

Tuning and Synthesis of 1DF IMC for Uncertain Processes

Objectives of the Chapter

- Introduce the concept of process uncertainty and explore its effect on IMC system stability and performance.
- Present a tuning method for adjusting the IMC filter time constant that accomplishes a desired relative stability for all processes in a predefined uncertainty set.
- Explore the effect of uncertainty on controller design and model selection.

Prerequisite Reading

Chapter 3, “One-Degree of Freedom Internal Model Control”

Appendix A, “Review of Basic Concepts”

Appendix B, “Frequency Response Analysis”

7.1 INTRODUCTION

Chapter 3 discusses the design and tuning of a linear IMC controller when the linear model used in the IMC system is assumed to be a perfect representation of the process. This chapter treats the realistic situation that the process model is not the same as the process. Generally, the greatest contribution to the mismatch between the model and the process in a linear IMC system is the fact that the model and controller are linear while the process is nonlinear and time varying. While it is possible to design nonlinear IMC systems (Kravaris and Kantor, 1990), such control systems are not yet widely used in industry because the improved process performance over a *well-designed and well-tuned* linear control system does not usually justify the time and expense necessary to design and maintain a nonlinear control system. Of course, the key question is, how does one achieve a well-designed and well-tuned linear IMC system when the actual process is nonlinear? In order to accomplish this objective, we will approximate the nonlinear process as a set of linear processes with constant coefficients. Because the process is nonlinear, the parameters of the local, linear descriptions of the process change over time due to changes in operating point. The process operating point changes due to both external disturbances, such as changes in feed composition and ambient conditions, and internal changes, such as heat exchanger fouling and catalyst aging. Approximating the process as a set of linear processes with constant coefficients ignores the behavior of the process during parameter changes and focuses instead on the behavior of the process about all its steady-state operating points. This approximation is useful in that it allows us to use the powerful tools of linear mathematics to carry out control system analysis and design. Such an approximation is reasonable provided that the disturbances are such that the process spends most of the time operating about steady states rather than moving from one steady state to another.

Control system responses to setpoint changes and disturbances change as the local description of the process changes. Therefore, in tuning a control system, one must consider the entire range of possible responses rather than focusing on a single response. Figure 7.1 shows a typical range of responses of a well-tuned control system to step setpoint changes for a linear process at different operating points (i.e., with different values for the local process parameters). Based on such a range of responses, we can qualitatively define IMC controller tuning and synthesis objectives. Our IMC tuning objective is to select the smallest IMC filter time constant for which no setpoint response overshoots the setpoint by more than a specified amount and no response becomes too oscillatory. Our IMC controller synthesis objective is to choose both the IMC controller and the process model so as to speed up the slowest closed loop responses as much as possible without violating our overshoot and relative stability (i.e., not too oscillatory) objectives.

It is quite difficult to make the above qualitative time domain tuning and synthesis objectives sufficiently precise so as to be useful in obtaining numerical values for the IMC controller filter time constant, and model parameters. To illustrate one of the difficulties, notice that it is hard to select the slowest response in Figure 7.1. Curve 3 is the slowest response up to 20 time units; curve 4 yields the slowest response between 20 time units and 50 time units. After 50 time units, both curves approach steady state at the same rate.

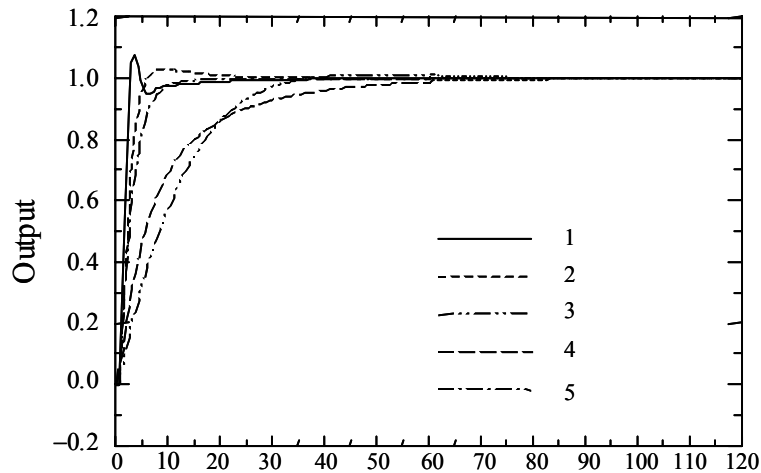


Figure 7.1 Responses to a step setpoint change at different operating points.

It turns out to be much easier to develop quantitative controller tuning and synthesis objectives and procedures for achieving such objectives in the frequency domain (i.e., in the domain of open-loop and closed-loop frequency responses). There is substantial literature on H_∞ frequency domain methods for the analysis and synthesis of control systems for processes described by sets of linear, constant coefficient systems. Kwakernaak (1993) gives a good, relatively brief overview of these methods for single-input single-output (SISO) systems. The texts by Morari and Zafiriou (1989), Doyle et al. (1992), and Dorato et al. (1992) provide more complete expositions. Unfortunately, all of the aforementioned texts require substantial expertise from the reader in order to understand and apply the H_∞ methods presented. In addition, the methods of these authors usually require various approximations before they can be applied to typical chemical process descriptions. For example, dead times must be replaced with finite dimensional approximations (e.g., Padé approximations). We have elected to present a related but simpler approach to tuning and synthesis, which we call Mp tuning and synthesis. Mp tuning aims to find the smallest IMC filter time constant that assures that (1) the magnitude of all closed-loop frequency responses between output and setpoint have magnitudes less than a specified value (usually taken as 1.05) at any frequency, and (2) any oscillations in the curve of the maximum magnitude of all closed-loop frequency responses do not have peaks higher than specified (usually about 0.1) from the highest adjoining valley. As we will discuss later, a control system tuned in this manner will usually satisfy the qualitative time domain tuning objectives discussed previously. Also, a by-product of the Mp tuning procedure is an estimate of the speed of response of the fastest and slowest responses of the control system for all processes in the set of possible processes (i.e., in the uncertainty set).

The next section discusses various process uncertainty descriptions. Section 7.3, which follows, presents the M_p tuning algorithm, describes a method for mitigating the fact that our uncertainty descriptions are themselves not known very precisely, and describes an inverse tuning algorithm that finds an uncertainty region over which a given controller will perform as specified. The two sections following M_p tuning provide justification for the tuning algorithm. Section 7.4 gives the conditions under which any M_p specification greater than one is achievable. Section 7.5 discusses the theoretically important property of *robust stability*. Robust stability means that the control system is stable for all processes in the uncertainty set. Any practical control system must be robustly stable. Since the conditions imposed on the process, model, and controller in order to safely apply the M_p tuning algorithm will automatically be met in most practical situations, those readers interested mainly in applications can skim Sections 7.4 and 7.5, paying attention only to Table 7.3, which limits selection of the model gain in order to be able to achieve M_p specifications arbitrarily close to one.

M_p synthesis, in Section 7.6, addresses the question of what controller and model to choose for the IMC system when the process can be any in the uncertainty set. The criterion that we select for choosing the IMC controller and model is that they speed up the slowest closed-loop responses as much as possible. It turns out that process uncertainty has a profound influence on both controller design and model selection. An important observation in this regard is that the traditional engineering approach of fitting a first-order plus dead time model to high-order overdamped processes, and designing the controller based on that model, often yields a control system that *performs better* than a system *based on a process model of the correct order* when other process parameters such as gain and dead time are sufficiently uncertain.

An important application of M_p tuning and synthesis applied to IMC systems is to convert the resulting IMC controller into an equivalent PID controller, using the methods described in Chapter 6. PID controllers are by far the most widely used industrial control systems and are likely to remain so for the foreseeable future. The IMCTUNE software provided with this text permits the user to automatically obtain PID parameters from the tuned IMC controller.

Another potentially important application of M_p tuning and synthesis is to determine the limits of linear, fixed parameter, control system performance. Such limits determine what incentive, if any, exists for the implementation of more complex nonlinear and adaptive control systems (Åström, 1995; Kravaris and Kantor, 1990; Seborg et al., 1989; Ljung 1987). The aim of such control systems is to improve the speed and quality of the control system response by substantially reducing process/model mismatch and by basing controller design and parameters on a nonlinear or an updated linear model. Such approaches have yet to be widely applied in the process industries, probably because the perceived benefits do not yet justify the added complexity. Further, even nonlinear and adaptive models are approximations to the actual process, and uncertainty in the process parameters will still need to be accounted for in the control system design and tuning. However, a discussion of the tuning of nonlinear and adaptive controllers to accommodate model uncertainty is beyond the scope of this text.

7.2 PROCESS UNCERTAINTY DESCRIPTIONS

The tuning and synthesis techniques in the following sections require frequency domain descriptions of the process uncertainty set. There are two convenient descriptions of the uncertainty set: (1) bounds on transfer function parameters and (2) bounds on the gain and phase of transfer functions over all frequencies of interest. Unfortunately, neither of these uncertainty descriptions is readily available from process data or from first principles. However, process engineers can often estimate ranges for the parameters of simple process models based on a combination of process observations and an understanding of how the process operates. Further, the uncertainty inherent in such estimates can be at least partially accounted for using multiple uncertainty regions, as discussed in Section 7.2.2. Gain and phase bounds as functions of frequency are usually more difficult for plant operating personnel to estimate. However, with some effort, such bounds can be obtained from input-output tests on the plant at different operating points. One advantage of gain and phase bounds is that they do not require a priori postulation of a model. While uncertainty bounds cannot be obtained with precision, they are nonetheless useful for obtaining safe controller tunings and in improving controller performance.

7.2.1 Parametric Uncertainty

The set of transfer functions with uncertain parameters, Π , is defined as

- Π = the set of all transfer functions, $p(s, \beta(\alpha))$, with the vector of parameters α lying in the set S_α .
- S_α = set of all vectors α with $\underline{\alpha}_i \leq \alpha_i \leq \bar{\alpha}_i$
- where $\bar{\alpha}_i$ = upper-bound on the parameter α_i
- $\underline{\alpha}_i$ = lower-bound on the parameter α_i
- $\beta(\alpha)$ = a vector of parameters which are continuous functions of the vector α .

In addition, we will also require that

$$p(0, \beta(\alpha)) > 0 \quad \text{or} \quad < 0 \quad \text{for all } \alpha \in S_\alpha. \quad (7.1)$$

The restriction given by Eq. (7.1) means that all process gains in the uncertainty set Π have the same sign. Such processes are called integral controllable because the restriction given by Eq. (7.1) is a necessary condition for a no offset IMC controller¹ to be stable for any process in Π . Integral controllability is discussed more completely in Section 7.4.1. Typical examples of parametric uncertainty descriptions follow.

¹ Recall that IMC controller has no offset if $q(0) = \tilde{p}^{-1}(0)$.

Example 7.1 A FOPDT Process with Uncorrelated Uncertainty

$$p(s) = \frac{K e^{-\tau s}}{\tau s + 1}, \quad \underline{K} \leq K \leq \bar{K}, \quad \underline{\tau} \leq \tau \leq \bar{\tau}, \quad \underline{T} \leq T \leq \bar{T}, \quad (7.2)$$

where $\underline{\sigma} \equiv$ the lower-bound on any parameter, σ ,
 $\bar{\sigma} \equiv$ the upper-bound on any parameter, σ .

◆

In Example 7.1 all three parameters vary independently. However, it quite often happens that one parameter depends on another, in Example 7.2.

Example 7.2 A FOPDT Process with Correlated Uncertainty

$$p(s) = \frac{K e^{-\tau s}}{\tau s + 1}, \quad \underline{K} \leq K \leq \bar{K}, \quad T(K) = \underline{T} + 2(K - \underline{K}) \quad (7.3)$$

◆

In Example 7.2 there is only one uncertain parameter, K . The function $\beta(\alpha)$ in the definition of the uncertainty set Π was included to accommodate correlated uncertainty such as that in the example. A common source of correlation between the parameters of a transfer function is the use of first principles modeling to obtain the transfer function. Consider Example 7.3.

Example 7.3 The Two-Tank Process

For Figure 7.2, we wish to obtain the transfer function between changes in inflow (Δq_i) and changes in the level of fluid in tank 1 (Δh_1). The differential equations that relate the flows and levels are

$$\begin{aligned} A_1 \frac{dh_1(t)}{dt} &= q_i - q_1; & q_1 &= C_v \sqrt{h_1 - h_2}, \\ A_2 \frac{dh_2(t)}{dt} &= q_1 - q_2; & q_2 &= C_v \sqrt{h_2}. \end{aligned} \quad (7.4)$$

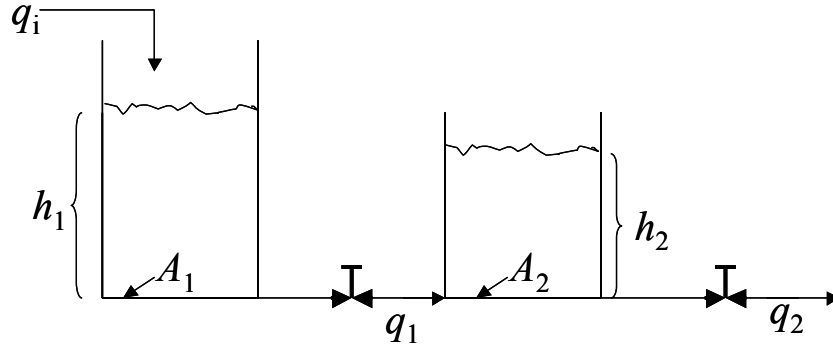


Figure 7.2 A two-tank process.

Linearizing the equations about the steady-state defined by \bar{q}_i , and solving for the desired transfer function gives

$$\frac{\Delta h_1(s)}{\Delta q_i(s)} = \frac{K(\tau_d s + 1)}{(\tau^2 s^2 + 2\zeta\tau s + 1)}, \quad (7.5)$$

where

$$\begin{aligned} K &= 4\bar{q}_i / C_V^2; & \tau_d &= A_2\bar{q}_i / C_V^2 \\ \tau &= 2(\sqrt{A_1 A_2})\bar{q}_i / C_V^2; & \zeta &= (2A_1 + A_2) / (2\sqrt{A_1 A_2}). \end{aligned}$$

Notice that of the four parameters in Eq. (7.5), three are uncertain and all three depend on only one uncertain parameter, \bar{q}_i . Once we have an estimate of the range of variation of \bar{q}_i , the uncertain process is completely specified. ♦

7.2.2 Frequency Domain Uncertainty Bounds

Most methods for treating process uncertainty accommodate frequency domain uncertainty bounds much more readily than parametric uncertainty bounds (Morari and Zafiriou, 1989; Doyle et al., 1992; Dorato et al., 1992). The general form of frequency domain uncertainty bounds for an uncertain process, $p(s)$, is given by

$$\underline{M}(\omega) \leq |p(i\omega)| \leq \bar{M}(\omega), \quad (7.6a)$$

$$\underline{\phi}(\omega) \leq \text{Angle } p(i\omega) \leq \bar{\phi}(\omega), \quad (7.6b)$$

where $\overline{M}(\omega)$ and $\underline{M}(\omega)$ = upper and lower magnitude bounds on $p(i\omega)$,
 $\overline{\phi}(\omega)$ and $\underline{\phi}(\omega)$ = upper and lower phase bounds the angle of $p(i\omega)$.

Values for the upper and lower magnitude and phase bounds as functions of frequency can be obtained by multiple identifications of the process frequency response at different operating points. More commonly, the bounds are either estimated based on experience or obtained from parametric uncertainty bounds. However, frequency response bounds obtained from parametric bounds are always more conservative uncertainty descriptions than the original parametric bounds, and therefore lead to more sluggish control system designs. To see why this is so, consider the following first-order process with an uncertain time constant:

$$p(s) = 1/(\tau s + 1); \quad 1 \leq \tau \leq 5. \quad (7.7)$$

The frequency response of Eq. (7.7) is

$$|p(i\omega)| = (\tau^2 \omega^2 + 1)^{-1/2}, \quad (7.8a)$$

$$\text{Angle } p(i\omega) = -\tan^{-1} \tau \omega. \quad (7.8b)$$

The set of all possible gain and phases of $p(i\omega)$ as the time constant τ ranges between one and five are given by the shaded areas of Figure 7.3, which are obtained from the maximum and minimum of Equations (7.8a) and (7.8b) and are given by Eq. (7.9).

$$\overline{M}(\omega) = (\omega^2 + 1)^{-1/2}, \quad (7.9a)$$

$$\underline{M}(\omega) = (25\omega^2 + 1)^{-1/2}, \quad (7.9b)$$

$$\overline{\phi}(\omega) = -\tan^{-1} \omega, \quad (7.9c)$$

$$\underline{\phi}(\omega) = -\tan^{-1} 5\omega. \quad (7.9d)$$

The uncertainty bounds given by Eq. (7.9) are not the same as those given by Eq. (7.8) because the bounds given by Eq. (7.9) allow the magnitude and phase of $p(i\omega)$ to vary independently, whereas for the process given by Eq. (7.7), the magnitude and phase of $p(i\omega)$ are related through Eq. (7.8). For example, at a frequency of one radian/unit time, Eq. (7.9) shows that the magnitude can vary between .196 and .707, while the phase can take on any value between -45.0° and -78.7° (see Fig. 7.3). According to Eq. (7.8), when the magnitude is $1/(\tau^2 + 1)^{1/2}$, the phase is $-\tan^{-1} \tau$ (e.g., if $\tau = 1$ the magnitude is .707 and the phase is -45.0°). Therefore, the magnitude and phase bounds given by Eq. (7.9) describe more processes than those given by Eq. (7.8). That is, the uncertainty set given by Eq. (7.9) is larger than that given by Eq. (7.8). As we shall see in Section 7.3.2, the larger the uncertainty set, the more sluggish the controller must be in order to meet closed-loop specifications. Therefore, a controller designed using the gain and phase bounds given by

Eq. (7.9) generally will be more sluggish than a controller designed using the original parametric bounds.

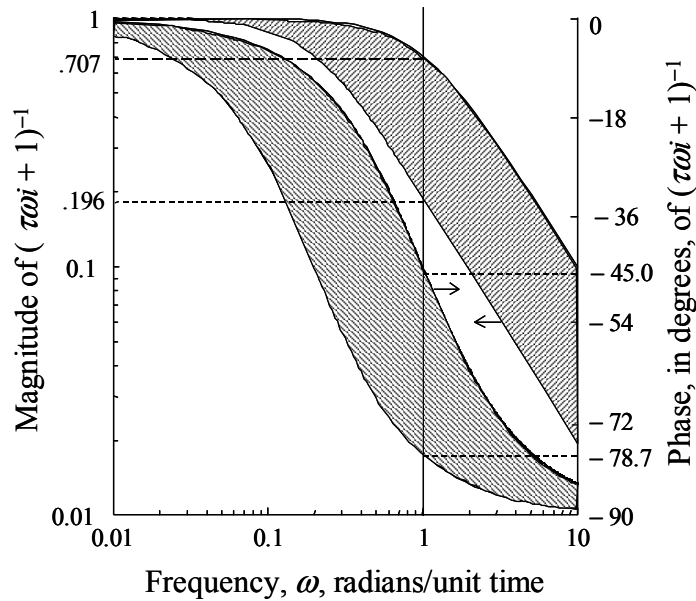


Figure 7.3 Frequency response of $1/(\tau s + 1)$ for $1 \leq \tau \leq 5$.

Many of the uncertainty descriptions in the literature make use of magnitude only uncertainty bounds as given by Eq. (7.6a). In such cases the implicit assumption is that the phase of the uncertain process can be anywhere within $\pm 360^\circ$. The advantage of such descriptions is that they sometimes lead to convex optimization problems for tuning or designing the controller. The disadvantage is that the resulting uncertainty set is even larger than that using gain and phase bounds, and therefore the final control system is likely to be significantly more sluggish than needed for uncertainty due only to parametric variations.

Because parametric uncertainty bounds generally lead to the least conservative controller tunings, and are usually the easiest to obtain, the next section deals only with parametric uncertainty. However, the methodology (but not the IMCTUNE software) also treats frequency domain bounds, should these be useful in particular situations.

7.3 MP TUNING

7.3.1 The Problem Statement

The aim in tuning any control system is to achieve desirable time domain closed-loop performance, such performance being measured by the speed of response, how oscillatory it is, and how much the response overshoots the setpoint. One can estimate such time domain performance measures most easily from the closed-loop frequency responses between output and setpoint. This frequency response is often called the complementary sensitivity function. The maximum magnitude of the complementary sensitivity function, which we will refer to as the M_p , generally gives a good indication of the overshoot to setpoint changes and/or the magnitude of oscillations in the time response. The frequency at which the maximum occurs is generally a good indication of frequency of the time domain oscillations. Finally, the inverse of the “break frequency” is a good estimate of the time constant of the fastest time domain response. The “break frequency” is usually taken as the intersection of the asymptote to the high-frequency portion of the frequency response with a horizontal line of magnitude one. This definition assumes a closed-loop gain of one (i.e., an integral control system). The justification for the foregoing statements is that the magnitude of the *closed-loop* frequency responses between output and setpoint for most control systems can be reasonably approximated by the magnitude of the frequency response of a second-order system of the form $1/(\tau^2 s^2 + 2\zeta\tau s + 1)$. For such a second-order system, the fractional overshoot to step inputs can be directly related to the maximum peak of the magnitude of its frequency response through the relationship:

$$OS = e^{-\pi M_p(1 - (1 - 1/M_p^2)^{.5})}, \quad M_p > 1, \quad (7.10)$$

where

$OS \equiv$ fractional overshoot

\equiv (maximum change in output – change in setpoint)/change in setpoint.

$M_p \equiv$ maximum magnitude of the frequency response.

The damping ratio ζ of the second-order response is related to the M_p by

$$\zeta = ((1 - (1 - 1/M_p^2)^{.5})/2)^{.5}. \quad (7.11)$$

Figures 7.4a and 7.4b show the time and frequency responses for $1/(\tau^2 s^2 + 2\zeta\tau s + 1)$ with $\tau = 1$ and $\zeta = .5$. Note that the unity gain arises because we are generally dealing with closed-loop systems with no offset.

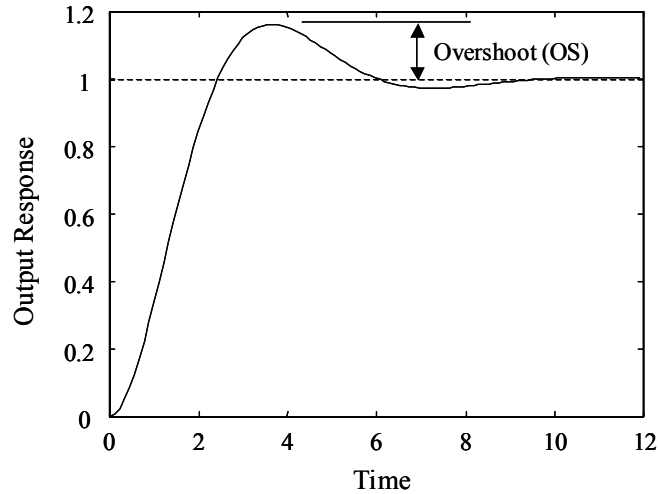


Figure 7.4a Response of $1/(s^2 + s + 1)$ to a unit step setpoint change.

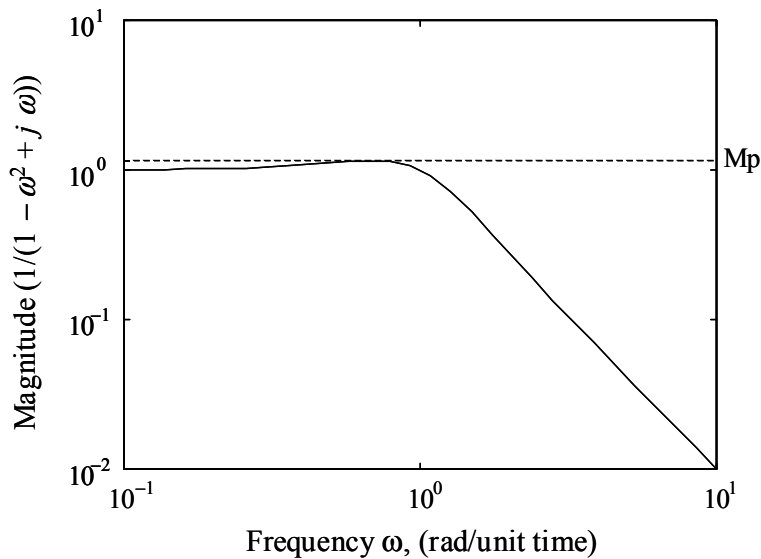


Figure 7.4b Frequency response of $1/(s^2 + s + 1)$.

The overshoot (OS) and damping ratio ζ are plotted versus Mp in Figure 7.5. As can be seen from Figure 7.5, an Mp of 1.05 corresponds to a 10% overshoot and a damping ratio of about 0.6. Higher values of Mp yield greater overshoots and lower damping ratios.

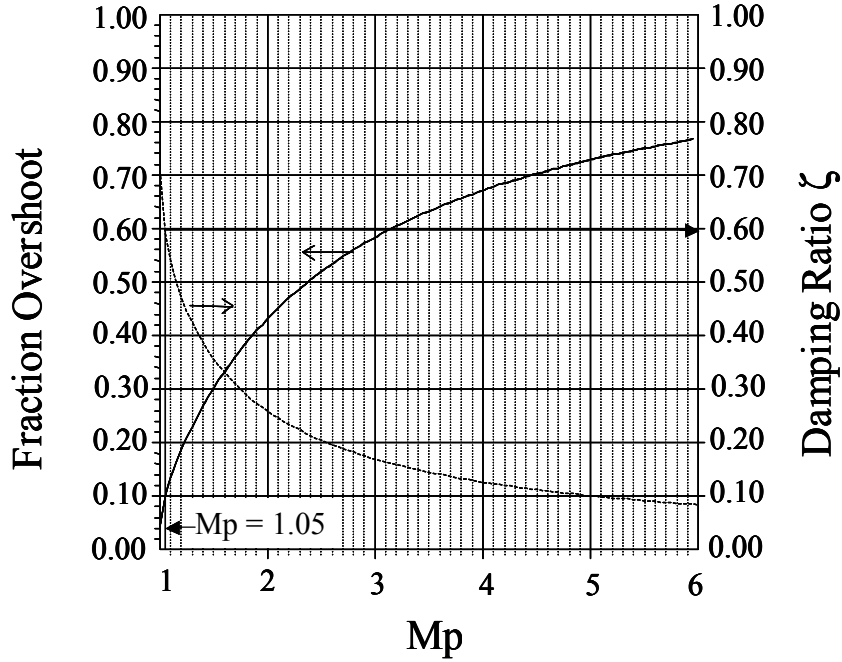


Figure 7.5 Overshoot and damping ratio versus M_p for $1/(s^2 + 2\zeta s + 1)$.

The M_p tuning problem is to find the smallest IMC filter time constant that assures that no closed-loop frequency response, from setpoint to output, will have more than the specified M_p (i.e., the specified maximum peak). The numerical value specified will approximately limit the maximum overshoot, as given by Figure 7.5. A typical specification is an M_p of 1.05, which leads to worst case overshoots of about 10%.

A formal statement of the M_p tuning problem follows:

Select the filter time constant ε^* for the IMC controller $q(s, \varepsilon)$ so that the magnitude of the complementary sensitivity function $CS(i\omega)$ is equal to or less than a specified M_p for all processes, $p(s)$, in a predefined set Π . For at least one process in Π , the magnitude of $CS(i\omega)$ must equal the specified M_p at one or more frequencies. That is,

$$|CS(i\omega, \varepsilon^*)| \leq M_p \quad \forall p(s) \in \Pi, \text{ and } \forall \omega \quad (7.12a)$$

where $\forall \equiv$ for all; $\in \equiv$ contained in, and

$$|CS(i\omega^*, \varepsilon^*)| = M_p \quad (7.12b)$$

for at least one $p \in \Pi$ and some frequencies, $\omega_1^*, \omega_2^*, \dots$.

The motivation for the condition in Eq. (7.12b) is to avoid introducing any conservatism into the tuning beyond that of the specification.

For the IMC configuration of Figure 7.6 (reproduced from Chapter 3, Figure 3.1), the closed-loop transfer function between output and setpoint (i.e., the complementary sensitivity function) is given by

$$CS(s, \varepsilon) = \frac{y(s)}{r(s)} = \frac{p(s)q(s)}{(1 + (p(s) - \tilde{p}(s))q(s, \varepsilon))}, \quad (7.13)$$

where $p(s)$ = any plant in the set of allowable processes Π
 $\tilde{p}(s)$ = a nominal model

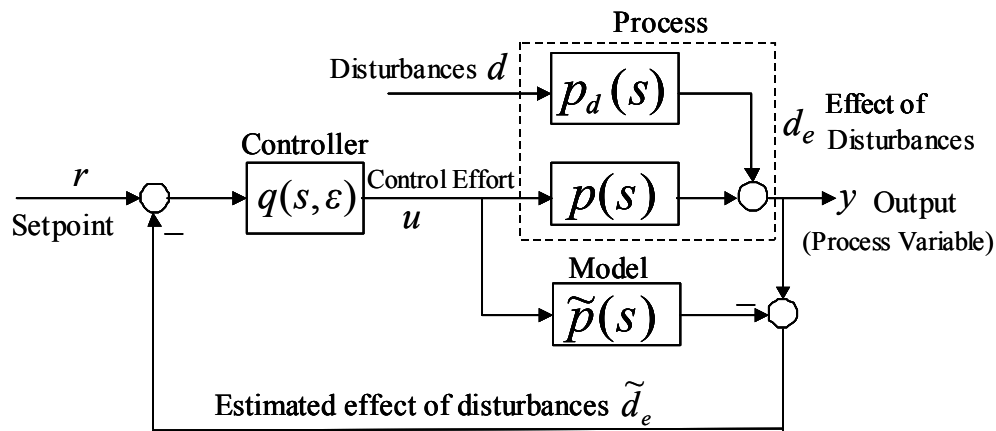


Figure 7.6 1DF IMC system.

The problem statement of Eq. (7.12) and above can be reformulated as the following optimization problem:

Find ε^* such that

$$\max_{\alpha, \omega} |CS(i\omega, \varepsilon^*, \alpha)| = M_p, \quad (7.14)$$

where α = vector of uncertain process parameters $\in \Pi$,

ω = frequency.

The solution of Eq. (7.14) yields values for the filter time constant ε^* , which satisfies Eq. (7.12) as well as values of the parameters α^* and ω^* , which solve the maximization problem given by Eq. (7.14).

The Mp tuning problem can be solved for very general processes, with uncertain parameters, using the IMCTUNE software associated with this text, and described in

Appendix G. The next section contains a brief description of the tuning algorithm used by the IMCTUNE software. Section 7.3.2 contains an example of M_p tuning, solved using IMCTUNE. Section 7.4 provides the theoretical justification for the M_p tuning algorithm presented in this section.

Occasionally, the M_p tuning procedure results in an upper-bound frequency response with one or more peaks where the valley to peak height is high enough to cause undesirable time domain oscillations. To avoid such time domain oscillations, we recommend restricting the magnitude of the upper-bound frequency domain peaks to not more than 0.1 above the highest adjoining valley. The IMCTUNE software checks the magnitude of any frequency domain peaks and adjusts the filter time constant to keep such peaks below a user specified value (the default value is 0.1).

7.3.2 The IMCTUNE Algorithm for Solving the M_p Tuning Problem

The following algorithm is implemented in MATLAB 5.3.1 for parametric uncertainty in the IMCTUNE software associated with this text.

Input data

1. A process model.
2. An uncertainty description in terms of upper- and lower-bounds on process parameters.
3. An initial value for the filter time constant. The default is the value of filter time constant that satisfies the maximum noise amplification specification (see item 5).
4. An M_p specification and tolerance. The defaults are $M_p = 1.05$ and Tolerance = ± 0.005 .
5. The maximum allowable high frequency controller noise amplification (i.e., $|q(\infty, \varepsilon)/q(0, \varepsilon)|$). The default is 20 (see Chapter 3).
6. Upper- and lower-bounds of the frequency range for the optimization. The defaults are
 Low frequency: reciprocal of 10 times the largest time constant or dead time
 High frequency: 1,000 times the low frequency
 Number of points and scale for plotting: 30, logarithmic
7. Upper- and lower-bounds of the frequency range for plotting. The defaults are
 Low Frequency: one-tenth the break frequency (see Figure 7.6)
 High Frequency: 100 times the break frequency
 Number of points and scale for plotting: 30, logarithmic

Output Data

1. The filter time constant that achieves the specified $M_p \pm$ Tolerance or the specified Maximum Peak Height \pm Tolerance.
2. Curves of the upper- and lower-bounds of the magnitude of the complementary sensitivity function for the computed value of the filter time constant.
3. Tables listing which processes give the upper- and lower-bound at each frequency.
4. Output and control effort time responses for the constrained IMC system with the specified process parameters in the uncertainty sets.
5. PID parameters induced by the IMC controller and process model.
6. Output and control effort time responses for the constrained PID control system with the specified process parameters in the uncertainty sets.
7. Output and control effort time responses for the constrained model state feedback IMC system with the specified process parameters in the uncertainty sets.

Stryczek et al. (2000) give a detailed description of the Mp tuning algorithm.

Example 7.4 Tuning an Uncertain FOPDT Process

Find the filter time constant ε for the process given by Eq. (7.15), and model and controller given by Equations (7.16a) and (7.16b).

$$p(s) = \frac{K e^{-\tau s}}{\tau s + 1}, \quad (7.15)$$

where

$$0.2 \leq K \leq 1; \quad 8 \leq \tau \leq 14; \quad .4 \leq T \leq 1.$$

Choosing a mid-range model and following the IMC controller design methods of Chapter 3 gives a model and IMC controller of

$$\tilde{p}(s) = \frac{.6e^{-.7s}}{(11s+1)}, \quad (7.16a)$$

$$q(s) = \frac{11s+1}{.6(\varepsilon s+1)}. \quad (7.16b)$$

To achieve an Mp specification of 1.05, the IMCTUNE software computes a value of 2.81 for the filter time constant ε^* . Using this filter time constant and IMCTUNE to compute the lower-bound of the complementary sensitivity function over all uncertain parameters, and requesting a plot of both upper- and lower-bounds, produces the curves in Figure 7.7.

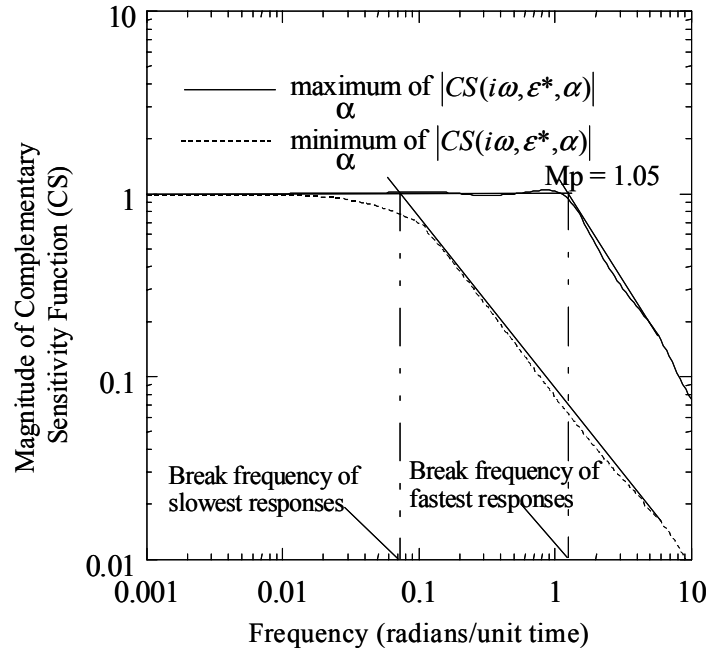


Figure 7.7 Upper- and lower-bounds of the complementary sensitivity function.

7.3.3 Interpretation of the Results of M_p Tuning

Figure 7.7 can be used to estimate the range of closed-loop time responses. The reciprocal of the break frequency of the upper-bound curve is an estimate of the time constant for the fastest responses, while the reciprocal of the lower-bound curve is an estimate of the time constant of the slowest responses. The break frequencies² of the upper-bound and lower-bound frequency responses are approximately 1.2 and .07. The reciprocal of these break frequencies (1/1.2 and 1/.07, respectively) provides estimates of the fastest and slowest time constants of the closed-loop, which are .83 and 14. Settling times³ should therefore be about 2.5 time units for the fastest responses and about 43 time units for the slowest responses. Comparison of these estimates with Figure 7.1, which gives the closed-loop time responses for several processes within the set given by Eq. (7.15), shows that the foregoing

² The break frequency is the intersection of the asymptote to the high frequency portion of the frequency response with the magnitude = 1 line.

³ The settling time is usually taken as three time constants.

estimates are quite accurate. The various process responses shown in Figure 7.1 were obtained from IMCTUNE and correspond to the parameters given in Table 7.1.

Table 7.1 Process Parameters for Figure 7.1 and the Process Given by Eq. (7.15).

curve no.	process gain, K	time constant, τ	dead time, T
1	1	8	1
2	1	14	1
3	.2	14	.4
4	.2	8	.4
5	.6	11	.7

It must be emphasized that the upper- and lower-bound frequency response curves in Figure 7.7 are not the frequency responses of any single process. At any frequency, the curves in Figure 7.7 represent the maximum and minimum of the complementary sensitivity function over the process parameters K , τ , and T . Therefore, each point on the curve could represent a different process. Usually, however, the upper-bound and lower-bound frequency responses come from only a small, finite subset of the possible process parameters. For example, the process parameters for the lower-bound curve in Figure 7.7 are given in Table 7.2.

Table 7.2 Process Parameters Associated with the Lower-Bound Curve in Figure 7.7.

frequency	process gain, K	time constant, τ	dead time, T
0.001 to 0.0033	0.2	14	1.00
0.0044 to 0.0862	0.2	8	.400
0.116 to 4.10	0.2	14	.400
5.52	0.2	14	.870
7.43	0.2	14	.636
10	0.2	14	.472

The parameters associated with the maximum peak of 1.05 are $[K, \tau, T] = [1, 8, 1]$, which yields the response given by curve 1 in Figure 7.1. There is a local maximum with an Mp of about 1.02 in the upper-bound curve at a frequency of about 0.1. The process parameters associated with this local maximum are $[K, \tau, T] = [1, 14, 1]$, which yields the response given by curve 2 in Figure 7.1.

As we shall see in Section 7.5, the shape of the upper-bound and lower-bound curves for the complementary sensitivity function and their associated break frequencies depend on the choice of the model. The Mp synthesis problem is to find the model and controller that give a lower-bound break frequency as high as possible for the specified Mp. Such a model and controller will provide the *fastest possible slowest responses* for the postulated uncertainty.

The above discussion assumes that the upper-bound curve of the complementary sensitivity function is similar to the response of a second-order system of the form $1/(\tau^2 s^2 + 2\zeta\tau s + 1)$. However, not all complementary sensitivity functions behave in this ideal manner. Indeed, even the upper-bound curve in Figure 7.7 is not quite that of a second-order system. Close inspection of the upper-bound in Figure 7.7 shows that it has a relative maximum of 1.0224, a relative minimum of 0.9771, and a global maximum of 1.0498. Second-order systems do not exhibit such relative maxima and minima. For systems with such oscillatory frequency responses, we must add the condition that the magnitude of the frequency response not have any peaks higher than specified (usually about 0.1) from the highest adjoining valley. For example, in Figure 7.7, the upper-bound curve has a peak at a frequency of .871, which is .0727 units higher than its highest adjoining valley, which is .9771 (.0727 = 1.0498 – .9771). In this case the peak is less than 0.1, so it does not influence the computed filter time constant of 2.81. If the peak had been higher than 0.1, then we would have increased the filter time constant beyond 2.81 to bring it down to 0.1. Section 7.5 contains some examples in which it is necessary to increase the filter time constant because of peaking, even though the Mp is less than 1.05.

7.3.4 Use of Multiple Uncertainty Regions to Account for Uncertain Uncertainty

Generally, the range of variation of the uncertain parameters α in an uncertain process, $p(s, \beta(\alpha))$ (see Section 7.2.1), is itself uncertain. This is particularly true when the variations of the parameters α arise due to the fact that the true process is nonlinear, as in Eq. (7.4), and the set Π contains the local linearizations of the process about all possible operating points, as in Eq. (7.5). In such situations, it is sometimes advantageous to view the set of uncertain parameters S_α as being composed of several overlapping sets, $S_{\alpha_j}, j=1\dots$ where the set S_{α_j} is defined as

$$S_{\alpha_j} \equiv \text{the set of all vectors } \alpha_j \text{ with } \underline{\alpha}_{j,i} \leq \alpha_{j,i} \leq \bar{\alpha}_{j,i}.$$

The $\underline{\alpha}_{j,i}$ and $\bar{\alpha}_{j,i}$ are ordered so that $\bar{\alpha}_{j,i} < \bar{\alpha}_{j+1,i}$ and $\underline{\alpha}_{j,i} > \underline{\alpha}_{j+1,i}$. A pictorial representation of the boundaries of the sets S_{α_j} is given in Figure 7.8. Notice that S_{α_1} is contained in S_{α_2} , and S_{α_2} is contained in S_{α_3} . The shaded areas in Figure 7.7 represent the difference between S_{α_3} and S_{α_2} (i.e., $S_{\alpha_3} - S_{\alpha_2}$) and between S_{α_2} and S_{α_1} (i.e., $S_{\alpha_2} - S_{\alpha_1}$). As we shall see in Example 7.5, by assigning different tuning objectives to the various shaded

areas of Figure 7.8 it is possible to account somewhat for uncertainty regarding process parameter ranges.

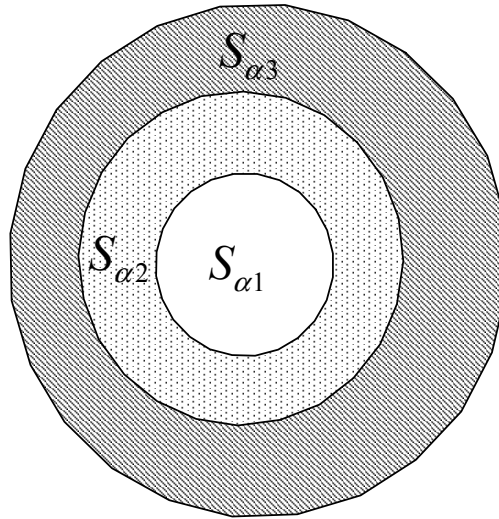


Figure 7.8 Multiple uncertainty sets.

Example 7.5 Use of Multiple Uncertainty Regions

Let us return to Example 7.4, where the uncertain process was given by Eq. (7.15):

$$p(s) = \frac{K e^{-Ts}}{\tau s + 1},$$

where

$$S_{\alpha_1} \text{ is } 0.2 \leq K \leq 1; \quad 8 \leq \tau \leq 14; \quad .4 \leq T \leq 1; \text{ and } \underline{\alpha}_1 \equiv [.2 \ 8 \ .4], \quad \bar{\alpha}_1 \equiv [1 \ 14 \ 1].$$

Let us assume that the specified upper-bound of the parameters, $\bar{\alpha}_1$, are either not known very precisely, or may be violated for relatively brief periods of time. It is known, however, that the parameters never exceed 1.5, 20, and 1.5 for the gain, time constant, and dead time. That is,

$$S_{\alpha_2} \text{ is: } \underline{\alpha}_2 = [.2 \ 8 \ .4], \quad \bar{\alpha}_2 = [1.5 \ 20 \ 1.5]. \quad (7.17a)$$

The mid-range model for the above is

$$[\tilde{K}, \tilde{\tau}, \tilde{T}] = [.85, 14, .95]. \quad (7.17b)$$

If we insist that no process in the enlarged uncertainty set S_{a2} given by Eq. (7.17a) have an Mp greater than 1.05, and the maximum peak to valley height of 0.1, then IMCTUNE yields a filter time constant of 6.08 using a mid-range model of Eq. (7.17b) and its associated controller ($q(s) = (14s+1)/.85(6.08s+1)$). The slowest closed-loop time constant is now about 37, compared to slowest closed-loop time constant of about 14 for the original uncertainty set model and controller (see Example 7.4). For the larger uncertainty set, the maximum peak height of 0.1 is the binding constraint and the Mp is only 1.02.

Now let us assume that for the region given by $(S_{a2} - S_{a1})$ (i.e., $1 \leq K \leq 1.5$; $14 \leq \tau \leq 20$; $1 \leq T \leq 1.5$), we are willing to accept overshoots of 50% and a fairly oscillatory response (e.g., $\zeta = .2$). The Mp specification then becomes 2.4 in that region, and the maximum peak height specification must be 1.5 or higher, because the maximum peak height must be more than the Mp specification minus one. The filter time constant obtained using the same model and controller is 3.2 (i.e., $[\tilde{K}, \tilde{\tau}, \tilde{T}] = [.85, 14, .95]$; $q(s) = (14s+1)/.85(3.2s+1)$). The slowest closed-loop time constant for the foregoing tuning is about 25 units. Since a slowest time constant of 25 is slower than the slowest loop time constant of 14 for the previous design (see Example 7.4), we should expect that using the foregoing model and controller will yield an Mp smaller than 1.05 for the original uncertainty bounds, and this is indeed the case. Keeping the model and controller the same yields an Mp of 1.0 for the original uncertainty set. Thus, by separating the uncertainty region into two regions and allowing greater overshoots in the outer region, we have sped up the slowest responses from a closed-loop time constant of 37 to one of 25, or in terms of settling times, from about 110 units to 75 units.

The fact that the Mp for the current controller is only 1 means that if we were willing to tolerate even more overshoot in the outer uncertainty region (i.e., in $(S_{a2} - S_{a1})$), then the filter time constant could be further reduced, thereby speeding up the slowest responses.

♦

7.3.4 The Inverse Tuning Problem

An alternate method of dealing with uncertainty about process parameter ranges is to specify what nominal closed-loop response time is acceptable, and then calculate the range of parameter variation that can be accommodated by a controller tuned to give that nominal response. This is equivalent to specifying the IMC controller filter time constant and calculating an uncertainty range that will yield that filter time constant for the specified Mp. Of course, it is necessary to recognize that actual response times will vary, possibly substantially, from the nominal (i.e., perfect model) response time. Further, the uncertainty range for a given filter time constant is not unique, because when more than one parameter varies, there is a multiple infinity of parameter ranges that yield the same filter time constant to achieve a specified Mp.

To associate an uncertainty range with an IMC controller filter time constant, it is necessary to specify a scalar function that tells how the model parameters vary within the uncertainty region. One convenient way of relating the parameters is to insist that all uncertain parameters vary by the same percentage from specified mid-range parameter values. Usually, the specified mid-range for parameter values will be the same as the model parameters. However, the IMCTUNE software permits the mid-range parameters supplied by the user to differ from the model parameters.

Example 7.6 Inverse Tuning

Again, we return to the process given by Eq. (7.15). The model parameters are as before (i.e., $[\tilde{K} \ \tilde{\tau} \ \tilde{T}] = [.6 \ 11 \ .7]$). Now, however, rather than calculating the IMC filter time constant to achieve an Mp of 1.05 for the uncertain parameter ranges given in Eq. (7.15), we shall assume that no uncertainty ranges are given, and our problem is to compute the uncertainty range for K , τ , and T that will yield a perfect model response time of about 8.4 time units to a unit step setpoint change. The foregoing problem specification is equivalent to specifying an IMC filter time constant, ε , of 2.8 (i.e., $2.8 = 8.4/3$).

The IMCTUNE software calculates parameter ranges of $.35 \leq K \leq .85$; $6.5 \leq \tau \leq 15.5$; $.41 \leq T \leq .99$ for an Mp of 1.05 and an ε of 2.8. Notice that, except for the dead time, the calculated ranges are not the same as those for the process given by Eq. (7.15). This is because the calculated ranges have the same percent variation from the mid-range parameters ($\pm 41\%$), while in the original problem the gain varies by 66% and the time constant and dead time vary by $\pm 27\%$ from their mid-range values of .6 and 11.

◆

The next two sections provide the theoretical justification for the Mp tuning algorithm. Those readers willing to accept the algorithm on faith may skip or skim all but the restrictions given by Table 7.3. However, those readers who would like some insight into why the algorithm works should at least skim Sections 7.4 and 7.5.

7.4 CONDITIONS FOR THE EXISTENCE OF SOLUTIONS TO THE MP TUNING PROBLEM

The Mp tuning methodology described in the previous section implicitly assumes that (1) if all closed-loop complementary sensitivity functions have finite magnitudes, then the control system is stable for all processes in the uncertainty set, and (2) there always exists a filter time constant, which will cause the Mp for the complementary sensitivity function to lie below any preset limit greater than 1 for all processes in the uncertainty set. The following subsections give the conditions under which the foregoing statements are indeed true. As a prerequisite, however, we first present an abbreviated version of the Nyquist stability criterion discussed in Appendix B, Section B.4.

7.4.1 Statement of the Nyquist Stability Criterion

The number of zeros, Z , of the term $1 + kg(s)$ inside a closed contour, D , in the complex plane is equal to the number of poles of $1 + kg(s)$ inside of the closed contour D plus the number of clockwise (counterclockwise) encirclements of the point $(-1/k, 0)$ by the phasor $g(s)$ as s moves in a clockwise (counterclockwise) direction around the closed contour D . That is,

$$Z = N + P$$

where

Z = # of zeros of $1 + kg(s)$ inside D ,

N = # of encirclements of $(-1/k, 0)$ point by $g(s)$ as s moves once around D ,

P = # of poles of $1 + kg(s)$ inside D .

The control system of Figure 7.9 is stable if, and only if, the contour D encloses the entire right half of the s -plane and the number of zeros, Z , of the characteristic equation $1 + kg(s)$, as calculated above, is zero.⁴

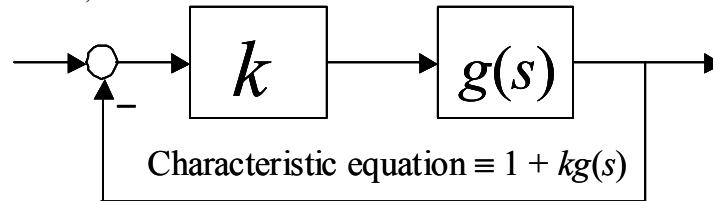


Figure 7.9 Feedback diagram for the Nyquist stability criterion.

If $g(s)$ has a pole at $s = 0$, then the contour D is usually taken as shown in Figure 7.10, and the radius δ is made to approach zero so that the contour encloses the entire right half of the s -plane.

⁴ An excellent, intuitive proof of the Nyquist stability theorem can be found in Van de Vegte (1986).

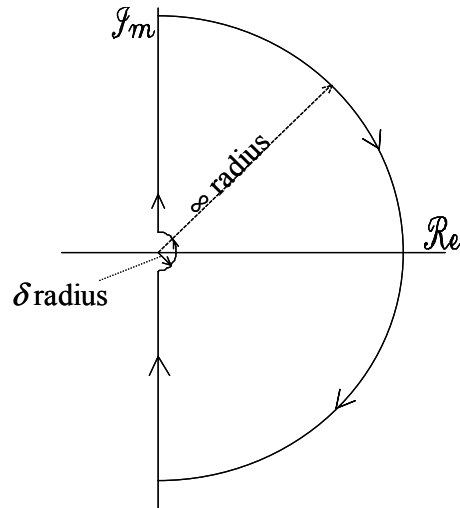


Figure 7.10 Nyquist D contour when there is a pole at the origin.

7.4.2 Integral Controllability

Any control system that forces the process output to exactly track the setpoint in the steady state (i.e., one that has no off-set) must have an integrator in the control loop. In IMC the required integration comes from the positive unity feedback loop formed by the controller and the model when the controller gain is the inverse of the model gain (see Chapter 6). For an uncertain process, the question arises as to what limits, if any, exist on the range of uncertain process parameters so that an integrating control system is stable over all possible process parameters. To address this question, Morari (1985) introduced the concept of integral controllability for uncertain processes. While the concept applies to general multivariable uncertain processes, here we shall use it only for SISO systems. Morari's definition of integral controllability follows

Integral Controllability: The open-loop stable uncertain system $h(s)$ in Figure 7.11 is called integral controllable if there exists a $k^* > 0$ such that the control system is stable for all k in the range $0 < k \leq k^*$.

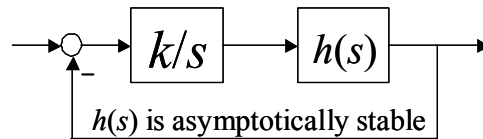


Figure 7.11 Feedback control diagram for integral controllability.

The uncertain system $h(s)$ in Figure 7.11 includes the process and actuator dynamics as well as a portion of the controller dynamics. Also, all the processes in $h(s)$ are assumed to be asymptotically stable.⁵ (Such a system can have no poles in the right half of the s -plane or along the imaginary axis.)

An uncertain process that is integral controllable is one for which it is always possible to design a stable offset-free control system for all processes in the uncertainty set.

Integral Controllability Theorem: An uncertain system $h(s)$ is integral controllable if, and only if, the gains of $h(s)$, (i.e., $h(0)$) are positive for all plants in the uncertainty set.

The condition that all gains of $h(s)$ be positive effectively requires that all process gains $p(0)$ in the uncertainty set have the same sign and, of course, are never zero. If all process gains are negative, then the controller gain must also be negative. In Figure 7.11, the negative controller gain is included in $h(s)$ so that $h(0)$ is positive for all processes in the uncertainty set.

Morari uses the Nyquist Stability Criterion to prove the above theorem. To illustrate the nature of the proof, we consider the system $h(s) = h(0)/(s + 1)^3$ with $h(0)$ uncertain. The Nyquist diagram for $g(s) = h(s)/s$ for $h(0)$ equal one, and the D contour taken as shown in Figure 7.10 is given in Figure 7.12.

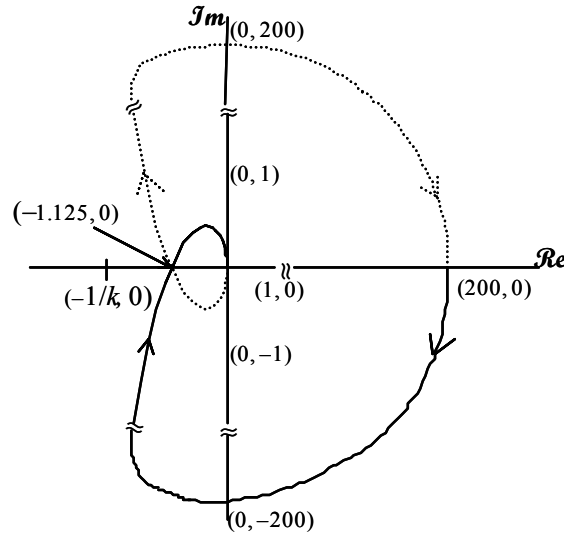


Figure 7.12 Nyquist diagram for $1/s(s + 1)^3$ as s travels about the contour, shown in Figure 7.10, with $\delta = .00492$.

The Nyquist diagram in Figure 7.12 is drawn with the central portion of the diagram (about the origin) much enlarged relative to the outer portion of the diagram. Also, the

⁵ An unforced dynamic system is said to be asymptotically stable if it tends to the origin from any initial conditions.

diagram is distorted at the break points where the scale changes so that the diagram appears to be continuous, as it actually is, rather than discontinuous, as it would appear if two scales were rigorously adhered to. The point $(\mathcal{R}e, \mathcal{I}m) = (200, 0)$ on the Nyquist diagram corresponds to the point $s = (\delta, 0)$ on the D contour of Figure 7.11, with $\delta = .00492$. As s moves along the curve $s = \delta e^{j\phi}$, with ϕ increasing from 0° to $+90^\circ$, the Nyquist diagram moves along the solid line. After ϕ reaches 90° , s moves along the curve $s = i\omega$ and the corresponding portion of the Nyquist diagram moves through the break in scale to the origin, as shown by the solid line in Figure 7.12. Along the infinite semicircle $1/s(s+1)^3$ is zero. The dotted line in Figure 7.12 corresponds to the value of $1/s(s+1)^3$ as s moves up the imaginary axis from $-\infty$, and then along the quarter circle back to $s = (\delta, 0)$.

Figure 7.12, shows that there will be no encirclements of the point $(-1/k, 0)$ provided that $-1/k$ is less than -1.125 which means that k must be greater than zero, but less than $1/1.125$ (i.e., .889). It does not matter how small δ is taken to be, since smaller values of δ simply increase the radius of the right half plane semicircle in Figure 7.12. Changes in the gain $h(0)$ do not affect the general shape of the Nyquist diagram, provided all gains are positive. As long as $h(0)$ is greater than zero, changes in gain and time constants only change the point at which the Nyquist curve intersects the negative real axis. The intersection point is always finite, and so there always exists a k^* such that the control system is stable for $0 < k \leq k^*$. Should any of the gains of $h(0)$ be negative, however, then the Nyquist diagram is rotated about the imaginary axis, as shown in Figure 7.13.

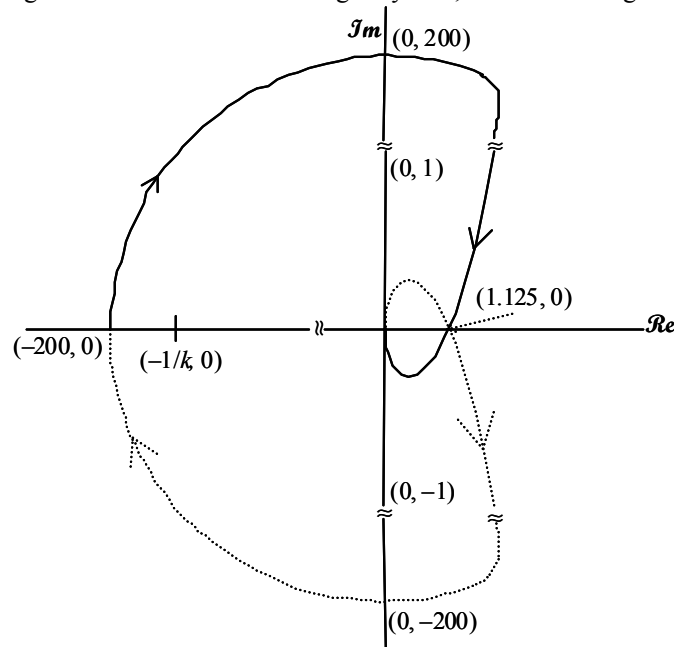


Figure 7.13 Nyquist diagram for $-1/s(s+1)^3$ as s travels about the D contours shown in Figure 7.10 with $\delta = 0.00492$ and $h(0)$ negative.

From Figure 7.13, it is clear that if k is positive, the $-1/k$ point will always be encircled in a clockwise direction at least once if δ is allowed to approach zero, as it must if the D contour is to enclose the entire right half plane.

The above analysis can be repeated for any stable process and control system where the open-loop transmission has a pole at the origin. Morari uses an argument very similar to the foregoing in proving that the necessary and sufficient conditions for integral controllability are that the gain of $h(s)$ does not change sign.

For an IMC system, the equivalent definition of integral controllability is that an open-loop stable system is integral controllable if, and only if, there exists a filter time constant $\varepsilon^* > 0$ such that the closed-loop IMC system is stable for all finite ε greater than or equal to ε^* . Again, integral controllability requires that the process gain not change sign. To prove this statement, it is adequate to recognize the correspondence of an IMC system to an integral control system when the controller gain is the inverse of the model gain (see Chapter 6). However, we shall determine the necessary and sufficient conditions for integral controllability directly by application of the Nyquist criterion to the IMC system.

The characteristic equation of an IMC system such as that given by Figure 3.1 in Chapter 3 is given by (also see Eq. 3.6)

$$\text{Characteristic Equation} = (1 + (p(s) - \tilde{p}(s))q(s)). \quad (7.18)$$

This characteristic equation will have no right half-plane zeros only if the Nyquist diagram of $(p(s) - \tilde{p}(s))q(s)$ does not encircle the -1 point as s traverses a D contour which travels from $-i\infty$ to $+i\infty$ and then clockwise around the infinite semicircle back to $-i\infty$. We will investigate the behavior of Eq. (7.18) for filter time constants larger than ε^* . To start, we note that for low enough frequencies, the frequency response of all processes in the uncertainty set Π can be approximated by their steady-state gains. That is

$$p(i\omega) \cong p(0) \text{ and } \tilde{p}(i\omega) \cong \tilde{p}(0) \text{ for } \omega < \delta. \quad (7.19)$$

Therefore, for the controller filter time constant large enough so that

$$\varepsilon\delta \gg 1, \quad (7.20)$$

the terms pq and $\tilde{p}q$ can be approximated as

$$pq(s, \varepsilon) \cong \frac{p(0)/\tilde{p}(0)}{(\varepsilon s + 1)^r} \text{ and } \tilde{p}q(s, \varepsilon) \cong \frac{1}{(\varepsilon s + 1)^r} \quad s = i\omega, \quad \omega < \delta. \quad (7.21)$$

Substituting Eq. (7.21) into Eq. (7.18) gives

$$\text{Characteristic Equation} = 1 + [p(0)/\tilde{p}(0) - 1]/(i\varepsilon\omega + 1)^r; \quad 0 \leq \omega < \delta. \quad (7.22)$$

The stability of the control system is therefore completely determined by whether or not the function $[p(0)/\tilde{p}(0) - 1]/(i\varepsilon\omega + 1)^r$ encircles the -1 point on the Nyquist diagram as $\varepsilon\omega$ ranges from zero to large values. This function cannot encircle the -1 point so long as

the ratio $p(0)/\tilde{p}(0)$ is greater than zero. However, if any process gains are such that $p(0)/\tilde{p}(0)$ is less than zero, then the Nyquist diagram will lie to the left of the -1 point at a frequency of zero and will spiral in a clockwise manner towards the origin as frequency increases. Thus, the Nyquist diagram will encircle the $(-1, 0)$ point, and the control system will be unstable. The necessary and sufficient conditions for integral controllability of an IMC system are therefore that all process gains must have the same sign and *the model gain must have the same sign as the process gains*.

7.4.3 Necessary and Sufficient Conditions for the Existence of a Solution to the Mp Tuning Problem for Any Mp Specification Greater than One

In this section we give the conditions under which any Mp specification can be achieved by choosing the IMC filter time constant large enough. As in Section 7.4.2, we consider all filter time constants, ε , greater than some ε^* , where ε^* is chosen so that $\varepsilon^*\delta$ is much greater than one and δ is small enough so that Eq. (7.19) is satisfied. Then the frequency response of the complementary sensitivity function $CS(i\omega, \varepsilon)$ (see Eq. 7.13) can be well approximated as follows:

$$CS(i\omega, \varepsilon) \cong \frac{(p(0)/\tilde{p}(0))/(i\varepsilon\omega + 1)^r}{1 + [(p(0)/\tilde{p}(0) - 1)/(i\varepsilon\omega + 1)^r]}, \quad 0 \leq \omega < \delta \quad (7.23)$$

$$\cong [(i\Omega + 1)^r - 1]\Phi + 1]^{-1}, \quad 0 \leq \Omega \leq \varepsilon\delta \gg 1, \quad (7.24)$$

where

$$\Omega \equiv \varepsilon\omega, \quad \Phi \equiv p(0)/\tilde{p}(0)$$

The term $\varepsilon\delta$ in Eq. (7.24) is much greater than one because ε is chosen to be greater than ε^* and $\varepsilon^*\delta$ is much greater than one by choice of ε^* .

The behavior of $CS(i\Omega)$ given by Eq. (7.24) depends only on the ratio of process gain to model gain (i.e., $p(0)/\tilde{p}(0)$). The upper-bound on the magnitude of $CS(i\Omega)$ (i.e., its M_p) will be less than one over all Ω greater than zero, only if the ratio of $p(0)$ to $\tilde{p}(0)$ (i.e., Φ) is less than that shown in Table 7.3. Notice that for processes higher than second-order, it may not be possible to achieve an Mp arbitrarily close to one using the mid-range gain for the model if the range of possible process gains is large. For example, if the relative order r is 3 and the range of process gains is from 1 to 5, then the mid-range gain is three, and the ratio of $p(0)$ to $\tilde{p}(0)$ can be as high as $5/3$. For this case, the minimum achievable Mp is 1.015. Choosing a model gain of $10/3$ allows one to achieve any Mp greater than 1 by choosing the filter time constant ε large enough. Of course, if the actual Mp specification is

1.05, then selecting a model gain of 3 will not cause a problem in meeting the Mp specification.

Table 7.3 Upper Limits on the Ratio of Process to Model Gain for Which It Is Possible to Achieve Mp Specifications as Close to One as Desired.

filter order, n	upper limit of the gain ratio $p(0)/\tilde{p}(0)$
1	∞
2	2
3	1.5
4	1.34
5	1.25

7.4.4 Justification for the Choice of Complementary Sensitivity Function in Mp Tuning

For the single-degree of freedom control system studied in this chapter, Mp tuning is based on the complementary sensitivity function rather than on the sensitivity function, which is commonly used in the literature to address issues of controller design with uncertain processes (Doyle et al., 1992). The sensitivity function $S(s)$ is the transfer function between the process output and the disturbance (see Figure 3.1). If the disturbance does not pass through a lag (i.e., if $P_d = 1$), then the sensitivity function is given by

$$S(s) \equiv \frac{(1 - \tilde{p}(s)q(s))}{1 + (p(s) - \tilde{p}(s))q(s)}. \quad (7.25)$$

From the definition of the complementary sensitivity function given by Eq. (7.13), the sum of the complementary sensitivity function and the sensitivity function given by Eq. (7.25) is one. That is,

$$CS(s) + S(s) = 1. \quad (7.26)$$

Therefore, from Equations (7.23) and (7.26), the expression for the sensitivity function for very large filter time constants is:

$$S(i\omega, \varepsilon) \equiv \frac{1 - 1/(i\varepsilon\omega + 1)^r}{1 + [p(0)/\tilde{p}(0) - 1]/(i\varepsilon\omega + 1)^r}, \quad 0 \leq \omega < \delta. \quad (7.27)$$

The upper-bound of the magnitude of the sensitivity function given by Eq. (7.27) over all frequencies increases as the maximum of the ratio of $p(0)/\tilde{p}(0)$ increases. Therefore the maximum value of $|S(i\omega, \varepsilon)|$ is a minimum when the model gain $\tilde{p}(0)$ is chosen as the

maximum process gain in the uncertainty set. In this case, $\max p(0)/\tilde{p}(0) = 1$, and the maximum of the sensitivity function is 1.15, 1.28, and 1.38 for $r = 2, 3$, and 4, respectively. Thus, unlike the situation for the complementary sensitivity function, there is no choice of model gain for which the maximum magnitude of the sensitivity function approaches one for large filter time constants. The corollary of this fact is that there is no single value for the maximum of the sensitivity function that corresponds to a specified overshoot for the time domain response to a step setpoint change. For this reason, it is more convenient to use the complementary sensitivity function to tune IMC controllers for good setpoint responses.

7.5 ROBUST STABILITY

This section introduces a theorem that states the conditions under which the Mp tuning algorithm in Section 7.3 yields stable control systems for all processes in the uncertainty set. A precise statement of the theorem and its proof can be found in Brosilow and Leitman (2001). The reason that such a theorem is required is that, by itself, a finite Mp does not guarantee stability. For example, the system $1/(-s + 1)$ has an Mp of one (i.e., upper-bound of $|1/(-i\omega + 1)|$ is 1), but is unstable. Similarly, a closed-loop control system for a stable process can have a finite Mp and yet the control system can be unstable. Consider the process and model given in Example 7.7.

Example 7.7 A Finite Mp, but an Unstable Closed-loop

$$p(s) = Ke^{-s} \quad 3 \leq K \leq 4 \quad (7.28a)$$

$$\tilde{p}(s) = e^{-s} \quad (7.28b)$$

$$q(s) = 1/(0.5s + 1) \quad (7.28c)$$

The Mp of the closed-loop system is 6.35, but *a Nyquist analysis of the control system for any process in the uncertainty set shows that the control system is unstable*. There are two right half plane poles for all processes with gains between 3 and 4. What's going on? We notice that the model gain is not in the uncertainty set, which is somewhat strange, but why should this matter? To answer such questions, we need the robust stability results from Brosilow and Leitman (2001), which we paraphrase below.

◆

7.5.1 A Frequency Domain Robust Stability Theorem for Infinite Dimensional Systems

A closed-loop transfer function, $H(s, \alpha)$, given by

$$H(s, \alpha) \equiv \frac{g(s, \alpha)}{1 + h(s, \alpha)}, \quad (7.29)$$

is stable for all parameter vectors, α , contained in the parameter set Π if $H(s, \alpha)$ is stable for at least one parameter in the set Π and if

$$\max_{\alpha \in \Pi} |H(i\omega, \alpha)| \text{ is finite } \forall \omega \geq 0 \quad (7.30)$$

and the parameter set Π and the transfer functions $g(s, \alpha)$ and $h(s, \alpha)$ must satisfy the following conditions:

1. The parameter set Π is an *open, connected* subset of \mathbb{R}^n .⁶
Since Π is *connected*, for any two points $\tilde{\alpha}$ and $\tilde{\beta}$ in Π , there is a path σ in Π from $\tilde{\alpha}$ to $\tilde{\beta}$; that is, there is a continuous function $\sigma : [0, 1] \rightarrow \Pi$ such that $\sigma(0) = \tilde{\alpha}$ and $\sigma(1) = \tilde{\beta}$. Since Π is *open*, a path σ from $\tilde{\alpha}$ to $\tilde{\beta}$ can be taken arbitrarily smooth.
2. There is a *connected, open* set $\hat{\Pi}$ in C^n such that $\hat{\Pi} \cap \mathbb{R}^n = \Pi$ and the functions $g(s, \alpha)$, and $h(s, \alpha)$ have meromorphic extensions to $C^+ \times \hat{\Pi}$ ⁷ also denoted by g and h , where \mathbb{R}^n and C^n are real and complex n space and C^+ is the open complex half plane in C .
3. For each parameter $\alpha \in \Pi$, the function $h(s, \alpha)$ is not constant on C^+ .⁸
4. For each parameter $\alpha \in \Pi$, the function $h(s, \alpha)$ is real-valued or ∞ on the positive real axis $\{s : \text{Re}(s) > 0, \text{Im}(s) = 0\}$.
5. The functions $g(s, \alpha)$, and $h(s, \alpha)$ are (jointly) continuous (in both s and α) functions from $\bar{P}^+ \times \Pi$ into P , where P is the complex projective sphere (Riemann sphere) and \bar{P}^+ represents the closure in P of the open hemisphere P^+ .

⁶ C^n denotes complex Euclidean n -space and \mathbb{R}^n denotes its real Euclidean n -subspace.

⁷ If $h(s, \alpha)$ is *meromorphic* for real parameters α in the sense of convergent power series expansions, this extension is always possible.

⁸ If $h(s, \alpha)$ is the Laplace Transform of some real-valued *integrable* function and it is constant at α , it must be identically zero at α .

Conditions (2), (3), and (4) are regularity or consistency conditions on g and h . However, conditions (1) and (5) on the parameter set and the continuity of g and h , especially on the imaginary axis, are structurally necessary for robust stability to hold.

A connected set is one in which there exists a path \mathcal{P} between any two elements in the set, say, $\hat{\alpha}$ and $\hat{\beta}$. That is, there is a continuous function \mathcal{P} such that $\mathcal{P}(0) = \hat{\alpha}$ and $\mathcal{P}(1) = \hat{\beta}$. The parameter set given by Eq. (7.15) is a particularly simple example of a connected set. It is not only connected, but also simply connected in that any closed curve in the set can be continuously contracted into a point. Figure 7.14 shows a connected set (shaded) that is not a simply connected set because of the hole in its center.

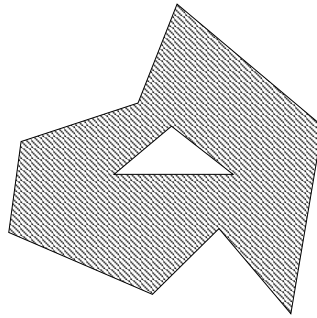


Figure 7.14 A connected set (but not simply connected).

We illustrate the concept of joint continuity by means of a transfer function that is not jointly continuous at the point $a = 0, s = 0$. Consider

$$h(s, a) = a(a - s) / (a(1 - a) + s(1 + a)) \quad -1 \leq a \leq 1. \quad (7.31)$$

At the point $a = 0, s \neq 0$, the value of the function is zero, even for very small s , since $h(s, 0) = 0/s$. For the point $s = 0, a \neq 0$, the value of the function is $a/(1 - a)$, which approaches one as a approaches zero. Therefore, $h(s, a)$ is not continuous at $s = a = 0$. Further, the point $a = 0$ cannot be removed from the uncertainty set because, if it were, the uncertainty set would not be connected. (The uncertainty set is just the section of the real line from -1 to 1 .) As a consequence of the foregoing, the robust stability theorem cannot be applied to Eq. (7.31). We will have more to say about Eq. (7.31) in Chapter 8 in the section on the design and tuning of controllers for unstable, uncertain systems.

Conclusion of Example 7.7

Example 7.7 satisfies all of the conditions of the robust stability theorem. Application of the Nyquist criterion for a process that has a gain of 3 shows that the control system is unstable for this process. Since the M_p is finite ($M_p = 6.36$) we can conclude that the control system is unstable for all processes in the uncertainty set, because if there were a process for which the control system is stable, the robust stability theorem tells us that the M_p would be infinite. Indeed, if the uncertainty set is expanded so that it includes the model

gain (e.g., $1 \leq K \leq 4$), then all processes with gains sufficiently near 1 are stable (because there is effectively no modeling error), and the M_p for such an uncertain system is indeed infinite.

◆

For inherently stable processes in which the model parameters lie inside the connected uncertainty set, there will always be a set of processes for which the control system is stable. Therefore, if the M_p is finite, we can conclude that the control system is stable for all parameters in the uncertainty set. Of course, strictly speaking, we should check on all the conditions required by the robust stability theorem. However, it will be the very rare engineering model of a stable system that does not satisfy these conditions. We emphasize that the foregoing applies only to inherently stable processes. For inherently unstable processes, the control system can be unstable for all processes in the neighborhood of the model, while the control system is stable when the process equals the model. In such cases, the closed-loop transfer function is usually not jointly continuous at the point where the process parameters equal the model parameters. For inherently unstable processes, it is usually best not to consider the perfect model point to be in the uncertain parameter set, provided its (conceptual) removal does not create a disconnected set, as happened with Eq. (7.31).

7.5.2 A Heuristic Proof of the Robust Stability Theorem for Inherently Stable Processes

Let us assume that the robust stability theorem for inherently stable processes is false. Then, the M_p is finite for a system where there is a parameter vector β for which the system is stable and another parameter vector α for which the system is unstable. Now, for the parameter α , the Nyquist diagram of $(p(s) - \tilde{p}(s))q(s)$ must encircle the -1 point, since the control system is unstable. Similarly, for the parameter β the Nyquist diagram must not encircle the -1 point, since the control system is stable. Since the parameter set is open and connected, and the closed-loop transfer function is jointly continuous, there is a smooth path from α to β lying entirely inside the parameter set such that there is at least one parameter, say, γ , such that the Nyquist diagram of $(p(s) - \tilde{p}(s))q(s)$ passes right through the $(-1, 0)$ point. So, there must also be a frequency $\hat{\omega}$ such that $1 + (p(i\hat{\omega}, \gamma) - \tilde{p}(i\hat{\omega}, \gamma))q(i\hat{\omega}) = 0$. Therefore, the maximum magnitude of the closed-loop frequency response over all parameters is not bounded. This contradicts our hypothesis that the robust stability theorem for inherently stable processes is false.

7.6 MP SYNTHESIS

Mp synthesis addresses the effect of uncertainty on controller design and model selection. The best model and controller combination is that which shifts the lower-bound break frequency furthest to the right without increasing the Mp, thereby speeding up the slowest closed-loop responses without increasing the overshoot to a step setpoint change. Example 7.8 explores the effect of changing just the controller and then changing both the controller and the model. We shall see that the effects are quite substantial when there is even a modest uncertainty in the model parameters.

Example 7.8 An Overdamped Second-Order Process Plus Dead Time

$$\text{Process:} \quad p(s) = \frac{K}{(8s+1)(6s+1)} e^{-Ts}, \quad 5 \leq K, T \leq 15 \quad (7.32a)$$

$$\text{Model:} \quad \tilde{p}(s) = \frac{10}{(8s+1)(6s+1)} e^{-10s} \quad (7.32b)$$

$$\text{Controller 1:} \quad q(s) = \frac{(8s+1)(6s+1)}{10(28.6s+1)^2} \quad (7.32c)$$

$$\text{Controller 2:} \quad q(s) = \frac{(14s+1)}{10(31.6s+1)} \quad (7.32d)$$

Controller 1, given by Eq. (7.32c), is the normal inverse of the invertible part of the model multiplied by a second-order filter to make the controller realizable. Controller 2, given by Eq. (7.32d) is the inverse of an approximate model with the original second-order lag replaced by a first-order lag whose time constant is the sum of the time constants of the second-order lag. Both controllers have been tuned to give an Mp of 1.05.

Figure 7.15a gives the upper-bound and lower-bound curves of the complementary sensitivity function for control systems that use the controllers 1 and 2. The model is that of Eq. (7.32b) for both control systems. Notice that both the upper-bound and lower-bound responses for Controller 2, Eq. (7.32d), are shifted to the right of those for Controller 1, Eq. (7.32c), implying that the fastest and slowest time domain responses for Controller 2 will be faster than those for Controller 1. Figure 7.15b bears this out.

Changing the model given by Eq. (7.32b) to $\tilde{p} = 10e^{-10s} / (14s+1)$, and using the controller given by Eq. (7.32d), does not change the control system responses significantly from those given in Figures 7.15a and 7.15b.

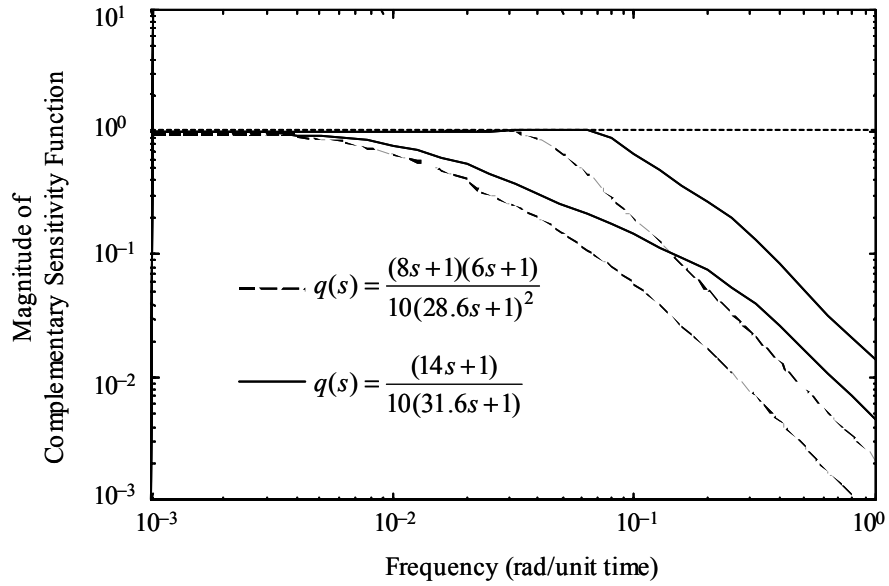


Figure 7.15a Comparison of the upper- and lower-bounds of the closed-loop frequency responses for the system of equation series (7.32).

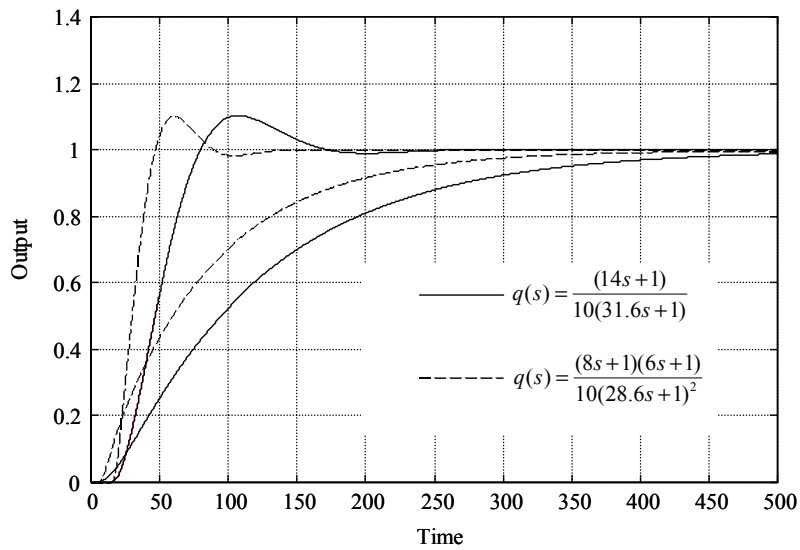


Figure 7.15b Fastest and slowest responses (i.e., $K=15$ and 5 , respectively) to a step setpoint change for the control systems of equation series (7.32).

Standard engineering practice for the control of overdamped systems such as that given by Eq. (7.31a) is to fit a first-order model of the form of Eq. (7.33a) to the output response to a step change in the control effort, and then to tune the controller based on that model (Seborg et al., 1989).

$$\tilde{p}(s) = \frac{K}{(\tau s + 1)} e^{-(T + \Delta)s}, \quad (7.33a)$$

where T = the estimated process dead time,

Δ = an additional dead time arising from the fitting procedure,

τ = an estimate of a first-order time constant which allows the approximate model step response to match that of the higher order process.

Using the reaction curve method (Seborg et al., 1989) yields $(T + \Delta) = 12$, $\tau = 20$, and of course $K = 10$. Tuning the control system using this model and its model inverse controller yields a filter time constant of 38.6. The resulting controller is

$$q = \frac{(20s + 1)}{10(38.6s + 1)}. \quad (7.33b)$$

The model and a controller given by Equations (7.33a) and (7.33b) does not perform quite as well as the controller and model given by Equations (7.32b) and (7.32d). It turns out that the above model parameters do not give a very good fit to a step change in the control effort. A much better fit is obtained with $\Delta = 4$ and $\tau = 11$. The responses with this model and its associated controller are insignificantly different from those of Equations (7.32b) and (7.32d) (i.e., the solid line responses in Figure 7.15b).

The above models and controllers are all mid-range in the sense that the model gain is the mid-range gain, and the dead time is either the mid-range dead time or the dead time obtained by fitting the mid-range model. However, one can obtain performance that is slightly better than that obtained with Equations (7.32b) and (7.32d) with either of the following models and controllers.

$$\tilde{p}(s) = \frac{15}{(8s + 1)(6s + 1)} e^{-15s} \quad q(s) = \frac{(8s + 1)(6s + 1)}{15(6.07s + 1)^2}, \quad (7.34)$$

$$\tilde{p}(s) = \frac{15}{(8s + 1)(6s + 1)} e^{-15s} \quad q(s) = \frac{(14s + 1)}{15(11s + 1)}. \quad (7.35)$$

The control systems of Equations (7.34) and (7.35) yield effectively the same frequency and time responses. Therefore Figures 7.16a and 7.16b show only the responses of the control system of Eq.(7.35).

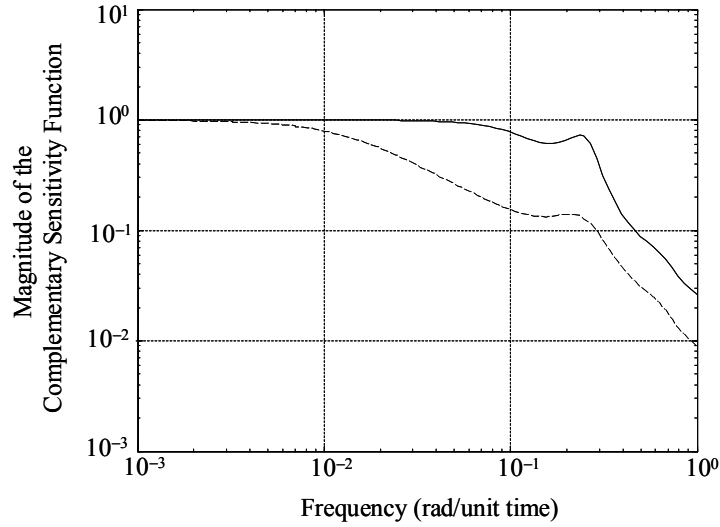


Figure 7.16a Upper-bound and lower-bound frequency responses for the process of Eq. (7.32a) and the control system of Eq. (7.35).

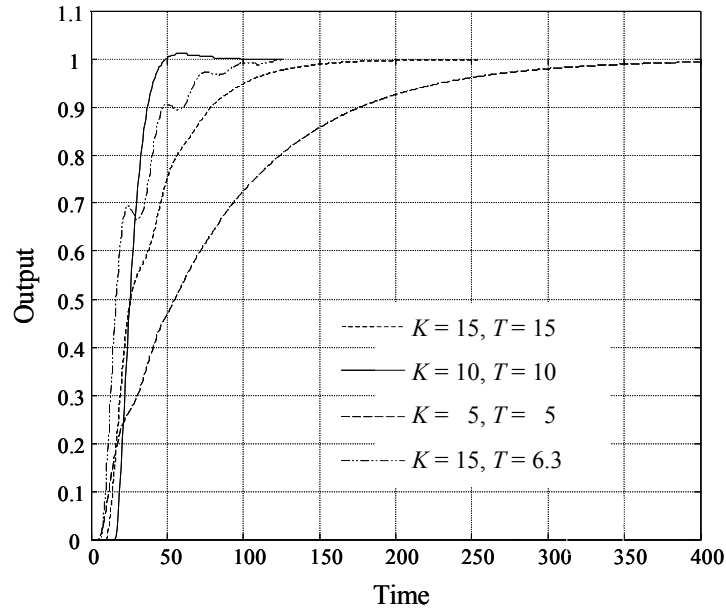


Figure 7.16b Time responses for the process of Eq. (7.32a) and the control system of Eq. (7.35).

Notice that in Figure 7.16a the Mp is 1, and the criterion that establishes the controller filter time constant is that no peak to valley height be greater than 0.1. The process at the local peak in Figure 7.16b has $K = 15$ and $T = 6.3$. This process yields the somewhat oscillatory response shown in Figure 7.16b. Also, comparison of the responses of mid-range process (i.e., $K = 10$, $T = 10$) using the control system of Eq. (7.35) with the control system of Equations (7.32b) and (7.32d), where the model is perfect, shows no significant difference. *Thus, for this example, there is no advantage to using the mid-range model, even if the process operates around the mid-range (i.e., $K = 10$, $T = 10$) far more often than at $K = 15$, $T = 15$.*

Changing the model in Eq. (7.35) to $\tilde{p}(s) = 15e^{-1.5s}/(14s+1)$ and retuning gives $q(s) = (14s+1)/15(10.6s+1)$. This controller model pair has an Mp of 1.05, and the lower-bound curve of the complementary sensitivity function actually lies slightly to the right of that in Figure 7.16a. Once again, the simpler model/controller is preferred.

As a check on the sensitivity of the control systems given by the equation series (7.32) to the assumed uncertainty bounds, we increased the upper- and lower-bounds by $\pm 20\%$ to \overline{K} , $\overline{T} = 18$ and \underline{K} , $\underline{T} = 4$. This change causes worst-case overshoots of 30% for Eq. (7.32c) and a 35 % overshoot for Eq. (7.32d). The settling time of the slowest responses increases to 500 and 450 units respectively. Thus, the tuned control systems are not overly sensitive to modest errors in estimating the parameter uncertainty ranges.

The PID controllers that have the same performance as Equations (7.34) and (7.35) are respectively: $K_c = 0.0412$, $\tau_I = 16.8$, $\tau_D = 4.41$, and $K_c = 0.0422$, $\tau_I = 16.5$, $\tau_D = 4.42$.

PID controllers with the same performance as Equations (7.32c) and (7.32d) are respectively: $K_c = 0.00383$, $\tau_I = 2.57$, $\tau_D = 6.27$, and $K_c = 0.0338$, $\tau_I = 14.0$, $\tau_D = 1.73$.

If the above results extend to other similar processes, and we believe that they do, then they provide a strong justification for standard engineering practice of fitting a first-order lag plus dead time model to high order overdamped processes when there are significant process parameter variations. A reasonable question, however, is how much parameter variation is significant? For the above process, it turns out that the first-order model and controller performs as well as or better than a second-order model and controller for gain and dead time variations greater than $\pm 25\%$. We conjecture that the break points for going from a higher order to a lower order controller occur at uncertainty levels where the tuned higher order controller becomes a lag (i.e., $|q(i\omega)| \leq 1$ for $\omega \geq 0$).

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Example 7.9 A Second-Order Process that Varies from Underdamped to Overdamped

This example consists of a second-order system with uncertainty in the time constant and damping ratio. The uncertain process has the following representation:

$$S = \left\{ p(s) \mid p(s) = \frac{2}{(\tau^2 s^2 + 2\zeta\tau s + 1)} \right\}, \quad 3 \leq \tau \leq 9, 0.5 \leq \zeta \leq 1.1. \quad (7.36)$$

The large uncertainty on the damping ratio makes the system underdamped for some plants ($\zeta < 1$) and overdamped for others ($\zeta > 1$).

Rather than varying the IMC controller separately from the model, as in the previous example, here we vary both simultaneously, with the controller taken as the inverse of the model so as to reduce the search space. Therefore, the controller model pair that is obtained is not necessarily the best possible. We start by designing the IMC controller using the mid-range plant so that we can use it as a reference to compare with other designs. The IMC model and controller are

$$\tilde{p}(s) = \frac{2}{(36s^2 + 9.6s + 1)}, \quad q(s) = \frac{36s^2 + 9.6s + 1}{2(\varepsilon s + 1)^2} \quad (7.37)$$

The filter time constant of the IMC controller $q(s)$ is chosen so that the robust performance condition is met:

$$|CS(i\omega)| \leq 1.05 \quad \forall p \in S, \forall \omega, \quad (7.38)$$

$$|CS(i\omega_c)| = 1.05 \quad \text{for some } p \in S \text{ and some frequency, } \omega_c. \quad (7.39)$$

Figure 7.17 illustrates that the conditions given by Equations (7.38) and (7.39) are satisfied when epsilon is 10.96.

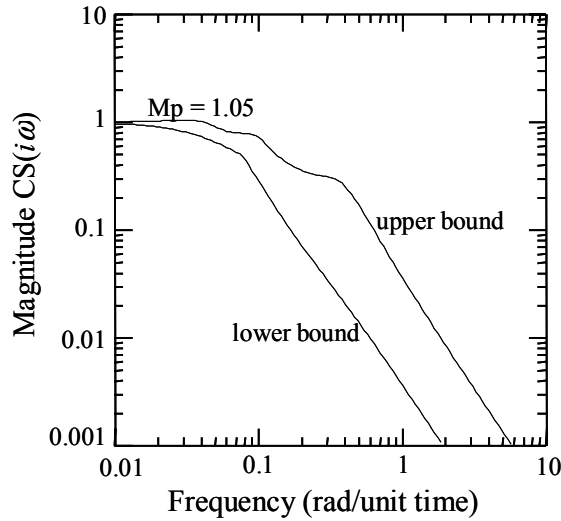


Figure 7.17 Maximum and minimum of $|CS(i\omega)| \forall p \in S$ when $\varepsilon = 10.96$, $\tilde{\tau} = 6$ and $\tilde{\zeta} = 0.8$.

From Figure 7.17 we observe that the lower-bound curve break frequency is about 0.052 rad/sec and the maximum peak of $|CS(i\omega)| \forall p \in S$ is given by the plant with $\tau = 9$ and $\zeta = 1.1$. A lower-bound break frequency of 0.057 implies a settling time of 57 seconds for the slowest plant. This is in agreement with the closed-loop step responses shown in Figure 7.18. This figure also shows that the maximum overshoot to a unit step in setpoint change is 10.48%, produced by the upper extreme plant (i.e., $\tau = 9$ and $\zeta = 1.1$).

To improve the speed of response of the control system, we choose another model in the set S to attempt to shift the lower-bound break frequency to the right, thereby speeding up the slowest closed-loop responses while maintaining $|CS(i\omega)| \leq 1.05$. The fastest design is obtained with a model that has a time constant of 9 and a natural damping ratio of 0.8. Figure 7.19 shows the upper- and lower-bounds on the closed-loop frequency responses. Figure 7.20 shows the closed-loop time responses to a unit-step setpoint change for several different plants. The maximum overshoot is 5.21%, which is produced by the upper extreme plant, and the slowest settling time is about 32 seconds.

Using the optimal model rather than the mid-range model reduces the slowest settling time from 52 seconds to 32 seconds. Moreover, since the maximum overshoot is 5.21%, the filter time constant can be further reduced, thereby further speeding the process response.

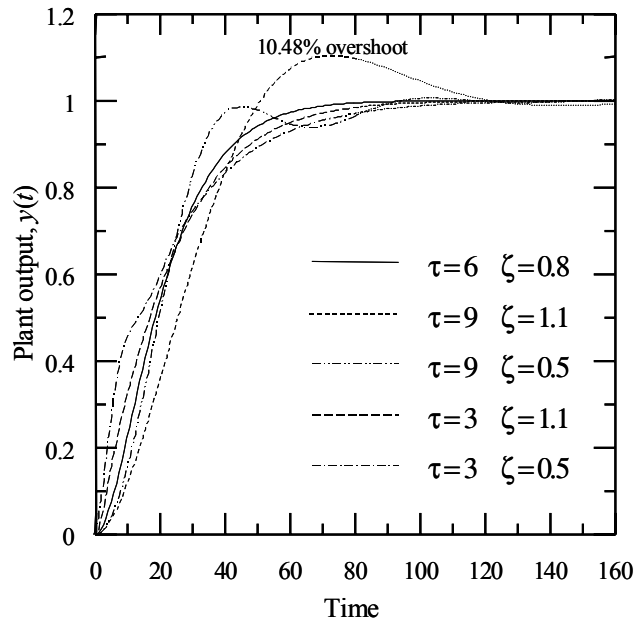


Figure 7.18 Closed-loop response to a unit-step change in setpoint for some plants in S when $\varepsilon = 10.96$, $\tilde{\tau} = 6$ and $\tilde{\zeta} = 0.8$.

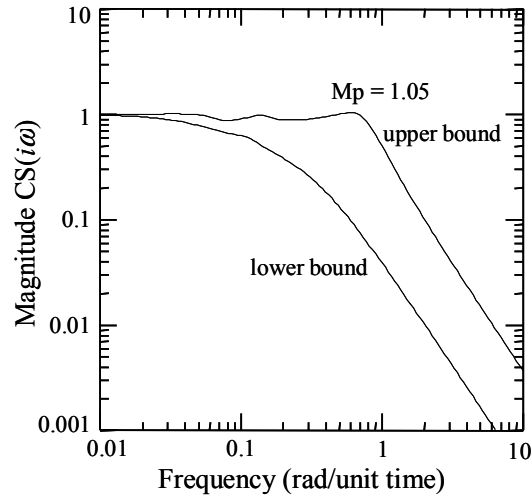


Figure 7.19 Maximum and minimum of $|CS(i\omega)| \forall p \in S$ when $\varepsilon = 4.93$, $\tilde{\tau} = 9$, $\tilde{\zeta} = 0.8$.

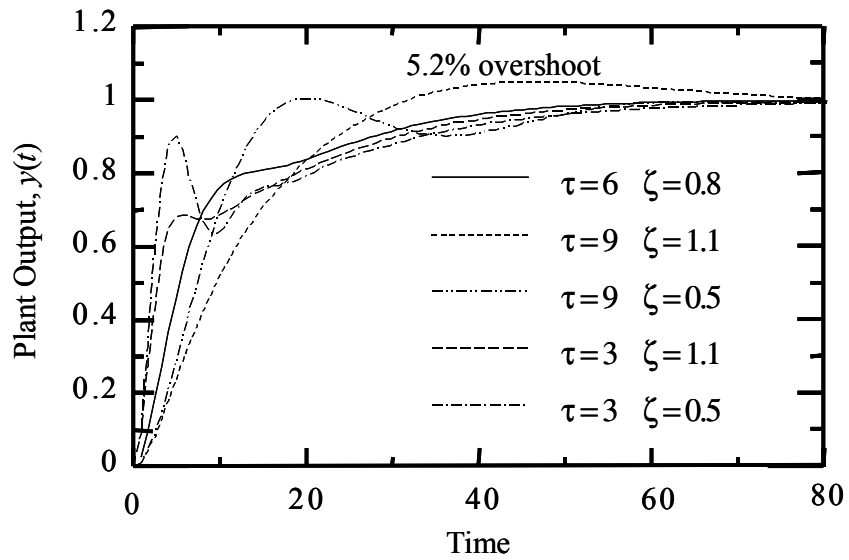


Figure 7.20 Closed-loop response to a unit-step change in setpoint for some plants in S when $\varepsilon = 4.927$, $\tilde{\tau} = 9$ and $\tilde{\zeta} = 0.8$.

7.7 SOFTWARE FOR MP TUNING AND SYNTHESIS

The IMCTUNE software supplied with this text was used to generate all the figures displayed in this section. Appendix G provides a description of this software.

7.8 SUMMARY

Process uncertainty is described by allowing the process to have any parameter values in a predefined set. Tuning an IMC system for an uncertain process (Mp tuning) is carried out by finding the IMC controller filter time constant that yields a specified upper-bound on the maximum magnitude of the closed-loop transfer function between the output and the setpoint (i.e., on the complementary sensitivity function). A common specification, or Mp, for the magnitude of the complementary sensitivity function is 1.05 because this specification usually limits the maximum overshoot to step setpoint changes to about 10%. When, as is usually the case, parametric uncertainty bounds cannot be obtained precisely, then we recommend the use of multiple uncertainty regions, with different Mp specifications in each region, as described and illustrated in Section 7.3.4 and Example 7.5.

The robust stability theorem of Section 7.5 provides the theoretical foundation for Mp tuning. This theorem states that if the Mp is finite, then the control system for an inherently stable process is stable for any process parameter vector in the uncertain set. For the theorem to be applicable, the uncertain parameter set must be connected. (That is, there has to be a path between any two parameters that lies completely within the uncertainty set.) In addition, an uncertain process gain must not change sign within the uncertain set if a no offset control system is to be stable. The process model gain must obey the restrictions in Table 7.3 in order to be able to achieve an arbitrarily small overshoot to a step setpoint change for a large enough filter time constant.

Mp synthesis seeks to find an optimum controller and model for the IMC system. The model and controller are optimal in the sense that they maximize the speed of the slowest closed-loop responses subject to maintaining the specified Mp (i.e., the specified relative stability). An example of a second-order overdamped process with dead time shows that:

1. For gain and dead time uncertainty greater than $\pm 25\%$, a first-order controller performs as well as or better than a model inverse controller. The degree of improvement increases as the uncertainty increases.
2. The traditional engineering approach of fitting a first-order plus dead time model to high-order overdamped processes, and designing the controller based on that model, yields as good or better control system than that based on a process model of the correct order when the process parameters are sufficiently uncertain.
3. Limited numerical experiments, along with the analysis leading to Table 7.3, indicates that the optimum model gain is the upper-bound gain for inherently stable overdamped processes. This seems to be true even if the process operating point is most likely to be about its mid-range gain.

Problems

7.1 In each of the Problems in Chapter 3, assume that the gain and dead time (if any) are nominal values and that actual values of gain and dead time vary independently about the nominal value by $\pm 20\%$. What filter time constant is required to achieve an Mp of 1.05?

7.2 If in problem 7.1, the gain and dead time vary together, rather than independently what filter time constant is required to achieve an Mp of 1.05? That is, the gain K varies as $\bar{K}(1-x) \leq K \leq \bar{K}(1+x)$, and the dead time T varies as $\bar{T}(1-x) \leq T \leq \bar{T}(1+x)$, where x takes values between $\pm .2$.

7.3 Find an optimal model for the following processes and compare the performance of each against a control system using a mid-range model.

$$\text{a. } p(s) = \frac{e^{-s}}{\tau^2 s^2 + 2\zeta\tau s + 1} \quad \begin{array}{l} 3 \leq \tau \leq 9 \\ .5 \leq \zeta \leq 1.1 \end{array}$$

$$\text{b. } p(s) = \frac{e^{-s}}{\tau^2 s^2 + 2\zeta\tau s + 1} \quad \begin{array}{l} 3 \leq \tau \leq 9 \\ .1 \leq \zeta \leq 1.1 \end{array}$$

$$\text{c. } p(s) = \frac{-\tau s + 1}{s + 1} e^{-s} \quad .5 \leq \tau \leq 1.5$$

7.4 How many zeros does the following process have in the right half plane as K ranges from 1 to 10?

$$p(s) = \frac{3(s-1)}{(s+1)} - \frac{K(s-1)e^{-s}}{(s+1)^2}$$

Hint: Convert p into a form that allows application of the Nyquist theorem

7.5 Sketch the slowest expected time response to a step setpoint change for a control system designed so that the fastest response to a step setpoint change does not overshoot by more than 20% for the following process:

$$p(s) = \frac{K(s^2 - s - 1)e^{-Ts}}{(3s+1)(2s+1)(s+1)}, \quad 3 \leq K \leq 7, \quad 2 \leq T \leq 6.$$

7.6 Derive Equations (7.13) and (7.25).

7.7 Show that for a second-order process, any Mp specification greater than 1 can be achieved, provided that $p(0)/\tilde{p}(0) < 2$.

The following questions assume familiarity with the material in Chapter 6.

7.8 For each of the problems in Chapter 3, assume that the gain and dead time (if any) are nominal values and that actual values of gain and dead time vary independently about their nominal values. How much variation is necessary before there is no advantage in using an IMC system over a PID control system?

7.9 Is it feasible to tune a PID controller so that the slowest closed-loop response for the process described below has a time constant of 100 minutes or less?

$$p_1(s) = \frac{K(s-1)e^{-Ts}}{(s+1)(3s+1)}, \quad 2 \leq K \leq 6, \quad 2 \leq T \leq 6$$

If so, what is one such tuning? If not, what is the best response that one can achieve, and what are the PID parameters for such a response? Show how you arrived at your result.

7.10 Over what range of process gains, K , is the closed-loop control system stable for the following process and controller?

$$\text{process: } \frac{K(-2s+1)}{(2s+1)(5s-1)}$$

$$\text{PID controller: } 2.37 \left(1 + \frac{1}{13.7s} + \frac{1.65s}{.082s+1} \right)$$

7.11 Your boss would like you to explore the possibility of speeding up a critical control loop by a factor of 2. The current PID controller tunings give a closed-loop response that can be well approximated by a first-order lag plus dead-time where the lag time constant is 20 minutes and the dead time is 15 minutes. That is

$$\frac{\Delta y(s)}{\Delta r} \cong \frac{e^{-15s}}{20s+1}, \quad \text{where } \Delta y = \text{change in output, } \Delta r = \text{change in setpoint}$$

The process model associated with the above closed-loop response is believed to be of the form

$$\frac{\Delta y(s)}{\Delta m(s)} \cong \frac{10e^{-15s}}{15s+1} \quad \text{where } \Delta m = \text{change in control effort}$$

The process operators are opposed to retuning the controller because they believe that the process gain can vary from 6 to 14. What recommendation would you give your boss and why? What PID controller tunings (i.e., gain, and integral and derivative time constants) would you recommend?

7.12 Design a PID controller for the following process using a nominal gain \hat{K} of 10 for the model, and a specification that no process response to a setpoint change should overshoot the setpoint by more than 10%.

$$\frac{y(s)}{m(s)} = \frac{K(2s^3 - 3s^2 - 1)e^{-s}}{(s^2 + s + 1)(s + 3)^2}, \quad 5 \leq K \leq 15,$$

Over what range of process gains, K , is the system stable?

7.13 Select a PID controller for the process given by

$$p(s) = \frac{K(1 + 5e^{-3s})}{(2s + 1)^2} \quad \text{for } 1 \leq K \leq 5.$$

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