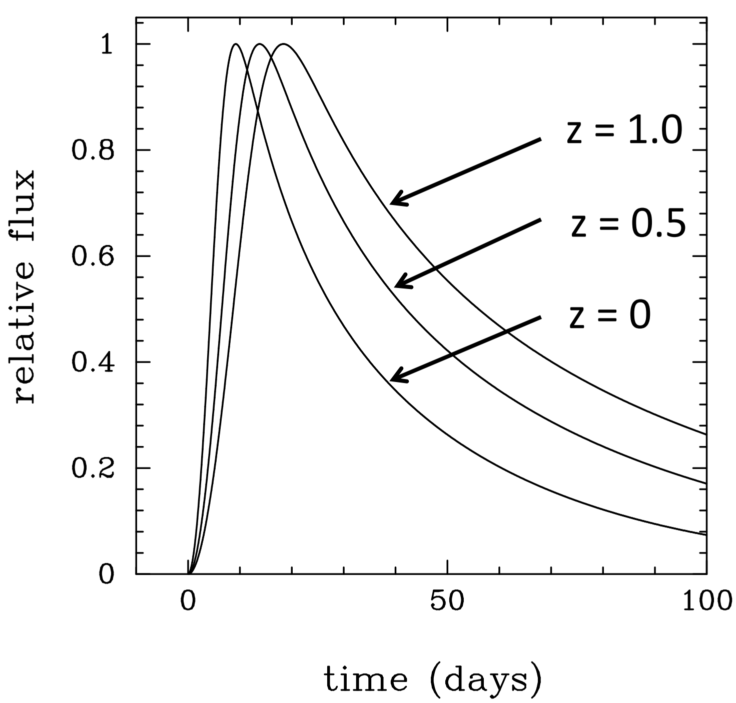
(i) A distant source emits light at time t_1, which is received on Earth at the present time t_0. Show that in a Friedmann–Robertson–Walker cosmology the light received on Earth is redshifted according to

$$1 + z = \frac{R(t_0)}{R(t_1)} ,$$

where $R(t)$ is the scale factor. [5]

Discuss briefly how observations of Type Ia supernovae have been used to provide evidence that the Universe is accelerating. [4]

The diagram below shows the bolometric light curves of three Type Ia supernovae with redshifts $z = 0$, 0.5 and 1.0. Give an explanation for the differences in the shapes of these curves. [1]



(ii) Consider a spatially-flat Friedmann–Roberston–Walker universe with zero cosmological constant. At early times, assume the universe is radiation dominated with uniform density ρ_γ. Suppose also that there is a sub-dominant homogeneous scalar field, ϕ, which obeys the equation of motion

$$\ddot{\phi} + 3H\dot{\phi} = -\frac{dV(\phi)}{d\phi} ,$$

where overdots denote differentiation with respect to time t, H is the Hubble parameter and $V(\phi)$ is the scalar field potential. If $V(\phi) = A/\phi^\alpha$, where A and α are positive constants, show that the equation of motion admits the solution

$$\phi(t) = \left(\frac{\alpha(\alpha+2)^2 A t^2}{(\alpha+6)} \right)^{1/(2+\alpha)} . \qquad (*)$$ [8]

The density of the scalar field is (in units with $c = \hbar = 1$)

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi) .$$

Show that if the solution $(*)$ applies,

$$\rho_\phi / \rho_\gamma \propto t^{4/(2+\alpha)}$$

and so the scalar field will eventually dominate the density of the universe. [4]

Assume that at late times the scalar field dominates the density of the universe and that it evolves slowly [$|\ddot{\phi}| \ll 3|H\dot{\phi}|$ and $\dot{\phi}^2/2 \ll V(\phi)$]. Show that the scale factor evolves as

$$\ln R \propto \left(t^{4/(4+\alpha)} + \text{const.} \right) .$$ [6]

Compare this evolution of the scale factor to that in a universe dominated by a cosmological constant Λ. [2]

TURN OVER...

Question 4Y - Structure and Evolution of Stars

(i) Describe in a few sentences the main properties of globular clusters and explain how they help us understand stellar evolution. [4]

Sketch the observed colour–magnitude diagram of a typical globular cluster, label its main features and discuss its relevance to stellar evolution. [4]

Why are planetary nebulae rarely seen in globular clusters? [2]

(ii) A white dwarf star may be modelled as an isothermal degenerate core with temperature T_c, mass M_c, and molecular weight μ_c, which cools and loses energy at a rate

$$L = -\frac{3}{2}\frac{\mathcal{R}M_c}{\mu_c}\frac{dT_c}{dt},$$

overlaid by a thin non-degenerate envelope where the dependence of the opacity κ on temperature T and density ρ follows Kramers' law:

$$\kappa = \frac{A\rho}{T^{3.5}},$$

where A is a constant. The transition density from core to envelope is given by $\rho_t = CT_c^{3/2}$, where C is a constant. Using equations of stellar structure for the envelope, show the following:

(a) in the envelope, pressure and density are related by

$$P \propto \rho^{(n+1)/n}$$

with $n = 3.25$; [6]

(b) the luminosity depends on the core temperature as $L \propto T_c^{3.5}$; and [2]

(c) the luminosity decreases with time as $L \propto t^{-7/5}$. [2]

Explain how you would verify empirically that $L \propto t^{-7/5}$ holds for real white dwarfs. [6]

Suppose that in the Milky Way white dwarfs are formed at a constant rate and that we are able to see all white dwarfs to a given distance from the Earth. How would you expect the number of white dwarfs per luminosity bin to vary with luminosity? [2]

Is this what is observed? If not, give a plausible interpretation of the discrepancy. [2]

Question 5Z - Statistical Physics

(i) A bosonic gas consisting of indistinguishable, non-interacting particles is in thermal equilibrium at temperature T and chemical potential μ. Starting with the single-state partition function for a system with a variable number of particles, show that the mean number of particles of energy E_i is

$$n_i = \frac{g_i}{e^{(E_i-\mu)/(k_\mathrm{B}T)} - 1}\,,$$

where g_i is the number of single-particle states of energy E_i. [10]

(ii) Assume that the bosonic gas of Part (i) consists of N non-relativistic, spin-0 particles of mass m confined within a box of volume V. Derive an expression for the number of single-particle states, $G(E)$, with energy less than E in the limit that E is much larger than the energy difference between the states. [5]

The pressure of the bosonic gas is

$$P = \frac{2}{3V}\int_0^\infty \frac{dG}{dE}\frac{E}{e^{(E-\mu)/(k_\mathrm{B}T)} - 1}\,dE\,.$$

Show that, if $e^{\mu/(k_\mathrm{B}T)} \ll 1$, the equation of state of the bosonic gas is

$$PV = Nk_\mathrm{B}T\left[1 - \left(\frac{T_B}{T}\right)^{3/2}\right]\,,$$

where, as part of your answer, you should provide an explicit expression for T_B in terms of quantities already given. [10]

What physical phenomenon is hinted at by the existence of the characteristic temperature T_B? [2]

Comment on the behaviour of the chemical potential μ in the limits of high and low temperatures. [3]

[*You may assume that* $\int_0^\infty x^n e^{-x}\,dx = \Gamma(n+1)$, *where* $\Gamma(5/2) = \frac{3}{2}\Gamma(3/2)$ *and* $\Gamma(3/2) = \sqrt{\pi}/2$.]

TURN OVER...

Question 6X - Principles of Quantum Mechanics

(i) The creation and annihilation operators for a particle of mass μ are defined by

$$a^\dagger = \left(\frac{\mu\omega}{2\hbar}\right)^{\frac{1}{2}} \left(x - i\frac{p}{\mu\omega}\right) \quad \text{and} \quad a = \left(\frac{\mu\omega}{2\hbar}\right)^{\frac{1}{2}} \left(x + i\frac{p}{\mu\omega}\right) ,$$

where x is a position operator, p is the momentum operator in the direction of x and ω is an angular frequency. Show that the commutator $[a, a^\dagger] = 1$. [3]

Write the Hamiltonian

$$H = \frac{p^2}{2\mu} + \frac{1}{2}\mu\omega^2 x^2$$

for a harmonic oscillator in terms of a and $a^\dagger$. [3]

Suppose there exists a stationary state $|n\rangle$ with energy $E_n > \hbar\omega$. Show that there must also exist states $|n-1\rangle \propto a|n\rangle$ and $|n+1\rangle \propto a^\dagger|n\rangle$ with energies $E_n - \hbar\omega$ and $E_n + \hbar\omega$, respectively. [4]

(ii) Using the results of Part (i), deduce the quantised energy levels of the quantum harmonic oscillator. [4]

Define the number operator N, with eigenvalue n for a normalized eigenvector $|n\rangle$, and express it in terms of the creation and annihilation operators $a^\dagger$ and a. [2]

By considering the expectation value of N, confirm that $n \geq 0$. [2]

Show that

$$\left(a^\dagger\right)^m a^m |n\rangle = \begin{cases} \frac{n!}{(n-m)!}|n\rangle & \text{if } m \leq n, \\ 0 & \text{if } m > n. \end{cases}$$ [8]

By considering the action on an arbitrary basis vector deduce that

$$\sum_{m=0}^{\infty} \frac{1}{m!}(-1)^m \left(a^\dagger\right)^m a^m = |0\rangle\langle 0| \,.$$ [4]

Question 7Y - Stellar Dynamics and the Structure of Galaxies

(i) Two stars orbiting each other have masses m_1 and m_2 and position vectors $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$, respectively. Write down the equations obeyed by $\ddot{\mathbf{r}}_1$ and $\ddot{\mathbf{r}}_2$. [2]

Show that the equation obeyed by the relative position vector $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ is the same as that obeyed by a particle under the influence of an immovable mass $M = m_1 + m_2$. [2]

Show that the orbital angular momentum $\mathbf{J}$ (about the centre of mass) of the binary system may be written

$$\mathbf{J} = \mu\, \mathbf{r} \times \dot{\mathbf{r}}\,,$$

where μ is the reduced mass. [6]

(ii) A particle of mass m (with $m \ll M_\odot$) is orbiting under the influence of the Sun's gravity alone. The particle is at position (r, ϕ) in polar coordinates, has orbital energy mE, orbital angular momentum mh and the orbit is of the form

$$\frac{1}{r} = \frac{GM_\odot}{h^2}(1 + e\cos\phi)\,,$$

where e is the eccentricity. Show that for gravitationally-bound orbits

$$E = -\frac{GM_\odot}{2a}\,,$$

where $a > 0$ is a constant, and give a geometric interpretation of a. [10]

An alien spaceship enters the Solar System on a parabolic orbit. When it reaches perihelion it briefly switches on retrorockets that change its velocity from V to λV, where $0 \leq \lambda < 1$. Show that the new orbit has eccentricity given by

$$e = |2\lambda^2 - 1|\,.$$ [8]

For what values of λ does the point at which the rocket is fired remain the perihelion of the new orbit? [2]

TURN OVER...

Question 8Z - Topics in Observational Astrophysics

(i) An area of the sky is imaged on to a noise-free detector. The light from a star falls in a small patch that counts a total of $Q + B$ photons on average in an exposure time T, where Q photons come from the star and B photons come from the background sky. A similar patch receives light from the sky only and is used to subtract the sky background so that Q can be estimated. Assuming that the errors in the two measurements are Poisson-distributed and independent, show that the signal-to-noise ratio Z of the estimate of Q is given by

$$Z = \frac{Q}{\sqrt{Q + 2B}} \, .$$ [5]

Show further that to attain a signal-to-noise ratio Z the required exposure time T is given by

$$T = \frac{Z^2(R_Q + 2R_B)}{R_Q^2} \, ,$$

where R_Q and R_B are the average photon arrival rates for the star and the sky, respectively. [5]

(ii) A spectrometer on an 8-metre diameter telescope records the spectrum of a star with magnitude $V = 22$. The spectral resolution is 0.1 nm and a signal-to-noise ratio of Z is obtained in an exposure time of 18 000 s at a wavelength of 550 nm. Only 10 % of the photons available in the telescope's aperture are recorded. Calculate the photon detection rate from the star for a single spectral-resolution element. [You may assume that a star with $V = 0$ delivers $1.02 \times 10^7\,\mathrm{m^{-2}\,s^{-1}}$ photons in a 0.1 nm wavelength interval.] [3]

Using the results in Part (i), determine the signal-to-noise ratio Z, ignoring detector noise and assuming the sky is completely dark. [2]

In practice, the star's light is collected by an optical fibre of 2 arcsec diameter and the sky spectrum is obtained with a similar fibre. The sky has a brightness of $V = 21.5\,\mathrm{arcsec^{-2}}$. Calculate the photon detection rate in the sky spectrum for a single spectral element. [5]

The measurements of the star and the sky photon counts in this spectral element both have a root-mean-square error of 50 photons due to detector noise. Determine the signal-to-noise ratio Z for the star, now accounting for both the detector noise and the subtraction of the sky background. [5]

What is the dominant source of error in the measured stellar spectrum? [5]

END OF PAPER

UNIVERSITY OF CAMBRIDGE

NST2AS NATURAL SCIENCES TRIPOS Part II

Thursday 4 June 2020

ASTROPHYSICS - PAPER 2

Candidates may attempt not more than six questions.

Each question is divided into Part **(i)** *and Part* **(ii)**, *which may or may not be related. Candidates may attempt either or both Parts.*

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Question 1Z - Relativity

(i) What properties does the metric satisfy in local inertial coordinates centred on some spacetime event P? [3]

Discuss the physical significance of local inertial coordinates in relation to the equivalence principle. [7]

(ii) Starting from the definition of the Riemann curvature tensor,

$$\nabla_\mu \nabla_\nu v_\rho - \nabla_\nu \nabla_\mu v_\rho = R_{\mu\nu\rho}{}^{\tau} v_\tau$$

for arbitrary dual vectors v_μ, show that in local inertial coordinates at some point P, the components of the Riemann tensor are

$$R_{\mu\nu\rho\tau} = \frac{1}{2} \left(\partial_\mu \partial_\tau g_{\nu\rho} + \partial_\nu \partial_\rho g_{\mu\tau} - \partial_\mu \partial_\rho g_{\nu\tau} - \partial_\nu \partial_\tau g_{\mu\rho} \right) ,$$

where $g_{\mu\nu}$ is the metric tensor. [9]

Consider a spacetime in which the line element in the vicinity of the point P, with coordinates $x^\mu = 0$, is

$$ds^2 = \left(1 + L^{-2} \eta_{\rho\tau} x^\rho x^\tau \right)^{-2} \eta_{\mu\nu} dx^\mu dx^\nu ,$$

where L is a constant with dimensions of length and $\eta_{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1)$ is the Minkowski metric. Show that at the event P,

$$R_{\mu\nu\rho\tau} = \frac{4}{L^2} \left(g_{\mu\rho} g_{\nu\tau} - g_{\mu\tau} g_{\nu\rho} \right)$$

in any coordinate system. [8]

Calculate the Ricci scalar, defined by $R = R_{\mu\nu}{}^{\nu\mu}$, at P. [3]

Question 2Y - Astrophysical Fluid Dynamics

(i) Briefly explain what is meant by a barotropic equation of state. [1]

Consider an unmagnetised ideal gas with a polytropic equation of state $P = K\rho^\gamma$ that, in equilibrium, is static and homogeneous with density ρ_0 and pressure P_0. Starting from the basic fluid equations, show that pressure perturbations δP satisfy

$$\frac{\partial^2 \delta P}{\partial t^2} - c_s^2 \nabla^2 \delta P = 0\,,$$

where you should relate c_s to P_0, γ and ρ_0. [9]

(ii) Consider a spherical bubble of density ρ_b and radius r_b embedded in the ideal gas of Part (i). The bubble interior can be treated as an ideal gas with a polytropic equation of state and in pressure balance with the surrounding medium. Further, consider small spherically-symmetric oscillations of the bubble with angular frequency ω. Show that the pressure perturbations driven by the bubble into the surrounding gas can take the form

$$\delta P = \frac{p_1}{kr} e^{i(kr - \omega t)}\,,$$

where p_1 is a constant, and derive the corresponding dispersion relation. [7]

Show that the corresponding velocity perturbations are

$$\delta v_r = \frac{k\,\delta P}{\omega \rho_0}\left(1 + \frac{i}{kr}\right)\,.$$

[4]

Suppose that the (equilibrium) density of the bubble is much lower than that of the surroundings ($\rho_b \ll \rho_0$). Explain why, provided the oscillation frequency is much below some critical threshold ($\omega \ll \omega_{\rm th}$), we can consider the interior of the bubble to possess a uniform density and pressure. [3]

Write down an approximate expression for $\omega_{\rm th}$. [1]

By considering velocity and pressure perturbations at the bubble surface, determine the oscillation frequency and damping rate that result if an initial, spherically-symmetric, perturbation of the bubble radius is allowed to evolve naturally. [5]

TURN OVER...

Question 3X - Introduction to Cosmology

(i) Consider a universe composed of matter with a uniform density ρ and isotropic pressure P. Energy conservation in general relativity requires

$$\frac{d(\rho R^3)}{dR} = -\frac{3}{c^2} P R^2 \, ,$$

where R is the scale factor. If the matter is in thermal equilibrium at temperature T, show that

$$\frac{d}{dR}\left[\frac{(\rho + P/c^2)R^3}{T}\right] = 0$$

if

$$\frac{d(P/c^2)}{dT} = \frac{(\rho + P/c^2)}{T} \, . \qquad (*)$$

[2]

For particles of type i in thermal equilibrium, the energy density and pressure are

$$\rho c^2 = g_i \frac{4\pi}{h^3} \int f E p^2 \, dp \, , \qquad P = c^2 g_i \frac{4\pi}{3h^3} \int f \frac{p^4}{E} \, dp \, ,$$

where $E^2 = p^2 c^2 + m_i^2 c^4$, m_i is the rest mass, g_i is the number of spin states and f is the distribution function

$$f = \left[\exp\left(\frac{E - \mu_i}{k_{\rm B} T}\right) \pm 1\right]^{-1} ,$$

where μ_i is the chemical potential and the plus and minus signs are for fermions and bosons, respectively. If the chemical potential is zero, show that

$$\frac{\partial f}{\partial T} = -\frac{E}{T}\frac{\partial f}{\partial E} \, .$$

[2]

Hence show that $(*)$ is satisfied and give a physical interpretation of this result. [6]

(ii) Show that in thermal equilibrium at temperatures $k_{\rm B} T \ll m_p c^2$, the neutron-to-proton ratio is

$$\frac{n}{p} \approx \left(\frac{m_n}{m_p}\right)^{3/2} \exp\left(-\xi_{\nu_e} - \frac{Q}{k_{\rm B} T}\right) ,$$

where $Q = (m_n - m_p)c^2$ and $(\mu_n - \mu_p) = -\xi_{\nu_e} k_{\rm B} T$ are the differences of the neutron and proton rest mass energies and chemical potentials, respectively. [You may use expressions given in Part (i) without proof.] [5]

Discuss the physical significance of the term ξ_{ν_e}. [3]

How would a small positive value of ξ_{ν_e} alter the helium abundance produced in big bang nucleosythesis? [2]

If the neutrino chemical potentials are non-zero, with values $\mu_\alpha = \xi_\alpha k_\mathrm{B} T_\nu$ (where α denotes the neutrino flavour, ν_e, ν_μ or ν_τ, and T_ν is the neutrino temperature), show that the neutrino contribution to the energy density at the time of nucleosynthesis is

$$\rho_\nu c^2 = \frac{7\pi^5}{5}\frac{k_\mathrm{B}^4 T_\nu^4}{h^3 c^3}\left[1 + \sum_\alpha \left(\frac{10}{7\pi^2}\xi_\alpha^2 + \frac{5}{7\pi^4}\xi_\alpha^4\right)\right] .$$ [7]

The observed helium abundance leads to the approximate constraint $|\xi_{\nu_e}| \lesssim 0.1$. If neutrino oscillations lead to equilibration of the chemical potentials, $\xi_{\nu_e} = \xi_{\nu_\mu} = \xi_{\nu_\tau} = \xi$, show that neutrino degeneracy alters the effective number of neutrino species by

$$\Delta N_\nu \approx \frac{90}{7\pi^2}\xi^2 \lesssim 1.2 \times 10^{-2} .$$ [3]

[*You may assume that*

$$\int_0^\infty y^3 \left[(e^{y-x}+1)^{-1} + (e^{y+x}+1)^{-1}\right] dy = \frac{7\pi^4}{60}\left(1 + \frac{30}{7\pi^2}x^2 + \frac{15}{7\pi^4}x^4\right) .]$$

TURN OVER...

Question 4Y - Structure and Evolution of Stars

(i) The Sun's luminosity is produced by the conversion of hydrogen to helium. In the process 0.7 % of the hydrogen's rest mass energy is released. Estimate the number of hydrogen nuclei which are consumed each second. [3]

The nuclear reactions involved in the conversion release three electron neutrinos per hydrogen nucleus consumed. Furthermore, because the neutrino collision cross-section is very small, you can assume that all the neutrinos produced will escape from the interior of the Sun. Estimate the total number of neutrinos which pass, per day, through a 30-meter diameter neutrino detector on Earth. [3]

Only about five electron neutrinos are detected per day by the detector, which is roughly one third of the number expected based on the calculated neutrino flux and the detection efficiency (which is accurately known). What might be an explanation for this discrepancy? [4]

(ii) A star is fully radiative. Suppose that the energy generation rate, $\mathcal{E}$, is independent of radial coordinate r. Show that the luminosity $L = \mathcal{E}m$ throughout the star, where m is the enclosed mass. [3]

Hence show that temperature T and pressure P satisfy

$$\frac{dT}{dP} = \frac{3\kappa\mathcal{E}}{16\pi acGT^3},$$

where κ is the opacity and $ac/4 = \sigma$, where σ is the Stefan–Boltzmann constant. [5]

If the opacity κ is also constant, as is the case for electron scattering, show that

$$T^4 = T_0^4 + \frac{3\kappa\mathcal{E}}{4\pi acG}(P - P_0),$$

where T_0 and P_0 are the temperature and pressure at the surface. [3]

Hence show that $P \to \mathcal{C}T^4$ (where $\mathcal{C}$ is a constant) in the interior of the star where $T \gg T_0$. [1]

A system is termed a 'polytrope of index n' if the dependence of pressure on density is of the form $P = K\rho^{(n+1)/n}$, where K is a constant. Use the results above to show that a star, where the total pressure is the sum of gas pressure (assuming an ideal gas) and radiation pressure, can be described as a polytrope of index $n = 3$, provided that the energy generation rate satisfies the inequality

$$\mathcal{E} < \frac{4\pi cG}{\kappa}.$$

[8]

Question 5Z - Statistical Physics

(i) A ferromagnet has a magnetization order parameter m and temperature T. The free energy is given by

$$F(T,m) = F_0(T) + \frac{a}{2}(T - T_c)m^2 + \frac{b}{4}m^4\,,$$

where a, b and T_c are positive constants. Sketch F as a function of m at both high and low temperatures and find the equilibrium value of the magnetization in each case. [10]

(ii) Evaluate the free energy of the stable equilibrium state(s) of the ferromagnet described in Part (i) as a function of temperature for both $T > T_c$ and $T < T_c$. [4]

Hence compute the entropy and heat capacity for both $T > T_c$ and $T < T_c$. [6]

Determine the jumps in the entropy and heat capacity as the system transitions through the $T = T_c$ point and identify the order of the phase transition. [3]

After imposing a background magnetic field B, the free energy becomes

$$F(T,m) = F_0(T) + Bm + \frac{a}{2}(T - T_c)m^2 + \frac{b}{4}m^4\,.$$

Explain graphically why the system undergoes a first-order phase transition at low temperatures as B changes sign. [3]

The *spinodial* point occurs when the meta-stable equilibrium ceases to exist. Determine the temperature T of the spinodial point as a function of T_c, a, b and B. [4]

TURN OVER...

Question 6X - Principles of Quantum Mechanics

(i) A spin-1/2 particle at rest in a magnetic field $\mathbf{B}$ is described by the Hamiltonian $H = -\hbar\gamma \mathbf{B}\cdot\boldsymbol{\sigma}/2$, where γ is a constant and the three components of $\boldsymbol{\sigma}$ are the Pauli matrices. Consider the case where $\mathbf{B} = (B_x, 0, B_z)$ is independent of time, and the particle is in the state $|\uparrow\rangle$ at time $t = 0$, where $\sigma_3|\uparrow\rangle = |\uparrow\rangle$. Show that the probability of finding the particle in the state $|\downarrow\rangle$, where $\sigma_3|\downarrow\rangle = -|\downarrow\rangle$, at time t is

$$P(t) = \frac{B_x^2}{B^2}\sin^2\left(\frac{\gamma B t}{2}\right), \qquad (*)$$

where $B = \sqrt{B_x^2 + B_z^2}$. [Hint: write $\mathbf{B}$ as $B\hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ is a unit vector, and construct the time-evolution operator.] [10]

[*The Pauli matrices are*

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

which you may assume satisfy $\sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij}I$.]

(ii) A system subject to a small time-dependent perturbation is described by the Hamiltonian $H(t) = H_0 + \delta H(t)$, where H_0 is time-independent. Let the orthonormal eigenstates of H_0 be $\{|\phi_r\rangle\}$, with eigenvalues E_r. If the system is in the state $|\phi_n\rangle$ at time $t = 0$, show that to leading order in $\delta H(t)$ the probability of finding the particle in the different state $|\phi_{n'}\rangle$ at time t is

$$P(n \to n') = \left|\frac{-i}{\hbar}\int_0^t dt'\, e^{-i(E_n - E_{n'})t'/\hbar}\langle\phi_{n'}|\delta H(t')|\phi_n\rangle\right|^2.$$ [7]

The particle in Part (i) is now placed in a magnetic field

$$\mathbf{B} = (A\cos\omega t, A\sin\omega t, B_z),$$

where the amplitude A of the time-dependent components is small compared to the constant B_z. Show that the probability of finding the particle in the state $|\downarrow\rangle$ at time t when it is in the state $|\uparrow\rangle$ at $t = 0$ is approximately

$$P(t) \approx \left(\frac{\gamma A}{\omega + \gamma B_z}\right)^2 \sin^2\left(\frac{(\omega + \gamma B_z)t}{2}\right)$$

for $|A| \ll |B_z + \omega/\gamma|$. [10]

Compare this result for $\omega = 0$ to the exact result $(*)$ in Part (i). [3]

Question 7Y - Stellar Dynamics and the Structure of Galaxies

(i) The gravitational potential of an infinitesimally-thin disc lying in the $z = 0$ plane with axisymmetric surface density $\Sigma(R)$ can be written in the form

$$\Phi(R, z) = \int_0^\infty f(k)\, e^{-k|z|}\, J_0(kR)\, dk\,,$$

where

$$f(k) = -2\pi G \int_0^\infty \Sigma(R) J_0(kR)\, R\, dR$$

and J_0 is the 0th order Bessel function of the first kind. For the case $\Sigma(R) = \Sigma_0\, R_0/R$, where Σ_0 and R_0 are constants, show that the circular velocity of a test particle orbiting in such a disc can be written as

$$v_c(R) = \sqrt{\frac{G\, m(\leq R)}{R}}\,,$$

where $m(\leq R)$ is the mass enclosed within radius R. [10]

[*For the Bessel functions of the first kind $J_0(x)$ and $J_1(x)$, you may assume $dJ_0(x)/dx = -J_1(x)$ and*

$$\int_0^\infty J_0(bx)\, dx = \int_0^\infty J_1(bx)\, dx = \frac{1}{b}$$

for any constant $b > 0$.]

(ii) Show that the axisymmetric gravitational potential

$$\Phi(R, z) = -\frac{GM}{\sqrt{R^2 + (a + |z|)^2}}\,,$$

where a and M are positive constants, obeys $\nabla^2\Phi = 0$ everywhere, except in the $z = 0$ plane. [5]

Deduce that the potential is generated by an infinitesimally-thin disc in the $z = 0$ plane with surface density

$$\Sigma(R) = \frac{Ma}{2\pi} \frac{1}{(R^2 + a^2)^{3/2}}\,. \qquad [5]$$

Calculate the circular velocity $v_c(R)$ of a test particle orbiting in the disc at radius R. [4]

TURN OVER...

Calculate the mass $m(\leq R)$ enclosed within R and deduce the total mass of the disc. [4]

Comment on the fact that $v_c^2(R) \neq G\,m(\leq R)/R$. [2]

[*In cylindrical polar coordinates* (R, ϕ, z),

$$\nabla^2 f = \frac{1}{R}\frac{\partial}{\partial R}\left(R\,\frac{\partial f}{\partial R}\right) + \frac{1}{R^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}.$$]

Question 8Z - Topics in Observational Astrophysics

(i) The observed rate of gamma ray bursts (GRBs) at low redshifts is $0.44\,\mathrm{Gpc}^{-3}\,\mathrm{yr}^{-1}$. The space density of typical galaxies is $3 \times 10^{-3}\,\mathrm{Mpc}^{-3}$. Assuming that a GRB within 2 kpc of the Earth is dangerous (i.e., causes mass extinctions) and that our galaxy is a uniform disc of 30 kpc diameter, estimate the rate of dangerous GRBs. [6]

How many has Earth experienced in its lifetime? [2]

How would the GRBs actually cause mass extinctions? [2]

(ii) A pulsar is observed over a period of time and its spin period P is seen to be increasing. Assuming that the entire gamma-ray luminosity L of the pulsar is driven by the loss of rotational energy, derive an expression for the time derivative $\dot{P}$ in terms of P, L and the pulsar's moment of inertia I. [8]

If $I = KMR^2$ (where K is a constant, M is the mass of the pulsar and R is its radius), derive an expression for the distance of the pulsar in terms of P, $\dot{P}$, K, M, R and F_{obs}, where F_{obs} is the observed gamma-ray flux. [4]

Calculate the pulsar's distance for $M = 2.1\mathrm{M}_\odot$, $K = 0.2$, $R = 10\,\mathrm{km}$, $P = 0.5\,\mathrm{s}$, $\dot{P} = 10^{-15}$ and $F_{\mathrm{obs}} = 2.2 \times 10^{-15}\,\mathrm{W\,m^{-2}}$. [4]

Comment on the plausible values M and R might have and calculate the range of distance to the pulsar that this implies. [2]

What mechanism might be responsible for spinning down the pulsar? [2]

END OF PAPER

UNIVERSITY OF CAMBRIDGE

NST2AS NATURAL SCIENCES TRIPOS Part II

Friday 5 June 2020

ASTROPHYSICS - PAPER 3

Candidates may attempt not more than six questions.

Each question is divided into Part **(i)** *and Part* **(ii)**, *which may or may not be related. Candidates may attempt either or both Parts.*

The number of marks for each question is the same, with Part **(ii)** *of each question carrying twice as many marks as Part* **(i)**. *The approximate number of marks allocated to each component of a question is indicated in the right margin. Additional credit will be given for a substantially complete answer to either Part.*

Write on **one side** *of the paper only and* **begin each answer on a separate sheet.** *Please ensure that your candidate number is written at the top of each sheet and number the pages within each answer.*

Answers must be uploaded to Moodle as separate PDF files, named with your candidate number followed by an underscore and **X, Y, Z**, *according to the letter associated with each question. (For example,* **3X and 6X** *should be in one file named 7850T_X.pdf and* **2Y, 4Y and 7Y** *in another file.) The first page of each such file should be a title page bearing your candidate number, the appropriate letter* **X, Y, Z**, *and a list of the questions (including whether Part* **(i)**, *Part* **(ii)** *or both) attempted.*

A master cover sheet bearing your candidate number and listing all Parts of all questions attempted must also be completed and uploaded as a separate PDF file.

STATIONERY REQUIREMENTS	SPECIAL REQUIREMENTS
Script Paper	*Astrophysics Formulae Booklet*
Master Cover Sheet	*Approved Calculators Allowed*

You should spend three hours working on this paper (plus any pre-agreed individual adjustment). Downloading and uploading times should not be included in the allocated exam time.

Question 1Z - Relativity

(i) The Schwarzschild line element outside a spherical mass M is

$$ds^2 = c^2\left(1 - \frac{2\mu}{r}\right)dt^2 - \left(1 - \frac{2\mu}{r}\right)^{-1}dr^2 - r^2 d\theta^2 - r^2\sin^2\theta d\phi^2\,,$$

where $\mu = GM/c^2$. Show that the wordline of a free, massive particle moving in the equatorial plane ($\theta = \pi/2$) satisfies

$$\left(1 - \frac{2\mu}{r}\right)\dot{t} = k\,,$$
$$r^2\dot{\phi} = h\,,$$
$$c^2\left(1 - \frac{2\mu}{r}\right)\dot{t}^2 - \left(1 - \frac{2\mu}{r}\right)^{-1}\dot{r}^2 - r^2\dot{\phi}^2 = c^2\,,$$

where k and h are constants and overdots denote differentiation with respect to the particle's proper time τ. [8]

Give physical interpretations of the constants k and h. [2]

(ii) Show that circular orbits at coordinate radius r (with $r > 3\mu$) of the Schwarzschild spacetime of Part (i) satisfy

$$\mu c^2 = \frac{h^2}{r}\left(1 - \frac{3\mu}{r}\right) \qquad \text{and} \qquad k = \left(1 - \frac{2\mu}{r}\right)\left(1 - \frac{3\mu}{r}\right)^{-1/2}.$$ [8]

Hence, or otherwise, show that

$$\frac{d\phi}{dt} = \pm\left(\frac{\mu c^2}{r^3}\right)^{1/2},$$

and comment on this result in relation to that in Newtonian theory. [5]

Two particles, each of mass m, travel in oppositely-directed circular orbits in the equatorial plane of the Schwarzschild geometry at radius r. If at some instant the particles collide, show that the total energy available in the zero-momentum frame of the collision is

$$E_{\text{tot}} = 2mc^2\left(\frac{1 - 2\mu/r}{1 - 3\mu/r}\right)^{1/2}.$$ [7]

Question 2Y - Astrophysical Fluid Dynamics

(i) Consider a barotropic steady-state flow in the z-direction through a pipe with a cross-sectional area $A(z)$ that varies along its length. Suppose that the fluid velocity $u_z(z)$ is uniform over the cross-section of the flow. Starting from the momentum equation, and taking care to highlight where each assumption is employed, show that

$$(u^2 - c_s^2)\frac{\partial \ln u}{\partial z} = c_s^2 \frac{\partial \ln A}{\partial z} \,, \qquad (*)$$

where you should define c_s appropriately. [7]

Discuss briefly the implications of $(*)$ for the design of a rocket-engine nozzle. [3]

(ii) Show that the pressure profile of a plane-parallel atmosphere of an ideal isothermal gas in a uniform gravitational field $\mathbf{g} = -g_0\hat{\mathbf{z}}$ is $P = P_0 e^{-z/z_0}$, where $z_0 = c_s^2/g_0$ and c_s is the isothermal sound speed. [3]

Consider a jet of low-density gas launched subsonically upwards from the base of this atmosphere with an injection velocity of u_0. After reaching a steady state, the jet propagates through the atmosphere with a cross-sectional area $A(z)$ and velocity $u(z)$. Making reference to $(*)$ from Part (i), briefly discuss the two possible types of flow that can be established. [3]

Assuming that the jet material remains isothermal with isothermal sound speed $c_{s\mathrm{j}}$, write down Bernoulli's constant for this flow. You can neglect the effect of gravity on the jet dynamics. [2]

Consider the situation whereby this isothermal jet makes a transition from subsonic to supersonic flow. Show that the location of the sonic transition is

$$z_s = \frac{z_0}{2}\left(1 - \frac{u_0^2}{c_{s\mathrm{j}}^2}\right) .$$

As part of your derivation, carefully state the boundary condition at the interface between the jet and the surrounding atmosphere. [8]

Sketch graphs of $u(z)$ and $A(z)$ and highlight the behaviour around $z = z_s$ as well as the behaviour for $z \gg z_0$. [4]

TURN OVER...

Question 3X - Introduction to Cosmology

(i) The Fourier transform of the comoving peculiar velocity field $\mathbf{u_k}(t)$ is related to the Fourier transform of the density field $\delta_\mathbf{k}(t)$ in linear perturbation theory by the equation of continuity

$$\frac{d\delta_\mathbf{k}}{dt} + i\mathbf{k}\cdot\mathbf{u_k} = 0\,.$$

Assume that the velocity field can be expressed as the gradient with respect to comoving coordinates $\mathbf{x}$ of a velocity potential ψ, $\mathbf{u} = \boldsymbol{\nabla}\psi$. By Fourier transforming ψ,

$$\psi = \sum_\mathbf{k} \psi_\mathbf{k} e^{i\mathbf{k}.\mathbf{x}}\,,$$

show that

$$\mathbf{u_k} = \frac{i\mathbf{k}}{|\mathbf{k}|^2} f(t) H \delta_\mathbf{k}\,,$$

where H is the Hubble parameter and $f(t)$ is related to the linear growth rate of perturbations $D(t)$ by

$$f(t) = \frac{R}{D}\frac{dD}{dR}\,,$$

where $R(t)$ is the scale factor. [7]

Discuss briefly how peculiar velocities can be used to test cosmological models. [3]

(ii) Using the results of Part (i), show that the two-point correlation function of the peculiar velocity field at time t,

$$\xi_u(x) = \langle \mathbf{u}(\mathbf{x}') \cdot \mathbf{u}(\mathbf{x}' + \mathbf{x})\rangle\,,$$

where $x = |\mathbf{x}|$, is given by

$$\xi_u(x) = \frac{1}{2\pi^2}\int P_u(k)\frac{\sin kx}{kx} k^2\, dk\,,$$

where $P_u(k)$ is the velocity power spectrum,

$$P_u(k) = \frac{VH^2 f^2(t)\langle|\delta_\mathbf{k}|^2\rangle}{|\mathbf{k}|^2}\,,$$

$k = |\mathbf{k}|$, and the perturbations are assumed to be periodic in a large box of volume V. [11]

Discuss briefly the relevance of this result to the observation of the dipole anisotropy of the cosmic microwave background radiation. [3]

Assuming a power spectrum of fluctuations

$$\langle|\delta_{\mathbf{k}}|^2\rangle \propto k^n \,,$$

show that the root-mean-square bulk velocity within a sphere of radius r_s scales as

$$V_{\rm rms} \propto r_s^{-(1+n)/2} \,. \qquad [6]$$

[*You may assume that for periodic fluctuations in a large box of volume* V*,*

$$\frac{1}{V}\int e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}}\, d^3\mathbf{x} = \delta_{\mathbf{k}\mathbf{k}'}\,, \qquad \sum_{\mathbf{k}} g(\mathbf{k}) = \frac{V}{(2\pi)^3}\int g(\mathbf{k})\, d^3\mathbf{k}\,,$$

for an arbitrary function $g(\mathbf{k})$*.*]

TURN OVER...

Question 4Y - Structure and Evolution of Stars

(i) Assume that the light curve of a supernova (SN) is dominated by the energy released in the radioactive decay of an isotope with decay constant $\lambda = \ln 2/\tau_{1/2}$, where $\tau_{1/2}$ is the isotope half-life. Show that the slope of the light curve is

$$\frac{d \log_{10} L}{dt} = -0.434\lambda \,. \qquad [3]$$

Derive an analogous expression for the change in *bolometric magnitude* with time. [2]

Assume that $0.075 M_\odot$ of $^{56}_{27}$Co is produced in a core-collapse SN explosion. The half-life of $^{56}_{27}$Co is $\tau_{1/2} = 77.7$ days, and the energy released by the decay of one $^{56}_{27}$Co atom is 3.72 MeV. If $^{56}_{27}$Co is the dominant isotope responsible for the SN light curve, estimate the SN luminosity:

(a) immediately after the formation of cobalt; and [3]

(b) one year after the explosion. [2]

(ii) Explain what is meant by the Kelvin–Helmholtz timescale τ_{KH}. [1]

Use the virial theorem to deduce the dependence of τ_{KH} on the stellar mass M, radius R, and luminosity L. [4]

The mass-loss rate in massive stars can be approximated by the expression

$$\dot{M} = \phi \frac{v_{\mathrm{esc}}}{c} \frac{LR}{GM} \,,$$

where v_{esc} is the escape velocity and ϕ is an efficiency factor ($0.1 \lesssim \phi \lesssim 1$). How does the mass-loss timescale, τ_{ml}, compare with τ_{KH}? [You may assume the escape velocity from the stellar surface is $v_{\mathrm{esc}} \simeq (2GM/R)^{1/2}$.] [5]

Show that the rate of energy supply required to sustain a mass-loss rate $\dot{M}$ is a very small fraction of L. [5]

Find the relation between τ_{ml} and the nuclear timescale of the star, τ_{nuc}, and show that for massive stars $\tau_{\mathrm{ml}} < \tau_{\mathrm{nuc}}$. [You may assume that typical values for v_{esc}/c are in the range 10^{-3}–10^{-2}, and you should assume a reasonable value for the nuclear efficiency.] [5]

Question 5Z - Statistical Physics

(i) A box of volume V contains particles with uniform number density n in kinetic equilibrium such that the mean particle speed is $\bar{v}$. Show that the number of particles striking the sides of the box per unit area per unit time is

$$F = \frac{1}{4} n\bar{v}\,. \qquad [5]$$

A small hole of area $A_{\rm hole}$ is made in the box at time $t = 0$, when the number density is n_0. Assuming that a vacuum is maintained outside the box, and the temperature of the gas in the box is maintained constant, find $N(t)$, the total number of particles in the box as a function of time, for $t > 0$. [5]

(ii) Consider an ideal, classical, monatomic gas of N atoms occupying the volume $z > 0$ above an area A. The gas is in thermal equilibrium at temperature T and is in a uniform gravitational field in the z-direction, $\mathbf{g} = -g\hat{\mathbf{z}}$. Write down the partition function and hence determine the mean energy per atom in the gas, taking the zero of the gravitational potential energy to be at $z = 0$. [8]

What is the probability, $p(z)dz$, that an atom is located at a height between z and $z + dz$? [6]

A planet has a radius $R = 6000\,\mathrm{km}$ and an atmosphere of molecular nitrogen with temperature $T = 300\,\mathrm{K}$. By considering the probability density $p(z)$ and making any assumptions and (crude) approximations necessary, estimate the minimum planetary mass required for this nitrogen atmosphere to be retained for at least 100 Myr. [You may assume that the mass of a nitrogen molecule is $28\,m_{\rm u}$.] [6]

TURN OVER...

Question 6X - Principles of Quantum Mechanics

(i) Let $\mathbf{J}$ be angular momentum operators and the state $|j\ m\rangle$ be the normalised eigenstate of $\mathbf{J}^2$ and J_z with quantum numbers j and m, respectively. Define the ladder operators

$$J_{\pm} = J_x \pm iJ_y\,.$$

Show that

$$J_{\pm}|j\ m\rangle = \hbar\sqrt{(j \mp m)(j \pm m + 1)}|j\ m \pm 1\rangle\,.$$ [10]

(ii) Two particles with angular momenta $\mathbf{J}_1$ and $\mathbf{J}_2$ are combined to give a system with total angular momentum $\mathbf{J} = \mathbf{J}_1 + \mathbf{J}_2$. Suppose the particles have angular momentum quantum numbers j_1 and j_2, respectively, with associated normalised eigenstates $|j_1\ m_1\rangle$ and $|j_2\ m_2\rangle$. If the quantum numbers of the total angular momentum $\mathbf{J}$ are j and m, corresponding to normalised eigenstates $|j\ m\rangle$, what are the possible values of j and m? [2]

How can a state with $j = m = j_1 + j_2$ be constructed? [2]

For the combined particle system, when both the angular momenta $j_1 = j_2 = 1$ show that the combined state $|2\ 0\rangle$ is

$$|2\ 0\rangle = \sqrt{\frac{1}{6}}\left(|1\ 1\rangle|1\ -1\rangle + |1\ -1\rangle|1\ 1\rangle\right) + \sqrt{\frac{2}{3}}|1\ 0\rangle|1\ 0\rangle\,.$$ [5]

Find similar expressions for the combined states $|1\ 0\rangle$ and $|0\ 0\rangle$ and determine their symmetry properties under interchange of the two particles. [8]

When the combined system is in the state $|0\ 0\rangle$ what is the probability that measurements of the z-component of angular momentum of either constituent particle returns $\hbar$? [3]

Question 7Y - Stellar Dynamics and the Structure of Galaxies

(i) State what is meant by the distribution function $f(\mathbf{x}, \mathbf{v}, t)$ for a system of stars (where the phase-space variables $\mathbf{x}$ and $\mathbf{v}$ are position and velocity, respectively) and give an interpretation of the collisionless Boltzmann equation. [4]

By taking moments of the collisionless Boltzmann equation, derive the Jeans equations

$$\frac{\partial \nu}{\partial t} + \frac{\partial(\nu\,\overline{v_i})}{\partial x_i} = 0$$

and

$$\frac{\partial}{\partial t}(\nu\,\overline{v_i}) + \frac{\partial(\nu\,\overline{v_i v_j})}{\partial x_i} + \nu\frac{\partial \Phi}{\partial x_j} = 0\,,$$

where Φ is the gravitational potential and

$$\nu = \int f\,d^3\mathbf{v}\,, \qquad \overline{v_i} = \frac{1}{\nu}\int f v_i\,d^3\mathbf{v}\,, \qquad \overline{v_i v_j} = \frac{1}{\nu}\int f v_i v_j\,d^3\mathbf{v}\,.$$ [6]

(ii) A distribution function that gives rise to an anisotropic velocity distribution of a spherically-symmetric stellar system is

$$F(\mathcal{E}, L) = L^{-2\beta}\,f(\mathcal{E})\,,$$

where $\mathcal{E}$ is the relative energy, L is the modulus of the angular momentum vector, f is an arbitrary function and $\beta < 1$. By introducing spherical polar coordinates in velocity space show that in this case the velocity distribution satisfies

$$\overline{v_\theta^2}/\overline{v_r^2} = 1 - \beta\,.$$ [16]

Describe the stellar orbits for the case $\beta = 0$ and in the limits $\beta \to 1$ and $\beta \to -\infty$. [4]

[*You may use that*

$$2\int_0^{\pi/2} \cos^{2p-1}\alpha\,\sin^{2q-1}\alpha\,d\alpha = \sqrt{\pi}\,\Gamma(p)\,\Gamma(q)/\Gamma(p+q)\,,$$

where $\Gamma(x)$ *is the usual Gamma function satisfying* $x\,\Gamma(x) = \Gamma(x+1)$.]

TURN OVER...

Question 8Z - Topics in Observational Astrophysics

(i) A point mass M is at a distance d from the Earth whilst a background star is at a distance $2d$. The small angle between these two objects at the Earth is β. A light ray from the star passes to within a distance h from the point mass and is gravitationally deflected by an angle

$$\alpha = \frac{4GM}{c^2 h}\,.$$

Show that an observer on Earth generally sees two images of the star, which are at apparent angular distances θ_1 and θ_2 from the point mass, where θ_1 and θ_2 are the roots of the equation

$$\theta^2 - \beta\theta - \frac{2GM}{c^2 d} = 0\,. \qquad [10]$$

(ii) According to the unified model of active galactic nuclei (AGN), the central black hole and its associated region of broad-line emitting gas reside within a dusty torus. An observer will, depending on the orientation of the torus with respect to the line of sight, be able to see either the broad line region (and then classify the AGN as a Type I Seyfert) or else will be prevented from doing so by the torus (then classifying the AGN as a Type II Seyfert). The ratio of Type I to Type II Seyferts is 1:4. In Seyfert Is, a burst of continuum emission from close to the black hole is followed by a brightening in the broad emission lines after about a week. The width of the broad lines is commonly ascribed to orbital motion of clouds in the black hole's potential. The width of the CIV line at 154.9 nm is 1 nm in the rest frame. It is found, however, that broad emission lines can be detected in Seyfert IIs when observed in polarised light. The polarised light is interpreted as being emission from the broad line region that has been scattered into the observer's line of sight. Assume that the line widths in these systems are similar to those in Seyfert Is. Use the above information to estimate the following:

(a) the typical opening angle for the dust torus; [4]

(b) the distance of the broad-line emitting region from the black hole; and [4]

(c) the mass of the black hole. [4]

Furthermore, explain:

(d) what, if anything, you can deduce about the orientation of the orbits of the broad-line emitting clouds; and [2]

(e) why, according to this model, Seyfert II AGN have physically longer radio jets on average (as seen in the plane of the sky). [2]

The dusty torus is located just outside the broad-line emitting region and can be modelled as containing spherical silicate grains of radius $1\,\mu$m and density $3 \times 10^3\,\mathrm{kg\,m^{-3}}$, whose cross section for the absorption of radiation is equal to their geometrical cross section. Assuming a gas-to-dust ratio that is typical of the interstellar medium (100:1), determine a lower limit on the torus mass. Would this limit increase or decrease if the grains were smaller? [4]

END OF PAPER

UNIVERSITY OF CAMBRIDGE

NST2AS NATURAL SCIENCES TRIPOS Part II

Monday 8 June 2020

ASTROPHYSICS - PAPER 4

Candidates may attempt not more than six questions.

Each question is divided into Part **(i)** *and Part* **(ii)**, *which may or may not be related. Candidates may attempt either or both Parts.*

The number of marks for each question is the same, with Part **(ii)** *of each question carrying twice as many marks as Part* **(i)**. *The approximate number of marks allocated to each component of a question is indicated in the right margin. Additional credit will be given for a substantially complete answer to either Part.*

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STATIONERY REQUIREMENTS	SPECIAL REQUIREMENTS
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Master Cover Sheet	*Approved Calculators Allowed*

You should spend three hours working on this paper (plus any pre-agreed individual adjustment). Downloading and uploading times should not be included in the allocated exam time.

Question 1Z - Relativity

(i) Consider the static, cylindrically-symmetric spacetime with line element

$$ds^2 = c^2 dt^2 - dr^2 - f^2(r) d\phi^2 - dz^2 \,, \qquad (*)$$

where $x^\mu = (ct, r, \phi, z)$ are cylindrical coordinates with $0 \leq \phi < 2\pi$ and $f(r)$ is some arbitrary function. Show that the spacetime is symmetric under Lorentz boosts along the z-axis. [4]

Given that the only non-zero connection coefficients are

$$\Gamma^r_{\phi\phi} = -ff' \quad \text{and} \quad \Gamma^\phi_{r\phi} = \Gamma^\phi_{\phi r} = f'/f \,,$$

where primes denote differentiation with respect to r, explain why the Riemann curvature tensor has only one independent component (e.g., $R_{r\phi r}{}^{\phi}$) and show that

$$R_{r\phi r}{}^{\phi} = -f''/f \,.$$ [6]

(ii) Use the Einstein field equation to show that the energy–momentum tensor of the matter that generates the spacetime $(*)$ in Part (i) has components of the form

$$T_{\mu\nu} = \mathrm{diag}\left(\rho c^2, 0, 0, -\rho c^2\right) \,,$$

where the (proper) density $\rho(r)$ is determined by

$$\frac{f''}{f} = -\frac{8\pi G}{c^2}\rho \,.$$ [8]

Give a physical interpretation of this form of the energy–momentum tensor. [3]

Suppose now that ρ is constant, with value ρ_0, for $r \leq r_0$, and is zero for $r > r_0$. Assuming that $f(r) \to r$ as $r \to 0$, determine the function $f(r)$ in terms of ρ_0 and r_0 for both $r \leq r_0$ and $r > r_0$. [You may assume that f and f' are continuous at $r = r_0$.] [6]

It can be shown show that in the limit $\sqrt{8\pi G\rho_0} r_0/c \ll 1$, and for $r \gg r_0$, the line element $(*)$ is approximately

$$ds^2 \approx c^2 dt^2 - dr^2 - \left(1 - \frac{8G\mu}{c^2}\right) r^2 d\phi^2 - dz^2 \,,$$

where $\mu = \pi r_0^2 \rho_0$. By considering the coordinate transformation

$$\bar{\phi} = \left(1 - 4G\mu/c^2\right)\phi \,,$$

or otherwise, give a physical interpretation of this exterior line element. [3]

Question 2Y - Astrophysical Fluid Dynamics

(i) Consider a shock front propagating adiabatically in an ideal gas with an equation of state $P = (\gamma - 1)\epsilon$, where P is pressure and ϵ is the internal energy density. Let P_1 and ρ_1 be the pre-shock pressure and density, respectively, and P_2 and ρ_2 the similar post-shock quantities. In the rest frame of the shock, let u_1 be the velocity of the pre-shock gas and u_2 be the velocity of the post-shock gas (both oriented perpendicularly to the shock front). Starting from the basic fluid equations, derive the three Rankine–Hugoniot jump conditions connecting these quantities. [10]

(ii) Now consider a shock front propagating in an isothermal gas such that the (isothermal) sound speed is c_s on each side of the shock. Using the relevant Rankine–Hugoniot jump conditions, show that

$$\frac{\rho_2}{\rho_1} = \frac{P_2}{P_1} = \frac{u_1}{u_2} = \mathcal{M}^2 \, ,$$

where $\mathcal{M} = u_1/c_\mathrm{s}$. [7]

Briefly and qualitatively compare and contrast the density jump in the cases of a strong shock propagating into: (a) a gas with an isothermal equation of state; and (b) a gas with an adiabatic equation of state, as in Part (i). [2]

A spherical cloud with radius $R_c = 10^{16}\,\mathrm{m}$, density $\rho_c = 1 \times 10^{-21}\,\mathrm{kg\,m^{-3}}$ and temperature $T_c = 100\,\mathrm{K}$ is embedded in a uniform patch of the interstellar medium (ISM) with temperature $T_\mathrm{ISM} = 10^4\,\mathrm{K}$. The cloud and the ISM are initially in pressure equilibrium, and the gas in both can be taken to be pure neutral atomic hydrogen with $\gamma = 5/3$. A shock from a nearby supernova passes through the ISM with a velocity $V = 200\,\mathrm{km\,s^{-1}}$. Calculate the velocity of the shock driven into the cloud, and explain why the overall effect is to crush the cloud from all sides. [9]

What will be the final density of the cloud and approximately how long will it take to reach this density? [2]

[*You may assume without proof that the Rankine–Hugoniot jump conditions for a strong adiabatic shock propagating into a gas with a γ-law equation of state give*

$$\frac{\rho_2}{\rho_1} \approx \frac{\gamma+1}{\gamma-1} \, , \qquad \textit{and} \qquad \frac{P_2}{P_1} \approx \frac{2\gamma}{\gamma+1}\mathcal{M}_1^2 \, ,$$

where $\mathcal{M}_1 = u_1/c_\mathrm{s1} \gg 1$ and c_s1 is the sound speed in the pre-shock gas.]

TURN OVER...

Question 3X - Introduction to Cosmology

(i) The distribution of hydrogen ions in the hot atmosphere of a rich cluster of galaxies can be approximated as

$$n_p(r) = \frac{n_0}{(1+r/r_0)^2}\,,$$

with $n_0 = 5\times10^4\,\mathrm{m}^{-3}$ and $r_0 = 20\,\mathrm{kpc}$. The hot atmosphere has a temperature of $T = 4\times10^7\,\mathrm{K}$ and the energy-loss rate per unit volume via radiative cooling can be approximated as

$$\lambda \approx \left(3\times10^{-40}\,\mathrm{J\,K^{-1/2}\,m^3\,s^{-1}}\right) T^{1/2}[n_p(r)]^2\,.$$

Estimate the radius $r_{\rm cool}$ at which the cooling time is less than the Hubble time $t_0 = 13.8\,\mathrm{Gyr}$. [You may assume that the cluster atmosphere is composed of fully-ionized hydrogen.] [5]

Estimate the baryonic mass in units of $\mathrm{M}_\odot$ contained within $r_{\rm cool}$. [3]

Briefly describe the observational evidence that suggests such large gas cooling rates do not actually occur in the cores of galaxy clusters. What physical processes might prevent the gas from cooling? [2]

[*You may assume that*

$$\int_0^y \frac{x^2}{(1+x)^2}\,dx = y - 2\ln(1+y) + \frac{y}{1+y}\,.\Big]$$

(ii) The mean comoving density in black holes can be related to the quasar luminosity function by

$$\rho_{\rm BH} = \frac{1}{\epsilon c^2}\int \mathcal{L}\phi_Q(\mathcal{L},t)\,d\mathcal{L}dt\,, \qquad (*)$$

where $\phi_Q(\mathcal{L},t)d\mathcal{L}$ is the comoving space density of quasars at time t with luminosities in the range $(\mathcal{L},\mathcal{L}+d\mathcal{L})$ and ϵ is the efficiency of conversion of the mass accreted by black holes into radiation. The luminosity of a quasar at redshift z is related to the observed bolometric flux, S, by $\mathcal{L} = 4\pi D_L(z)^2 S$, where $D_L(z)$ is the luminosity distance. Show that in a Friedmann–Robertson–Walker universe $(*)$ can be written as

$$\rho_{\rm BH} = \frac{4\pi}{\epsilon c^3}\int S\mathcal{N}_Q(S,z)(1+z)\,dSdz\,, \qquad (**)$$

where $\mathcal{N}_Q(S,z)dSdz$ is the number of quasars per unit solid angle within the flux range $(S,S+dS)$ and redshift range $(z,z+dz)$. [Take the scale factor to be unity at the present.] [10]

Counts of optically-selected quasars suggest the following model:

$$\mathcal{N}_Q(S,z)dSdz = f(S)\delta(z-z_c)\frac{dS}{S_*}dz\,,$$

with

$$f(S) = \begin{cases} A(S_*/S)^{\beta_1+1} & S \le S_*\,, \\ A(S_*/S)^{\beta_2+1} & S > S_*\,, \end{cases}$$

where $z_c = 2$, $\beta_1 = 0.75$, $\beta_2 = 2.12$ and $AS_* = 3\times 10^{-11}\,\mathrm{W\,m^{-2}\,sr^{-1}}$. Show by evaluating (∗∗) that the mass density in black holes is

$$\rho_{\mathrm{BH}} \approx 3\times 10^{13}\left(\frac{0.1}{\epsilon}\right)\mathrm{M_\odot\,Gpc^{-3}}\,.$$ [6]

Comment on this result and discuss briefly its possible relevance for galaxy formation. [4]

TURN OVER...

Question 4Y - Structure and Evolution of Stars

(i) Explain what is meant by the Strömgren sphere around hot stars and derive an expression for its radius $r_{\rm strom}$ in terms of the number of ionizing photons emitted by the star per unit time, Q_*, the number density n of the ambient interstellar medium (ISM) and the (temperature-dependent) recombination coefficient $\alpha(T)$. [4]

Stars A and B have the same effective temperature, but the radius of star A is twice that of star B. Given that the ambient density of the ISM around star A is twice that of the ISM around star B, would you expect the Strömgren sphere of star A to be larger or smaller than that of star B? [3]

Use simple physical arguments to explain why in photographs that represent colours in the same way as the human eye does, regions of on-going star formation generally appear to have a red colour. [3]

(ii) If energy transport within a star is by radiative diffusion, the luminosity $L(r)$ at some radius r within the star can be written as

$$L(r) = -4\pi r^2 \, \frac{16\sigma}{3} \, \frac{T^3(r)}{\rho(r)\kappa(r)} \, \frac{dT(r)}{dr} \, ,$$

where ρ is the density, T is the temperature, σ is the Stefan–Boltzmann constant, and the opacity κ is here assumed to be given by

$$\kappa(r) \propto \rho(r) T^{-3.5}(r) \, .$$

From these two equations show, using homology arguments, that as a pre-main-sequence star contracts, its luminosity changes with temperature according to the relation

$$L \propto T_{\rm eff}^{4/5} \, . \qquad [14]$$

The path taken by a contracting star as it approaches the main sequence in the H-R diagram is called the Henyey track. Detailed computer calculations show that $L \propto T_{\rm eff}^{4/5}$ is a satisfactory approximation of Henyey tracks of massive stars but becomes a progressively poorer fit to the tracks of stars with masses $M \lesssim 1.5 {\rm M}_\odot$. What conclusions can you draw from this statement? [6]

Question 5Z - Statistical Physics

(i) The pressure, P, entropy, S, internal energy, E, and Helmholtz free energy, $F = E - TS$, of a fixed mass of gas can be considered as functions of the volume, V, and temperature, T. Prove the following thermodynamic identities: [10]

(a)
$$P = -\left.\frac{\partial F}{\partial V}\right|_T ;$$

(b)
$$\left.\frac{\partial S}{\partial V}\right|_T = \left.\frac{\partial P}{\partial T}\right|_V ;$$

(c)
$$\left.\frac{\partial E}{\partial V}\right|_T = T\left.\frac{\partial P}{\partial T}\right|_V - P \, .$$

(ii) Define the heat capacity at constant volume, C_V. [2]

Using results from Part (i), show that for an adiabatic change

$$C_V dT + T \left.\frac{\partial P}{\partial T}\right|_V dV = 0 \, .$$ [4]

An imperfect gas of N atoms in a volume V obeys the van der Waals equation of state

$$\left(P + \frac{aN^2}{V^2}\right)(V - Nb) = Nk_{\mathrm{B}}T \, ,$$

where a and b are positive constants. Give brief explanations of the physical origins of the terms involving a and b. [4]

For this gas, show that

$$\left.\frac{\partial C_V}{\partial V}\right|_T = 0 \, ,$$

and deduce that C_V is a function of T alone. [5]

It can further be shown that C_V is independent of T. Given this, show that during an adiabatic expansion of the imperfect gas

$$T(V - Nb)^{Nk_{\mathrm{B}}/C_V} = \text{const.}$$ [5]

TURN OVER...

Question 6X - Principles of Quantum Mechanics

(i) Briefly explain what is meant by the *density operator* ρ. [2]

A one-dimensional harmonic oscillator of mass μ with Hamiltonian $H = p^2/(2\mu) + \mu\omega^2 x^2/2$, where x and p are the position and momentum operators, respectively, is described by the density operator

$$\rho_\beta = \frac{e^{-\beta H}}{\mathrm{Tr}(e^{-\beta H})} \,,$$

for some constant $\beta > 0$. Find the expectation value of H, expressing your result in terms of the oscillation frequency ω and β. [6]

Give a physical interpretation of ρ_β. [2]

(ii) Let $\mathbf{J}$ be angular momentum operators and $|j\, m\rangle$ the normalised eigenstates of $\mathbf{J}^2$ and J_3 with eigenvalues $j(j+1)\hbar^2$ and $m\hbar$, respectively. Consider the unitary operator $U(\theta) = e^{-i\theta J_2/\hbar}$. By considering derivatives of $U(\theta)J_iU^{-1}(\theta)$, or otherwise, show that [8]

$$\begin{aligned} U(\theta)J_1U^{-1}(\theta) &= J_1\cos\theta - J_3\sin\theta \,, \\ U(\theta)J_3U^{-1}(\theta) &= J_1\sin\theta + J_3\cos\theta \,. \end{aligned}$$

Give a physical interpretation of the operator $U(\theta)$. [2]

Show that the state $U(\pi/2)|j\, m\rangle$ is an eigenstate of $\mathbf{J}^2$ and J_1 and give the eigenvalues. [3]

When acting on $j = 1/2$ states, show that

$$U(\pi/2) = \frac{1}{\sqrt{2}}\left(I - i\frac{2J_2}{\hbar}\right) ,$$

where I is the identity operator. [5]

Hence construct the $j = 1/2$ eigenstates of J_1 in terms of those of J_3. [2]

[*You may assume the following action of the ladder operators, $J_\pm = J_1 \pm iJ_2$:*

$$J_\pm|j\, m\rangle = \hbar\sqrt{(j \mp m)(j \pm m + 1)}|j\, m \pm 1\rangle \,.$$]

Question 7Y - Stellar Dynamics and the Structure of Galaxies

(i) A particle orbits in a spherically-symmetric gravitational potential $\Phi(r)$ with specific angular momentum h. Using polar coordinates (r, ϕ) in the orbital plane, the equation of motion is

$$\frac{d^2u}{d\phi^2} + u = -\frac{1}{h^2}\frac{d\Phi}{du}\,,$$

where $u = 1/r$. Consider the potential

$$\Phi(r) = -GM\left(\frac{1}{r} + \frac{a}{r^2}\right),$$

where M and a are constants. Show that

$$u = C\cos\left(\frac{\phi - \phi_0}{K}\right) + \frac{GMK^2}{h^2}\,,$$

where C, K and ϕ_0 are constants, is a solution of the equation of motion and determine the constant K. [6]

Comment on the shape of the orbits. [1]

Explain the term integral of motion and list five integrals of motion of orbits in this potential. [3]

(ii) The distribution function $f(\mathbf{x}, \mathbf{v})$ for a steady-state stellar system with spherical symmetry, and total mass M, is given by

$$f(\mathcal{E}) = \begin{cases} A\,\mathcal{E}^{7/2} & \mathcal{E} > 0\,, \\ 0 & \mathcal{E} \le 0\,, \end{cases}$$

where the relative energy $\mathcal{E} = \Psi(r) - v^2/2$ (with $v = |\mathbf{v}|$ the particle speed and $r = |\mathbf{x}|$ the radial distance), $\Psi(r)$ is the relative gravitational potential with $\Psi(\infty) = 0$ and A is a constant. Show that the number density of stars is

$$\nu(r) = \frac{8\sqrt{2}}{9}\,\pi\, I_{10}\, A\, \Psi^5(r)\,,$$

where $I_n = \int_0^{\pi/2} \sin^n\theta\, d\theta$. [You may wish to use $I_{n+1} = nI_{n-1}/(n+1)$.] [9]

Hence write down Poisson's equation for this system and show that a solution is

$$\Psi(r) = \frac{\beta\,\Psi_0}{\sqrt{\beta^2 + r^2}}\,,$$

where Ψ_0 and β are constants. [6]

Calculate the mass M in terms of Ψ_0 and β. [5]

TURN OVER...

Question 8Z - Topics in Observational Astrophysics

(i) Describe and contrast the two observational techniques that have been used to detect most of the currently-known extrasolar planets and discuss their selection biases. [4]

A star's observed radial velocity, $V_{\rm star}$, shows a periodic variation of $\pm 1\,{\rm m\,s^{-1}}$, suggesting the presence of an orbiting planet. Estimate the mass of the planet for a Solar-mass star, and an orbital period of 1 year. [3]

Under what assumption would this be the true mass and not a lower limit? What other practical measurement would allow the true mass of the planet to be estimated? [3]

(ii) An extrasolar planet orbits the star 51 Peg. The star has a surface temperature of $T = 5700\,{\rm K}$ and a radius $R = 1.4\,{\rm R}_\odot$. The planet is in a circular orbit with radius $r = 0.05\,{\rm AU}$ and has a mass obeying the constraint

$$M \sin i = 0.46 M_{\rm Jup} ,$$

where i is the inclination angle of the planet's orbital axis to the line of sight and $M_{\rm Jup}$ is the mass of Jupiter. Estimate the temperature of the planet assuming that it absorbs all the radiation incident on its atmosphere. Clearly state any assumption that you have made concerning the rotation of the planet. [10]

Assuming the planet has the same radius as Jupiter, compute a lower limit to the escape velocity, $v_{\rm esc}$, from its surface. [4]

Compute the thermal velocity, $v_{\rm th}$, of atoms in the planet's atmosphere. [4]

Can gas typically overcome the gravitational field of the planet and escape? [2]

[*The mass of Jupiter is* $M_{\rm Jup} = 2 \times 10^{27}\,{\rm kg}$ *and its average density is* $\rho = 1300\,{\rm kg\,m^{-3}}$.]

END OF PAPER