Two-Way Analysis of Variance

Example 1: Paints are commonly applied to metal surfaces. An experiment was performed to investigate the effect of paint primers (3 levels: three different primers: 1,2,3), and the application method (2 levels: two application methods: spray (S), dip(D)) on the paint adhesion force (adhesion: the response variable).

The data is given as the following. This is a common form to have the data supplied, and usually the format required to read it into statistical software.

adhesion	primer	application
4.0	1	D
4.5	1	D
4.3	1	D
5.6	2	D
4.9	2	D
5.4	2	D
3.8	3	D
3.7	3	D
4.0	3	D
5.4	1	S
4.9	1	S
5.6	1	S
5.8	2	S
6.1	2	S
6.3	2	S
5.5	3	S
5.0	3	S
5.0	3	S

The response variable is adhesion, and the two factors are primer and application . Note the factors are indicator variables that simply tell us which treatment was applied.

It is often convenient to rearrange these data in a two-way table

		application	
		D	S
	1	4.0,4.5,4.3	5.4,4.9.5.6
primer	2	5.6,4.9,5.4	5.8,6.1,6.3
	3	3.8,3.7,4.0	5.5,5.0,5.0

Here, it is noted that there is a 3 observations, or replicates, per cell (hence K=3), and there are three levels of factor A (primer) so I=3, and two levels of factor B so J=2.

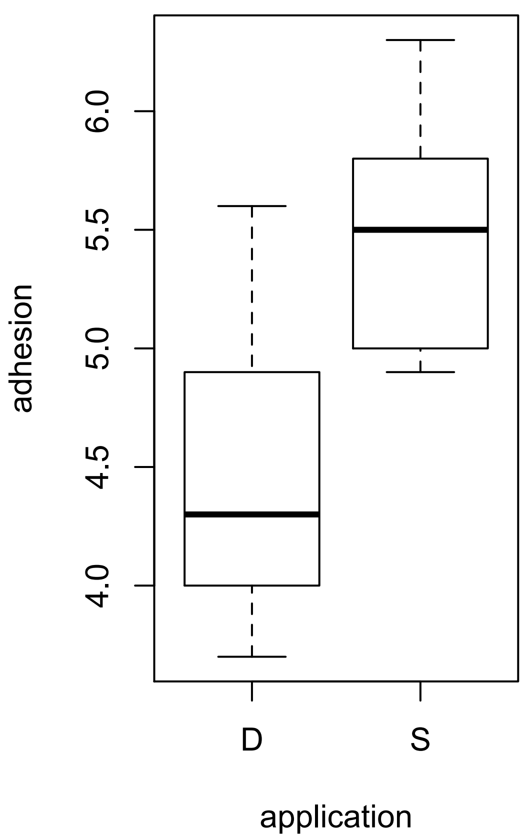
To apply the ANOVA formulas in the class notes we would need to compute:

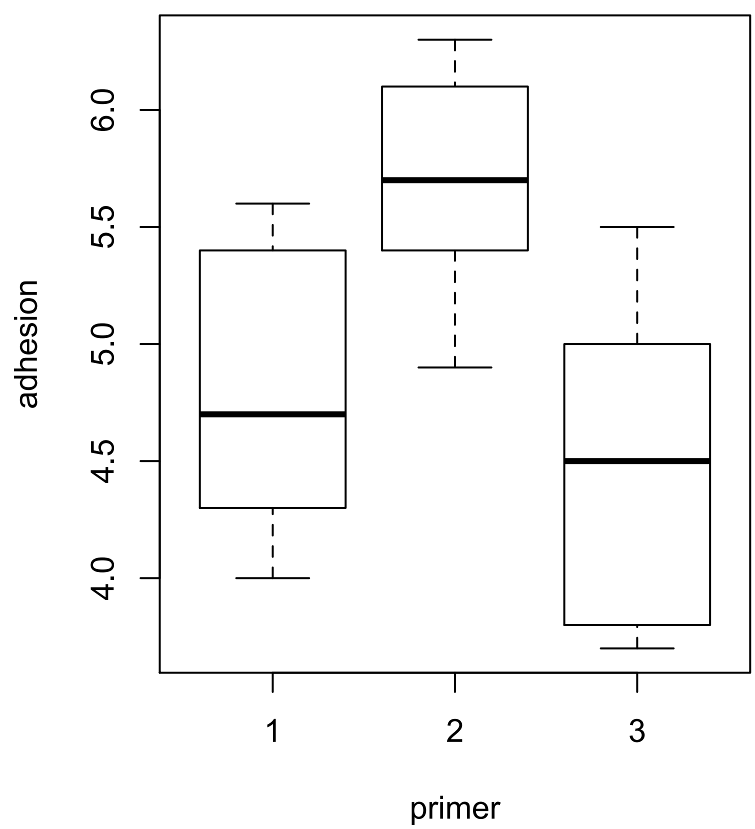
- the cell means: the average of each of the three values for the 9 cells in the above table.
- the grand mean: the overall mean of all the 18 data points
- the marginal means: the means of the rows, i.e. for primers 1,2 and 3; and the means of the columns, i.e. for applications D and S.

However, if the are using statistical software these will be done as part of the two-way ANOVA analysis procedure.

The first step in any analysis should be to do the appropriate plots of the data. For Two-Way ANOVA these include the following:

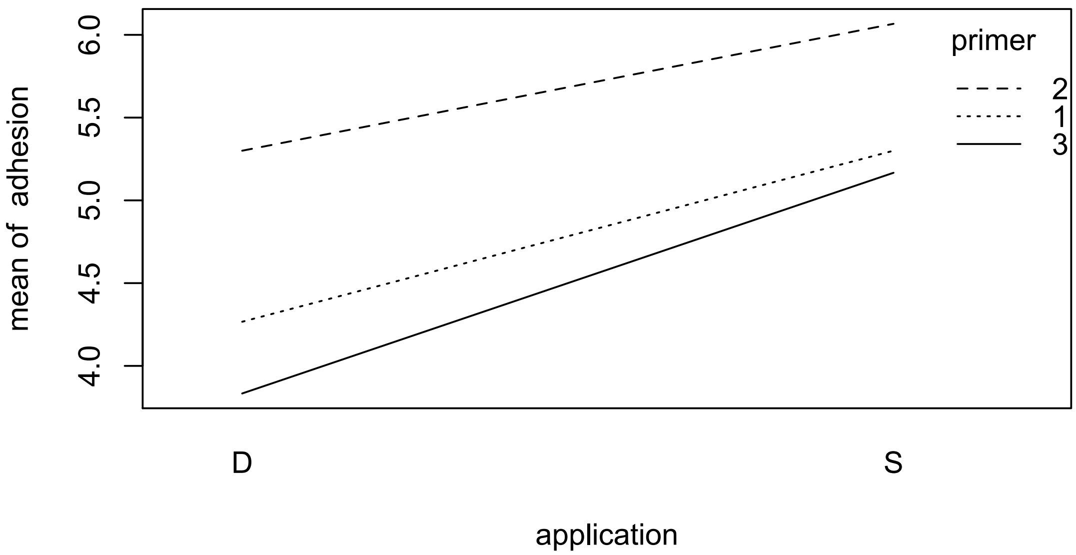
1) Data plots or box plots for the response (adhesion) vs each of the factors (application and primer) at their different levels

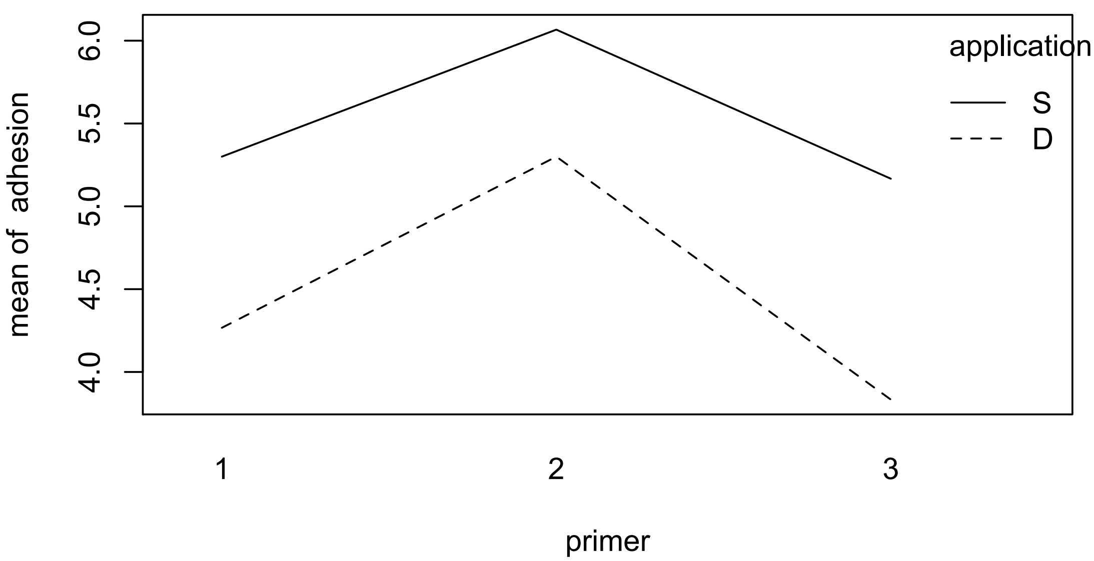




The above box and whisker plots give a sense of how much variation occurs in the response due to changes in the factors. This is important since the purpose of two-way ANOVA is to test for differences in the means of the treatment combinations. However, it is difficult to diagnose significance of the differences just from examining a plot.

2) Interaction plots that show how the cell means vary with respect to both factors:





Interaction plots tell us whether the factors vary independently of one another (an additive ANOVA model), or whether the factors are dependent. If the factors are dependent (the response is determined by the interaction of the level of one factor with the level of the other factor). The central diagnostic for no interaction is whether the lines in the interaction plots are parallel. In the case of interaction, the lines will follow different patterns and tend to cross one another. For this example, it appears there is no interaction, but this will be formally tested below.

The two-way ANOVA procedure is usually carried out by statistical software (e.g. MINITAB or R). To look at just the main effects (or the two factors) the following models is fit (see course notes for details)

$$Y_{ijk} = \mu + \alpha_i + \beta_j + \varepsilon_{ijk}$$

The output of the statistical software will look like:

```
Analysis of Variance Table

Response: adhesion
            Df Sum Sq Mean Sq F value    Pr(>F)
primer       2 4.5811  2.2906  26.119 1.884e-05 ***
application  1 4.9089  4.9089  55.975 2.960e-06 ***
Residuals   14 1.2278  0.0877
```

The P-values indicate that both factors are highly significant. This means that both the primer type and the application method both significantly affect the paint adhesion.

To formally test whether or not there is interaction (does the paint adhesion due to a primer type influenced by which application method is used? Or do primer type and application method interact to affect paint adhesion), we fit the two-way ANOVA with interaction model

$$Y_{ijk} = \mu + \alpha_i + \beta_j + \gamma_{ij} + \varepsilon_{ijk}$$

This yields the following results:

```
Analysis of Variance Table

Response: adhesion
                   Df Sum Sq Mean Sq F value    Pr(>F)
primer              2 4.5811  2.2906 27.8581 3.097e-05 ***
application         1 4.9089  4.9089 59.7027 5.357e-06 ***
primer:application  2 0.2411  0.1206  1.4662    0.2693
Residuals          12 0.9867  0.0822
```

The P-value for the interaction term is 0.27, and hence interaction of the factors is deemed not significant. This is consistent with what we expect from our earlier examination of the interaction plots.

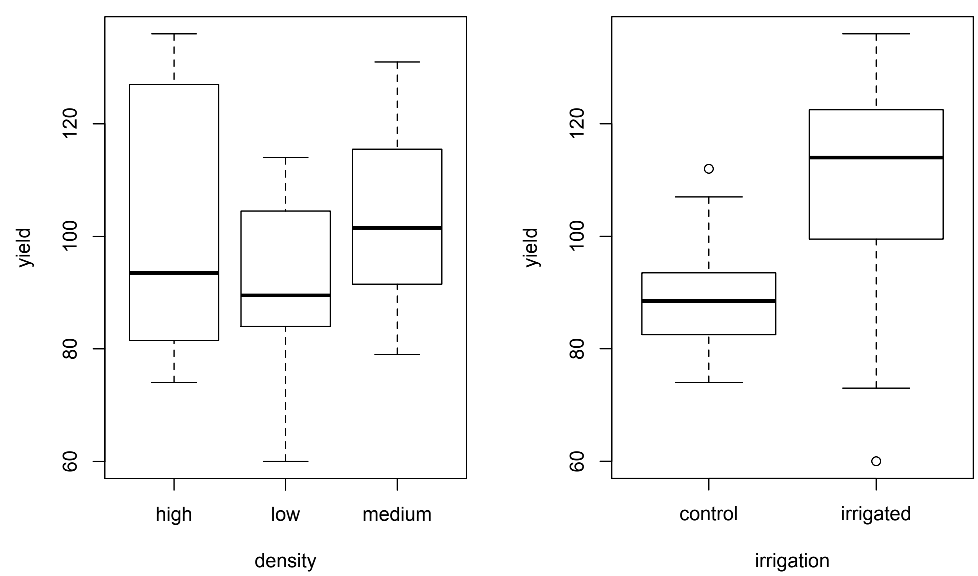
Note that in general the main effects must also be significant, before we can conclude the interaction is significant.

EXAMPLE 2: Agricultural researchers designed an experiment to look at crop yield in a number of plots in a research farm. Crop yield was recorded as a function of irrigation (2 levels: irrigated or not), sowing density (3 levels: low, medium, high), fertilizer application (3 levels: low, medium, high). The data set is the following:

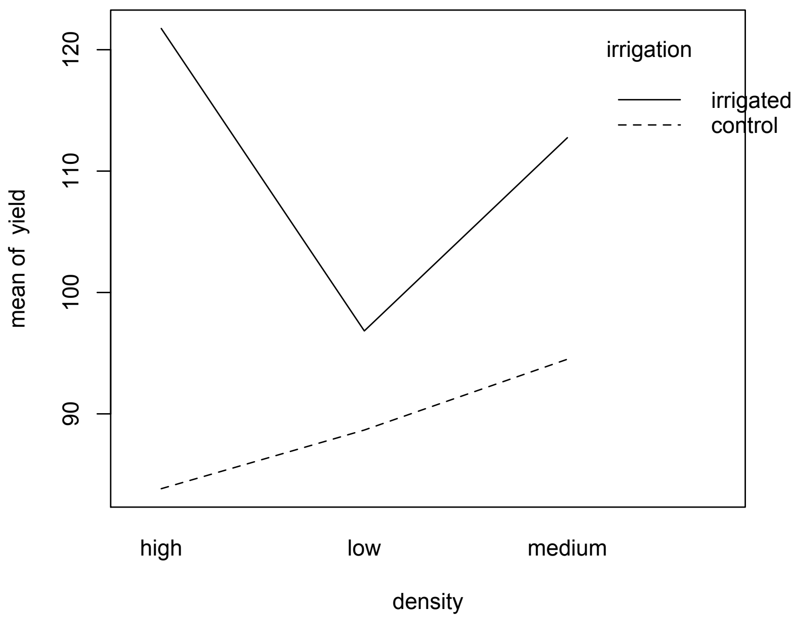
	yield	block	irrigation	density	fertilizer
1	90	A	control	low	N
2	95	A	control	low	P
3	107	A	control	low	NP
4	92	A	control	medium	N
5	89	A	control	medium	P
6	92	A	control	medium	NP
7	81	A	control	high	N
8	92	A	control	high	P
9	93	A	control	high	NP
10	80	A	irrigated	low	N
11	87	A	irrigated	low	P
12	100	A	irrigated	low	NP
13	121	A	irrigated	medium	N
14	110	A	irrigated	medium	P
15	119	A	irrigated	medium	NP
16	78	A	irrigated	high	N
17	98	A	irrigated	high	P
18	122	A	irrigated	high	NP
19	83	B	control	low	N
20	80	B	control	low	P
21	95	B	control	low	NP
22	98	B	control	medium	N
23	98	B	control	medium	P
24	106	B	control	medium	NP
25	74	B	control	high	N
26	81	B	control	high	P
27	74	B	control	high	NP
28	102	B	irrigated	low	N
29	109	B	irrigated	low	P
30	105	B	irrigated	low	NP
31	99	B	irrigated	medium	N
32	94	B	irrigated	medium	P
33	123	B	irrigated	medium	NP
34	136	B	irrigated	high	N
35	133	B	irrigated	high	P
36	132	B	irrigated	high	NP
37	85	C	control	low	N
38	88	C	control	low	P
39	88	C	control	low	NP
40	112	C	control	medium	N
41	104	C	control	medium	P
42	91	C	control	medium	NP
43	82	C	control	high	N
44	78	C	control	high	P
45	94	C	control	high	NP
46	60	C	irrigated	low	N
47	104	C	irrigated	low	P
48	114	C	irrigated	low	NP
49	90	C	irrigated	medium	N
50	118	C	irrigated	medium	P
51	113	C	irrigated	medium	NP

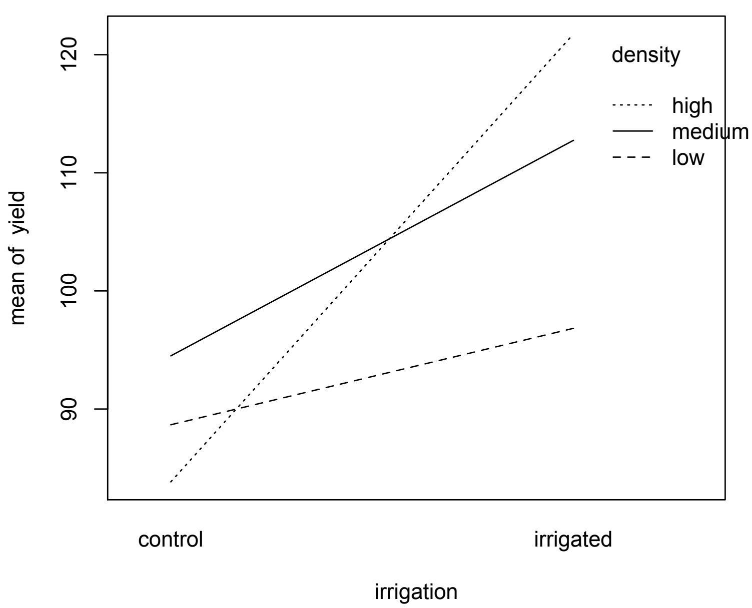
52	119	C	irrigated	high	N
53	122	C	irrigated	high	P
54	136	C	irrigated	high	NP
55	86	D	control	low	N
56	78	D	control	low	P
57	89	D	control	low	NP
58	79	D	control	medium	N
59	86	D	control	medium	P
60	87	D	control	medium	NP
61	85	D	control	high	N
62	89	D	control	high	P
63	83	D	control	high	NP
64	73	D	irrigated	low	N
65	114	D	irrigated	low	P
66	114	D	irrigated	low	NP
67	109	D	irrigated	medium	N
68	131	D	irrigated	medium	P
69	126	D	irrigated	medium	NP
70	116	D	irrigated	high	N
71	136	D	irrigated	high	P
72	133	D	irrigated	high	NP

For this analysis, we will focus only on the response (crop yield) as it is influenced by two factors: density and irrigation. These data are plotted below



Interaction plots are given below, and suggest significant interaction between density and irrigation in their effect on crop yield.


irrigation
irrigated
control
mean of yield
120
110
100
90
high
low
medium
density

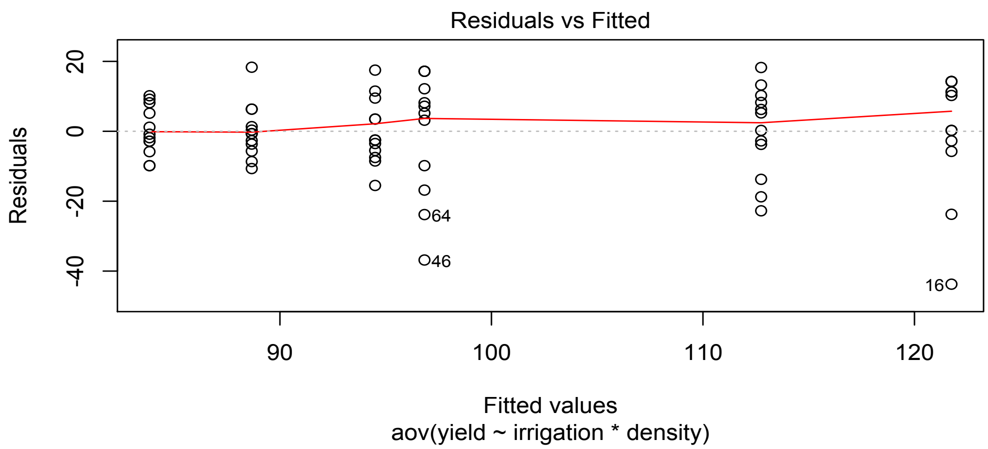

density
high
medium
low
mean of yield
120
110
100
90
control
irrigated
irrigation

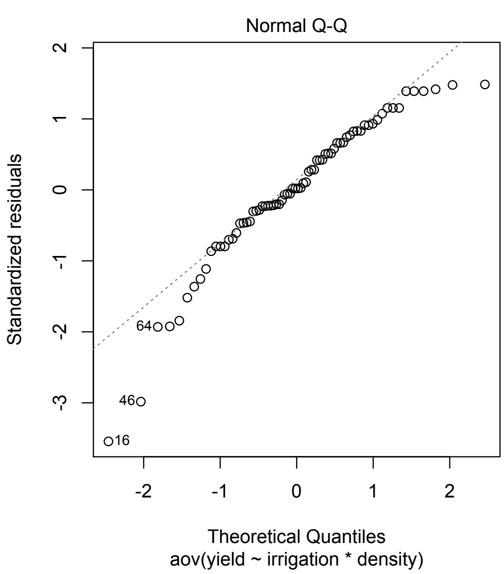
The output from a Two-Way ANOVA that includes interaction is:

	Df	Sum Sq	Mean Sq	F value	Pr(>F)	
irrigation	1	8278	8278	49.785	1.29e-09	***
density	2	1758	879	5.288	0.007412	**
irrigation:density	2	2747	1374	8.261	0.000628	***
Residuals	66	10974	166			

The analysis suggests that crop yield is significant affected by irrigation, sowing density, and their interaction.

As a final step of any analysis we should check whether or not the assumptions of the statistical model are satisfied. The main ones we can check graphically are: (i) equality of variance; and (ii) normality of the residuals. The plots are given below.





It appears that the residuals have a relatively constant variance, and that they follow roughly a normal distribution (though they have some deviations from a straight line in the tails, which is a common feature in the analysis of real data). We conclude that the assumptions are roughly satisfied, and hence the inferential conclusions we have made are reasonable.

EXAMPLE:

Two-Way ANOVA

With Blocking

(from Devore)

A consumer product-testing organization wished to compare the annual power consumption for five different brands of dehumidifier. Because power consumption depends on the prevailing humidity level, it was decided to monitor each brand at four different levels ranging from moderate to heavy humidity (thus blocking on humidity level). Within each level, brands were randomly assigned to the five selected locations. The resulting observations (annual kWh) appear in Table 11.2, and the ANOVA calculations are summarized in Table 11.3.

Table 11.2 Power Consumption Data for Example 11.5

Treatments (brands)	Blocks (humidity level) 1	2	3	4	$x_{i\cdot}$	$\bar{x}_{i\cdot}$
1	685	792	838	875	3190	797.50
2	722	806	893	953	3374	843.50
3	733	802	880	941	3356	839.00
4	811	888	952	1005	3656	914.00
5	828	920	978	1023	3749	937.25
$x_{\cdot j}$	3779	4208	4541	4797	17,325	
$\bar{x}_{\cdot j}$	755.80	841.60	908.20	959.40		866.25

Table 11.3 ANOVA Table for Example 11.5

Source of Variation	df	Sum of Squares	Mean Square	f
Treatments (brands)	4	53,231.00	13,307.75	$f_A = 95.57$
Blocks	3	116,217.75	38,739.25	$f_B = 278.20$
Error	12	1671.00	139.25	
Total	19	171,119.75		

Since $F_{.05,4,12} = 3.26$ and $f_A = 95.57 \geq 3.26$, H_0 is rejected in favor of H_a. Power consumption appears to depend on the brand of humidifier. To identify significantly different brands, we use Tukey's procedure. $Q_{.05,5,12} = 4.51$ and $w = 4.51\sqrt{139.25/4} = 26.6$.

EXAMPLE:

Two-Way ANOVA

With Interaction

(from Devore)

Lightweight aggregate asphalt mix has been found to have lower thermal conductivity than a conventional mix, which is desirable. The article "Influence of Selected Mix Design Factors on the Thermal Behavior of Lightweight Aggregate Asphalt Mixes" (*J. of Testing and Eval.*, 2008: 1–8) reported on an experiment in which various thermal properties of mixes were determined. Three different binder grades were used in combination with three different coarse aggregate contents (%), with two observations made for each such combination, resulting in the conductivity data (W/m· °K) that appears in Table 11.6.

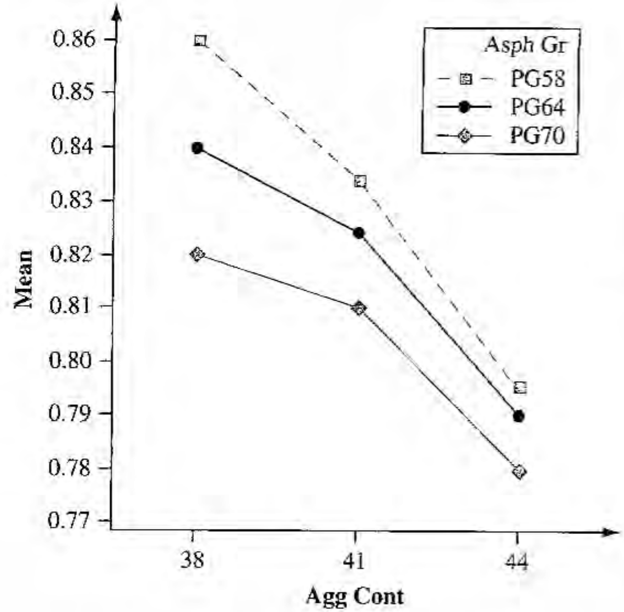
Table 11.6 Conductivity Data for Example 11.7

	Coarse Aggregate Content (%)			
Asphalt Binder Grade	**38**	**41**	**44**	$\bar{x}_{i\cdot\cdot}$
PG58	.835, .845	.822, .826	.785, .795	.8180
PG64	.855, .865	.832, .836	.790, .800	.8297
PG70	.815, .825	.800, .820	.770, .790	.8033
$\bar{x}_{\cdot j\cdot}$	.8400	.8227	.7883	

Here $I = J = 3$ and $K = 2$ for a total of $IJK = 18$ observations. The results of the analysis are summarized in the ANOVA table which appears as Table 11.7 (a table with additional information appeared in the cited paper).

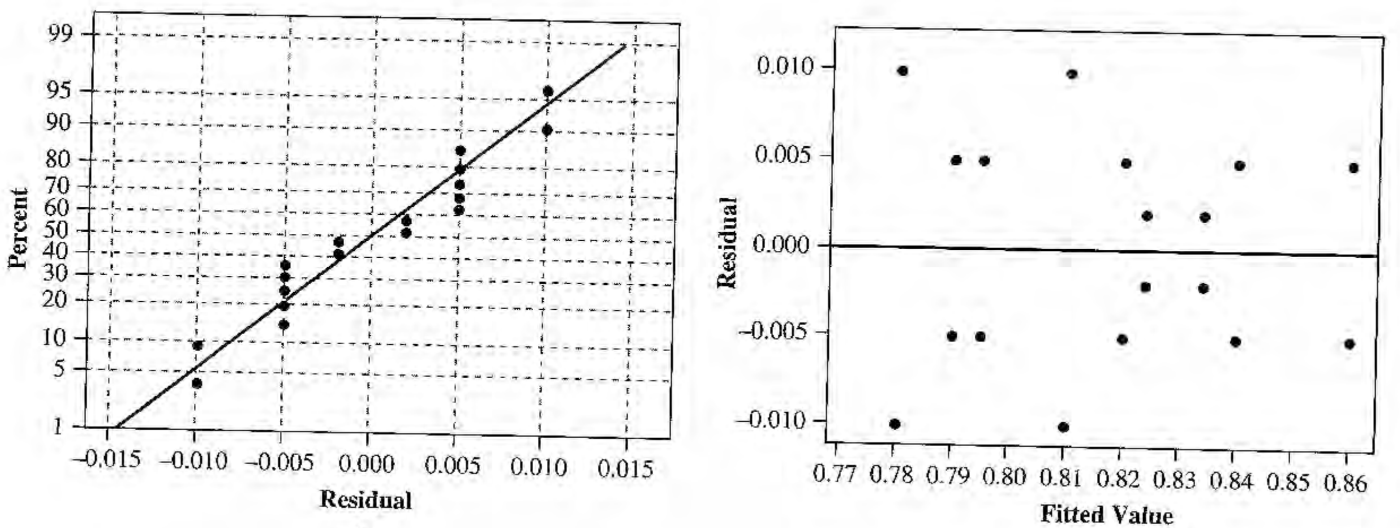
Table 11.7 ANOVA Table for Example 11.7

Source	DF	SS	MS	f	P
AsphGr	2	.0020893	.0010447	14.12	0.002
AggCont	2	.0082973	.0041487	56.06	0.000
Interaction	4	.0003253	.0000813	1.10	0.414
Error	9	.0006660	.0000740		
Total	17	.0113780			



(a)

Figure 11.5 Interaction Plots for the Asphalt Data of Exa



(a)

(b)