Tutorial 6 solution

1.

I. $a_n = 5a_{n-1} - 6a_{n-2}$ for n>=2, $a_0 = 1$, $a_1 = 0$

Solution -> $x^2 - 5x + 6 = 0$ is the characteristic equation of the above equation. Root of the equations are 2 and 3.

So $a_n = \alpha_1 2^n + \alpha_2 3^n$

For a_0 -> $1 = \alpha_1 + \alpha_2$ (1

For a_1 -> $0 = 2\alpha_1 + 3\alpha_2$ (2

After solving equation 1 and 2 , $\alpha_1 = 3$ and $\alpha_2 = -2$

$\mathbf{a_n = 3*2^n - 2*3^n}$

II. $a_n = -4a_{n-1} - 4a_{n-2}$ for n>=2, $a_0 = 0$, $a_1 = 4$

Solution -> $x^2 + 4x + 4 = 0$ is the characteristic equation of the above equation. Root of the equations are -2 only.

So $a_n = \alpha_1 (-2)^n + \alpha_2 n(-2)^n$

For a_0 -> $0 = \alpha_1$ (1

For a_1 -> $4 = -2\alpha_1 +- \alpha_2$ (2

After solving equation 1 and 2 , $\alpha_1 = 0$ and $\alpha_2 = -4$

$\mathbf{a_n = -4*n*(-2)^n}$

2. How many different messages can be transmitted in n microseconds using three different signal if one signal requires 1 microsecond for transmittal, the other two signals require 2 microseconds each for transmittal, and a signal in message is followed immediately by the next signal?

Solution Recurrence of the above problem is following

$a_n = 1a_{n-1} + 2a_{n-2}$

$x^2 - x - 2 = 0$ is the characteristic equation of the above equation. Root of the equations are 2 and -1 .

So $a_n = \alpha_1 (2)^n + \alpha_2 (-1)^n$

Initial conditions are $a_0 = 1$, $a_1 = 1$

For a_0 -> $1 = \alpha_1 + \alpha_2$ (1

For a_1 -> $1 = 2\alpha_1 - \alpha_2$ (2

After solving equation (1 and (2 , $\alpha_1 = 2/3$ and $\alpha_2 = 1/3$

$\mathbf{a_n = 2/3*2^n + 1/3*(-1)^n}$

3. A new employee at an exciting new software company starts with a salary of $50,000 and is promised that at the end of each year her salary will be double her salary of the previous year, with an extra increment of $10,000 for each year she has been with the company.

a) Construct a recurrence relation for her salary for her nth year of employment.

b) Solve this recurrence relation to find her salary for her nth year of employment.

Solution

a) Recurrence of above problem is following

$a_n = 2a_{n-1} + 10000$

Initial condition is , $a_1 = 50000$

b) $a_n = 2^1 a_{n-1} + 10000 \to 2^1(2a_{n-2} + 10000) + 10000 \to 2^2(2a_{n-3} + 10000) + (1+2)10000 \to 2^3(2a_{n-2} + 10000) + (1+2+4)10000 \to \ldots \to 2^{n-2}(2a_1 + 10000) + (1+2+4+\ldots+2^{n-3})10000$

$a_n = 2^{n-2}(2*50000 + 10000) + (1+2+4+\ldots+2^{n-3})10000$

$= 2^{n-1}*50000 + (1+2+4+\ldots+2^{n-3}+2^{n-2})10000$

$= 2^{n-1}*50000 + (2^{n-1}-1)10000$

$a_n = 2^{n-1}60000 - 10000$

4. Let $\mathbf{a_n}$ be the sum of first n perfect squares, that is , $\sum_{k=1}^{n} k^2$. Show that the sequence {an} satisfies the linear nonhomogeneous recurrence relation $a_n = a_{n-1} + n^2$ and the initial condition $a_1 = 1$.

Solution

The associate linear homogenous recurrence relation for a_n is

$a_n = a_{n-1}$

The solution of this homogenous recurrence relation are given by $a_n^{(h)} = c(1)^n = c$, where c is constant . To find all solutions of $a_n = a_{n-1} + n^2$ we need find a single particular solution. Because $F(n) = n^2 .(1)^n$ and s =1 is a root degree one of the characteristic equation of the associate linear homogenous recurrence relation, there is a particular solution of the form $n(p_2n^2 + p_1n + p_0)$

Inserting this into the recurrence relation gives $n(p_2n^2 + p_1n + p_0) = n(p_2(n-1)^2 + p_1(n-1) + p_0) + n^2$.

Solve the equation for the coefficient we get the particular solution

$a_n^{(p)} = n(n+1)(2n+1)/6$

Hence the solution is

$a_n = c + n(n+1)(2n+1)/6$

since $a_1 = 1$ it follows that c = 0; hence $a_n = n(n+1)(2n+1)/6$

5. Use generating functions to solve the recurrence relation $a_k = 5a_{k-1} - 6a_{k-2}$ with initial conditions $a_0 = 6$ and $a_1 = 30$.

Solution

$a_k = 18 \cdot 3^k - 12 \cdot 2^k$

6. Suppose that a valid codeword is an n digit number in decimal; notation containing an even number of 0s. Let $\mathbf{a_n}$ denote the number of valid codeword of length n. Find the recurrence relation of $\mathbf{a_n}$. Use generating functions to find an explicit formula for $\mathbf{a_n}$.

Solution

Note that $a1 = 9$ because there are 10 one-digit strings, and only one, namely, the string 0, is not valid.A recurrence relation can be derived for this sequence by considering how a valid *n*-digit string can be obtained from strings of $n - 1$ digits. There are two ways to form a valid string with *n* digits from a string with one fewer digit.

First, a valid string of *n* digits can be obtained by appending a valid string of $n - 1$ digits with a digit other than 0. This appending can be done in nine ways. Hence, a valid string with *n*

digits can be formed in this manner in 9*an*−1 ways. Second, a valid string of *n* digits can be obtained by appending a 0 to a string of length
n − 1 that is not valid. (This produces a string with an even number of 0 digits because the invalid string of length *n* − 1 has an odd number of 0 digits.) The number of ways that this can be done equals the number of invalid *(n* − 1*)*-digit strings. Because there are 10*n*−1 strings of length *n* − 1, and *an*−1 are valid, there are 10*n*−1 − *an*−1 valid *n*-digit strings obtained by appending an invalid string of length *n* − 1 with a 0.

Because all valid strings of length *n* are produced in one of these two ways, it follows that there are

a_n **= $9a_{n-1}$ + (10_{n-1}- a_{n-1})**

we multiply both sides of the recurrence relation by x^n ,to obtain

a_n **= $8a_{n-1}$** x^n **+ 10_{n-1}** x^n

let $G(x) = \sum_{n=0}^{\infty} a_n x^n$ be the generating function of the sequence a_0,a_1,a_2…… We sum both sides of the last equations starting with n =1. To find that

$$\begin{aligned} G(x) - 1 = \sum_{n=1}^{\infty} a_n x^n &= \sum_{n=1}^{\infty} (8a_{n-1}x^n + 10^{n-1}x^n) \\ &= 8\sum_{n=1}^{\infty} a_{n-1}x^n + \sum_{n=1}^{\infty} 10^{n-1}x^n \\ &= 8x\sum_{n=1}^{\infty} a_{n-1}x^{n-1} + x\sum_{n=1}^{\infty} 10^{n-1}x^{n-1} \\ &= 8x\sum_{n=0}^{\infty} a_n x^n + x\sum_{n=0}^{\infty} 10^n x^n \\ &= 8xG(x) + x/(1-10x), \end{aligned}$$

There for we have

$$G(x) - 1 = 8xG(x) + x/(1-10x).$$

Expanding the right hand side of the eqation into partial fractions gives

$$G(x) = \frac{1}{2}\left(\frac{1}{1-8x} + \frac{1}{1-10x}\right).$$

This is equivalent to

$$G(x) = \frac{1}{2}\left(\sum_{n=0}^{\infty} 8^n x^n + \sum_{n=0}^{\infty} 10^n x^n\right)$$

$$= \sum_{n=0}^{\infty} \frac{1}{2}(8^n + 10^n)x^n.$$

$$a_n = \frac{1}{2}(8^n + 10^n).$$

7. Consider the nonhomogeneous linear recurrence relation $a_n = 3\, a_{n-1} + 2^n$.
   I. Show that $a_n = -2^{n+1}$ is a solution of this recurrence relation.
   II. Find all the solutions of this recurrence relation.
   III. Find the solution with $a_0 = 1$

   Solution

   I $3a_{n-1} + 2n = 3(-2)^n + 2^n = 2^n(-3 + 1) = -2^{n+1} = a_n$
   II $a_n = \alpha 3^n - 2^{n+1}$
   III $a_n = 3^{n+1} - 2^{n+1}$

8. For the recurrence given bellow answer the following question

$$T(n) = 7T\left(\frac{n}{2}\right) - 6T\left(\frac{n}{4}\right), T(1) = 2, T(2) = 7$$

   I. Does recurrence for T(n) involve previous terms which are within a fixed range of n?
   II. If not , use the domain transform such that for given terms T(f(n)), T(f(f(n)))… , select a function g(m) which hold the property f(g(m)) = g(m-1).
   III. Solve the transformed recurrence with initial conditions written above.
   IV. Find the solution of original recurrence using inverse transformation.

Solution

In this case $f(n) = n/2$ and by selecting $g(m) = 2^m$ we obtain
$f(g(m)) = g(m)/2 = 2^m/2 = 2^{m-1} = g(m-1)$.
By equating n with 2^m we obtain

$$T(2^m) = 7T\left(\frac{2^m}{2}\right) - 6T\left(\frac{2^m}{4}\right) = 7T(2^{m-1}) - 6T(2^{m-2}),$$

Let $S(m) = T(2^m)$. The recurrence for S(m) is
$S(m) = 7S(m - 1) - 6S(m - 2)$, $S(0) = T(1) = 2$, $S(1) = T(2) = 7$.
Note that the initial conditions for S(m) were also transformed. This recurrence was solved in
Example 1; its solution is $S(m) = 6^m + 1$. The inverse of $g(m) = 2^m$ is $g^{-1}(n) = \log_2 n$. The solution for T(n) is
$T(n) = S(g^{-1}(n)) = S(\log_2 n) = 6^{\log_2 n} + 1 = n^{\log_2 6} + 1$.

9. For the recurrence given bellow , answer the following question

$$T(2) = 2.\frac{T(n-1)^3}{T(n-2)^2}, T(0) = 2, T(1) = 2$$

   I. Is the recurrence linear?
   II. If not, apply the range transformations to the recurrence to make it linear. Idea is for given relation for T(n) in terms of T(n-1) , T(n-2),……., T(n-k), find a function f(x) such that f(T(n)) is a linear combination of f(T(n-1)),……, f(T(n-k)).
   III. Solve the transformed recurrence with initial condition written above.
   IV. Find the solution of original recurrence using the inverse transformation.

Solution

In this case by selecting $f(x) = \log_2 x$ we obtain

$f(T(n)) = \log_2 T(n)$

$= \log_2 2 + \log_2(T(n-1)^3) - \log_2 T(n-2)^2$

$= 3 \log_2 T(n-1) - 2 \log_2 T(n-2) + 1$

$= 3f(T(n-1)) - 2f(T(n-2)) + 1.$

If we let W(n) = f(T(n)) and the we obtain the following recurrence for W(n)

$W(n) = 3W(n-1) - 2W(n-2) + 1,$

which has characteristic equation:

$(x-1)(x-2)(x-1)^{0+1} = (x-1)^2(x-2) = 0.$

The general form of the solution is:

$W(n) = c_1 1^n + c_2 n 1^n + c_3 2^n.$

Since there are three constants, one additional initial value must be obtained form the recurrence for solving the constants.

$$T(2) = 2.\frac{T(1)^3}{T(0)^2}=2.\frac{2^3}{2^2} = 4$$

The values for the first three terms of T(n) must be transformed into the corresponding values for

W(n) using f().

$W(0) = f(T(0)) = \log_2 T(0) = \log_2 2 = 1$

$W(1) = f(T(1)) = \log_2 T(1) = \log_2 2 = 1$

$W(2) = f(T(2)) = \log_2 T(2) = \log_2 4 = 2$

Constraints for constants:

$W(0) = 1 = c_1 + c_3$

$W(1) = 1 = c_1 + c_2 + 2c_3$

$W(2) = 2 = c_1 + 2c_2 + 4c_3$

Solution for constants: $c_1 = 0$, $c_2 = -1$, and $c_3 = 1$.

The solution to W(n)'s recurrence is $2^n - n$.

The final step is to transform the solution for W(n) into the solution for T(n) using the inverse

of f(). The inverse of $f(x) = \log_2 x$ is $f^{-1}(y) = 2^y$ so the solution for T(n) is

$T(n) = f^{-1}(W(n)) = 2^{W(n)} = 2^{2^n - n} = 2^{2^n}/2^n$

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