FINITENESS OBSTRUCTIONS AND EULER CHARACTERISTICS OF CATEGORIES

THOMAS M. FIORE, WOLFGANG LÜCK, AND ROMAN SAUER

Abstract. We introduce notions of finiteness obstruction, Euler characteristic, L^2-Euler characteristic, and Möbius inversion for wide classes of categories. The finiteness obstruction of a category Γ of type (FP) is a class in the projective class group $K_0(R\Gamma)$; the Euler characteristic and L^2-Euler characteristic are respectively its $R\Gamma$-rank and L^2-rank. We also extend the second author's K-theoretic Möbius inversion from finite categories to quasi-finite categories. Our main example is the proper orbit category, for which these invariants are established notions in the geometry and topology of classifying spaces for proper group actions. Baez-Dolan's groupoid cardinality and Leinster's Euler characteristic are special cases of the L^2-Euler characteristic. Some of Leinster's results on Möbius-Rota inversion are special cases of the K-theoretic Möbius inversion.

0. Introduction and statement of results

The Euler characteristic is one the earliest and most elementary homotopy invariants. Though purely combinatorially defined for finite simplicial complexes as the alternating sum of the numbers of simplices in each dimension, the Euler characteristic has remarkable connections to geometry. For example, for closed connected orientable surfaces M, the Euler characteristic determines the genus: $g = 1 - \frac{1}{2}\chi(M)$. For such M, if $\chi(M)$ is negative, then M admits a hyperbolic metric. More substantially, the celebrated Gauss–Bonnet Theorem computes the Euler characteristic in terms of curvature. A further example of geometry in the Euler characteristic is provided by the Hopf-Singer conjecture.

Of course, Euler characteristics are not only defined for finite simplicial complexes or manifolds, but also for a great variety of objects, such as equivariant spaces, orbifolds, or finite posets. Baez-Dolan considered in [2] an Euler characteristic (groupoid cardinality) for finite groupoids and certain infinite ones, such as the groupoid of finite sets. Leinster and Berger-Leinster have considered Euler characteristics not just of finite posets and groupoids, but more generally of finite categories in [13] and [7]. If a finite category admits both a weighting and coweighting, then it admits an Euler characteristic in the sense of Leinster.

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In the present paper, we define Euler characteristics for wide classes of categories, provide a unified conceptual framework in terms of finiteness obstructions and projective class groups, and extract geometric and algebraic information from our invariants in certain cases. This obstruction-theoretic framework works well for both finite and infinite categories. Our main example is the proper orbit category of a group G. In this case, our invariants are established geometric invariants of the classifying space for proper G-actions. We also extend the second author's K-theoretic Möbius inversion from finite EI-categories to quasi-finite EI-categories (a category Γ is said to be EI if each endomorphism in Γ is an isomorphism). The K-theoretic Möbius inversion does not require the categories in question to be skeletal, unlike the Möbius inversion of Leinster [13]. Several of the results of [13] are special cases.

Our point of departure is the theory of projective modules over a category and the associated projective class group. Let Γ be a small category, and R an associative commutative ring with unit. An *$R\Gamma$-module* is a functor from Γ^{op} to the abelian category of left R-modules. If Γ is a group G viewed as a one-object category, then an $R\Gamma$-module is nothing more than a right RG-module. The category $\mathsf{MOD}\text{-}R\Gamma$ of $R\Gamma$-modules is an abelian category, and therefore we automatically have the notions of projective $R\Gamma$-module, chain complexes of $R\Gamma$-modules, and resolutions of $R\Gamma$-modules. The finiteness obstruction, whenever it exists, lives in the *projective class group* $K_0(R\Gamma)$, which is the free abelian group on the isomorphism classes of finitely generated projective $R\Gamma$-modules modulo short exact sequences. We say that Γ is *of type (FP$_R$)* if the constant $R\Gamma$-module $\underline{R}\colon \Gamma^{\mathrm{op}} \to R\text{-}\mathsf{MOD}$ admits a resolution by finitely generated projective $R\Gamma$-modules in which only finitely many of the $R\Gamma$-modules are nonzero. If Γ is of type (FP$_R$), the *finiteness obstruction* $o(\Gamma; R) \in K_0(R\Gamma)$ is the alternating sum of the classes of modules appearing in a finite resolution of $\underline{R}$. For example, if Γ is a finite group then $\underline{\mathbb{Q}}$ is itself a projective $\mathbb{Q}\Gamma$-module and provides us with a finite resolution of $\underline{\mathbb{Q}}$. The basics of $R\Gamma$-modules and finiteness obstructions are discussed in Sections 1 and 2.

To obtain the Euler characteristic and the L^2-Euler characteristic from the finiteness obstruction, we use Lück's Splitting of K_0 [15, Theorem 10.34 on page 196], and two notions of rank for $R\Gamma$-modules: the $R\Gamma$-rank $\mathrm{rk}_{R\Gamma}$ and the L^2-rank $\mathrm{rk}^{(2)}_{\Gamma}$. In the case that every endomorphism in Γ is an isomorphism, that is, Γ is an EI-category, Lück constructed in [15] the natural *splitting* isomorphism

$$S\colon K_0(R\Gamma) \to \operatorname{Split} K_0(R\Gamma) := \bigoplus_{\overline{x} \in \mathrm{iso}(\Gamma)} K_0(R\operatorname{aut}(x))$$

and its natural inverse E, called *extension*. In Section 3 we recall the splitting (S, E), and prove that S remains a left inverse to E in the more general case of directly finite Γ. Let S_x denote the $\overline{x}$-component of S and let $U(\Gamma)$ denote the free abelian group on the isomorphism classes of objects of Γ. The *$R\Gamma$-rank* of a finitely generated $R\Gamma$-module M is the element $\mathrm{rk}_{R\Gamma} M \in U(\Gamma)$

which is $\mathrm{rk}_R\left(S_xM \otimes_{R\,\mathrm{aut}(x)} R\right)$ at $\overline{x} \in \mathrm{iso}(\Gamma)$. This induces a homomorphism $\mathrm{rk}_{R\Gamma}\colon K_0(R\Gamma) \to U(\Gamma)$. If Γ is of type (FP_R), we define the *functorial Euler characteristic* $\chi_f(\Gamma; R)$ to be the image of the finiteness obstruction $o(\Gamma; R)$ under $\mathrm{rk}_{R\Gamma}$. We may think of the sum of the components of $\chi_f(\Gamma; R)$ as the Euler characteristic of Γ. Indeed, if R is Noetherian, every endomorphism in Γ is an isomorphism, and Γ is of type (FP_R), then the sum of the components of $\chi_f(\Gamma; R)$ is $\chi(B\Gamma; R)$. For example, if Γ is a finite group, then $\chi_f(\Gamma; \mathbb{Q})$ is 1, and so is the rational Euler characteristic. In Section 4 we treat the $R\Gamma$-rank $\mathrm{rk}_{R\Gamma}$, the functorial Euler characteristic $\chi_f(\Gamma; R)$, and the Euler characteristic $\chi(\Gamma; R)$.

To obtain the L^2-Euler characteristic from the finiteness obstruction using the splitting functor S_x and the L^2-rank $\mathrm{rk}_\Gamma^{(2)}$, we need some elementary theory of finite von Neumann algebras. For a group G, the *group von Neumann algebra of* G is the algebra of G-equivariant bounded operators $\ell^2(G) \to \ell^2(G)$, which we denote by $\mathcal{N}(G)$. If G is a finite group, $\mathcal{N}(G)$ is simply the group ring $\mathbb{C}G$. The *von Neumann dimension for* $\mathcal{N}(G)$*-modules* is the unique function $\dim_{\mathcal{N}(G)}$ satisfying Hattori-Stallings rank, additivity, cofinality, and continuity as recalled in Theorem 5.2. In the case of a finite group G, the von Neumann dimension of a $\mathbb{C}G$-module is the complex dimension divided by $|G|$. The L^2*-rank* of a finitely generated $\mathbb{C}\Gamma$-module M is the element $\mathrm{rk}_\Gamma^{(2)} M \in U(\Gamma) \otimes_{\mathbb{Z}} \mathbb{R}$ which is $\dim_{\mathcal{N}(\mathrm{aut}(x))}\left(S_xM \otimes_{\mathbb{C}\,\mathrm{aut}(x)} \mathcal{N}(\mathrm{aut}(x))\right)$ at $\overline{x} \in \mathrm{iso}(\Gamma)$. This induces a homomorphism $\mathrm{rk}_\Gamma^{(2)}\colon K_0(R\Gamma) \to U(\Gamma) \otimes_{\mathbb{Z}} \mathbb{R}$. If Γ is of type $(\mathrm{FP}_{\mathbb{C}})$, the *functorial* L^2*-Euler characteristic* $\chi_f^{(2)}(\Gamma)$ is the image of the finiteness obstruction $o(\Gamma; \mathbb{C})$ under $\mathrm{rk}_\Gamma^{(2)}$. The L^2*-Euler characteristic* $\chi^{(2)}(\Gamma)$ is the sum of the components of $\chi_f^{(2)}(\Gamma)$. For example, if Γ is a finite groupoid of type $(\mathrm{FP}_{\mathbb{C}})$, its functorial L^2-Euler characteristic has at $\overline{x}$ the value $1/|\,\mathrm{aut}(x)|$, and the L^2-Euler characteristic is the sum of these. This agrees with the *groupoid cardinality* of Baez-Dolan [2] and also Leinster's Euler characteristic in the case of finite groupoids. In Section 5 we review the necessary prerequisites from the theory of finite von Neumann algebras, and introduce the L^2-rank $\mathrm{rk}_\Gamma^{(2)}$, the functorial L^2-Euler characteristic $\chi_f^{(2)}(\Gamma)$, and the L^2-Euler characteristic $\chi^{(2)}(\Gamma)$. These are defined for categories of type (L^2), a slightly weaker requirement than type $(\mathrm{FP}_{\mathbb{C}})$.

The invariants we introduce in this paper have many desirable properties. The finiteness obstruction, functorial Euler characteristic, Euler characteristic, functorial L^2-Euler characteristic, and L^2-Euler characteristic are all invariant under equivalence of categories and are compatible with finite products, finite coproducts, and homotopy colimits (see [12] for the compatibility with homotopy colimits). Moreover, the L^2-Euler characteristic is compatible with isofibrations and coverings between finite groupoids (see Subsection 5.5). The L^2-Euler characteristic coincides with the classical L^2-Euler characteristic in the case of a group, for finite groups this is $\chi^{(2)}(G) = \frac{1}{|G|}$. Another advantage of the L^2-Euler characteristic is that it is closely related to the geometry and topology of the classifying space for proper G-actions, a topic to which we return in Section 8.

After this treatment of finiteness obstructions and various Euler characteristics, we turn in Section 6 to our next main result: the generalization of the second author's K-theoretic Möbius inversion to quasi-finite EI-categories. We introduce the *restriction-inclusion splitting* $\operatorname{Res}\colon K_0(R\Gamma) \rightleftarrows \operatorname{Split} K_0(R\Gamma)\colon I$ in Subsection 6.1. The *K-theoretic Möbius inversion*

$$\mu\colon \operatorname{Split} K_0(R\Gamma) \rightleftarrows \operatorname{Split} K_0(R\Gamma)\colon \omega$$

compares the splitting (Res, I) with the splitting (S, E) in Theorem 6.22. See Subsection 6.2 for the definition of (μ, ω) in terms of chains in Γ and hom-sets of Γ. A computationally useful biproduct of the comparison via Möbius inversion is the equation

$$S\left(o(\Gamma; R)\right) = \mu\left(\left(o(\widehat{\operatorname{aut}(x)}; R)\right)_{\overline{x} \in \operatorname{iso}(\Gamma)}\right)$$

for Γ of type (FP). For example, this enables us to compute in Theorem 6.23 the finiteness obstruction and Euler characteristics of a finite EI-category in terms of chains. The K-theoretic Möbius inversion is also compatible with the L^2-rank $\operatorname{rk}_\Gamma^{(2)}$ and the pair $(\overline{\mu}^{(2)}, \overline{\omega}^{(2)})$ as in Subsection 6.3. All of these splittings and homomorphisms are illustrated explicitly for G-H-bisets in Subsection 6.4. The rest of Section 6 compares and contrasts the invariants for Γ and Γ^{op}, which can generally be quite different. Important special cases are *Möbius-Rota inversion* for a finite partially ordered set (Example 6.24), *Leinster's Möbius inversion* for a finite skeletal category with trivial endomorphisms (Example 6.25), and *rational Möbius inversion* for a finite, skeletal, free EI-category (Example 6.33).

In Section 7 we recall the groupoid cardinality of Baez-Dolan [2] and the Euler characteristic of Leinster [13] and make comparisons. The groupoid cardinality coincides with the L^2-Euler characteristic for finite groupoids. Leinster's Euler characteristic coincides with the L^2-Euler characteristic for finite, free, skeletal EI-categories. Here "free" is not meant in the usual category-theoretic sense, but rather in the sense of group actions. We say that a category Γ is *free* if the left $\operatorname{aut}(y)$-action on $\operatorname{mor}(x, y)$ is free for every two objects $x, y \in \operatorname{ob}(\Gamma)$. If Γ is not free, then $\chi^{(2)}(\Gamma)$ could very well be different from Leinster's Euler characteristic of Γ (see Remark 7.4). Our invariants are more sensitive than Leinster's Euler characterstic. For example, Leinster's Euler characteristic for finite categories only depends on the set of objects $\operatorname{ob}(\Gamma)$ and the orders $|\operatorname{mor}_\Gamma(x, y)|$. As such, it cannot distinguish between the group $\mathbb{Z}/2\mathbb{Z}$ and the two-element monoid consisting of the identity and an idempotent. The finiteness obstruction and the L^2-Euler characteristic can distinguish these. Leinster's Euler characteristic cannot distinguish between Γ and Γ^{op}, while the functorial Euler characteristic, the functorial L^2-Euler characteristic, and the L^2-Euler characteristic can.

In Section 7 we also explain how to construct weightings in the sense of Leinster from finite free resolutions of the constant $R\Gamma$-module $\underline{R}$ as well as from finite Γ-CW-models for the classifying Γ-space. Several of the weightings in Leinster's article [13] arise in this way.

As mentioned at the outset, Euler characteristics of spaces and manifolds contain geometric information, such as genus, curvature, or evidence of a hyperbolic metric. Similarly, the Euler characteristics of certain categories contain geometric and algebraic information. The topic of Section 8 is our main example: the *proper orbit category of a group* G, denoted $\underline{\mathtt{Or}}(G)$. Its objects are the homogeneous sets G/H for finite subgroups H of G, and its morphisms are the G-equivariant maps between such homogeneous sets. The invariants of the category $\underline{\mathtt{Or}}(G)$ are closely related to the equivariant invariants of a model $\underline{E}G$ for the classifying space for proper G-actions. Namely, if the model $\underline{E}G$ is a finitely dominated G-CW-complex, then our category-theoretic finiteness obstruction $o(\underline{\mathtt{Or}}(G);\mathbb{Z})$ agrees with the equivariant finiteness obstruction of $\underline{E}G$. If the model $\underline{E}G$ is even a finite G-CW-complex, then both the functorial Euler characteristic $\chi_f(\underline{\mathtt{Or}}(G);\mathbb{Z})$ and the functorial L^2-Euler characteristic $\chi_f^{(2)}(\underline{\mathtt{Or}}(G))$ agree with the equivariant Euler characteristic of $\underline{E}G$. Examples of groups G with finite models $\underline{E}G$ include hyperbolic groups, groups that act simplicially cocompactly and properly by isometries on a CAT(0)-space, mapping class groups, the group of outer automorphisms of a finitely generated free group, finitely generated one-relator groups, and cocompact lattices in connected Lie groups.

In addition to these geometric aspects of our invariants in the case of the category $\underline{\mathtt{Or}}(G)$, we also have interesting algebraic consequences of the K-theoretic Möbius inversion and its compatibility with the L^2-rank. For example, if the category $\underline{\mathtt{Or}}(G)$ is of type (FP) and satisfies condition (I) of Condition 6.26, then the functorial L^2-Euler characteristic of $\underline{\mathtt{Or}}(G)$ is the L^2-Möbius inversion of the L^2-Euler characteristics of Weyl groups associated to finite $H < G$:

$$\chi_f^{(2)}(\underline{\mathtt{Or}}(G)) = \overline{\mu}^{(2)}\left(\big(\chi^{(2)}(W_G H)\big)_{(H),|H|<\infty}\right).$$

More substantially, for finite G we deduce the *Burnside ring congruences*, which distinguish the image of the character map

$$\mathrm{ch} = \mathrm{ch}^G\colon U(\underline{\mathtt{Or}}(G)) \to \bigoplus_{(H)} \mathbb{Z}.$$

Here $U(\underline{\mathtt{Or}}(G))$ is the free abelian group on the set of isomorphism classes of objects in $\underline{\mathtt{Or}}(G)$, we identify $U(\underline{\mathtt{Or}}(G))$ with the Burnside ring $A(G)$, and the direct sum of $\mathbb{Z}$'s is over the conjugacy classes (H) of subgroups of the finite group G. The character map counts H-fixed points, namely, for any finite G-set S we have $\mathrm{ch}(S) = \big(|S^H|\big)_{(H)}$. An element ξ lies in the image of ch if and only if the integral congruence

$$\nu(\xi)_{(H)} \equiv 0 \mod |W_G H|$$

holds for every conjugacy class (H) of subgroups of G (the matrix ν is specified in Subsection 8.4). We finish Section 8 by working out everything explicitly for the infinite dihedral group.

The last two sections of the paper are explicit examples. In Section 9 we consider a small example of a category which is not EI and calculate its various K-theoretic morphisms: the splitting functor S, the extension functor E, the restriction functor Res, and the homomorphism ω. In Section 10 we consider a category $\mathbb{A}$ which does not satisfy property (FP). Leinster considered this category in Example 1.11.d of [13] and proved that it does not admit a weighting. We prove that $\mathbb{A}$ does not satisfy property (FP), classify the finitely generated projective $R\mathbb{A}$-modules, and compute the projective class group $K_0(R\mathbb{A})$, the Grothendieck group of finitely generated $\mathbb{Q}\mathbb{A}$-modules $G_0(\mathbb{Q}\mathbb{A})$, and the homology $H_n(B\mathbb{A};R) = H_n(\mathbb{A};R)$.

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Contents

1. Basics about modules over a category

Throughout this section let Γ be a small category and let R be an associative commutative ring with unit. We explain some basics about modules over a category. More details can be found in [15, Section 9]. An *$R\Gamma$-module* is a functor from Γ^{op} into the abelian category of left R-modules. This is a natural generalization of the notion of right RG-module for a group G. The category of $R\Gamma$-modules forms an abelian category $\mathsf{MOD}\text{-}R\Gamma$. An object of $\mathsf{MOD}\text{-}R\Gamma$ is projective if and only if it is a direct summand in an $R\Gamma$-module which is free on a collection of sets indexed by $\mathrm{ob}(\Gamma)$. Given a functor $F\colon \Gamma_1 \to \Gamma_2$, we have induction and restriction functors $\mathrm{ind}_F\colon \mathsf{MOD}\text{-}R\Gamma_1 \rightleftarrows \mathsf{MOD}\text{-}R\Gamma_2\colon \mathrm{res}_F$, and these are adjoint. We also introduce in this section the projective class group $K_0(R\Gamma)$, which provides a home for the finiteness obstruction $o(\Gamma; R)$. The *projective class group* $K_0(R\Gamma)$ is the free abelian group on the isomorphism classes of finitely generated projective $R\Gamma$-modules modulo short exact sequences. The induction functor induces a homomorphism of projective class groups, as does the restriction functor, provided F is admissible.

Definition 1.1 (Modules over a category)**.** A *(contravariant) $R\Gamma$-module* is a contravariant functor $\Gamma \to R\text{-}\mathsf{MOD}$ from Γ to the abelian category of R-modules. A *morphism of $R\Gamma$-modules* is a natural transformation of such functors. Denote by $\mathsf{MOD}\text{-}R\Gamma$ the category of (contravariant) $R\Gamma$-modules.

Example 1.2 (Modules over group rings)**.** Let G be a group. Let $\widehat{G}$ be the associated groupoid with one object and G as its set of morphisms with the obvious composition law. Then the category $\mathsf{MOD}\text{-}R\widehat{G}$ of contravariant $R\widehat{G}$-modules agrees with the category of right RG-modules, where RG is the group ring of G with coefficients in R.

Example 1.3. Let Γ be the category having one object and the natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$ as morphisms with the obvious composition law. Then $\mathsf{MOD}\text{-}R\Gamma$ is the category whose objects are endomorphisms of R-modules and whose set of morphisms from an endomorphism f to an endomorphism g is given by the set of commutative diagrams

$$\begin{array}{ccc} M & \xrightarrow{f} & M \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle u} \\ N & \xrightarrow[g]{} & N \end{array}$$

If one replaces $\mathbb{N}$ by $\mathbb{Z}$ and endomorphisms by automorphisms, the corresponding statement holds.

The (standard) structure of an abelian category on R-MOD induces the structure of an abelian category on MOD-$R\Gamma$ in the obvious way, namely objectwise. In particular, the notion of a projective $R\Gamma$-module is defined. Namely, an $R\Gamma$-module P is *projective* if for every surjective $R\Gamma$-morphism $p\colon M \to N$ and $R\Gamma$-morphism $f\colon P \to N$ there exists an $R\Gamma$-morphism $\overline{f}\colon P \to M$ such that $p\circ\overline{f} = f$, where p is called *surjective* if for any object $x \in \Gamma$ the R-homomorphism $p(x)\colon M(x) \to N(x)$ is surjective.

Consider an object x in Γ. For a set C we denote by RC the free module with C as basis, i.e., the R-module of maps with finite support from C to R. Denote by

$$R\operatorname{mor}(?,x) \quad \text{for } x \in \operatorname{ob}(\Gamma) \tag{1.4}$$

the $R\Gamma$-module which sends an object y to the R-module $R\operatorname{mor}(y,x)$, and a morphism $u\colon y \to z$ to the R-map induced by the morphism of sets $\operatorname{mor}(z,x) \to \operatorname{mor}(y,x)$ that maps $v\colon z \to x$ to $v \circ u\colon y \to x$.

Lemma 1.5. *Let M be any $R\Gamma$-module. Consider any element $\alpha \in M(x)$. Then there is precisely one map of $R\Gamma$-modules*

$$F_\alpha\colon R\operatorname{mor}(?,x) \to M$$

such that $F_\alpha(x)\colon R\operatorname{mor}(x,x) \to M(x)$ sends id_x to α.

Proof. This is a direct application of the Yoneda Lemma. Given $u\colon y \to x$, define $F_\alpha(u) := M(u)(\alpha)$. □

Since Γ is by assumption small, its objects form a set denoted by $\operatorname{ob}(\Gamma)$. An $\operatorname{ob}(\Gamma)$*-set* C is a collection of sets $C = \{C_x \mid x \in \operatorname{ob}(\Gamma)\}$ indexed by $\operatorname{ob}(\Gamma)$. A *morphism of* $\operatorname{ob}(\Gamma)$*-sets* $f\colon C \to D$ is a collection of maps of sets $\{f_x\colon C_x \to D_x \mid x \in \operatorname{ob}(\Gamma)\}$. Denote by $\operatorname{ob}(\Gamma)$-$\mathsf{SETS}$ the category of $\operatorname{ob}(\Gamma)$-sets. We obtain an obvious *forgetful functor*

$$F\colon \mathsf{MOD}\text{-}R\Gamma \to \operatorname{ob}(\Gamma)\text{-}\mathsf{SETS}.$$

Let

$$B\colon \operatorname{ob}(\Gamma)\text{-}\mathsf{SETS} \to \mathsf{MOD}\text{-}R\Gamma$$

be the functor sending an $\operatorname{ob}(\Gamma)$-set C to the $R\Gamma$-module

$$B(C) := \bigoplus_{x\in\operatorname{ob}(\Gamma)} \bigoplus_{C_x} R\operatorname{mor}(?,x).$$

We call $B(C)$ the *free $R\Gamma$-module with basis the* $\operatorname{ob}(\Gamma)$*-set* C. This name is justified by the following consequence of Lemma 1.5 and the universal property of the direct sum.

Lemma 1.6. *We obtain a pair of adjoint functors by (B,F).*

Lemma 1.6 implies that the abelian category $\mathsf{MOD}\text{-}R\Gamma$ has enough projectives. Namely, any free $R\Gamma$-module is projective and for any $R\Gamma$-module M there is a surjective morphism of $R\Gamma$-modules $B(F(M)) \to M$, given by the adjoint of $\mathrm{id}\colon F(M) \to F(M)$. Therefore the standard machinery of homological algebra applies to $\mathsf{MOD}\text{-}R\Gamma$. We also conclude that an $R\Gamma$-module is projective if and only if it is a direct summand in a free $R\Gamma$-module.

An $\mathrm{ob}(\Gamma)$-set C is *finite* if the cardinality of $\coprod_{x\in\mathrm{ob}(\Gamma)} C_x$ is finite. An $R\Gamma$-module M is *finitely generated* if and only if there is a finite $\mathrm{ob}(\Gamma)$-set C together with a surjective $R\Gamma$-morphism $B(C) \to M$. An $R\Gamma$ module is finitely generated projective if and only if it is a direct summand in free $R\Gamma$-module $B(C)$ for a finite $\mathrm{ob}(\Gamma)$-set C.

Definition 1.7. If $M\colon \Gamma^{\mathrm{op}} \to R\text{-}\mathsf{MOD}$ and $N\colon \Gamma \to R\text{-}\mathsf{MOD}$ are functors, then the *tensor product* $M \otimes_{R\Gamma} N$ is the quotient of the R-module

$$\bigoplus_{x\in\mathrm{ob}(\Gamma)} M(x) \otimes_R N(x)$$

by the R-submodule generated by elements of the form

$$(M(f)m) \otimes n - m \otimes (N(f)n)$$

where $f : x \to y$ is a morphism in Γ, $m \in M(y)$, and $n \in N(x)$. The tensor product is an R-module, not an $R\Gamma$-module.

Definition 1.8 (Projective class group)**.** The *projective class group* $K_0(R\Gamma)$ is the abelian group whose generators $[P]$ are isomorphism classes of finitely generated projective $R\Gamma$-modules and whose relations are given by expressions $[P_0]-[P_1]+[P_2]=0$ for every exact sequence $0 \to P_0 \to P_1 \to P_2 \to 0$ of finitely generated projective $R\Gamma$-modules.

Given a functor $F\colon \Gamma_1 \to \Gamma_2$, *induction with* F is the functor

$$\mathrm{ind}_F\colon \mathsf{MOD}\text{-}R\Gamma_1 \;\to\; \mathsf{MOD}\text{-}R\Gamma_2 \tag{1.9}$$

which sends a contravariant $R\Gamma_1$-module $M = M(?)$ to the contravariant $R\Gamma_2$-module $M(?) \otimes_{R\Gamma_1} R\,\mathrm{mor}_{\Gamma_2}(??, F(?))$ which is the tensor product over $R\Gamma_1$ with the $R\Gamma_1$-$R\Gamma_2$-bimodule $R\,\mathrm{mor}_{\Gamma_2}(??, F(?))$ (see [15, 9.15 on page 166] for more details). The functor ind_F respects direct sums and satisfies $\mathrm{ind}_F(R\,\mathrm{mor}_{\Gamma_1}(?, x)) = R\,\mathrm{mor}_{\Gamma_2}(??, F(x))$ for every $x \in \mathrm{ob}(\Gamma_1)$. Hence ind_F sends finitely generated projective $R\Gamma_1$-modules to finitely generated projective $R\Gamma_2$-modules and induces a homomorphism

$$F_*\colon K_0(R\Gamma_1) \;\to\; K_0(R\Gamma_2), \tag{1.10}$$

which depends only on the natural isomorphism class of F. Given functors $F_0\colon \Gamma_0 \to \Gamma_1$ and $F_1\colon \Gamma_1 \to \Gamma_2$, the functors of abelian categories $\mathrm{ind}_{F_1\circ F_0}$ and $\mathrm{ind}_{F_1} \circ \mathrm{ind}_{F_0}$ are naturally isomorphic and hence $(F_1 \circ F_0)_* = (F_1)_* \circ (F_0)_*$.

Given a functor $F\colon \Gamma_1 \to \Gamma_2$, *restriction with* F is the functor of abelian categories

$$\operatorname{res}_F\colon \mathsf{MOD}\text{-}R\Gamma_2 \to \mathsf{MOD}\text{-}R\Gamma_1, \quad M \mapsto M \circ F. \tag{1.11}$$

It is exact and sends the constant $R\Gamma_2$-module $\underline{R}$ to the constant $R\Gamma_1$-module $\underline{R}$. In general it does not send a finitely generated projective $R\Gamma_2$-module to a finitely generated projective $R\Gamma_1$-module. We call F *admissible* if res_F sends a finitely generated projective $R\Gamma_2$-module to a finitely generated projective $R\Gamma_1$-module. The question when F is admissible is answered in [15, Proposition 10.16 on page 187]. If F is admissible, it induces a homomorphism

$$F^*\colon K_0(R\Gamma_2) \to K_0(R\Gamma_1). \tag{1.12}$$

The following is proved in [15, 9.22 on page 169] and is based on the fact that res_F is the same as the functor $-\otimes_{R\Gamma_2} R\operatorname{mor}_{\Gamma_2}(F(?), ??)$.

Lemma 1.13. *Given a functor $F\colon \Gamma_0 \to \Gamma_1$, we obtain an adjoint pair of functors* $(\operatorname{ind}_F, \operatorname{res}_F)$.

2. The finiteness obstruction of a category

After the introduction to $R\Gamma$-modules in Section 1, we can now define the finiteness obstruction of a category in terms of chain complexes and establish its basic properties. Since $\mathsf{MOD}\text{-}R\Gamma$ is abelian, we can talk about $R\Gamma$-chain complexes. In the sequel all chain complexes C_* will satisfy $C_i = 0$ for $i \leq -1$. A *finite projective $R\Gamma$-chain complex* P_* is an $R\Gamma$-chain complex such there exists a natural number N with $P_n = 0$ for $n > N$ and each $R\Gamma$-module P_i is finitely generated projective. Let M be an $R\Gamma$-module. A *finite projective $R\Gamma$-resolution* of M is a finite projective $R\Gamma$-chain complex P_* satisfying $H_i(P_*) = 0$ for $i \geq 1$ together with an isomorphism of $R\Gamma$-modules $M \xrightarrow{\cong} H_0(P_*)$. If P_* can be chosen as a finite free $R\Gamma$-chain complex, we call it a *finite free $R\Gamma$-resolution*.

If the constant $R\Gamma$-module $\underline{R}\colon \Gamma^{\mathrm{op}} \to R\text{-}\mathsf{MOD}$ with value R admits a finite projective $R\Gamma$-resolution or a finite free $R\Gamma$-resolution, we say that Γ is *of type (FP$_R$)* or *of type (FF$_R$)* respectively. Examples of categories of type (FP$_R$) are: any finite group of order invertible in R, any finite groupoid Γ such that $\operatorname{aut}_\Gamma(x)$ is invertible in R for each object x, and any finite category in which every endomorphism is an isomorphism and $\operatorname{aut}_\Gamma(x)$ is invertible in R for each object x. Any category Γ which admits a finite Γ-CW-model for $E\Gamma$ is of type (FF$_R$), in particular any category with a terminal object is of type (FF$_R$).

If Γ is of type (FP$_R$), we define the *finiteness obstruction* $o(\Gamma; R) \in K_0(R\Gamma)$ to be the alternating sum of the classes $[P_n]$ appearing in a finite projective resolution. If G is a finitely presented group of type (FP$_{\mathbb{Z}}$), then the finiteness obstruction is the same as Wall's finiteness obstruction $o(BG) \in K_0(\mathbb{Z}G)$.

Type (FP) and the finiteness obstruction have all the properties one could hope for. Any category equivalent to a category of type (FP) is also of type (FP), and the induced map of an equivalence preserves the finiteness obstruction. If Γ_1 and Γ_2 are of type (FP), then so are $\Gamma_1 \times \Gamma_2$ and $\Gamma_1 \amalg \Gamma_2$, and the finiteness obstructions behave accordingly. Restriction along admissible functors preserves type (FP) and finiteness obstructions, as does induction along left adjoints. In [12] we prove that type (FP) and the finiteness obstruction are compatible with homotopy colimits.

Definition 2.1 (Finiteness obstruction of a module)**.** Let M be an $R\Gamma$-module which possesses a finite projective resolution. Define its *finiteness obstruction*

$$o(M) := \sum_{n \geq 0} (-1)^n \cdot [P_n] \ \in K_0(R\Gamma)$$

for any choice P_* of a finite projective $R\Gamma$-resolution of M.

This definition is a special case of [15, Definition 11.1 on page 211]. It is indeed independent of the choice of finite projective resolution. If P is a finitely generated projective $R\Gamma$-module, then of course $o(P) = [P]$. Given an exact sequence $0 \to M_0 \to M_1 \to M_2 \to 0$ of $R\Gamma$-modules such that two of them possess finite projective resolutions, then all three possess finite projective resolutions and we get in $K_0(R\Gamma)$

$$o(M_0) - o(M_1) + o(M_2) \ = \ 0. \qquad (2.2)$$

All this follows for instance from [15, Chapter 11].

Definition 2.3 (Type (FP) and (FF) for categories)**.** We call a category Γ *of type (FP$_R$) or briefly (FP)* if the constant functor $\underline{R}\colon \Gamma^{\mathrm{op}} \to R\text{-}\mathsf{MOD}$ with value R defines a contravariant $R\Gamma$-module which possesses a finite projective resolution.

We call a category Γ *of type (FF$_R$) or briefly (FF)* if $\underline{R}$ possesses a finite free resolution.

If G is a group and $\widehat{G}$ is the groupoid with one object and G as automorphism group of this object, then the notions (FP) and (FF) for $\widehat{G}$ of Definition 2.3 agree with the classical notions (FP) and (FF) for the group G (see [9, page 199]).

Example 2.4 (Finite groups of invertible order are of type (FP))**.** Let G be a finite group whose order is invertible in the ring R. Then the RG-map $RG \to \underline{R}$,

$$\sum_{g \in G} r_g g \mapsto \sum_{g \in G} r_g$$

admits a right inverse, namely $1 \mapsto \frac{1}{|G|} \sum_{g \in G} g$. The trivial RG-module $\underline{R}$ is then a direct summand of a free RG-module, and is therefore projective. A finite projective resolution of $\underline{R}$ is simply the identity $\underline{R} \to \underline{R}$. The group G and category $\widehat{G}$ are of type (FP$_R$).

Example 2.5 (Finite EI-categories with automorphism groups of invertible order are of type (FP))**.** We may extend Example 2.4 to certain categories. If Γ is a category in which every endomorphism is an automorphism, $\mathrm{aut}(x)$ is invertible in R for every object x, the category Γ has only finitely many isomorphism classes of objects, and $\mathrm{mor}_\Gamma(x,y)$ is finite for all objects x and y, then Γ is of type (FP_R). This will follow from Lemma 6.15 (v).

Example 2.6 (Categories Γ with a finite Γ-CW-model for $E\Gamma$ are of type (FF_R))**.** If Γ is a category which admits a finite Γ-CW-model X for the classifying Γ-space $E\Gamma$, then the cellular R-chains of X form a finite free resolution of the constant $R\Gamma$-module $\underline{R}$. For example, the categories $\{1 \leftarrow 0 \rightarrow 2\}$ and $\{a \rightrightarrows b\}$ admit finite models, as does the poset of non-empty subsets of $[q] = \{0, 1, \ldots, q\}$. Every category with a terminal object also admits a finite model. (Our paper [12] recalls the Γ-CW-complexes of [11] in the context of Euler characteristics and homotopy colimits.)

Definition 2.7 (Finiteness obstruction of a category)**.** Define the *finiteness obstruction with coefficients in* R of a category Γ of type (FP)

$$o(\Gamma; R) := o(\underline{R}) \ \in K_0(R\Gamma)$$

to be the finiteness obstruction $o(\underline{R})$ of Definition 2.1. We also use the notation $[\underline{R}]$, or simply $[R]$, to denote the finiteness obstruction $o(\Gamma; R)$.

The notation $[\underline{R}]$ for the finiteness obstruction is quite natural, for in Example 2.4 the module $\underline{R}$ is projective, and the alternating sum of Definition 2.1 is merely $[\underline{R}]$. However, in general, the module $\underline{R}$ may not be projective.

The homomorphism F_* of (1.10) depends only on the natural isomorphism class of F. Hence F_* is bijective if F is an equivalence of categories. In general ind_F is not exact and $\mathrm{ind}_F \underline{R}$ is not isomorphic to $\underline{R}$. However, this is the case if F is an equivalence of categories. This implies

Theorem 2.8 (Invariance of the finiteness obstruction under equivalence of categories)**.** *Let Γ_1 and Γ_2 be two categories such that there exists an equivalence $F\colon \Gamma_1 \to \Gamma_2$ of categories.*

Then Γ_1 is of type (FP) if and only if Γ_2 is of type (FP). In this case the isomorphism induced by F

$$F_*\colon K_0(R\Gamma_1) \xrightarrow{\cong} K_0(R\Gamma_2)$$

maps $o(\Gamma_1; R)$ to $o(\Gamma_2; R)$.

Moreover, Γ_1 is of type (FF) if and only if Γ_2 is of type (FF).

One easily checks

Theorem 2.9 (Restriction)**.** *Suppose that $F\colon \Gamma_1 \to \Gamma_2$ is an admissible functor and Γ_2 is of type (FP).*

Then Γ_1 is of type (FP) and the homomorphism $F^*\colon K_0(R\Gamma_2) \to K_0(R\Gamma_1)$ sends $o(\Gamma_2; R)$ to $o(\Gamma_1; R)$.

Theorem 2.10 (Left adjoints and induction)**.** *Suppose for the functors $F\colon \Gamma_1 \to \Gamma_2$ and $G\colon \Gamma_2 \to \Gamma_1$ that they form an adjoint pair (G, F). Suppose that Γ_1 is of type (FP).*

Then Γ_2 is of type (FP) and

$$F_*(o(\Gamma_1; R)) \quad = \quad o(\Gamma_2; R).$$

Proof. Recall that ind_F agrees with $- \otimes_{R\Gamma_1} R\,\mathrm{mor}_{\Gamma_2}(??, F(?))$ and res_G agrees with $- \otimes_{R\Gamma_1} R\,\mathrm{mor}_{\Gamma_1}(G(??), ?)$. The adjunction (G, F) (see Lemma 1.13) implies that $\mathrm{res}_G = \mathrm{ind}_F$. Hence G is admissible. We conclude from Theorem 2.9

$$F_*(o(\Gamma_1; R)) = G^*(o(\Gamma_1; R)) = o(\Gamma_2; R).$$ □

Example 2.11 (Category with a terminal object)**.** Suppose that Γ has a terminal object x. Then the constant $R\Gamma$-module $\underline{R}$ with value R agrees with the free $R\Gamma$-module $R\,\mathrm{mor}(?, x)$. Hence Γ is of type (FF) and the finiteness obstruction satisfies

$$o(\Gamma; R) = [R\,\mathrm{mor}(?, x)] \in K_0(R\Gamma).$$

Let $i\colon \{*\} \to \Gamma$ be the inclusion of the trivial category which has precisely one morphism and sends the only object in $\{*\}$ to x. Then the induced map

$$i_*\colon K_0(R) = K_0(R\{*\}) \to K_0(R\Gamma)$$

sends $[R]$ to $o(\Gamma; R)$. This follows also from Theorem 2.10 taking $F = i$ and G to be the obvious projection.

Example 2.12 (Wall's finiteness obstruction)**.** Let G be a group. Let $\widehat{G}$ be the groupoid with one object and G as morphism set with the composition law coming from the group structure. Because of Example 1.2 the group G is of type (FP) in the sense of homological algebra (see [9, page 199]) if and only $\widehat{G}$ is of type (FP) in the sense of Definition 2.3, and the projective class group $K_0(\mathbb{Z}G)$ of the group ring $\mathbb{Z}G$ agrees with $K_0(\mathbb{Z}\widehat{G})$ introduced in Definition 1.8.

Suppose that G is of type (FP) and finitely presented. Then there is a model for BG which is finitely dominated (see [9, Theorem 7.1 in VIII.7 on page 205]) and Wall (see [31] and [32]) has defined its finiteness obstruction

$$o(BG) \in K_0(\mathbb{Z}G).$$

It agrees with the finiteness obstruction $o(\widehat{G}; \mathbb{Z})$ of Definition 2.7.

The elementary proof of the next result is left to the reader.

Theorem 2.13 (Coproduct formula for the finiteness obstruction)**.** *Let Γ_1 and Γ_2 be categories of type (FP). Then their disjoint union $\Gamma_1 \amalg \Gamma_2$ has type (FP) and the inclusions induce an isomorphism*

$$K_0(R\Gamma_1) \oplus K_0(R\Gamma_2) \xrightarrow{\cong} K_0(R(\Gamma_1 \amalg \Gamma_2))$$

which sends $(o(\Gamma_1), o(\Gamma_2))$ to $o(\Gamma_1 \amalg \Gamma_2)$.

Let x be any object of Γ. Denote by $\operatorname{aut}(x)$ the group of automorphisms of x. We often abbreviate the associated group ring by

$$R[x] \quad := \quad R[\operatorname{aut}(x)]. \tag{2.14}$$

Example 2.15 (The finiteness obstruction of a finite groupoid)**.** Let $\mathcal{G}$ be a finite groupoid, i.e., a (small) groupoid such that $\operatorname{iso}(\mathcal{G})$ and $\operatorname{aut}(x)$ for any object $x \in \operatorname{ob}(\mathcal{G})$ are finite. Then Γ is of type (FP) if and only if for every object $x \in \operatorname{ob}(\mathcal{G})$ $|\operatorname{aut}(x)| \cdot 1_R$ is a unit in R.

Suppose that $\mathcal{G}$ is of type (FP). Then the trivial $R[x]$-module R is finitely generated projective and defines a class $[R]$ in $K_0(R[x])$ for every object $x \in \operatorname{ob}(\mathcal{G})$. We obtain from Theorem 2.8 and Theorem 2.13 a decomposition

$$K_0(R\mathcal{G}) = \bigoplus_{\overline{x} \in \operatorname{iso}(\Gamma)} K_0(R[x]).$$

The finiteness obstruction $o(\mathcal{G})$ has under the decomposition above the entry $[R] \in K_0(R[x])$ for $x \in \operatorname{iso}(\Gamma)$.

Let Γ_1 and Γ_2 be two small categories. Then their product $\Gamma_1 \times \Gamma_2$ is a small category. Since R is commutative, the tensor product $\otimes_R$ defines a functor

$$\otimes_R \colon \mathsf{MOD}\text{-}R\Gamma_1 \times \mathsf{MOD}\text{-}R\Gamma_2 \to \mathsf{MOD}\text{-}R(\Gamma_1 \times \Gamma_2).$$

Namely, put $(M \otimes_R N)(x, y) = M(x) \otimes_R N(y)$. Obviously

$$(M_1 \oplus M_2) \otimes_R (N_1 \oplus N_2) \;\cong\; (M_1 \otimes_R N_1) \oplus (M_1 \otimes_R N_2) \oplus (M_2 \otimes_R N_1) \oplus (M_2 \otimes_R N_2),$$

and for $x_1 \in \operatorname{ob}(\Gamma_1)$ and $x_2 \in \operatorname{ob}(\Gamma_2)$ we obtain isomorphisms of $R(\Gamma_1 \times \Gamma_2)$-modules

$$R\operatorname{mor}_{\Gamma_1}(?, x_1) \otimes_R R\operatorname{mor}_{\Gamma_2}(??, x_2) \cong R\operatorname{mor}_{\Gamma_1 \times \Gamma_2}\bigl((?, ??), (x_1, x_2)\bigr).$$

Hence we obtain a well-defined pairing

$$\otimes_R \colon K_0(R\Gamma_1) \otimes_{\mathbb{Z}} K_0(R\Gamma_2) \to K_0(R(\Gamma_1 \times \Gamma_2)), \quad [P_1] \otimes [P_2] \to [P_1 \otimes_R P_2]. \tag{2.16}$$

Theorem 2.17 (Product formula for the finiteness obstruction)**.** *Let* Γ_1 *and* Γ_2 *be categories of type (FP).*

Then $\Gamma_1 \times \Gamma_2$ *is of type (FP) and we get*

$$o(\Gamma_1 \times \Gamma_2; R) \quad = \quad o(\Gamma_1; R) \otimes_R o(\Gamma_2; R)$$

under the pairing (2.16).

Proof. Let P^i_* be a finite projective resolution of $\underline{R}$ over $\mathsf{MOD}\text{-}R\Gamma_i$ for $i = 1, 2$. The evaluation of a projective $R\Gamma_i$-module at an object is projective and hence flat as R-module since this is obviously true for $R\operatorname{mor}_{\Gamma_i}(?, x)$ and every projective $R\Gamma_i$-module is a direct sum in a free one. Hence the $R(\Gamma_1 \times \Gamma_2)$-chain complex $P^1_* \otimes_R P^2_*$ is a projective $R\Gamma_1 \times R\Gamma_2$-resolution of $\underline{R}$. Now an easy calculation (see [15, 11.18 on page 227] shows

$$o(\Gamma_1 \times \Gamma_2; R) = o(P^1_* \otimes_R P^2_*) = o(P^1_*) \otimes_R o(P^2_*) = o(\Gamma_1; R) \otimes_R o(\Gamma_2; R). \quad \square$$

Example 2.18. Let Γ be the category which has precisely one object x and two morphisms $\mathrm{id}_x\colon x \to x$ and $p\colon x \to x$ such that $p \circ p = p$. Given an R-module M, let $I_i(M)$ for $i = 0, 1$ be the contravariant $R\Gamma$-module which sends $p\colon x \to x$ to $i \cdot \mathrm{id}_M\colon M \to M$. Given any $R\Gamma$-module N, we obtain an isomorphism of $R\Gamma$-modules

$$f\colon I_0\big(\ker(N(p))\big) \oplus I_1\big(\mathrm{im}(N(p))\big) \xrightarrow{\cong} N$$

from the inclusions of $\ker(N(p))$ and $\mathrm{im}(N(p))$ to $N(x)$. This isomorphism is natural in N and respects direct sums. If $N = R\,\mathrm{mor}(?, x)$, we have $\ker(N(p)) \cong \mathrm{im}(N(p)) \cong R$. Hence $I_i(R)$ is a finitely generated projective $R\Gamma$-module for $i = 0, 1$. This implies that N is a finitely generated projective $R\Gamma$-module if and only if $\ker(N(p))$ and $\mathrm{im}(N(p))$ are finitely generated projective R-modules. Hence we obtain an isomorphism

$$K_0(R\Gamma) \xrightarrow{\cong} K_0(R) \oplus K_0(R), \quad [P] \mapsto \big([\ker(P(p))], [\mathrm{im}(P(p))]\big).$$

Its inverse sends $([P_0], [P_1])$ to $[I_0(P_0) \oplus I_1(P_1)]$. The constant $R\Gamma$-module $\underline{R}$ agrees with $I_1(R)$. Hence the category Γ is of type (FP) and the finiteness obstruction $o(\Gamma; R)$ is sent under the isomorphism above to the element $(0, [R])$.

3. Splitting the projective class group

In this section we will investigate the projective class group $K_0(R\Gamma)$. In the case that every endomorphism in Γ is an isomorphism, we construct the natural *splitting* isomorphism

$$S\colon K_0(R\Gamma) \to \mathrm{Split}\, K_0(R\Gamma) := \bigoplus_{\overline{x} \in \mathrm{iso}(\Gamma)} K_0(R\,\mathrm{aut}(x))$$

and its natural inverse E, called *extension*. This is the Splitting of $K_0(R\Gamma)$, a part of [15, Theorem 10.34 on page 196]. If Γ is merely directly finite rather than EI, we still have $S \circ E = \mathrm{id}_{\mathrm{Split}\, K_0(R\Gamma)}$ and the naturality of S, though S is no longer bijective. The splitting functor S_x of (3.3) and the extension functor E_x of (3.4) respect direct sums and send epimorphisms to epimorphisms. The extension functor E_x sends free $R\,\mathrm{aut}(x)$-modules to free $R\Gamma$-modules. If Γ is directly finite, the restriction functor S_x sends free $R\Gamma$-modules to free $R\,\mathrm{aut}(x)$-modules and respects finitely generated and projective. The relationship between EI-categories, directly finite categories, and Cauchy complete categories is clarified in Lemma 3.13.

Recall that a ring is called *directly finite* if for two elements $r, s \in R$ we have the implication $rs = 1 \implies sr = 1$. Therefore we define

Definition 3.1 (Directly finite category)**.** A category is called *directly finite* if for any two objects x and y and morphisms $u\colon x \to y$ and $v\colon y \to x$ the implication $vu = \mathrm{id}_x \implies uv = \mathrm{id}_y$ holds.

Lemma 3.2 (Invariance of direct finiteness under equivalence of categories)**.** *Suppose* Γ_1 *and* Γ_2 *are equivalent categories. Then* Γ_1 *is directly finite if and only if* Γ_2 *is directly finite.*

Proof. Suppose $F\colon \Gamma_1 \to \Gamma_2$ is fully faithful and essentially surjective, that Γ_1 is directly finite, and $vu = \mathrm{id}_x$ in Γ_2. Then we can extend to a commutative diagram

$$\begin{array}{ccccccc} & & & \overset{\mathrm{id}_x}{\frown} & & & \\ x & \xrightarrow[u]{} & y & \xrightarrow[v]{} & x & \xrightarrow[u]{} & y \\ {\scriptstyle m}\downarrow\cong & & {\scriptstyle n}\downarrow\cong & & \cong\downarrow{\scriptstyle m} & & \cong\downarrow{\scriptstyle n} \\ F(a) & \xrightarrow[F(f)]{} & F(b) & \xrightarrow[F(g)]{} & F(a) & \xrightarrow[F(f)]{} & F(b). \end{array}$$

Hence $F(g \circ f) = \mathrm{id}_{F(a)}$, and $g \circ f = \mathrm{id}_a$. The direct finiteness of Γ_1 then implies $f \circ g = \mathrm{id}_b$. Together with the commutativity of the two right squares above, this implies $u \circ v = \mathrm{id}_y$, so that Γ_2 is also directly finite. $\square$

Let M be any $R\Gamma$-module and let x be any object. Denote by $\mathrm{aut}(x)$ the group of automorphisms of x. As in 2.14, we abbreviate the associated group ring by $R[x] := R[\mathrm{aut}(x)]$. Define an R-module S_xM by the cokernel of the map of R-modules

$$S_xM := \mathrm{coker}\left(\bigoplus_{\substack{u\colon x \to y \\ u \text{ is not an isomorphism}}} M(u)\colon \bigoplus_{\substack{u\colon x \to y \\ u \text{ is not an isomorphism}}} M(y) \;\to\; M(x) \right).$$

In other words, S_xM is the quotient of the R-module $M(x)$ by the R-submodule generated by all images of R-module homomorphisms $M(u)\colon M(y) \to M(x)$ induced by all non-invertible morphisms $u\colon x \to y$ in Γ. One easily checks that the right $R[x]$-module structure on $M(x)$ coming from functoriality induces a right $R[x]$-module structure on S_xM. Thus we obtain a functor called *splitting functor at* $x \in \mathrm{ob}(\Gamma)$

$$S_x\colon \mathsf{MOD}\text{-}R\Gamma \to \mathsf{MOD}\text{-}R[x], \tag{3.3}$$

where $\mathsf{MOD}\text{-}R[x]$ denotes the category of right $R[x]$-modules. Define a functor, called *extension functor at* $x \in \mathrm{ob}(\Gamma)$*,*

$$E_x\colon \mathsf{MOD}\text{-}R[x] \to \mathsf{MOD}\text{-}R\Gamma \tag{3.4}$$

by sending an $R[x]$-module N to the $R\Gamma$-module $N \otimes_{R[x]} R\,\mathrm{mor}(?, x)$.

Lemma 3.5 (Extension/Splitting, direct sums, and free/projective modules)**.**

(i) *The functor* E_x *respects direct sums. It sends epimorphisms to epimorphisms. It sends a free* $R[x]$*-module with the set* C *as basis to the free* $R\Gamma$*-module with the* $\mathrm{ob}(\Gamma)$*-set* D *as basis, where* $D_x = C$ *and* $D_y = \emptyset$ *for* $y \neq x$*. It respects finitely generated and projective;*

(ii) *We have $S_y \circ E_x = 0$, if x and y are not isomorphic. For every projective right $R[x]$-module P we have a surjective map of $R[x]$-modules, natural in P and compatible with direct sums*

$$\sigma_P \colon P \to S_x \circ E_x(P);$$

(iii) *The functor S_x respects direct sums. It sends epimorphisms to epimorphisms and sends finitely generated $R\Gamma$-modules to finitely generated $R[x]$-modules;*

(iv) *Suppose that Γ is directly finite. Then S_x sends a free $R\Gamma$-module with the $\mathrm{ob}(\Gamma)$-set C as basis to the free $R[x]$-module with $\coprod_{y \in \mathrm{ob}(\Gamma), \overline{y}=\overline{x}} C_y$ as basis and respects finitely generated and projective. Further, σ_P appearing in assertion (ii) is bijective for every projective right $R[x]$-module P.*

Proof. (i) Obviously E_x is compatible with direct sums. It sends epimorphisms to epimorphisms since tensor products are right exact. We have

$$E_x(R[x]) = R[x] \otimes_{R[x]} R\,\mathrm{mor}(?, x) = R\,\mathrm{mor}(?, x).$$

(ii) Suppose that x and y are not isomorphic. Let P be an $R[x]$-module. Consider an element $p \otimes u \in E_xP(y) = P \otimes_{R[x]} R\,\mathrm{mor}(y, x)$. Since x and y are not isomorphic, u is not an isomorphism. The element $p \otimes u$ lies in the image of the map induced by composition from the right with u

$$P \otimes_{R[x]} R\,\mathrm{mor}(x, x) \to P \otimes_{R[x]} R\,\mathrm{mor}(y, x),$$

a preimage is given by $p \otimes \mathrm{id}_x$. Hence $S_y \circ E_x(P) = 0$.

Define an $R[x]$-map $P \to P \otimes_{R[x]} \mathrm{mor}(x, x)$ by sending $p \in P$ to $p \otimes_{R[x]} \mathrm{id}_x$. Its composition with the canonical projection $P \otimes_{R[x]} \mathrm{mor}(x, x) \to S_x \circ E_x(P)$ yields an $R[x]$-map

$$\sigma_P \colon P \to S_x \circ E_x(P).$$

Obviously it is surjective, natural in P and compatible with direct sums.

(iii) This is obvious except that S_x respects finitely generated. We know already that $S_y R\,\mathrm{mor}(?, x) = 0$ if x and y are not isomorphic and that there is an epimorphism $R[x] \to S_x R\,\mathrm{mor}(?, x)$. Hence $S_x R\,\mathrm{mor}(?, y)$ is a finitely generated $R\,\mathrm{aut}(x)$-module for all $y \in \mathrm{ob}(\Gamma)$ and the claim follows.

(iv) Consider an endomorphism $u \colon x \to x$. It lies in the image of the map $\mathrm{mor}(x, x) \to \mathrm{mor}(x, x)$, $v \mapsto v \circ u$, a preimage is id_x. If u is an isomorphism, then there exists no morphism $v \colon x \to y$ such that v is not an isomorphism and u lies in the image of $\mathrm{mor}(x, x) \to \mathrm{mor}(x, x)$, $w \mapsto w \circ v$ since Γ is directly finite. This implies that

$$\sigma_{R[x]} \colon R[x] \xrightarrow{\cong} S_x \circ E_x(R[x]) = S_x R\,\mathrm{mor}(?, x)$$

is an isomorphism. Now assertion (iv) follows from compatibility with direct sums and the facts that an $R\Gamma$-module is projective if and only if it is a direct summand in a free $R\Gamma$-module and that S_x respects epimorphisms. □

Denote by $\operatorname{iso}(\Gamma)$ the set of isomorphism classes of objects of Γ. Choose for any class $\overline{x} \in \operatorname{iso}(\Gamma)$ a representative $x \in \overline{x}$. Define

$$\operatorname{Split} K_0(R\Gamma) \;:=\; \bigoplus_{\overline{x} \in \operatorname{iso}(\Gamma)} K_0(R[x]). \tag{3.6}$$

Provided that Γ is directly finite, we obtain from Lemma 3.5 homomorphisms

$$S\colon K_0(R\Gamma) \to \operatorname{Split} K_0(R\Gamma), \qquad [P] \mapsto \{[S_x P] \mid \overline{x} \in \operatorname{iso}(\Gamma)\}; \tag{3.7}$$

$$E\colon \operatorname{Split} K_0(R\Gamma) \to K_0(R\Gamma), \qquad \{[Q_x] \mid \overline{x} \in \operatorname{iso}(\Gamma)\} \mapsto \sum_{\overline{x} \in \operatorname{iso}(\Gamma)} [E_x Q_x], \tag{3.8}$$

and get

Lemma 3.9. *Suppose that Γ is directly finite. The composite $S \circ E$ is the identity. In particular S is split surjective.*

The group $\operatorname{Split} K_0(R\Gamma)$ is easier to understand than $K_0(R\Gamma)$ since its input are projective class groups over group rings. We will later explain that for an EI-category the maps E and S are bijective (see Theorem 3.14).

Definition 3.10. A category is an *EI-category* if every endomorphism is an isomorphism.

The EI-property is invariant under equivalence of categories.

Lemma 3.11. *Suppose Γ_1 and Γ_2 are equivalent categories. Then Γ_1 is an EI-category if and only if Γ_2 is an EI-category.*

Proof. Let Γ_1 be an EI-category, $F\colon \Gamma_1 \to \Gamma_2$ an equivalence of categories, and $b \in \operatorname{ob}(\Gamma_2)$. Then $b \cong F(a)$ for some $a \in \operatorname{ob}(\Gamma_1)$. We have isomorphisms of monoids

$$\operatorname{mor}_{\Gamma_1}(a,a) \cong \operatorname{mor}_{\Gamma_2}(F(a),F(a)) \cong \operatorname{mor}_{\Gamma_2}(b,b).$$

The first monoid is a group, and hence so is the last. □

Definition 3.12 (Cauchy complete category)**.** A category Γ is *Cauchy complete* if every idempotent splits, i.e., for every idempotent $p\colon x \to x$ there exists morphisms $i\colon y \to x$ and $r\colon x \to y$ with $r \circ i = \operatorname{id}_y$ and $i \circ r = p$.

Lemma 3.13. *Consider a category Γ. Consider the statements*

(i) *Γ is an EI-category;*
(ii) *Every idempotent $p\colon x \to x$ in Γ satisfies $p = \operatorname{id}_x$;*
(iii) *Γ is directly finite and Cauchy complete.*

Then (i) $\implies$ (ii) and (ii) $\iff$ (iii).

If $\operatorname{mor}(x,x)$ is finite for all $x \in \operatorname{ob}(\Gamma)$, then (i) $\iff$ (ii) $\iff$ (iii).

Proof. (i) $\implies$ (ii) If $p\colon x \to x$ is an idempotent, it is an endomorphism and hence an isomorphism. Hence $\mathrm{id}_x = p^{-1} \circ p = p^{-1} \circ p \circ p = \mathrm{id}_x \circ p = p$.

(ii) $\implies$ (iii) Consider morphisms $u\colon x \to y$ and $v\colon y \to x$ with $vu = \mathrm{id}_x$. Then $(uv)^2 = uvuv = u \circ \mathrm{id}_x \circ v = uv$ is an idempotent and hence by assumption $uv = \mathrm{id}_y$. Obviously Γ is Cauchy complete.

(iii) $\implies$ (ii) Consider an idempotent $p\colon x \to x$. Since Γ is Cauchy complete, we can choose morphisms $i\colon y \to x$ and $r\colon x \to y$ with $r \circ i = \mathrm{id}_y$ and $i \circ r = p$. Since Γ is directly finite, $p = i \circ r = \mathrm{id}_x$.

It remains to show (ii) $\implies$ (i) provided that $\mathrm{mor}(x, x)$ is finite for all objects $x \in \mathrm{ob}(\Gamma)$. Consider an endomorphism $f\colon x \to x$. Since $\mathrm{mor}(x, x)$ is finite, there exists integers $m, n \geq 1$ with $f^m = f^{m+n}$. This implies $f^m = f^{m+kn}$ for all natural numbers $k \geq 1$. Hence we get $f^m = f^{m+n}$ for some $n \geq 1$ with $n - m \geq 0$. Then

$$f^n \circ f^n = f^{2n} = f^{m+n} \circ f^{n-m} = f^m \circ f^{n-m} = f^n.$$

Hence f^n is an idempotent. Since then $f^n = \mathrm{id}$ for some $n \geq 1$, the endomorphism f must be an isomorphism. ☐

The next result is taken from [15, Theorem 10.34 on page 196].

Theorem 3.14 (Splitting of $K_0(R\Gamma)$ for EI-categories)**.** *If Γ is an EI-category, the group homomorphisms*

$$\begin{aligned} S\colon K_0(R\Gamma) &\to \mathrm{Split}\, K_0(R\Gamma), & [P] &\mapsto \{[S_x P] \mid \overline{x} \in \mathrm{iso}(\Gamma)\}; \\ E\colon \mathrm{Split}\, K_0(R\Gamma) &\to K_0(R\Gamma), & \{[Q_x] \mid \overline{x} \in \mathrm{iso}(\Gamma)\} &\mapsto \sum_{\overline{x} \in \mathrm{iso}(\Gamma)} [E_x Q_x], \end{aligned}$$

of (3.7) *and* (3.8) *are isomorphisms and inverse to one another. They are covariantly natural with respect to functors $F : \Gamma_1 \to \Gamma_2$ between EI-categories, that is*

$$(\mathrm{Split}\, F_*) \circ S^{R\Gamma_1} = S^{R\Gamma_2} \circ F_*$$

and

$$F_* \circ E^{R\Gamma_1} = E^{R\Gamma_2} \circ (\mathrm{Split}\, F_*).$$

The functor Split F_* *is defined in more detail in Lemma 3.15. Moreover, S and E are also contravariantly natural with respect to admissible functors $F : \Gamma_1 \to \Gamma_2$ between EI-categories, that is*

$$S^{R\Gamma_1} \circ F^* = \mathrm{Split}\, F^* \circ S^{R\Gamma_2}$$

and

$$E^{R\Gamma_1} \circ (\mathrm{Split}\, F^*) = F^* \circ E^{R\Gamma_2}.$$

Example 2.18 shows that the EI hypothesis on Γ in Theorem 3.14 is necessary for S and E to be bijections. Though the splitting homomorphism S is no longer an isomorphism in general, it is covariantly natural in the more general setting of directly finite categories.

Lemma 3.15. *Let Γ_1 and Γ_2 be directly finite categories and $F\colon \Gamma_1 \to \Gamma_2$ be a functor.*

Then the following diagram commutes

$$\begin{array}{ccc} K_0(R\Gamma_1) & \xrightarrow{F_*} & K_0(R\Gamma_2) \\ \downarrow{\scriptstyle S^{R\Gamma_1}} & & \downarrow{\scriptstyle S^{R\Gamma_2}} \\ \operatorname{Split} K_0(R\Gamma_1) & \xrightarrow[\operatorname{Split} F_*]{} & \operatorname{Split} K_0(R\Gamma_2) \end{array}$$

where the vertical maps have been defined in (3.7)*, the upper horizontal map is induced by induction with F, and the lower horizontal arrow is given by the matrix of homomorphisms*

$$\big((F_{\overline{x},\overline{y}})_*\big)_{\overline{x}\in \operatorname{iso}(\Gamma_1),\overline{y}\in\operatorname{iso}(\Gamma_2)}$$

where $(F_{\overline{x},\overline{y}})_*$ is trivial if $\overline{F(x)} \neq \overline{y}$ and given by induction with the group homomorphism $F_x\colon \operatorname{aut}_{\Gamma_1}(x) \to \operatorname{aut}_{\Gamma_2}(F(x))$, $f \mapsto F(f)$ for $\overline{y} = \overline{F(x)}$.

In particular, the commutativity of the diagram guarantees

$$S^{R\Gamma_2}_{F(x)} \circ F_* = F_x \circ S^{R\Gamma_1}_x.$$

Proof. For two objects x and y in Γ_1, let $\operatorname{mor}^{\cong}(x,y)$ be the set of isomorphisms from x to y. The covariant $R\Gamma_1$-module $R\operatorname{mor}^{\cong}(x,?)$ assigns to an object x the trivial R-module $\{0\}$ if $\overline{x} \neq \overline{y}$ and $R\operatorname{mor}^{\cong}(x,y)$ if $\overline{x} = \overline{y}$. The evaluation of $R\operatorname{mor}^{\cong}(x,?)$ at a morphism $f\colon y_1 \to y_2$ is given by

$$R\operatorname{mor}^{\cong}(x,y_1) \to R\operatorname{mor}^{\cong}(x,y_2), \quad g \mapsto f\circ g$$

if f is an isomorphism and $\overline{x} = \overline{y}$, and by the trivial R-homomorphism otherwise. This definition makes sense since Γ_1 is directly finite. Obviously $R\operatorname{mor}^{\cong}(x,?)$ is an $R\Gamma_1$-$R[x]$-bimodule. Hence we obtain a functor

$$\mathsf{MOD}\text{-}R\Gamma_1 \to \mathsf{MOD}\text{-}R[x], \quad P \mapsto P \otimes_{R\Gamma_1} R\operatorname{mor}^{\cong}(x,?).$$

It is naturally equivalent to the splitting functor S_x defined in (3.3). Namely, a natural equivalence is given by the $R[x]$-isomorphisms which are inverse to one another

$$S_xP \to P \otimes_{R\Gamma_1} R\operatorname{mor}^{\cong}(x,?), \quad \overline{p} \mapsto p \otimes \operatorname{id}_x.$$

and

$$P \otimes_{R\Gamma_1} R\operatorname{mor}^{\cong}(x,?) \to S_xP, \quad p\otimes f \mapsto \overline{P(f)(p)}.$$

Consider an $R\Gamma_1$-module P. Then we obtain for $y \in \operatorname{iso}(\Gamma_2)$ a natural isomorphism of $R[y]$-modules

$$\begin{aligned}
S_y \circ \operatorname{ind}_F P &= P \otimes_{R\Gamma_1} R\operatorname{mor}_{\Gamma_2}(??, F(?)) \otimes_{R\Gamma_2} R\operatorname{mor}^{\cong}_{\Gamma_2}(y, ??) \\
&= P \otimes_{R\Gamma_1} R\operatorname{mor}^{\cong}_{\Gamma_2}(y, F(?)) \\
&= P \otimes_{R\Gamma_1} \bigoplus_{\overline{x} \in \operatorname{iso}(\Gamma_1), \overline{F(x)} = \overline{y}} R\operatorname{mor}^{\cong}_{\Gamma_1}(x, ?) \otimes_{R[x]} R\operatorname{mor}^{\cong}_{\Gamma_2}(y, F(x)) \\
&= \bigoplus_{\overline{x} \in \operatorname{iso}(\Gamma_1), \overline{F(x)} = \overline{y}} P \otimes_{R\Gamma_1} R\operatorname{mor}^{\cong}_{\Gamma_1}(x, ?) \otimes_{R[x]} R\operatorname{mor}^{\cong}_{\Gamma_2}(y, F(x)) \\
&= \bigoplus_{\overline{x} \in \operatorname{iso}(\Gamma_1), \overline{F(x)} = \overline{y}} \operatorname{ind}_{F_x} \circ S_x P.
\end{aligned}$$

This finishes the proof of Lemma 3.15. □

4. The (functorial) Euler characteristic of a category

Depending on which notion of rank we choose for $R\Gamma$-modules, $\operatorname{rk}_{R\Gamma}$ vs. $\operatorname{rk}^{(2)}_{\Gamma}$, there are two possible ways to define (functorial) Euler characteristics. In this section, we start with the more elementary definition $\chi(\Gamma; R)$ and its functorial counterpart $\chi_f(\Gamma; R)$, both of which arise from $\operatorname{rk}_{R\Gamma}$. In Section 5 we take $R = \mathbb{C}$ and $\operatorname{rk}^{(2)}_{\Gamma}$ to treat the L^2-Euler characteristic $\chi^{(2)}(\Gamma)$ and its functorial counterpart $\chi^{(2)}_f(\Gamma)$.

To obtain the naive Euler characteristic, we use the splitting functor S_x as follows. The *$R\Gamma$-rank* of an $R\Gamma$-module M is an element of $U(\Gamma)$, the free abelian group on the isomorphism classes of objects of Γ. At $\overline{x} \in \operatorname{iso}(\Gamma)$, $\operatorname{rk}_{R\Gamma} M$ is $\operatorname{rk}_R(S_x M \otimes_{R\operatorname{aut}(x)} R)$. This induces a homomorphism $\operatorname{rk}_{R\Gamma}$ from $K_0(R\Gamma)$ to $U(\Gamma)$. If Γ is of type (FP), we define the *functorial Euler characteristic* $\chi_f(\Gamma; R)$ to be the image of the finiteness obstruction $o(\Gamma; R)$ under $\operatorname{rk}_{R\Gamma}$. The functorial Euler characteristic is compatible with equivalences between directly finite categories of type (FP). If Γ is an EI-category of type (FP) and R is Noetherian, then the sum of the components of $\chi_f(\Gamma; R)$ is equal to the alternating sum of the ranks of $H_i(B\Gamma; R)$, the familiar topological Euler characteristic $\chi(B\Gamma; R)$. If R is Noetherian and Γ is of type (FP), but not necessarily EI, then the image of the finiteness obstruction under $\operatorname{rk}_R \operatorname{pr}_*$ in (4.16) is $\chi(B\Gamma; R)$. Whenever $\chi(B\Gamma; R)$ exists, (for example in the two cases just mentioned), we think of it as the most naive definition of Euler characteristic of Γ, and we denote it by $\chi(\Gamma; R)$.

Each notion of Euler characteristic (χ vs. $\chi^{(2)}$) has its advantages and disadvantages. Both are invariant under equivalence of categories and are compatible with products, coproducts, and homotopy colimits (see [12] for the compatibility with homotopy colimits). The L^2-Euler characteristic is compatible with isofibrations and coverings between finite coverings (see Subsection 5.5). The naive definition $\chi(B\Gamma; R)$ may not exist in even the simplest cases, for example $\Gamma = \widehat{\mathbb{Z}_2}$

and $R = \mathbb{Z}_2$. For a finite discrete category (a set), both invariants return the cardinality. For a finite group G, we have $\chi(\widehat{G}; \mathbb{Q}) = 1$, while the L^2-Euler characteristic is $\chi^{(2)}(\widehat{G}) = \frac{1}{|G|}$. The groupoid cardinality of Baez-Dolan [2] and the Euler characteristic of Leinster [13] will occur as an L^2-Euler characteristic, see Section 7 for the comparison. The main advantages of our K-theoretic approach are that it also works for infinite categories and encompasses important examples such as the L^2-Euler characteristic of a group and the equivariant Euler characteristic of the classifying space $\underline{E}G$ for proper G-actions.

To begin the details of the naive Euler characteristic, suppose that we have specified the notion of a rank

$$\operatorname{rk}_R(N) \in \mathbb{Z} \tag{4.1}$$

for every finitely generated R-module such that $\operatorname{rk}(N_1) = \operatorname{rk}_R(N_0) + \operatorname{rk}_R(N_2)$ for any sequence $0 \to N_0 \to N_1 \to N_2 \to 0$ of finitely generated R-modules and $\operatorname{rk}_R(R) = 1$. If R is a commutative principal ideal domain, we will use $\operatorname{rk}_R(N) := \dim_F(F \otimes_R N)$ for F the quotient field of R.

We begin with the most naive definition of an Euler characteristic.

Definition 4.2 (The Euler characteristic of a category Γ)**.** Let Γ be a category. Let $B\Gamma$ be its classifying space, i.e., the geometric realization of its nerve. Suppose that $H_i(B\Gamma; R)$ is a finitely generated R-module for every $i \geq 0$ and that there exists a natural number d with $H_i(B\Gamma; R) = 0$ for $i > d$. Then we define the *Euler characteristic of* Γ to be

$$\chi(\Gamma; R) = \sum_{i \geq 0} (-1)^i \cdot \operatorname{rk}_R(H_i(B\Gamma; R)) \in \mathbb{Z}.$$

Example 4.3 (The Euler characteristic of a finite groupoid)**.** Let $\mathcal{G}$ be a finite groupoid, i.e., a (small) groupoid such that $\operatorname{iso}(\mathcal{G})$ and $\operatorname{aut}(x)$ for any object $x \in \operatorname{ob}(\mathcal{G})$ are finite. Consider $R = \mathbb{Q}$. Then the assumptions in Definition 4.2 are satisfied and

$$\chi(\mathcal{G}) = |\operatorname{iso}(\mathcal{G})|.$$

Notation 4.4 (The abelian group $U(\Gamma)$ and the augmentation homomorphism ϵ)**.** Let Γ be a category. We denote by $U(\Gamma)$ the free abelian group on the set of isomorphism classes of objects in Γ, that is

$$U(\Gamma) := \mathbb{Z} \operatorname{iso}(\Gamma).$$

The *augmentation homomorphism* $\epsilon \colon U(\Gamma) \to \mathbb{Z}$ sends every basis element of $\operatorname{iso}(\Gamma)$ to $1 \in \mathbb{Z}$. The augmentation homomorphism is a natural transformation from the covariant functor $U \colon \mathsf{CAT} \to \mathsf{ABELIAN\text{-}GROUPS}$ to the constant functor $\mathbb{Z}$, that is, for any functor $F\colon \Gamma_1 \to \Gamma_2$ the diagram

$$\begin{array}{ccc} U(\Gamma_1) & \xrightarrow{U(F)} & U(\Gamma_2) \\ \downarrow{\scriptstyle \epsilon} & & \downarrow{\scriptstyle \epsilon} \\ \mathbb{Z} & \xrightarrow[\mathrm{id}_{\mathbb{Z}}]{} & \mathbb{Z} \end{array} \tag{4.5}$$

commutes.

Definition 4.6 (Rank of a finitely generated $R\Gamma$-module)**.** Let M be a finitely generated $R\Gamma$-module M, define its *rank*

$$\mathrm{rk}_{R\Gamma}(M) := \big\{\mathrm{rk}_R(S_x M \otimes_{R[x]} R) \mid \overline{x} \in \mathrm{iso}(\Gamma)\big\} \quad \in U(\Gamma).$$

The rank $\mathrm{rk}_{R\Gamma}$ defines a homomorphism

$$\mathrm{rk}_{R\Gamma}\colon K_0(R\Gamma) \to U(\Gamma), \quad [P] \to \mathrm{rk}_{R\Gamma}(P). \tag{4.7}$$

It obviously factorizes over $S\colon K_0(R\Gamma) \to \mathrm{Split}\, K_0(R\Gamma)$. Define

$$\iota\colon U(\Gamma) \to K_0(R\Gamma), \quad (n_{\overline{x}})_{\overline{x}\in\mathrm{iso}(\Gamma)} \mapsto \sum_{\overline{x}\in\mathrm{iso}(\Gamma)} n_{\overline{x}} \cdot [R\,\mathrm{mor}(?, x)]. \tag{4.8}$$

This is the same as the composite

$$U(\Gamma) = \bigoplus_{\overline{x}\in\mathrm{iso}(\Gamma)} \mathbb{Z} \xrightarrow{\bigoplus_{\overline{x}\in\mathrm{iso}(\Gamma)} i_x} \bigoplus_{\overline{x}\in\mathrm{iso}(\Gamma)} K_0(R[x]) = \mathrm{Split}\, K_0(R\Gamma) \xrightarrow{E} K_0(R\Gamma),$$

where $i_x\colon \mathbb{Z} \to K_0(R[x])$ sends n to $n \cdot [R[x]]$ and E has been defined in (3.8).

Lemma 4.9 (Naturality of $\mathrm{rk}_{R\Gamma}$)**.** *The rank* $\mathrm{rk}_{R\Gamma}$ *is natural for functors* $F\colon \Gamma_1 \to \Gamma_2$ *between directly finite categories. In particular, we have a natural transformation*

$$\mathrm{rk}_{R-}\colon K_0(R-) \to U(-)$$

between covariant functors

$$K_0(R-), U(-)\colon \mathsf{DIR.FIN.\text{-}CAT} \to \mathsf{ABELIAN\text{-}GROUPS}.$$

Proof. The proof of [15, Proposition 10.44 (b) on page 202] for functors between EI-categories also works for functors between directly finite categories. The rank $\mathrm{rk}_{R\Gamma}$ is equal to $r \circ S$ where $r\colon \mathrm{Split}\, K_0(R\Gamma) \to U(\Gamma)$ is the direct sum of

$$K_0(R[x]) \to \mathbb{Z}$$
$$[P] \mapsto \mathrm{rk}_R(P \otimes_{R[x]} R)$$

over $\overline{x} \in \mathrm{iso}(\Gamma)$. By Lemma 3.15, the functor S is covariantly natural with respect to functors between directly finite categories. The functor r is also natural for such functors F, for if $F_x\colon \mathrm{aut}_{\Gamma_1}(x) \to \mathrm{aut}_{\Gamma_2}(Fx)$ is the restriction of F to $\mathrm{aut}_{\Gamma_1}(x)$ we have

$$P \otimes_{R[x]} R \cong \mathrm{ind}_{F_x}(P) \otimes_{R[Fx]} R. \qquad \square$$

Lemma 4.10. *Let* Γ *be a directly finite category.*

(i) *The composite*

$$U(\Gamma) \xrightarrow{\iota} K_0(R\Gamma) \xrightarrow{\mathrm{rk}_{R\Gamma}} U(\Gamma)$$

of the homomorphisms defined in (4.7) *and* (4.8) *is the identity;*

(ii) *Let* F *be a finitely generated free* $R\Gamma$*-module. Then*

$$F \cong \bigoplus_{\overline{x} \in \mathrm{iso}(\Gamma)} \bigoplus_{i=1}^{\mathrm{rk}_{R\Gamma}(F)_{\overline{x}}} R\,\mathrm{mor}(?;x).$$

In particular two finitely generated free $R\Gamma$*-modules* F_1 *and* F_2 *are isomorphic if and only if* $\mathrm{rk}_{R\Gamma}(F_1) = \mathrm{rk}_{R\Gamma}(F_2)$*;*

Proof. (i) This follows from Lemma 3.5.
(ii) Let F be a free $R\Gamma$-module. By definition it looks like

$$F = \bigoplus_{\overline{x} \in \mathrm{iso}(\Gamma)} \bigoplus_{I_x} R\,\mathrm{mor}(?,x)$$

for some index sets I_x. It is finitely generated if there exist natural numbers m_x and an epimorphism

$$f\colon \bigoplus_{\overline{x} \in \mathrm{iso}(\Gamma)} \bigoplus_{i=1}^{m_x} R\,\mathrm{mor}(?,x) \to \bigoplus_{\overline{x} \in \mathrm{iso}(\Gamma)} \bigoplus_{I_x} R\,\mathrm{mor}(?,x)$$

such that only finitely many m_x are different from zero. Lemma 3.5 implies that we obtain for every $\overline{x} \in \mathrm{iso}(\Gamma)$ an epimorphism $S_x f\colon \bigoplus_{i=1}^{m_x} R[x] \to \bigoplus_{I_x} R[x]$. This implies that each set I_x is finite and only finitely many of the sets I_x are not empty. Hence we can find for a finitely generated free $R\Gamma$-module F natural numbers n_x such that

$$F \cong \bigoplus_{\overline{x} \in \mathrm{iso}(\Gamma)} \bigoplus_{i=1}^{n_x} R\,\mathrm{mor}(?,x)$$

and only finitely many n_x are different from zero. Lemma 3.5 implies

$$\mathrm{rk}_{R\Gamma}(F)_{\overline{x}} = n_x.$$

In particular $\mathrm{rk}_{R\Gamma}(F)$ determines the isomorphism type of a finitely generated free $R\Gamma$-module F. □

Definition 4.11 (The functorial Euler characteristic of a category)**.** Suppose that Γ is of type (FP). Define the *functorial Euler characteristic of* Γ *with coefficients in* R

$$\chi_f(\Gamma;R) \in U(\Gamma)$$

to be the image of its finiteness obstruction $o(\Gamma;R) \in K_0(R\Gamma)$ of Definition 2.1 under the homomorphism $\mathrm{rk}_{R\Gamma}$ of (4.7).

The word functorial refers to the fact that the group, in which χ_f takes values, depends in a functorial way on Γ.

Example 4.12 (The functorial Euler characteristic of a finite groupoid)**.** Let $\mathcal{G}$ be a finite groupoid, i.e., a (small) groupoid such that $\operatorname{iso}(\mathcal{G})$ and $\operatorname{aut}(x)$ for any object $x \in \operatorname{ob}(\mathcal{G})$ are finite. Consider $R = \mathbb{Q}$. Then $U(\mathcal{G})$ is the abelian group generated by $\operatorname{iso}(\mathcal{G})$ and $\chi_f(\mathcal{G}) \in U(\mathcal{G})$ is given by the sum of the basis elements.

Theorem 4.13 (Invariance of the functorial Euler characteristic under equivalence of categories)**.** *Let $F\colon \Gamma_1 \to \Gamma_2$ be an equivalence of categories and suppose that Γ_1 is of type (FP) and directly finite. Then Γ_2 is of type (FP) and directly finite, and*

$$U(F)(\chi_f(\Gamma_1; R)) = \chi_f(\Gamma_2; R).$$

Proof. The category Γ_2 is of type (FP) and $F_*(o(\Gamma_1; R)) = o(\Gamma_2; R)$ by Theorem 2.8. The category Γ_2 is directly finite by Lemma 3.2. We have $U(F)(\chi_f(\Gamma_1; R)) = \chi_f(\Gamma_2; R)$ by the naturality of rk_{R-} in Lemma 4.9 and $F_*(o(\Gamma_1; R)) = o(\Gamma_2; R)$. □

Lemma 4.14. *Let Γ be a directly finite category. Suppose that Γ is of type (FF) (see Definition 2.3). Then the finiteness obstruction $o(\Gamma; R) \in K_0(R\Gamma)$ is the image of $\chi_f(\Gamma; R)$ under the homomorphism ι of* (4.8)*.*

Proof. This follows from the definitions in combination with Lemma 4.10. □

Obviously the functorial Euler characteristic $\chi_f(\Gamma; R)$ and the Euler characteristic $\chi(\Gamma; R)$ are weaker invariants than the finiteness obstruction and carry less information but they live in explicit abelian groups and are easier to compute.

Theorem 4.15 (The finiteness obstruction determines the Euler characteristic)**.** *Let Γ be a category of type (FP). Suppose that R is Noetherian. Denote by $\operatorname{pr}\colon \Gamma \to \{*\}$ the projection to the trivial category with precisely one morphism.*

Then the assumptions in Definition 4.2 are satisfied and the composite

$$K_0(R\Gamma) \xrightarrow{\operatorname{pr}_*} K_0(R\{*\}) = K_0(R) \xrightarrow{\operatorname{rk}_R} \mathbb{Z} \tag{4.16}$$

sends the finiteness obstruction $o(\Gamma; R)$ to the Euler characteristic $\chi(\Gamma; R)$.

Proof. Associated to a category Γ there is a classifying contravariant Γ-space $E\Gamma$ which is a Γ-CW-complex with the property that $E\Gamma$ evaluated at any object $x \in \operatorname{ob}(\Gamma)$ is contractible. We refer to Davis-Lück [11, Definition 1.2, Definition 3.2, Definition 3.8] for the definition of a contravariant Γ-space, a Γ-CW-complex (which is called free Γ-CW-complex there) and of the classifying Γ-space $E\Gamma$. The cellular $R\Gamma$-chain complex $C_*(X)$ with R coefficients of a Γ-CW-complex X is the composition of the functor given by X with the functor cellular chain complex with coefficients in R and has free $R\Gamma$-chain modules. The proof of the last fact is analogous to the proof of [15, Lemma 13.2 on page 260]. Since the evaluation of $E\Gamma$ at any object $x \in \operatorname{ob}(\Gamma)$ is contractible, the $R\Gamma$-module

$H_n(C_*(E\Gamma;R))$ is trivial for $n \neq 0$ and isomorphic to the constant $R\Gamma$-module $\underline{R}$ for $n = 0$. In particular $C_*(E\Gamma;R)$ is a projective $R\Gamma$-resolution of the constant $R\Gamma$-module $\underline{R}$. By assumption there exists a finite projective $R\Gamma$-resolution P_* of $\underline{R}$. By the fundamental lemma of homological algebra (see [15, Lemma 11.3 on page 212]) there exists an $R\Gamma$-chain homotopy equivalence $f_* \colon C_*(E\Gamma;R) \to P_*$. If $\mathrm{pr}\colon \Gamma \to \{*\}$ is the projection to the trivial category, we obtain an R-chain homotopy equivalence $\mathrm{ind}_{\mathrm{pr}} f_* \colon \mathrm{ind}_{\mathrm{pr}} C_*(E\Gamma;R) \to \mathrm{ind}_{\mathrm{pr}} P_*$. There is also the notion of an induction functor for contravariant Γ-spaces (see [11, Definition 1.8] and a natural isomorphism of R-chain complexes $\mathrm{ind}_{\mathrm{pr}} C_*(E\Gamma;R) \xrightarrow{\cong} C_*(\mathrm{ind}_{\mathrm{pr}} E\Gamma;R)$. The CW-complex $\mathrm{ind}_{\mathrm{pr}} E\Gamma$ is a model for $B\Gamma$ (see [11, Definition 3.10, page 225 and page 230]). Hence we obtain a chain homotopy equivalence

$$C_*(B\Gamma;R) \xrightarrow{\simeq} \mathrm{ind}_{\mathrm{pr}} P_*$$

and $\mathrm{ind}_{\mathrm{pr}} P_*$ is an R-chain complex such that every R-chain module is finitely generated projective and only finitely many R-chain modules are non-trivial. Since R is Noetherian, this implies that $H_i(\mathrm{ind}_{\mathrm{pr}} P_*)$ is finitely generated as an R-module for every $i \geq 0$ and that there is a natural number d with $H_i(\mathrm{ind}_{\mathrm{pr}} P_*) = 0$ for $i > d$. This implies that the same is true for the homology $H_*(B\Gamma;R)$. Our assumptions on the rank function rk_R of (4.1) imply

$$\begin{aligned}\sum_{i\geq 0}(-1)^i \cdot \mathrm{rk}_R(\mathrm{ind}_{\mathrm{pr}} P_i) &= \sum_{i\geq 0}(-1)^i \cdot \mathrm{rk}_R(H_i(\mathrm{ind}_{\mathrm{pr}} P_*)) \\ &= \sum_{i\geq 0}(-1)^i \cdot \mathrm{rk}_R(H_i(B\Gamma)) \\ &= \chi(\Gamma;R).\end{aligned}$$

Since the composite

$$K_0(R\Gamma) \xrightarrow{\mathrm{pr}_*} K_0(R\{*\}) = K_0(R) \xrightarrow{\mathrm{rk}_R} \mathbb{Z}$$

sends $o(\Gamma;R) = \sum_{i\geq 0}(-1)^i \cdot [P_i]$ to $\sum_{i\geq 0}(-1)^i \cdot \mathrm{rk}_R(\mathrm{ind}_{\mathrm{pr}} P_i)$, Theorem 4.15 follows. □

Example 4.17. Let Γ be the category appearing in Example 2.18. It contains idempotents different from the identity, is directly finite, and of type (FP). We have $U(\Gamma) = \mathbb{Z}$ and $\chi_f(\Gamma;R) = \chi(\Gamma;R) = 1$.

Corollary 4.18 (Invariance of the Euler characteristic under equivalence of categories)**.** *Let $F\colon \Gamma_1 \to \Gamma_2$ be an equivalence of categories and suppose that Γ_1 is of type (FP) and R is Noetherian. Then Γ_2 is of type (FP) and satisfies the assumptions of Definition 4.2. Moreover, Γ_1 and Γ_2 have the same Euler characteristic,*

$$\chi(\Gamma_1;R) = \chi(\Gamma_2;R).$$

Proof. The category Γ_2 is of type (FP) and $F_*(o(\Gamma_1; R)) = o(\Gamma_2; R)$ by Theorem 2.8. Theorem 4.15 then guarantees that Γ_2 satisfies the assumptions of Definition 4.2 and $\mathrm{rk}_R \,\mathrm{pr}_*\, o(\Gamma_2; R) = \chi(\Gamma_2; R)$. We have

$$\chi(\Gamma_1; R) = \mathrm{rk}_R \,\mathrm{pr}_*\, o(\Gamma_1; R) = \mathrm{rk}_R (\mathrm{pr} \circ F)_* o(\Gamma_1; R)$$
$$\chi(\Gamma_2; R) = \mathrm{rk}_R \,\mathrm{pr}_*\, o(\Gamma_2; R) = \mathrm{rk}_R \,\mathrm{pr}_*\, F_*(o(\Gamma_1; R)).$$

Or, of course, once we use Theorems 2.8 and 4.15 to conclude that Γ_2 admits an Euler characteristic in the sense of Definition 4.2, we may simply conclude that $\chi(\Gamma_1; R) = \chi(\Gamma_2; R)$ by the homotopy equivalence $B\Gamma_1 \simeq B\Gamma_2$. □

As we have seen, the Euler characteristic is determined by the finiteness obstruction. We can also obtain the Euler characteristic from the functorial Euler characteristic. For this we need the augmentation homomorphism introduced in Notation 4.4..

Corollary 4.19 (The functorial Euler characteristic determines the Euler characteristic)**.** *Let Γ be an EI-category of type (FP). Suppose that R is Noetherian. Denote by*

$$\epsilon\colon U(\Gamma) \to \mathbb{Z}$$

the augmentation homomorphism from Notation 4.4.

Then the assumptions in Definition 4.2 are satisfied and

$$\epsilon(\chi_f(\Gamma; R)) = \chi(\Gamma; R).$$

Proof. Because of Theorem 4.15 is suffices to show that the diagram

$$\begin{array}{ccc} K_0(R\Gamma) & \xrightarrow{\mathrm{rk}_{R\Gamma}} & U(\Gamma) \\ {\scriptstyle pr_*}\downarrow & & \downarrow{\scriptstyle \epsilon} \\ K_0(R\{*\}) = K_0(R) & \xrightarrow[\mathrm{rk}_R]{} & \mathbb{Z} \end{array}$$

commutes. However, this is precisely the naturality diagram for rk in Lemma 4.9. □

There is an obvious pairing coming from the natural bijection $\mathrm{iso}(\Gamma_1) \times \mathrm{iso}(\Gamma_2) \xrightarrow{\cong} \mathrm{iso}(\Gamma_1 \times \Gamma_2)$

$$\otimes\colon U(\Gamma_1) \otimes_{\mathbb{Z}} U(\Gamma_2) \quad \to \quad U(\Gamma_1 \times \Gamma_2) \tag{4.20}$$

Theorem 4.21 (Product formula for χ_f and χ)**.** *Let Γ_1 and Γ_2 be categories of type (FP). Suppose that the rank rk_R satisfies $\mathrm{rk}_R(M \otimes N) = \mathrm{rk}_R(M) \cdot \mathrm{rk}_R(N)$ for all finitely generated R-modules M and N.*

Then $\Gamma_1 \times \Gamma_2$ is of type (FP), we get for the functorial Euler characteristic

$$\chi_f(\Gamma_1 \times \Gamma_2; R) \quad = \quad \chi_f(\Gamma_1; R) \otimes \chi_f(\Gamma_2; R)$$

under the pairing (4.20)*, and we get for the Euler characteristic.*

$$\chi(\Gamma_1 \times \Gamma_2; R) = \chi(\Gamma_1; R) \cdot \chi(\Gamma_2; R).$$

Proof. Consider the diagram

$$\begin{array}{ccc} K_0(R\Gamma_1) \otimes K_0(R\Gamma_2) & \xrightarrow{\otimes_R} & K_0(R(\Gamma_1 \times \Gamma_2)) \\ {\scriptstyle \left(\mathrm{rk}_{R\Gamma_1} \circ S_{R\Gamma_1}\right) \otimes \left(\mathrm{rk}_{R\Gamma_2} \circ S_{R\Gamma_2}\right)} \downarrow & & \downarrow {\scriptstyle \mathrm{rk}_{R(\Gamma_1 \times \Gamma_2)} \circ S_{R(\Gamma_1 \times \Gamma_2)}} \\ U(\Gamma_1) \otimes U(\Gamma_2) & \xrightarrow{\otimes} & U(\Gamma_1 \times \Gamma_2) \end{array}$$

where the horizontal pairings have been introduced in (2.16) and (4.20), the homomorphisms S in (3.7) and the homomorphism $\mathrm{rk}_{R\Gamma}$ in (4.7). One easily checks that it commutes. Now the claim follows for χ_f from Theorem 2.17.

The claim for χ follows from the fact $B\Gamma_1 \times B\Gamma_2 = B(\Gamma_1 \times \Gamma_2)$ and the Künneth formula. □

5. The (functorial) L^2-Euler characteristic and L^2-Betti numbers of a category

In this section we introduce the (functorial) L^2-Euler characteristic and L^2-Betti numbers of a category. This requires some input from the theory of finite von Neumann algebras and their dimension theory which we briefly record next. For more information we refer for instance to [19] and [20].

In Subsection 5.1 we recall the group von Neumann algebra $\mathcal{N}(G)$ associated to a group G, the von Neumann dimension $\dim_{\mathcal{N}(G)}$ for right $\mathcal{N}(G)$-modules, its properties, and compatibility with induction and restriction for modules over group von Neumann algebras. For a finite group G, the von Neumann algebra $\mathcal{N}(G)$ is $\mathbb{C}G$ and the von Neumann dimension of a $\mathbb{C}G$-module is the complex dimension divided by $|G|$. For general G, the von Neumann algebra $\mathcal{N}(G)$ is a $\mathbb{C}G$-$\mathcal{N}(G)$-bimodule.

In Subsection 5.2 we recall the L^2-Euler characteristic $\chi^{(2)}(C_*)$ of an $\mathcal{N}(G)$-chain complex C_* as the alternating sum of the von Neumann dimensions of the homology groups, and discuss the relevant properties.

In Subsection 5.3 we define the L^2-Euler characteristic for categories of type (L^2) using the splitting functor S_x. A category Γ is *of type* (L^2) if the constant $\mathbb{C}\Gamma$-module $\underline{\mathbb{C}}$ admits a (not necessarily finite) projective $\mathbb{C}\Gamma$-resolution P_* such that the sum over all $\overline{x} \in \mathrm{iso}(\Gamma)$ of all von Neumann dimensions of the homology groups of all $\mathcal{N}(\mathrm{aut}(x))$-chain complexes $S_x P_* \otimes_{\mathbb{C}\,\mathrm{aut}(x)} \mathcal{N}(\mathrm{aut}(x))$ converges to a finite number. Any directly finite category of type $(\mathrm{FP}_{\mathbb{C}})$ is of type (L^2). For example, finite groupoids, finite posets, and more generally finite EI-categories are of type (L^2).

Let $U^{(1)}(\Gamma)$ be the set of absolutely convergent sequences on the index set $\mathrm{iso}(\Gamma)$. The *functorial L^2-Euler characteristic* $\chi_f^{(2)}(\Gamma) \in U^{(1)}(\Gamma)$ has at index $\overline{x}$ the number $\chi^{(2)}\big(S_x P_* \otimes_{\mathbb{C}\,\mathrm{aut}(x)} \mathcal{N}(\mathrm{aut}(x))\big)$, where P_* is a projective $\mathbb{C}\Gamma$-resolution of $\underline{\mathbb{C}}$. The *L^2-Euler characteristic* $\chi^{(2)}(\Gamma) \in \mathbb{R}$ is the sum of the sequence $\chi_f^{(2)}(\Gamma)$.

For example, if Γ is a finite groupoid, then $\chi_f^{(2)}(\Gamma)$ has at index $\overline{x}$ the value $1/|\operatorname{aut}(x)|$, and the L^2-Euler characteristic is the sum of these.

Like the naive Euler characteristic, the L^2-Euler characteristic comes from the finiteness obstruction in certain cases. However, for the L^2-Euler characteristic, we use the L^2-rank $\operatorname{rk}_\Gamma^{(2)}$ instead of the $R\Gamma$-rank $\operatorname{rk}_{R\Gamma}$. In Subsection 5.4 we define the L^2-rank and prove that $\operatorname{rk}_\Gamma^{(2)} o(\Gamma;\mathbb{C}) = \chi_f^{(2)}(\Gamma)$ whenever Γ is directly finite and of type $(\mathrm{FP}_\mathbb{C})$.

The L^2-Euler characteristic is compatible with covering maps and isofibrations between connected finite groupoids, as we prove in Subsection 5.5.

We now recall the prerequisites from the theory of finite von Neumann algebras and motivate its use.

5.1. **Group von Neumann algebras and their dimension theory.** The appearence of (group) von Neumann algebras and their dimension theory in our context stems from the task to assign some sort of rational- or real-valued dimension to projective modules over group rings (coming from automorphism groups in a category), which itself is needed to extract a number, namely the Euler characteristic, from the finiteness obstruction.

The well-known *Hattori-Stallings rank* $\mathrm{HS}(M)$ [9, Chapter IX, 2] of a finitely generated projective R-module M over an arbitrary ring R is a way to assign a "dimension" to M. However, $\mathrm{HS}(M)$ is not a number but an element in the quotient $R/[R,R]$ of R by the additive subgroup $[R,R]$ generated by all commutators $ab-ba$, $a,b\in R$. In order to get, say, a $\mathbb{C}$-valued invariant one needs an additive homomorphism $t\colon R\to\mathbb{C}$ satisfying the *trace property* $t(ab)=t(ba)$.

Consider the case of the complex group ring $R=\mathbb{C}G$ of a group G. The map $\operatorname{tr}_{\mathcal{N}(G)}\colon \mathbb{C}G\to\mathbb{C}$, the notation of which already anticipates a more general setup, is defined by

$$\operatorname{tr}_{\mathcal{N}(G)}\big(\sum_{g\in G}\lambda_g g\big)=\lambda_e$$

and satisfies the trace property, thus providing a notion of dimension for finitely generated projective $\mathbb{C}G$-modules. This dimension does not extend to arbitrary $\mathbb{C}G$-modules, which is a major drawback as we would like to define the dimension of certain homology groups of projective resolutions that are *not* projective anymore. Next we explain work of the second author that allows to define a dimension for all modules – if one works with the larger ring $\mathcal{N}(G)$, the group von Neumann algebra of G, instead.

Let $l^2(G)$ be the Hilbert space with Hilbert basis G; it consists of formal sums $\sum_{g\in G}\lambda_g\cdot g$ for complex numbers λ_g such that $\sum_{g\in G}|\lambda_g|^2<\infty$. The complex group ring $\mathbb{C}G$ is a dense subset of $l^2(G)$. In fact, $l^2(G)$ is the Hilbert space completion of the complex group ring $\mathbb{C}G$ with respect to the pre-Hilbert structure for which G is an orthonormal basis. Left and right multiplication with elements in G induce respectively isometric left and right G-actions on $l^2(G)$.

Definition 5.1 (Group von Neumann algebra)**.** The *group von Neumann algebra* of the group G

$$\mathcal{N}(G) = \mathcal{B}(l^2(G))^G$$

is the algebra of bounded operators that equivariant with respect to the right G-action. The *standard trace* on $\mathcal{N}(G)$ is defined by

$$\operatorname{tr}_{\mathcal{N}(G)} \colon \mathcal{N}(G) \to \mathbb{C}, \quad f \mapsto \langle f(e), e\rangle_{l^2(G)}.$$

The standard trace extends the definition on $\mathbb{C}G$ given earlier on. From now on we view $\mathcal{N}(G)$ simply as a ring, ignoring its functional-analytic origin. The latter is only important for the proof of our 'blackbox' Theorem 5.2 below. Modules over $\mathcal{N}(G)$ are understood in the purely algebraic sense.

Sending an element $g \in G$ to the isometric G-equivariant operator $l^2(G) \to l^2(G)$ given by left multiplication with $g \in G$ induces an embedding of $\mathbb{C}G$ into $\mathcal{N}(G)$ as a subring. In particular, we can view $\mathcal{N}(G)$ as a $\mathbb{C}G$-$\mathcal{N}(G)$-bimodule.

Theorem 5.2 (Properties of the dimension function)**.** *There exists a dimension function* $\dim_{\mathcal{N}(G)}$ *that assigns to every right* $\mathcal{N}(G)$*-module* M *a number, possibly infinite,*

$$\dim_{\mathcal{N}(G)}(M) \in [0,\infty] = \mathbb{R}_{\geq 0} \cup \{\infty\}$$

and satisfies the following properties:

(i) *Hattori-Stallings rank*
If M *is a finitely generated projective* $\mathcal{N}(G)$*-module, then*

$$\dim_{\mathcal{N}(G)}(M) = \sum_{i=1}^{n} \operatorname{tr}_{\mathcal{N}(G)}(a_{i,i}) \in [0,\infty),$$

where $A = (a_{i,j})$ *is any* (n,n)*-matrix over* $\mathcal{N}(G)$ *with* $A^2 = A$ *such that the image of the* $\mathcal{N}(G)$*-homomorphism* $\mathcal{N}(G)^n \to \mathcal{N}(G)^n$ *given by left multiplication with* A *is* $\mathcal{N}(G)$*-isomorphic to* M*;*

(ii) *Additivity*
If $0 \to M_0 \to M_1 \to M_2 \to 0$ *is an exact sequence of* $\mathcal{N}(G)$*-modules, then*

$$\dim_{\mathcal{N}(G)}(M_1) = \dim_{\mathcal{N}(G)}(M_0) + \dim_{\mathcal{N}(G)}(M_2),$$

where for $r, s \in [0,\infty]$ *we define* $r + s$ *by the ordinary sum of two real numbers if both* r *and* s *are not* ∞*, and by* ∞ *otherwise;*

(iii) *Cofinality*
Let $\{M_i \mid i \in I\}$ *be a cofinal system of submodules of* M*, i.e.* $M = \bigcup_{i \in I} M_i$ *and for two indices* i *and* j *there is an index* k *in* I *satisfying* $M_i, M_j \subseteq M_k$*. Then*

$$\dim_{\mathcal{N}(G)}(M) = \sup\{\dim_{\mathcal{N}(G)}(M_i) \mid i \in I\}.$$

Proof. See [19, Theorem 6.5 and Theorem 6.7 on page 239]. □

Let $i\colon H \to G$ be an injective group homomorphism. Then the induced injective ring homomorphism $i_*\colon \mathbb{C}H \to \mathbb{C}G$ extends to an injective ring homomorphism denoted in the same way $i_*\colon \mathcal{N}(H) \to \mathcal{N}(G)$.

Lemma 5.3. *Let $i\colon H \to G$ be an injective group homomorphism.*

(i) *The induction functor* $\operatorname{ind}_{i_*}\colon \mathsf{MOD}\text{-}\mathcal{N}(H) \to \mathsf{MOD}\text{-}\mathcal{N}(G)$ *sending M to $M \otimes_{\mathcal{N}(H)} \mathcal{N}(G)$ is faithfully flat, i.e., a sequence of $\mathcal{N}(H)$-modules $M_1 \to M_2 \to M_3$ is exact if and only if the induced sequence of $\mathcal{N}(G)$-modules* $\operatorname{ind}_{i_*} M_1 \to \operatorname{ind}_{i_*} M_2 \to \operatorname{ind}_{i_*} M_3$ *is exact;*

(ii) *If M is an $\mathcal{N}(H)$-module, then*
$$\dim_{\mathcal{N}(G)}(\operatorname{ind}_{i_*} M) = \dim_{\mathcal{N}(H)}(M);$$

(iii) *Suppose that the index* $[G : \operatorname{im}(H)]$ *of* $\operatorname{im}(H)$ *in G is finite. Then we get for every $\mathcal{N}(G)$-module M, if* res_{i_*} *denotes its restriction to an $\mathcal{N}(H)$-module by i_**
$$\dim_{\mathcal{N}(H)}(\operatorname{res}_{i_*} M) = [G : \operatorname{im}(H)] \cdot \dim_{\mathcal{N}(G)}(M),$$
where $[G : \operatorname{im}(H)] \cdot \infty$ *is defined to be* ∞*.*

Proof. See [19, Theorem 6.29 on page 253 and Theorem 6.54 (6) on page 266]. □

Here are some useful examples of the von Neumann dimension.

Example 5.4.

(i) (von Neumann dimension for finite groups)**.** Let G be a finite group. Then $\mathcal{N}(G) = \mathbb{C}G$ and we get for a $\mathbb{C}G$-module M
$$\dim_{\mathcal{N}(G)}(M) = \frac{1}{|G|} \cdot \dim_{\mathbb{C}}(M);$$
where $\dim_{\mathbb{C}}$ is the dimension of M viewed as a complex vector space.

(ii) (von Neumann dimension and permutation modules)**.** Let G be a (not necessarily finite) group and S a cofinite G-set, i.e., S is the disjoint union of homogeneous G-spaces $\coprod_{i\in I} G/L_i$ for finite I. By [16, Lemma 4.4], we have
$$\dim_{\mathcal{N}(G)}\bigl(\mathbb{C}S \otimes_{\mathbb{C}G} \mathcal{N}(G)\bigr) = \sum_{\substack{i\in I\\ |L_i|<\infty}} \frac{1}{|L_i|}.$$

(iii) (von Neumann dimension for $\mathbb{Z}$)**.** Let $G = \mathbb{Z}$. Then $\mathcal{N}(\mathbb{Z}) = L^\infty(S^1)$ by Fourier transformation. Under this identification we obtain that
$$\operatorname{tr}_{\mathcal{N}(\mathbb{Z})}\colon \mathcal{N}(\mathbb{Z}) \to \mathbb{C}, \quad f \mapsto \int_{S^1} f\, d\mu,$$
where μ is the probability Lebesgue measure on S^1.

Let $X \subseteq S^1$ be any measurable set and $\chi_X \in L^\infty(S^1)$ be its characteristic function. Since χ_X is an idempotent, its image P is a finitely generated projective $\mathcal{N}(\mathbb{Z})$-module, whose von Neumann dimension $\dim_{\mathcal{N}(\mathbb{Z})}(P)$ is the volume $\mu(X)$ of X. In particular any non-negative real number occurs as $\dim_{\mathcal{N}(\mathbb{Z})}(P)$ for some finitely generated projective $\mathcal{N}(\mathbb{Z})$-module P.

5.2. **The L^2-Euler characteristic and L^2-Betti numbers.** In this section we briefly recall some basic facts about L^2-Betti numbers and L^2-Euler characteristics. For more information we refer to [19, Section 6.6.1 on page 277ff].

Definition 5.5 (L^2-Betti numbers)**.** Let C_* be an $\mathcal{N}(G)$-chain complex. Define its *p-th L^2-Betti number*

$$b_p^{(2)}(C_*) := \dim_{\mathcal{N}(G)}(H_p(C_*)) \quad \in [0,\infty]$$

to be the von Neumann dimension of the $\mathcal{N}(G)$-module given by its p-th homology.

Definition 5.6 (L^2-Euler characteristic)**.** Let C_* be an $\mathcal{N}(G)$-chain complex. Define

$$h^{(2)}(C_*) := \sum_{p\geq 0} b_p^{(2)}(C_*) \quad \in [0,\infty].$$

If $h^{(2)}(C_*) < \infty$, define the *L^2-Euler characteristic*

$$\chi^{(2)}(C_*) := \sum_{p\geq 0} (-1)^p \cdot b_p^{(2)}(C_*) \quad \in \mathbb{R}.$$

Notice that $h^{(2)}(C_*)$ can be finite also in the case, where infinitely many L^2-Betti numbers are different from zero.

Lemma 5.7.

(i) *Let C_* be an $\mathcal{N}(G)$-chain complex. Suppose that $\sum_{p\geq 0}\dim_{\mathcal{N}(G)}(C_p)$ is finite. Then $h^{(2)}(C_*)$ is finite and $\sum_{p\geq 0}(-1)^p\cdot\dim_{\mathcal{N}(G)}(C_p) = \chi^{(2)}(C_*)$;*

(ii) *Let C_* and D_* be $\mathcal{N}(G)$-chain complexes which are $\mathcal{N}(G)$-homotopy equivalent. Then we get $b_p^{(2)}(C_*) = b_p^{(2)}(D_*)$ and $h^{(2)}(C_*) = h^{(2)}(D_*)$ and, provided that $h^{(2)}(C_*)$ is finite, $\chi^{(2)}(C_*) = \chi^{(2)}(D_*)$;*

(iii) *Let $0 \to C_* \to D_* \to E_* \to 0$ be an exact sequence of $\mathcal{N}(G)$-chain complexes. Suppose that two of the elements $h^{(2)}(C_*)$, $h^{(2)}(D_*)$, and $h^{(2)}(E_*)$ in $[0,\infty]$ are finite. Then this is true for all three and we obtain that*

$$\chi^{(2)}(C_*) - \chi^{(2)}(D_*) + \chi^{(2)}(E_*) = 0;$$

(iv) *Let $i\colon H \to G$ be an injective group homomorphism and let C_* be an $\mathcal{N}(H)$-chain complex. Then $h^{(2)}(C_*) = h^{(2)}(\operatorname{ind}_{i_*} C_*)$ and, provided that $h^{(2)}(C_*) < \infty$, we have $\chi^{(2)}(C_*) = \chi^{(2)}(\operatorname{ind}_{i_*} C_*)$;*

(v) *Let $i\colon H \to G$ be an injective group homomorphism with finite index $[G:H]$. Let C_* be an $\mathcal{N}(G)$-chain complex. Then*

$$h^{(2)}(\operatorname{res}_{i_*} C_*) = [G:H]\cdot h^{(2)}(\operatorname{ind}_{i_*} C_*)$$

and, provided that $h^{(2)}(C_*) < \infty$, we have

$$\chi^{(2)}(\operatorname{res}_{i_*} C_*) = [G:H]\cdot \chi^{(2)}(C_*).$$

Proof. ii) is obvious from the definition. The rest easily follows from Theorem 5.2 and Lemma 5.3. □

5.3. **The (functorial) L^2-Euler characteristic.** In the following Γ is always a small category. For every $x \in \mathrm{ob}(\Gamma)$ let

$$\mathcal{N}(x) := \mathcal{N}(\mathrm{aut}(x))$$

be the group von Neumann algebra of the automorphism group $\mathrm{aut}(x)$.

Recall that two projective $\mathcal{N}(G)$-resolutions P_* and Q_* of the constant $\mathbb{C}\Gamma$-module $\underline{\mathbb{C}}$ are $\mathbb{C}\Gamma$-chain homotopy equivalent and hence the $\mathbb{C}[x]$-chain complexes S_xP_* and S_xQ_* and the $\mathbb{C}[x]$-chain complexes $\mathrm{Res}_x\, P_*$ and $\mathrm{Res}_x\, Q_*$ are $\mathbb{C}[x]$-chain homotopy equivalent. Therefore the following definitions will be independent of the choice of a projective $\mathbb{C}\Gamma$-resolution of $\underline{\mathbb{C}}$.

Definition 5.8 (Type (L^2))**.** We call Γ *of type* (L^2) if for one (and hence all) projective $\mathbb{C}\Gamma$-resolutions P_* of the constant $\mathbb{C}\Gamma$-module $\underline{\mathbb{C}}$ we have

$$\sum_{\overline{x} \in \mathrm{iso}\,\Gamma} h^{(2)}\big(S_xP_* \otimes_{\mathbb{C}[x]} \mathcal{N}(x)\big) < \infty.$$

We shall see in Example 5.12 that any finite groupoid is of type (L^2). We shall also see in Theorem 5.22 that any directly finite category of type $(\mathrm{FP}_{\mathbb{C}})$ is of type (L^2).

Definition 5.9 (The functorial L^2-Euler characteristic of a category)**.** Suppose that Γ is of type (L^2). Define

$$U^{(1)}(\Gamma) := \left\{ \sum_{\overline{x} \in \mathrm{iso}(\Gamma)} r_{\overline{x}} \cdot \overline{x} \;\middle|\; r_{\overline{x}} \in \mathbb{R}, \sum_{\overline{x} \in \mathrm{iso}(\Gamma)} |r_{\overline{x}}| < \infty \right\} \subseteq \prod_{\overline{x} \in \mathrm{iso}(\Gamma)} \mathbb{R}.$$

Define the *functorial L^2-Euler characteristic of* Γ by

$$\chi_f^{(2)}(\Gamma) := \left\{ \chi^{(2)}\big(S_xP_* \otimes_{\mathbb{C}[x]} \mathcal{N}(x)\big) \mid \bar{x} \in \mathrm{iso}(\Gamma) \right\} \in U^{(1)}(\Gamma),$$

where P_* is a projective $\mathbb{C}\Gamma$-resolution of the constant $\mathbb{C}\Gamma$-module $\underline{\mathbb{C}}$.

The word functorial refers to the fact that the group, in which $\chi_f^{(2)}$ takes values, depends in a functorial way on Γ.

We can also get a real-valued invariant as follows.

Definition 5.10 (The L^2-Euler characteristic of a category)**.** Suppose that Γ is of type (L^2). Define the *L^2-Euler characteristic of* Γ to be the real number

$$\chi^{(2)}(\Gamma) := \sum_{\overline{x} \in \mathrm{iso}(\Gamma)} \chi^{(2)}\big(S_xP_* \otimes_{\mathbb{C}[x]} \mathcal{N}(x)\big).$$

Notice that this definition makes sense since the condition (L^2) ensures that the sum $\sum_{\overline{x} \in \mathrm{iso}(\Gamma)} \chi^{(2)}\big(S_xP_* \otimes_{\mathbb{C}[x]} \mathcal{N}(x)\big)$ is absolutely convergent.

Remark 5.11. In Definition 5.10, the L^2-Euler characteristic is defined to be the sum of the components of the functorial L^2-Euler characteristic. This is analogous to the situation for the ordinary Euler characteristic, namely Corollary 4.19 states that the Euler characteristic of an EI-category is the sum of the components of the functorial Euler characteristic, provided R is Noetherian and the category is of type (FP_R).

Example 5.12 (The (functorial) L^2-Euler characteristic of groupoids)**.** Let $\mathcal{G}$ be a (small) groupoid such that $\operatorname{aut}(x)$ for any object $x \in \operatorname{ob}(\mathcal{G})$ is finite and

$$\sum_{\overline{x} \in \operatorname{iso}(\mathcal{G})} \frac{1}{|\operatorname{aut}(x)|} < \infty. \tag{5.13}$$

Let P_* be any projective $\mathbb{C}\mathcal{G}$-resolution of $\underline{\mathbb{C}}$; a (not necessarily finite) projective resolution always exists. Since $\mathcal{G}$ is a groupoid, for every $x \in \operatorname{ob}\mathcal{G}$ and every $\mathbb{C}\mathcal{G}$-module M we have $S_x M = \operatorname{Res}_x M$. Thus S_x is exact. By Lemma 3.5 S_x respects projectives. Hence $S_x P_x$ is a projective $\mathbb{C}[x]$-resolution of the trivial module $\mathbb{C}$. Since $\operatorname{aut}(x)$ is finite, $\mathbb{C}$ is already a projective $\mathbb{C}[x]$-module. This implies that

$$H_i\big(S_x P_* \otimes_{\mathbb{C}[x]} \mathcal{N}(x)\big) = \begin{cases} \mathbb{C} \otimes_{\mathbb{C}[x]} \mathcal{N}(x) & i = 0 \\ 0 & i > 0. \end{cases}$$

Example 5.4 (i) and (5.13) yield that $\mathcal{G}$ is of type (L^2), the functorial L^2-Euler characteristic $\chi_f^{(2)}(\mathcal{G}) \in \prod_{\overline{x} \in \operatorname{iso}(\mathcal{G})} \mathbb{R}$ has at $\overline{x} \in \operatorname{iso}(\mathcal{G})$ the value $1/|\operatorname{aut}(x)|$, and

$$\chi^{(2)}(\mathcal{G}) = \sum_{\overline{x} \in \operatorname{iso}(\mathcal{G})} \frac{1}{|\operatorname{aut}(x)|}.$$

In particular, we can conclude that, for all finite groupoids, the L^2-Euler characteristic coincides with the Baez-Dolan groupoid cardinality and Leinster's Euler characteristic.

A concrete case of a groupoid satisfying our conditions is a skeleton $\mathcal{G}$ of the groupoid of nonempty finite sets. This groupoid has objects (isomorphic to) $\underline{1} = \{1\}$, $\underline{2} = \{1, 2\}$, $\underline{3} = \{1, 2, 3\}$, and so on. The morphisms are the permutations. This example was studied in [2]. The groupoid $\mathcal{G}$ is of type (L^2), and the functorial L^2-Euler characteristic has at the object $\underline{n}$ the value $1/|\operatorname{aut}(\underline{n})| = 1/n!$. The L^2-Euler characteristic is

$$\chi^{(2)}(\mathcal{G}) = \sum_{n \geq 1} \frac{1}{|S_n|} = \sum_{n \geq 1} \frac{1}{n!} = e.$$

Remark 5.14. If G is a group and $\widehat{G}$ denotes the groupoid with precisely one object and G as automorphism group of this object, then $\chi^{(2)}(\widehat{G})$ in the sense of Definition 5.10 agrees with the classical definition of the L^2-Euler characteristic $\chi^{(2)}(G)$ of a group which has been intensively studied in the literature (see for instance [19, Chapter 7]).

Lemma 5.15 (Invariance of L^2-Euler characteristic under equivalence of categories)**.**

(i) *Suppose* Γ_1 *and* Γ_2 *are equivalent categories. Then* Γ_1 *is both directly finite and of type* (L^2) *if and only if* Γ_2 *is both directly finite and of type* (L^2)*.*

(ii) *Let* $F\colon \Gamma_1 \to \Gamma_2$ *be an equivalence of categories. Suppose that* Γ_i *is both directly finite and of type* (L^2) *for* $i = 1, 2$.

Then the bijection

$$U^{(1)}(F)\colon U^{(1)}(\Gamma_1) \xrightarrow{\cong} U^{(1)}(\Gamma_2)$$

induced by F *sends* $\chi_f^{(2)}(\Gamma_1)$ *to* $\chi_f^{(2)}(\Gamma_2)$ *and we have*

$$\chi^{(2)}(\Gamma_1) = \chi^{(2)}(\Gamma_2).$$

Proof. We have already shown that the property of being directly finite depends only on the equivalence class of a category (see Lemma 3.2). So in the sequel we can assume that Γ_1 and Γ_2 are directly finite.

Let $F\colon \Gamma_1 \to \Gamma_2$ be an equivalence of categories. It induces a bijection

$$F_*\colon \operatorname{iso}(\Gamma_1) \xrightarrow{\cong} \operatorname{iso}(\Gamma_2), \quad \overline{x} \mapsto \overline{F(x)},$$

and thus a bijection

$$U^{(1)}(F)\colon U^{(1)}(\Gamma_1) \xrightarrow{\cong} U^{(1)}(\Gamma_2).$$

The induction functor ind_F associated to F is compatible with direct sums over arbitrary index sets and sends $\mathbb{C}\operatorname{mor}_{\Gamma_1}(?, x)$ to $\mathbb{C}\operatorname{mor}_{\Gamma_2}(?, F(x))$. Recall that a free $\mathbb{C}\Gamma_1$-module is of the form $\bigoplus_{i\in I} \mathbb{C}\operatorname{mor}_{\Gamma_1}(?, x_i)$ for some index set I and a collection of objects $\{x_i \mid i \in I\}$ of Γ_1. Hence ind_F sends projective $\mathbb{C}\Gamma_1$-modules to projective $\mathbb{C}\Gamma_2$-modules.

The equivalence F induces for every object x in Γ_1 an isomorphism of groups

$$F_x\colon \operatorname{aut}_{\Gamma_1}(x) \xrightarrow{\cong} \operatorname{aut}_{\Gamma_2}(F(x)), \quad f \mapsto F(f).$$

Next we construct for every object x in Γ_1 and projective $\mathbb{C}\Gamma_1$-module P a natural isomorphism of $\mathbb{C}[F(x)]$-modules

$$\alpha(P)\colon \operatorname{ind}_{F_x} \circ S_x P \xrightarrow{\cong} S_{F(x)} \circ \operatorname{ind}_F P.$$

Let $\alpha_1(P)\colon P(x) \to \big(\operatorname{ind}_F P\big)(F(x))$ be the map sending $p \in P(x)$ to the class of $p \otimes \operatorname{id}_{F(x)} \in P \otimes_{\mathbb{C}\Gamma_1} \mathbb{C}\operatorname{mor}_{\Gamma_2}(F(x), F(?))$. Let $\alpha_2(P)\colon P(x) \to S_{F(x)} \circ \operatorname{ind}_F P$ be the composite of $\alpha_1(P)$ with the canonical projection $\big(\operatorname{ind}_F P\big)(F(x)) \to S_{F(x)}\big(\operatorname{ind}_F P\big)$. One easily checks that $\alpha_2(P)$ factorizes over the canonical projection $P(x) \to S_x P$ to a map $\alpha_3(P)\colon S_x P \to S_{F(x)} \circ \operatorname{ind}_F P$. Since $\alpha_3(P)$ is equivariant with respect to the group homomorphism $F_x\colon \operatorname{aut}_{\Gamma_1}(x_1) \to \operatorname{aut}_{\Gamma_2}(F(x))$, it induces a $R[F(x)]$-homomorphism $\alpha(P)\colon \operatorname{ind}_{F_x} \circ S_x P \xrightarrow{\cong} S_{F(x)} \circ \operatorname{ind}_F P$. One easily checks that $\alpha(P)$ is natural in P and compatible with direct sums over arbitrary index sets. Since Γ_1 and Γ_2 are directly finite, $\alpha\big(\mathbb{C}\operatorname{mor}_{\Gamma_1}(?, y)\big)$ is bijective for every object y in Γ_1 (compare with the proof of Lemma 3.5 (iv)). Since every projective $\mathbb{C}\Gamma_1$-module is a direct summand in a direct sum of $\mathbb{C}\Gamma_1$-module of the form $\mathbb{C}\operatorname{mor}_{\Gamma_1}(?, y)$ for some object y, the map $\alpha(P)$ is an $\mathbb{C}[F(x)]$-isomorphism for every projective $\mathbb{C}\Gamma$-module P and every object x.

Fix an object x in Γ_1. The argument in the proof of Theorem 2.10 shows that the induction functor ind_F associated to F is an exact functor and sends $\underline{\mathbb{C}}$ to $\underline{\mathbb{C}}$. Let P_* be a free $\mathbb{C}\Gamma_1$-resolution of $\underline{\mathbb{C}}$. Then $\operatorname{ind}_F P_*$ is a free $\mathbb{C}\Gamma_2$-resolution of $\underline{\mathbb{C}}$. The various isomorphisms $\alpha(P_n)$ induce an isomorphism of $\mathbb{C}[F(x)]$-chain complexes

$$\alpha(P_*)\colon \operatorname{ind}_{F_x} \circ S_x P_* \xrightarrow{\cong} S_{F(x)} \circ \operatorname{ind}_F P_*.$$

We have for every $R[x]$-module M a canonical $\mathcal{N}(F(x))$-isomorphism

$$\big(\operatorname{ind}_{F_x} M\big) \otimes_{\mathbb{C}[F(x)]} \mathcal{N}(F(x)) \xrightarrow{\cong} \operatorname{ind}_{F_x}\big(M \otimes_{\mathbb{C}[x]} \mathcal{N}(x)\big).$$

If we apply $- \otimes_{\mathbb{C}[F(x)]} \mathcal{N}(F(x))$ to $\alpha(P_*)$ and use the isomorphisms above we obtain an isomorphism of $\mathcal{N}(F(x))$-chain complexes

$$\alpha^{(2)}(P_*)\colon \operatorname{ind}_{F_x}\big(S_x P_* \otimes_{\mathbb{C}[x]} \mathcal{N}(x)\big) \xrightarrow{\cong} \big(S_{F(x)} \circ \operatorname{ind}_F P_*\big) \otimes_{\mathbb{C}[F(x)]} \mathcal{N}(F(x)).$$

We conclude from Lemma 5.7 (ii)

$$h^2\big(S_x P_* \otimes_{\mathbb{C}[x]} \mathcal{N}(x)\big) \quad = \quad h^2\left(\big(S_{F(x)} \circ \operatorname{ind}_F P_*\big) \otimes_{\mathbb{C}[F(x)]} \mathcal{N}(F(x))\right)$$

and, provided that $h^2\big(S_x P_* \otimes_{\mathbb{C}[x]} \mathcal{N}(x)\big) < \infty$

$$\chi^2\big(S_x P_* \otimes_{\mathbb{C}[x]} \mathcal{N}(x)\big) \quad = \quad \chi^2\left(\big(S_{F(x)} \circ \operatorname{ind}_F P_*\big) \otimes_{\mathbb{C}[F(x)]} \mathcal{N}(F(x))\right).$$

Now Lemma 5.15 follows. □

Next we consider products of categories. Since $\operatorname{iso}(\Gamma_1 \times \Gamma_2) = \operatorname{iso}(\Gamma_1) \times \operatorname{iso}(\Gamma_2)$, we obtain a pairing

$$\text{(5.16)} \quad \otimes\colon U^{(1)}(\Gamma_1) \otimes U^{(1)}(\Gamma_2) \to U^{(1)}(\Gamma_1 \times \Gamma_2),$$
$$\sum_{\overline{x_1} \in \operatorname{iso}(\Gamma_1)} r_{\overline{x_1}} \cdot \overline{x_1} \otimes \sum_{\overline{x_2} \in \operatorname{iso}(\Gamma_2)} s_{\overline{x_2}} \cdot \overline{x_2} \mapsto \sum_{\overline{(x_1,x_2)} \in \operatorname{iso}(\Gamma_1 \times \Gamma_2)} r_{\overline{x_1}} s_{\overline{x_2}} \cdot \overline{(x_1, x_2)}.$$

Theorem 5.17 (Product formula for $\chi_f^{(2)}$ and $\chi^{(2)}$)**.** *Let Γ_1 and Γ_2 be categories of type (L^2).*

Then $\Gamma_1 \times \Gamma_2$ is of type (L^2), we get for the functorial L^2-Euler characteristic

$$\chi_f^{(2)}(\Gamma_1 \times \Gamma_2) \quad = \quad \chi_f^{(2)}(\Gamma_1) \otimes \chi_f^{(2)}(\Gamma_2)$$

under the pairing (5.16)*, and we get for the L^2-Euler characteristic*

$$\chi^{(2)}(\Gamma_1 \times \Gamma_2) = \chi^{(2)}(\Gamma_1) \cdot \chi^{(2)}(\Gamma_2).$$

Proof. If P_* is a projective $\mathbb{C}\Gamma_1$-resolution of the constant $\mathbb{C}\Gamma_1$-module $\underline{\mathbb{C}}$ and Q_* is a projective $\mathbb{C}\Gamma_2$-resolution of the constant $\mathbb{C}\Gamma_2$-module $\underline{\mathbb{C}}$, then $P_* \otimes Q_*$ is a projective $\mathbb{C}(\Gamma_1 \times \Gamma_2)$-resolution of the constant $\mathbb{C}(\Gamma_1 \times \Gamma_2)$-module $\underline{\mathbb{C}}$. Given $\overline{x} \in \operatorname{iso}(\Gamma_1)$ and $\overline{y} \in \operatorname{iso}(\Gamma_2)$, there is a canonical isomorphism of chain complexes over $\mathbb{C}[(x,y)] = \mathbb{C}[x] \otimes_{\mathbb{C}} \mathbb{C}[y]$

$$S_x P_* \otimes_{\mathbb{C}} S_y P_* = S_{(x,y)}(P_* \otimes_{\mathbb{C}} Q_*).$$

Since the Cauchy product of two absolutely convergent series of real numbers is again an absolutely convergent series, it suffices to show for two groups H and G, a projective $\mathbb{C}H$-chain complex C_* and a projective $\mathbb{C}G$-chain complex D_*, that for the projective $\mathbb{C}[G \times H]$-chain $C_* \otimes_{\mathbb{C}} D_*$ we have

$$\begin{array}{rcl} h^{(2)}(C_* \otimes_{\mathbb{C}} D_*) & < & \infty; \\ \chi^{(2)}(C_* \otimes_{\mathbb{C}} D_*) & = & \chi^{(2)}(C_*) \cdot \chi^{(2)}(D_*) \end{array}$$

provided that $h^{(2)}(C_*)$ and $h^{(2)}(D_*)$ are finite. The proof of this claim is the chain complex analogue of the proof of [19, Theorem 6.80 (6) on page 278]. □

5.4. **The finiteness obstruction and the (functorial) L^2-Euler characteristic.** Next we compare these definitions with the finiteness obstruction.

Definition 5.18 (L^2-rank of a finitely generated $\mathbb{C}\Gamma$-module)**.** Let M be a finitely generated $\mathbb{C}\Gamma$-module M. Define its *L^2-rank*

$$\operatorname{rk}_\Gamma^{(2)}(M) := \left\{ \dim_{\mathcal{N}(x)}(S_x M \otimes_{\mathbb{C}[x]} \mathcal{N}(x)) \mid \bar{x} \in \operatorname{iso}(\Gamma) \right\} \in U(\Gamma) \otimes_{\mathbb{Z}} \mathbb{R} = \bigoplus_{\operatorname{iso}(\Gamma)} \mathbb{R}.$$

The rank $\operatorname{rk}_\Gamma^{(2)}$ defines a homomorphism

$$\operatorname{rk}_\Gamma^{(2)} \colon K_0(\mathbb{C}\Gamma) \to U(\Gamma) \otimes_{\mathbb{Z}} \mathbb{R}, \quad [P] \to \operatorname{rk}_\Gamma^{(2)}(P) \tag{5.19}$$

since for a finitely generated $\mathbb{C}\Gamma$-module M the value of $S_x M$ is non-trivial only for finitely many elements $\overline{x} \in \operatorname{iso}(\Gamma)$ and the $\mathbb{C}\operatorname{aut}(x)$-module $S_x M$ is finitely generated for every $x \in \operatorname{ob}(\Gamma)$ (see Lemma 3.5).

If Γ is directly finite, then the map $\operatorname{rk}_\Gamma^{(2)}$ obviously factorizes over $S \colon K_0(\mathbb{C}\Gamma) \to \operatorname{Split} K_0(\mathbb{C}\Gamma)$.

Example 5.20. If H is a subgroup of G of finite index $[G : H]$, and i denotes the inclusion, then the diagram

$$\begin{array}{ccc} K_0(\mathbb{C}G) & \xrightarrow{\operatorname{rk}_G^{(2)}} & \mathbb{R} \\ {\scriptstyle i^*}\downarrow & & \downarrow{\scriptstyle [G:H]\cdot} \\ K_0(\mathbb{C}H) & \xrightarrow[\operatorname{rk}_H^{(2)}]{} & \mathbb{R} \end{array}$$

commutes.

Proof. It follows from existence of a $\mathbb{C}H$-isomorphism $\mathbb{C}G^n = \oplus_{G/H}\mathbb{C}H^n$ that the restriction i^*P of a finitely generated projective $\mathbb{C}G$-module P is a finitely generated projective $\mathbb{C}H$-module. So the left vertical map in the above diagram is well defined. It directly follows from the proof of [19, Theorem 6.54 (6) on page 266] that

$$(i^*P) \otimes_{\mathbb{C}H} \mathcal{N}(H) \cong \operatorname{res}_{i_*}\big(P \otimes_{\mathbb{C}G} \mathcal{N}(G)\big).$$

Now the assertion follows from Lemma 5.3 (iii). □

Remark 5.21 (L^2-rank of a finitely generated $R\Gamma$-module)**.** In Definition 5.18 we have defined the L^2-rank of a finitely generated $\mathbb{C}\Gamma$-module. If R is a subring of $\mathbb{C}$, we may analogously define the L^2-rank of a finitely generated $R\Gamma$-module M. Namely, we view $\mathcal{N}(x)$ as an $R\operatorname{aut}(x)$-$\mathcal{N}(x)$-bimodule via the embedding of rings $R\operatorname{aut}(x) \to \mathbb{C}\operatorname{aut}(x) \to \mathcal{N}(\operatorname{aut}(x))$ and then take $\dim_{\mathcal{N}(x)}(S_xM \otimes_{R[x]} \mathcal{N}(x))$ as the components of the L^2-rank of M. We will primarily be interested in the case $R = \mathbb{C}$, so we omit $\mathbb{C}$ from the notation $\operatorname{rk}_\Gamma^{(2)}$. Occasionally we will also consider $R = \mathbb{Q}$.

Theorem 5.22 (Relating the finiteness obstruction and the L^2-Euler characteristic)**.** *Suppose that Γ is a directly finite category of type (FP$_\mathbb{C}$). Then Γ is of type (L^2) and the image of the finiteness obstruction $o(\Gamma;\mathbb{C})$ (see Definition 2.7) under the homomorphism*

$$\operatorname{rk}_\Gamma^{(2)} \colon K_0(\mathbb{C}\Gamma) \to U(\Gamma) \otimes_\mathbb{Z} \mathbb{R} = \bigoplus_{\overline{x} \in \operatorname{iso}(\Gamma)} \mathbb{R}$$

defined in (5.19) *is* $\chi_f^{(2)}(\Gamma)$.

Proof. Since Γ is of type (FP$_\mathbb{C}$), we can find a finite projective $\mathbb{C}\Gamma$-resolution P_* of $\underline{\mathbb{C}}$. Hence S_xP_* is non-trivial only for finitely many objects x in Γ and a finite projective $\mathbb{C}[x]$-chain complex for all objects x in Γ by Lemma 3.5. Hence Γ is of type (L^2). Now apply Lemma 5.7 (i). □

Lemma 5.23. *Suppose that Γ is directly finite. Then:*

(i) *If F is a finitely generated free $\mathbb{C}\Gamma$-module, the rank $\operatorname{rk}_{\mathbb{C}\Gamma}(F)$ of Definition 4.6 and the rank $\operatorname{rk}_\Gamma^{(2)}(F)$ of Definition 5.18 agree;*

(ii) *The composite*

$$U(\Gamma) \xrightarrow{\iota} K_0(\mathbb{C}\Gamma) \xrightarrow{\operatorname{rk}_\Gamma^{(2)}} U(\Gamma) \otimes_\mathbb{Z} \mathbb{R}$$

of the homomorphisms defined in (4.8) *and* (5.19) *is the obvious inclusion* $U(\Gamma) \to U(\Gamma) \otimes_\mathbb{Z} \mathbb{R}$*;*

(iii) *Suppose that Γ is of type (FF) (see Definition 2.3). Then*

$$\chi_f^{(2)}(\Gamma) \in U^{(1)}(\Gamma)$$

of Definition 5.10 lies already in $U(\Gamma)$ and agrees with the functorial Euler characteristic $\chi_f(\Gamma;\mathbb{C}) \in U(\Gamma)$ defined in Definition 4.11 and we get

$$\iota\bigl(\chi_f^{(2)}(\Gamma;\mathbb{C})\bigr) = o(\Gamma;\mathbb{C}).$$

Moreover, if R is Noetherian, the real numbers $\chi(\Gamma;R)$ of Definition 4.2 and $\chi^{(2)}(\Gamma)$ of Definition 5.9 agree.

Proof. (i) This follows from Lemma 4.10 since for $\overline{y} = \overline{x}$ we have

$$\begin{aligned}\operatorname{rk}_\Gamma^{(2)}(\mathbb{C}\operatorname{mor}(?,x))_{\overline{y}} &= \dim_{\mathcal{N}(x)}\bigl(S_x\mathbb{C}\operatorname{mor}(?,x)\otimes_{\mathbb{C}[x]}\mathcal{N}(x)\bigr)\\ &= \dim_{\mathcal{N}(x)}\bigl(\mathcal{N}(x)\bigr) = 1 = \operatorname{rk}_{\mathbb{C}}(S_x\mathbb{C}\operatorname{mor}(?,x)\otimes_{\mathbb{C}[x]}\mathbb{C}) = \operatorname{rk}_{\mathbb{C}\Gamma}(\mathbb{C}\operatorname{mor}(?,x))_{\overline{y}}.\end{aligned}$$

and for $\overline{y} \neq \overline{x}$ we get

$$\operatorname{rk}_\Gamma^{(2)}(\mathbb{C}\operatorname{mor}(?,x))_{\overline{y}} = 0 = \operatorname{rk}_{\mathbb{C}\Gamma}(\mathbb{C}\operatorname{mor}(?,x))_{\overline{y}}$$

(ii) This follows from assertion (i) and Lemma 4.10 (i).

(iii) This follows from assertion (i), Lemma 4.14, Theorem 4.15, and Theorem 5.22. □

Remark 5.24. We conclude from Lemma 4.10 and Lemma 5.23 that for a directly finite category Γ of type (FF) the invariants $o(\Gamma;R)$ of Definition 2.7, $\chi_f(\Gamma;R)$ of Definition 4.11, and $\chi_f^{(2)}(\Gamma)$ of Definition 5.9 all carry the same information. If additionally R is Noetherian, the invariants $\chi(\Gamma;R)$ of Definition 4.2 and $\chi^{(2)}(\Gamma)$ of Definition 5.10 agree.

Remark 5.25. Recall that $\chi(\mathcal{C};\mathbb{Q})$ is the Euler characteristic of $B\mathcal{C}$. However, it is not true that $\chi^{(2)}(\mathcal{C})$ is related to the L^2-Euler characteristic $\chi^{(2)}\bigl(\widetilde{B\mathcal{C}};\mathcal{N}(\pi_1(B\mathcal{C}))\bigr)$ in the sense of [19, Definition 6.20]. We will compute $\chi^{(2)}(\underline{\mathsf{Or}}(D_\infty)) = 0$ in Subsection 8.5. On the other hand $B\underline{\mathsf{Or}}(D_\infty) = D_\infty\backslash\underline{E}D_\infty$ is contractible and hence $\chi^{(2)}\bigl(\widetilde{B\mathcal{C}};\mathcal{N}(\pi_1(B\mathcal{C}))\bigr) = \chi(B\underline{\mathsf{Or}}(D_\infty)) = 1$.

5.5. **Compatibility of the L^2-Euler characteristic with coverings and isofibrations.** Our next task is to show that the L^2-Euler characteristic is compatible with covering maps and isofibrations between connected finite groupoids.

In the context of groupoids, the role of a covering neighborhood is played by the star of an object. If $\mathcal{E}$ is a small groupoid and e is an object of $\mathcal{E}$, we denote by $St(e)$ the *star of* e, namely the set of all morphisms in $\mathcal{E}$ with domain e.

Definition 5.26 (Covering of a groupoid)**.** A functor $p\colon \mathcal{E} \to \mathcal{B}$ between connected small groupoids is a *covering* if it is surjective on objects and restricts to a bijection

$$St(e) \longrightarrow St(p(e))$$

for each object e of $\mathcal{E}$. We say that a covering p is *n-sheeted* if $|\operatorname{ob}(p^{-1}(b))| = n$ for all objects b of $\mathcal{B}$.

Recall that a small groupoid $\mathcal{E}$ is *finite* if $\mathrm{iso}(\mathcal{E})$ is finite and for any object $e \in \mathrm{ob}(\mathcal{E})$ the set $\mathrm{aut}(e)$ is finite.

Theorem 5.27. *Let $\mathcal{E}$ and $\mathcal{B}$ be finite connected groupoids. If $p\colon \mathcal{E} \to \mathcal{B}$ is an n-sheeted covering, then*

$$\chi^{(2)}(\mathcal{E}) = n\chi^{(2)}(\mathcal{B}).$$

Proof. We present two proofs, one counting morphisms and the other using the technology of the finiteness obstruction.

To prove the theorem by counting morphisms, we first reduce to the case where the base groupoid has only one object. If $b \in \mathcal{B}$ and $\mathcal{E}_b$ denotes the groupoid $p^{-1}(\widehat{\mathrm{aut}(b)})$, then the diagram

$$\begin{array}{ccc} \mathcal{E}_b & \hookrightarrow & \mathcal{E} \\ {\scriptstyle p|_{\mathcal{E}_b}}\downarrow & & \downarrow{\scriptstyle p} \\ \widehat{\mathrm{aut}(b)} & \hookrightarrow & \mathcal{B} \end{array}$$

commutes and the horizontal functors are equivalences of categories. The groupoid $\mathcal{E}_b$ is connected; for if $e, e' \in \mathcal{E}_b$, then $f\colon e \cong e'$ in $\mathcal{E}$, and $p(f) \in \mathrm{aut}(b)$, so $f \in \mathrm{mor}(\mathcal{E}_b)$. Moreover, $St_{\mathcal{E}_b}(e) \subseteq St_{\mathcal{E}}(e)$ for all $e \in \mathcal{E}_b$, $St_{\mathrm{aut}(b)}(b) \subseteq St_{\mathcal{B}}(b)$, and $p|_{\mathcal{E}_b}$ is an n-sheeted covering. By Theorem 2.8, Theorem 5.22, and the L^2-analogue of Corollary 4.19, the groupoids $\mathcal{E}_b$ and $\mathcal{E}$ have the same L^2-Euler characteristic, as do $\widehat{\mathrm{aut}(b)}$ and $\mathcal{B}$. Alternatively, we know from Example 5.12 directly that

$$\chi^{(2)}(\mathcal{E}_b) = \frac{1}{|\,\mathrm{aut}(e)|} = \chi^{(2)}(\mathcal{E})$$

$$\chi^{(2)}(\widehat{\mathrm{aut}(b)}) = \frac{1}{|\,\mathrm{aut}(b)|} = \chi^{(2)}(\mathcal{B}).$$

Thus, if the theorem holds in the case where the base groupoid has only one object, it holds in general.

Suppose now that $\mathcal{B}$ has only one object b, so that $\mathcal{B} = \widehat{\mathrm{aut}(b)}$. Then $\mathcal{E}$ has only n objects, say $e_1, \ldots, e_n$. Since $\mathcal{E}$ is a connected finite groupoid, all of its hom-sets have the same number of elements. Let $e \in \mathcal{E}$. We have

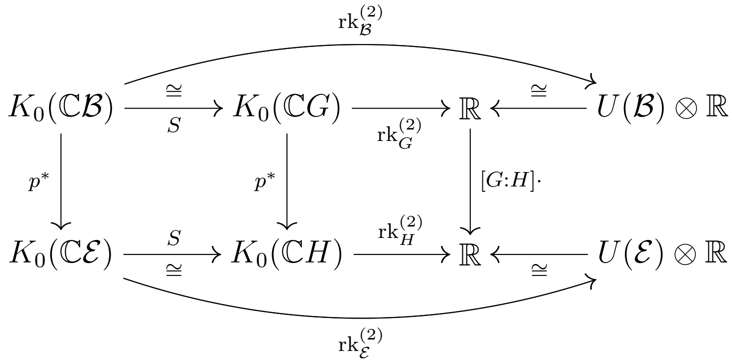
$$\begin{aligned} |\operatorname{aut}(b)| &= |St(e)| \\ &= |\bigcup_{i=1}^{n} \operatorname{mor}_{\mathcal{E}}(e, e_i)| \\ &= \sum_{i=1}^{n} |\operatorname{mor}_{\mathcal{E}}(e, e_i)| \\ &= \sum_{i=1}^{n} |\operatorname{aut}(e)| \\ &= n|\operatorname{aut}(e)|. \end{aligned} \tag{5.28}$$

In conclusion, $\chi^{(2)}(\mathcal{E}) = n\chi^{(2)}(\mathcal{B})$.

We may also prove Theorem 5.27 on the level of finiteness obstructions as follows, without reduction to the case of one object in the base groupoid.

The covering $p\colon \mathcal{E} \to \mathcal{B}$ is admissible in the sense that res_p sends a finitely generated projective $R\mathcal{B}$-module to a finitely generated projective $R\mathcal{E}$-module as a consequence of [15, Proposition 10.16 on page 187]. The set $\operatorname{Irr}(x, y)$ of irreducible morphisms $p(x) \to y$ is $\operatorname{mor}_{\mathcal{B}}(p(x), y)$, since $\mathcal{E}$ is a groupoid. Since $\mathcal{E}$ is finite, for a given $y \in \mathcal{B}$, the set $\operatorname{Irr}(x, y)$ is nonempty for only finitely many $\overline{x} \in \operatorname{iso}(\mathcal{E})$. Since $\mathcal{B}$ is finite, for each $x \in \mathcal{E}$ the $\operatorname{aut}_{\mathcal{E}}(x)$-set $\operatorname{Irr}(x, y)$ has only finitely many orbits. The action of $\operatorname{aut}_{\mathcal{E}}(x)$ is free because $\mathcal{B}$ is a groupoid and p is a covering: if $i \in \operatorname{mor}_{\mathcal{B}}(p(x), y)$ and $i \circ pm = i \circ pn$, then $pm = pn$ and $m = n$. Every morphism h in $\operatorname{mor}_{\mathcal{B}}(p(x), y)$ is irreducible, so clearly we have a factorization $f \circ p(g) = h$ with f irreducible. Any two factorizations $f \circ p(g) = h$ and $f' \circ p(g') = h$ with f and f' irreducible are related by the isomorphism $k := g' \circ g$.

We fix an $x \in \mathcal{E}$ and let $H = \operatorname{aut}_{\mathcal{E}}(x)$, $G = \operatorname{aut}_{\mathcal{B}}(p(x))$. The covering p induces an inclusion of H into G. Consider the following diagram.



The left square commutes by Theorem 3.14. The second square commutes by Example 5.20. The top and bottom diagrams commute by definition of $\operatorname{rk}^{(2)}$. Beginning in the upper lefthand corner, we have $o(\mathcal{B}; \mathbb{C}) \in K_0(\mathbb{C}\mathcal{B})$. By Theorem 2.9, we have $p^*(o(\mathcal{B};\mathbb{C})) = o(\mathcal{E};\mathbb{C})$. Two applications of Theorem 5.22 combined with the commutativity of the diagrams leads us to $\chi^{(2)}(\mathcal{E}) = [G : H] \cdot \chi^{(2)}(\mathcal{B})$. An argument similar to the one in (5.28) shows that $[G : H]$ is equal to the number of sheets n. □

Example 5.29. Let $\mathcal{E} = \{0 \leftrightarrow 1\}$ and let $\mathcal{B}$ be the category with one object and one nontrivial arrow, which is its own inverse. By Example 5.12, the L^2-Euler characteristics are $\chi^{(2)}(\mathcal{E}) = 1$ and $\chi^{(2)}(\mathcal{B}) = 1/2$. The unique covering $\mathcal{E} \to \mathcal{B}$ is 2-sheeted and we have

$$\chi^{(2)}(\mathcal{E}) = 2\chi^{(2)}(\mathcal{B}).$$

Corollary 5.30. *Any n-sheeted covering functor between connected finite groupoids is equivalent to the inclusion of an index n subgroup into a finite group. More precisely, if $p\colon \mathcal{E} \to \mathcal{B}$ is an n-sheeted covering between connected finite groupoids and $e \in \mathcal{E}$, then the diagram*

$$\begin{array}{ccc} \widehat{\operatorname{aut}(e)} & \hookrightarrow & \mathcal{E} \\ {\scriptstyle p|_{\widehat{\operatorname{aut}(e)}}} \downarrow & & \downarrow {\scriptstyle p} \\ \widehat{\operatorname{aut}(p(e))} & \hookrightarrow & \mathcal{B} \end{array}$$

commutes, the horizontal functors are equivalences of categories, the left vertical functor is mono, and $[\operatorname{aut}(p(e)) : p(\operatorname{aut}(e))] = n$.

Remark 5.31. Examples of covering functors are obtained from coverings of topological spaces: a covering of topological spaces induces a covering functor between the associated fundamental groupoids.

We next turn to compatibility of $\chi^{(2)}$ with isofibrations.

Definition 5.32 (Isofibration)**.** A functor $p\colon \mathcal{E} \to \mathcal{B}$ is an *isofibration* if for every isomorphism in $\mathcal{B}$ of the form $g\colon b \cong p(e)$ there is an isomorphism f in $\mathcal{E}$ such that $p(f) = g$

We remark that if $\mathcal{E}$ and $\mathcal{B}$ are groupoids, then isofibrations and Grothendieck fibrations coincide (because isomorphisms in the domain category are always cartesian arrows).

Theorem 5.33. *Let $p\colon \mathcal{E} \to \mathcal{B}$ be an isofibration between connected finite groupoids. If $b \in \mathcal{B}$ and $p^{-1}(b)$ is connected, then*

$$\chi^{(2)}(\mathcal{E}) = \chi^{(2)}(p^{-1}(b_0)) \cdot \chi^{(2)}(\mathcal{B}). \tag{5.34}$$

Proof. As in the proof of Theorem 5.27, we reduce to the case where the base groupoid has only one object. If $b \in \mathcal{B}$ and $\mathcal{E}_b$ denotes the groupoid $p^{-1}(\widehat{\operatorname{aut}(b)})$, then the diagram

$$\begin{array}{ccc} \mathcal{E}_b & \hookrightarrow & \mathcal{E} \\ {\scriptstyle p|_{\mathcal{E}_b}}\downarrow & & \downarrow{\scriptstyle p} \\ \widehat{\operatorname{aut}(b)} & \hookrightarrow & \mathcal{B} \end{array}$$

commutes, the horizontal functors are equivalences of categories, and $\mathcal{E}_b$ is connected. The fiber groupoid $p|_{\mathcal{E}_b}^{-1}(b)$ is the same as the fiber groupoid $p^{-1}(b)$, so $p|_{\mathcal{E}_b}^{-1}(b)$ is also connected. Since $\chi^{(2)}(\mathcal{E}) = \chi^{(2)}(\mathcal{E}_b)$ and $\chi^{(2)}(\mathcal{B}) = \chi^{(2)}(\widehat{\operatorname{aut}(b)})$, we have (5.34) if $\chi^{(2)}(\mathcal{E}_b) = \chi^{(2)}(p|_{\mathcal{E}_b}^{-1}(b)) \cdot \chi^{(2)}(\widehat{\operatorname{aut}(b)})$. We have reduced to the case where the base groupoid has only one object.

Suppose now that $\mathcal{B}$ has only one object b, so that $\mathcal{B} = \widehat{\operatorname{aut}(b_0)}$. For $e \in p^{-1}(b)$, we write simply p_e for the group homomorphism $\operatorname{aut}(e) \to \operatorname{aut}(b)$. Then p_e is surjective. If g is an automorphism of b, there exists an $f \colon e' \to e$ with $p(f) = g$. The connectivity of the fiber $p^{-1}(b)$ then gives us an isomorphism $h \colon e \to e'$, and an automorphism $f \circ h$ of e such that $p_e(f \circ h) = g$.

Finally,

$$\chi^{(2)}(\mathcal{E}) = \frac{1}{|\operatorname{aut}(e)|} = \frac{1}{|\ker p_e| \cdot |\operatorname{aut}(b)|} = \chi^{(2)}(p^{-1}(b)) \cdot \chi^{(2)}(\mathcal{B}). \qquad \square$$

6. Möbius inversion

We extend the K-theoretic Möbius inversion of [15, Chapter 16] from finite to quasi-finite EI-categories and apply it to the finiteness obstruction and the Euler characteristic of a category. Throughout this section let Γ be an EI-category (see Definition 3.10). We have already introduced the splitting (S, E) of $K_0(R\Gamma)$ in Theorem 3.14. Provided that Γ is a quasi-finite EI-category, we obtain a second splitting (Res, I) in Theorem 6.16. The K-theoretic Möbius inversion (μ, ω) will compare these two splittings in Theorem 6.22. As a consequence, in Theorem 6.23 we obtain explicit formulas for the various Euler characteristics of finite EI-categories. Important special cases of our K-theoretic Möbius inversion include Philip Hall's Möbius inversion formula for finite posets and Leinster's Möbius inversion formula for finite skeletal categories with only trivial endomorphisms. See Examples 6.24 and 6.25.

After treating the second splitting (Res, I) and the K-theoretic Möbius inversion (μ, ω) in Subsections 6.1 and 6.2, we turn to the relationship between the K-theoretic Möbius inversion (μ, ω) and the L^2-rank in Subsection 6.3. There we construct a pair of homomorphisms $\overline{\mu}^{(2)} \colon U(\Gamma) \otimes_{\mathbb{Z}} \mathbb{Q} \rightleftarrows U(\Gamma) \otimes_{\mathbb{Z}} \mathbb{Q} \colon \overline{\omega}^{(2)}$ that are inverse to one another if Γ is a quasi-finite free EI-category, and commute appropriately with (μ, ω) and $\operatorname{rk}_{\Gamma}^{(2)}$ as in Theorem 6.34. All of these homomorphisms and splittings are illustrated for G-H-bisets (viewed as two-object EI-categories) in Subsection 6.4.

In general, the finiteness obstruction and Euler characteristics of Γ^{op} are different from those of Γ, as we see in Subsection 6.5 with a biset example. However, in the case of a finite EI-category Γ, the groups $K_0(\mathbb{Q}\Gamma)$ and $K_0(\mathbb{Q}\Gamma^{\mathrm{op}})$ are isomorphic, and we say more about the respective splittings in Subsection 6.6.

In Section 6 we also introduce the proper orbit category $\underline{\mathsf{Or}}(G)$, an important quasi-finite, free EI-category to which we shall return in Section 8.

6.1. **A second splitting.** Given an object x in a (small) category Γ, define the *restriction functor at x*

$$\operatorname{Res}_x\colon \mathsf{MOD}\text{-}R\Gamma \to \mathsf{MOD}\text{-}R[x] \tag{6.1}$$

by evaluating an $R\Gamma$-module N at the object x. This functor is exact but does not respect finitely generated projective in general. Given an EI-category Γ, the *inclusion functor at x*

$$I_x\colon \mathsf{MOD}\text{-}R[x] \to \mathsf{MOD}\text{-}R\Gamma \tag{6.2}$$

sends a right $R[x]$-module M to the $R\Gamma$-module given by

$$I_xM(y) := \begin{cases} M \otimes_{R[x]} R\operatorname{mor}(y,x) & \text{if } \overline{y} = \overline{x}; \\ 0 & \text{if } \overline{y} \neq \overline{x}. \end{cases}$$

Notice that we need the EI-condition to ensure that this definition makes sense. This functor is compatible with direct sums, but does not respect finitely generated projective in general.

Lemma 6.3. *Let Γ be an EI-category. Then we obtain for every $x \in \operatorname{ob}(\Gamma)$ adjoint pairs of functors $(E_x, \operatorname{Res}_x)$ and (S_x, I_x), where E_x, Res_x, S_x and I_x are the functors defined in* (3.4), (6.1), (3.3) *and* (6.2).

Proof. See [15, Lemma 9.31 on page 171]. □

The EI-property ensures that we obtain a well-defined partial ordering on $\operatorname{iso}(\Gamma)$ by

$$\overline{x} \leq \overline{y} \iff \operatorname{mor}(x,y) \neq \emptyset. \tag{6.4}$$

Definition 6.5 (Length of an element)**.** Given an element $x \in \operatorname{iso}(\Gamma)$, define its *length*

$$l(\overline{x}) \in \{0,1,2,\ldots\} \amalg \{\infty\}$$

to be the supremum over the natural numbers n, for which there exists elements $\overline{x_n}$, $\overline{x_{n-1}}$, …,$\overline{x_0}$ in $\operatorname{iso}(\Gamma)$ with $\overline{x_n} < \overline{x_{n-1}} < \ldots < \overline{x_0}$ and $\overline{x_0} = \overline{x}$.

The length of $\overline{x}$ is zero if and only if every morphism with x as target is an isomorphism.

Definition 6.6 (Finite, quasi-finite, and free categories)**.** Let Γ be a (small) category.

We call Γ *quasi-finite* if for every $\overline{x} \in \operatorname{iso}(\Gamma)$ the set $\{\overline{y} \in \operatorname{iso}(\Gamma) \mid \overline{y} \leq \overline{x}\}$ is finite, and for every two objects $x, y \in \operatorname{ob}(\Gamma)$ the right $\operatorname{aut}(x)$-set $\operatorname{mor}(x, y)$ is proper and cofinite, i.e., every isotropy group under the right $\operatorname{aut}(x)$-action is finite and the quotient $\operatorname{mor}(x, y)/\operatorname{aut}(x)$ is finite.

We call Γ *finite* if $\operatorname{iso}(\Gamma)$ is finite and $\operatorname{mor}(x, y)$ is finite for every two objects $x, y \in \operatorname{ob}(\Gamma)$. A small category is finite if and only if it is equivalent to a category with finitely many objects and finitely many morphisms.

We call Γ *free* if the left $\operatorname{aut}(y)$-action on $\operatorname{mor}(x, y)$ is free for every two objects $x, y \in \operatorname{ob}(\Gamma)$.

One of our main examples for Γ will be the orbit category.

Definition 6.7 (Orbit category)**.** Let G be a group. The *orbit category* $\mathsf{Or}(G)$ has as objects homogeneous spaces G/H and as morphisms G-equivariant maps. The *proper orbit category*

$$\underline{\mathsf{Or}}(G) = \mathsf{Or}_{\mathcal{FIN}}(G),$$

sometimes also called the orbit category associated to the family $\mathcal{FIN}$ of finite subgroups, is defined to be the full subcategory of $\mathsf{Or}(G)$ consisting of objects G/H with finite H.

Lemma 6.8. *Let H and K be subgroups of a group G. If $g \in G$ and $g^{-1}Hg \subseteq K$, then we get a well-defined G-equivariant map*

$$R_g \colon G/H \longrightarrow G/K$$
$$g'H \longmapsto g'gK\ .$$

Every G-equivariant map $G/H \to G/K$ is of the form R_g. We have $R_g = R_{g'}$ if and only if $g^{-1}g' \in K$ holds. In particular, we have a bijection

$$\operatorname{mor}(G/H, G/K) \longrightarrow \{gK \mid g^{-1}Hg \subseteq K\} \qquad (6.9)$$
$$f \longmapsto f(1H)\ .$$

We also have $R_{g_2} \circ R_{g_1} = R_{g_1 g_2}$.

Proof. See [29, I.1.14] and [15, Lemma 1.31 on page 22]. □

Lemma 6.10. *The orbit category $\mathsf{Or}(G)$ is a free EI-category.*

Proof. A direct consequence of Lemma 6.8 is that the monoid $\operatorname{map}(G/H, G/H)$ is isomorphic to the Weyl group $N_G H/H$, so every endomorphism of $\mathsf{Or}(G)$ is an automorphism.

If G/H and G/K are two objects in $\mathsf{Or}(G)$, and $f \colon G/H \to G/K$ and $a \colon G/K \to G/K$ are G-equivariant maps, then $a \circ f = f$ implies $a = \operatorname{id}$ since f is surjective. Hence $\mathsf{Or}(G)$ is free. □

Lemma 6.11. *The proper orbit category $\underline{\mathtt{Or}}(G)$ is a quasi-finite and free EI-category.*

Proof. The proper orbit category $\underline{\mathtt{Or}}(G)$ is a full subcategory of the orbit category $\mathtt{Or}(G)$, which is a free EI-category, so $\underline{\mathtt{Or}}(G)$ is also a free EI-category.

For the quasi-finiteness, we first observe from the bijection (6.9) that

$$\operatorname{mor}(G/H, G/K) \neq \emptyset$$

if and only if H is G-conjugate to a subgroup of K. If H and H' are G-conjugate, then G/H and G/H' are isomorphic objects of $\mathtt{Or}(G)$. Thus for a fixed $\overline{G/K}$, the number of isomorphism classes $\overline{G/H}$ with $\operatorname{mor}(G/H, G/K) \neq \emptyset$ is at most the number of G-conjugacy classes of subgroups of K. Whenever K is a finite group, this number is finite. Thus, $\{\overline{G/H} \in \operatorname{iso}(\underline{\mathtt{Or}}(G)) \mid \overline{G/H} \leq \overline{G/K}\}$ is finite.

Continuing the notation of Lemma 6.8, consider a morphism $R_{g_2}\colon G/H \to G/K$ in $\underline{\mathtt{Or}}(G)$. Suppose $R_{g_1} \in \operatorname{aut}(G/H)$ fixes R_{g_2}. Then $R_{g_1g_2} = R_{g_2}$ and $g_1g_2K = g_2K$, so that $g_1 \in g_2Kg_2^{-1}$. But $g_2Kg_2^{-1}$ is finite, so there are only finitely many possibilities for g_1. Thus every isotropy group for the right $\operatorname{aut}(G/H)$-action on $\operatorname{mor}(G/H, G/K)$ is finite.

For objects G/H and G/K in $\underline{\mathtt{Or}}(G)$, the quotient $\operatorname{mor}(G/H, G/K)/\operatorname{aut}(G/H)$ is in bijective correspondence with

$$\{g_2K \mid g_2^{-1}Hg_2 \subseteq K\}/\sim \tag{6.12}$$

by Lemma 6.8, where $g_2K \sim g_1g_2K$ if $g_1 \in G$ and $g_1^{-1}Hg_1 \subseteq H$. Since H is finite, $g_1^{-1}Hg_1 \subseteq H$ implies $g_1^{-1}Hg_1 = H$. But (6.12) is in bijective correspondence with G-conjugates of H contained in K, of which there are only finitely many because K is finite. Thus the quotient $\operatorname{mor}(G/H, G/K)/\operatorname{aut}(G/H)$ is finite. □

Lemma 6.13.

(i) *Suppose for the EI-category Γ that for every $\overline{x} \in \operatorname{iso}(\Gamma)$ the set $\{\overline{y} \in \operatorname{iso}(\Gamma) \mid \overline{y} \leq \overline{x}\}$ is finite. Let M be a finitely generated $R\Gamma$-module M. Then*

$$\big\{\overline{x} \in \operatorname{iso}(\Gamma) \mid M(x) \neq 0\big\}$$

is finite;

(ii) *If Γ is a quasi-finite EI-category and of type (FP), then* $\operatorname{iso}(\Gamma)$ *is finite.*

Proof. (i) Choose a finite subset $I \subseteq \operatorname{iso}(\Gamma)$ and natural numbers $n_i \geq 1$ for each $i \in I$ such that there exists an epimorphism of $R\Gamma$-modules

$$\bigoplus_{i \in I} R\operatorname{mor}(?, x_i)^{n_i} \to M.$$

Then for every $\overline{y} \in \operatorname{iso}(\Gamma)$ with $M(y) \neq 0$ there is $i \in I$ with $\overline{y} \leq \overline{x_i}$. Since I is finite, $\big\{\overline{x} \in \operatorname{iso}(\Gamma) \mid M(x) \neq 0\big\}$ is finite.

(ii) This follows from assertion (i) applied to the constant module $\underline{R}$. □

Definition 6.14 (Length of a module)**.** The *length* $l(M) \in \{-1, 0, 1, 2 \ldots\} \amalg \{\infty\}$ of an $R\Gamma$-module M is defined to be -1 if M is zero and otherwise to be the supremum of the length of elements $\overline{x} \in \operatorname{iso}(\Gamma)$ with $M(x) \neq 0$.

If Γ is quasi-finite and hence $\{\overline{y} \in \operatorname{iso}(\Gamma) \mid \overline{y} \leq \overline{x}\}$ is finite for every $\overline{x} \in \operatorname{iso}(\Gamma)$, the length of $R\operatorname{mor}(?, x)$ is finite for every object $x \in \operatorname{ob}(\Gamma)$ and hence every finitely generated $R\Gamma$-module has finite length.

Lemma 6.15. *Suppose that Γ is a quasi-finite EI-category. Suppose for any morphism $f \colon x \to y$ in Γ that the order of the finite group $\{g \in \operatorname{aut}(x) \mid f \circ g = f\}$ is invertible in R.*

(i) *Consider $x \in \operatorname{ob}(\Gamma)$. Let M be an $R\Gamma$-module which is finitely generated projective or which possesses a finite projective $R\Gamma$-resolution respectively. Then the $R\operatorname{aut}(x)$-module $\operatorname{Res}_x M = M(x)$ is finitely generated projective or has a finite projective $R[x]$-resolution respectively;*

(ii) *Let M be an $R\Gamma$-module such that the set*
$$\big\{\overline{x} \in \operatorname{iso}(\Gamma) \mid M(x) \neq 0\big\}$$
is finite. Then M possesses a finite projective $R\Gamma$-resolution, if M has finite length and $\operatorname{Res}_x M$ possesses a finite projective $R[x]$-module for all $x \in \operatorname{ob}(\Gamma)$;

(iii) *Let $x \in \operatorname{ob}(\Gamma)$ and let N be an $R[x]$-module which possesses a finite projective $R[x]$-resolution. Then the $R\Gamma$-module $I_x N$ defined in* (6.2) *possesses a finite projective $R\Gamma$-resolution;*

(iv) *Γ is of type (FP) if and only if $\operatorname{iso}(\Gamma)$ is finite and for every object $x \in \operatorname{ob}(\Gamma)$ the trivial $R[x]$-module R is of type (FP) respectively;*

(v) *Let Γ be a finite EI-category. Assume that for every object x the order of the finite group $\operatorname{aut}(x)$ is invertible in R. Then an $R\Gamma$-module M possesses a finite projective resolution if for every object x the R-module $M(x)$ possesses a finite projective R-resolution. In particular Γ is of type (FP).*

Proof. (i) Since Res_x is exact, it suffices to show that $\operatorname{Res}_x R\operatorname{mor}(?, y) = R\operatorname{mor}(x, y)$ is a finitely generated projective $R[x]$-module for every $y \in \operatorname{ob}(\Gamma)$. This follows from the assumptions that the right $\operatorname{aut}(x)$-set $\operatorname{mor}(x, y)$ is a finite union of homogeneous $\operatorname{aut}(x)$-spaces of the form $H \backslash \operatorname{aut}(x)$ for finite $H \subseteq \operatorname{aut}(x)$ such that $|H| \cdot 1_R$ is a unit in R.

(ii) We do induction over the length of the $R\Gamma$-module M. The induction beginning $l = -1$ is trivial, the induction step from $l - 1$ to $l \geq 0$ done as follows.

If $0 \to M_1 \to M_1 \to M_3 \to 0$ is an exact sequence of $R\Gamma$-modules such that two of the $R\Gamma$-modules M_1, M_2 and M_3 possess finite projective $R\Gamma$-resolutions, then all three possess finite projective $R\Gamma$-resolutions (see [15, Lemma 11.6 on page 216]). Thus, using the Filtration Theorem (see [15, Theorem 16.8 on page 326]) and the induction hypothesis, it suffices to show for any object x of length l and any $R[x]$-module N which admits a finite projective $R[x]$-resolution that I_xN has a finite projective $R\Gamma$-resolution. Since I_x is exact, it is enough to consider the case $N = R[x]$. Consider the epimorphism $f\colon R\operatorname{mor}(?,x) \to I_x(R[x])$ sending id_x to $1_{R[x]} \otimes \mathrm{id}_x \in R[x] \otimes_{R[x]} R\operatorname{mor}(x,x) = I_x(R[x])$. Its kernel $\ker(f)$ is an $R\Gamma$-module of length $\le l-1$ and satisfies $\operatorname{Res}_y(\ker(f)) = R\operatorname{mor}(y,x) = \operatorname{Res}_y R\operatorname{mor}(?,x)$ for $\overline{y} < \overline{x}$ and $\operatorname{Res}_y(\ker(f)) = 0$ otherwise. Assertion (i) implies that $\operatorname{Res}_y(\ker(f))$ possesses a finite projective $R[y]$-resolution for all objects $y \in \operatorname{ob}(\Gamma)$. Hence $\ker(f)$ possesses a finite projective $R\Gamma$-resolution by induction hypothesis. This implies that $I_xR[x]$ possesses a finite projective $R\Gamma$-resolution. This finishes the proof of the induction step.

(iii) This follows directly from assertion (ii).

(iv) This follows directly from Lemma 6.13 (ii) and assertions (i) and (ii).

(v) Since $|\operatorname{aut}(x)|$ is invertible in R and finite, an $R[x]$-module possesses a finite projective $R[x]$-resolution if and only if it possesses a finite projective R-resolution. Now apply assertion (ii). ☐

Our main example for R will of course be $\mathbb{Q}$.

Theorem 6.16 (A second splitting of $K_0(R\Gamma)$)**.** *Suppose that Γ is a quasi-finite EI-category. Suppose for any morphism $f\colon x \to y$ in Γ that the order of the finite group $\{g \in \operatorname{aut}(x) \mid f \circ g = f\}$ is invertible in R.*

Then we obtain isomorphisms Res *and I which are inverse to one another.*

$$\begin{aligned} \operatorname{Res}\colon K_0(R\Gamma) &\to \operatorname{Split} K_0(R\Gamma), \qquad [P] \mapsto \{[\operatorname{Res}_x P] \mid \overline{x} \in \operatorname{iso}(\Gamma)\} \\ I\colon \operatorname{Split} K_0(R\Gamma) &\to K_0(R\Gamma), \qquad \{[Q_x] \mid \overline{x} \in \operatorname{iso}(\Gamma)\} \mapsto \sum_{\overline{x} \in \operatorname{iso}(\Gamma)} [I_xQ_x] \end{aligned}$$

Proof. Consider a finitely generated projective $R\Gamma$-module P. Then for any object $x \in \operatorname{ob}(\Gamma)$ the $R[x]$-module $\operatorname{Res}_x P$ possesses a finite projective $R[x]$-resolution (see Lemma 6.15 (i)) and hence defines an element in $K_0(R[x])$, namely its finiteness obstruction in the sense of Definition 2.1. Since Γ is by assumption quasi-finite and hence $\{\overline{y} \in \operatorname{iso}(\Gamma) \mid \overline{y} \le \overline{x}\}$ is finite for every object $x \in \operatorname{ob}(\Gamma)$, there are only finitely many elements $\overline{x} \in \operatorname{iso}(\Gamma)$ with $\operatorname{Res}_x P \ne 0$ by Lemma 6.13 (i). Hence we obtain a well-defined element

$$\operatorname{Res}([P]) := \{[\operatorname{Res}_x P] \mid \overline{x} \in \operatorname{iso}(\Gamma)\} \in \bigoplus_{\overline{x} \in \operatorname{iso}(\Gamma)} K_0(R[x]) = \operatorname{Split} K_0(R\Gamma).$$

Thus we obtain a homomorphism

$$\operatorname{Res}\colon K_0(R\Gamma) \to \operatorname{Split} K_0(R\Gamma).$$

Define

$$I\colon \operatorname{Split} K_0(R\Gamma) \to K_0(R\Gamma)$$

analogously using Lemma 6.15 (iii).

One obtains $\operatorname{Res} \circ I = \operatorname{id}$ from the fact that the functor $\operatorname{Res}_y \circ I_x \colon \mathsf{MOD}\text{-}R[x] \to \mathsf{MOD}\text{-}R[y]$ is naturally isomorphic to the identity functor if $x = y$ and is trivial if $\overline{x} \neq \overline{y}$. It remains to show that I is surjective. This is done by induction over the length, which is finite by Lemma 6.13 (i), of a finitely generated projective $R\Gamma$-module representing a class in $K_0(R\Gamma)$ using Lemma 6.15 and the Filtration Theorem (see [15, Theorem 16.8 on page 326]). □

6.2. **The K-theoretic Möbius inversion.**

Convention 6.17. Suppose for the remainder of this subsection that Γ is a quasi-finite EI-category and that for every morphism $f \colon x \to y$ in Γ the order of the finite group $\{g \in \operatorname{aut}(x) \mid f \circ g = f\}$ is invertible in R.

We obtain a well-defined homomorphism

$$\omega_{x,y} \colon K_0(R[x]) \to K_0(R[y]), \quad [P] \mapsto [P \otimes_{R[x]} R\operatorname{mor}(y,x)]$$

since the right $R[y]$-module $R\operatorname{mor}(y,x) = \operatorname{Res}_y R\operatorname{mor}(?,x)$ is finitely generated projective by Lemma 6.15 (i). Define

$$\omega \colon \operatorname{Split} K_0(R\Gamma) \to \operatorname{Split} K_0(R\Gamma) \tag{6.18}$$

by the matrix of homomorphisms

$$(\omega_{x,y})_{\overline{x},\overline{y} \in \operatorname{iso}(\Gamma)} \colon \bigoplus_{\overline{x} \in \operatorname{iso}(\Gamma)} K_0(R[x]) \to \bigoplus_{\overline{y} \in \operatorname{iso}(\Gamma)} K_0(R[y]).$$

This definition makes sense since for given $\overline{x} \in \operatorname{iso}(\Gamma)$ there are only finitely many $\overline{y} \in \operatorname{iso}(\Gamma)$ with $\omega_{x,y} \neq 0$.

Example 6.19. If $R = \mathbb{Q}$ and Γ is a finite skeletal category with trivial automorphism groups, then $K_0(\mathbb{Q}[x]) = \mathbb{Z}$ and $\omega_{x,y} = |\operatorname{mor}_\Gamma(y,x)|$ for all $x, y \in \operatorname{ob}(\Gamma)$. In this case of R and Γ, the matrix for ω is the transpose of the zeta function considered by Leinster in Section 1 of [13]. See also Example 6.25.

Definition 6.20 (l-chain in $\operatorname{iso}(\Gamma)$)**.** Let Γ be an EI-category. Given a natural number $l \geq 1$, an *l-chain in* $\operatorname{iso}(\Gamma)$ is a sequence $c = \overline{x_0} < \overline{x_1} < \cdots < \overline{x_l}$. Denote by $\operatorname{ch}_l(\Gamma)$ the set of l-chains in Γ.

Given two objects x and y, let $\operatorname{ch}_l(y,x)$ be the set of l-chains $c = \overline{x_0} < \overline{x_1} < \cdots < \overline{x_l}$ with $\overline{x_0} = \overline{y}$ and $\overline{x_l} = \overline{x}$. Define for an l-chain $c = \overline{x_0} < \overline{x_1} < \cdots < \overline{x_l}$ in $\operatorname{ch}_l(\overline{y},\overline{x})$ the $\operatorname{aut}(x)$-$\operatorname{aut}(y)$-biset

$$S(c) = \operatorname{mor}(x_{l-1},x) \times_{\operatorname{aut}(x_{l-1})} \operatorname{mor}(x_{l-2},x_{l-1}) \times_{\operatorname{aut}(x_{l-2})} \cdots \times_{\operatorname{aut}(x_1)} \operatorname{mor}(y,x_1)$$

for some choice of representatives $x_i \in \overline{x_i}$ for $0 < i < l-1$. (If $l = 1$ then $S(c)$ is to be understood as the $\operatorname{aut}(x)$-$\operatorname{aut}(y)$-biset $\operatorname{mor}(y,x)$.)

Define $\operatorname{ch}_0(\Gamma)$ to be $\operatorname{iso}(\Gamma)$. Define $\operatorname{ch}_0(y,x)$ to be empty if $\overline{x} \neq \overline{y}$ and to be $\overline{y}$ if $\overline{x} = \overline{y}$. If $\overline{x} = \overline{y}$, put $S(c) = \operatorname{mor}(y,x)$ for $c \in \operatorname{ch}_0(y,x)$.

Notice that the $\operatorname{aut}(x)$-$\operatorname{aut}(y)$-biset $S(c)$ is unique up to isomorphism of $\operatorname{aut}(x)$-$\operatorname{aut}(y)$-bisets. Since Γ is quasi-finite and hence for every two objects $x, y \in \operatorname{ob}(\Gamma)$ the right $\operatorname{aut}(y)$-set $\operatorname{mor}(y, x)$ is proper and cofinite, each set $S(c)$ is a proper cofinite right $\operatorname{aut}(y)$-set, and the $R[y]$-module $RS(c)$ is finitely generated projective. Hence we obtain a well-defined homomorphism for $c \in \operatorname{ch}_l(y, x)$

$$\mu_{x,y}(c)\colon K_0(R[x]) \to K_n(R[y]), \quad [P] \mapsto [P \otimes_{R[x]} RS(c)].$$

Define a homomorphism

$$\mu\colon \operatorname{Split} K_0(R\Gamma) \to \operatorname{Split} K_0(R\Gamma) \tag{6.21}$$

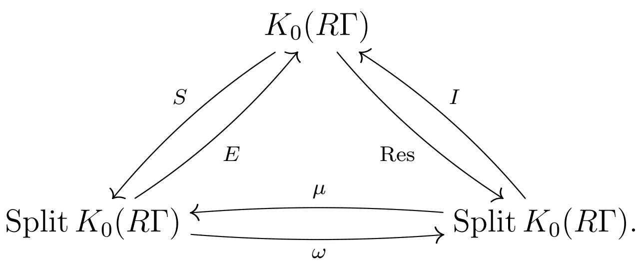
by the matrix of homomorphisms

$$\left(\sum_{l \geq 0} (-1)^l \cdot \sum_{c \in \operatorname{ch}_l(y,x)} \mu_{x,y}(c)\right)_{\overline{x}, \overline{y} \in \operatorname{iso}(\Gamma)} : \bigoplus_{\overline{x} \in \operatorname{iso}(\Gamma)} K_0(R[x]) \to \bigoplus_{\overline{y} \in \operatorname{iso}(\Gamma)} K_0(R[y]).$$

This definition makes sense since for given $\overline{x} \in \operatorname{iso}(\Gamma)$ there are only finitely many $\overline{y} \in \operatorname{iso}(\Gamma)$ with $\mu_{x,y} \neq 0$.

Theorem 6.22 (Two splittings and the K-theoretic Möbius inversion)**.** *Suppose that Γ is a quasi-finite EI-category. Suppose for any morphism $f\colon x \to y$ in Γ that the order of the finite group $\{g \in \operatorname{aut}(x) \mid f \circ g = f\}$ is invertible in R.*

(i) *Then we obtain pairs of inverse isomorphisms (S, E) (see Theorem 3.14), (Res, I) (see Theorem 6.16) and (ω, μ) (see (6.18) and (6.21)). They are compatible with one another in the sense that the following diagram commutes*



(ii) *Suppose that Γ is of type (FP), or, equivalently, that $\operatorname{iso}(\Gamma)$ is finite and for each object $x \in \operatorname{ob}(\Gamma)$ the trivial $R[x]$-module R possesses a finite projective $R[x]$-resolution. Let $\eta \in \operatorname{Split} K_0(R\Gamma)$ be the element whose component at $\overline{x} \in \operatorname{iso}(\Gamma)$ is given by the class $[R] \in K_0(R[x])$ of the trivial $R[x]$-module R. That is, the component of η at each $\overline{x}$ is the finiteness obstruction $o(\widehat{\operatorname{aut}(x)}; R) \in K_0(R\operatorname{aut}(x))$. Then*

$$S\left(o(\Gamma; R)\right) = \mu(\eta).$$

Proof. (i) We have already shown in Theorem 3.14 that S and E are inverse to one another and in Theorem 6.16 that Res and I are inverse to one another. Obviously $\omega = \operatorname{Res} \circ E$. Hence it remains to show that $\mu \circ \omega = \operatorname{id}$. This follows analogously to the argument at the end of the proof of [15, Theorem 16.27 on page 330].

(ii) This follows from assertion (i) and Lemma 6.15 (i) and(iv). Namely, $\mathrm{Res}_x[R] = [R]$, so $\mathrm{Res}[R] = \eta$, and $S\left(o(\Gamma; R)\right) = \mu \,\mathrm{Res}\left(o(\Gamma; R)\right) = \mu \,\mathrm{Res}[R] = \mu(\eta)$. □

We can now apply Möbius inversion to calculate the finiteness obstruction and Euler characteristics of finite EI-categories in terms of chains.

Theorem 6.23 (The finiteness obstruction and Euler characteristics of finite EI-categories)**.** *Suppose that* Γ *is a finite EI-category. Suppose that for every object* $x \in \mathrm{ob}(\Gamma)$ *the order of its automorphism group* $|\mathrm{aut}(x)|$ *is invertible in* R*. Then* Γ *is of type (FP) and we have:*

(i) *The image of the finiteness obstruction* $o(\Gamma; R)$ *under the isomorphism*

$$S\colon K_0(R\Gamma) \xrightarrow{\cong} \bigoplus_{\overline{y} \in \mathrm{iso}(\Gamma)} K_0(R[y])$$

has as component for $\overline{y} \in \mathrm{iso}(\Gamma)$ *the element in* $K_0(R[y])$ *given by*

$$\sum_{l \geq 0} (-1)^l \cdot \sum_{\overline{x} \in \mathrm{iso}(\Gamma)} \sum_{c \in \mathrm{ch}_l(y,x)} \bigl[R(\mathrm{aut}(x)\backslash S(c))\bigr],$$

where $\mathrm{aut}(x)\backslash S(c)$ *is the finite right* $\mathrm{aut}(y)$*-set obtained from the* $\mathrm{aut}(x)$*-*$\mathrm{aut}(y)$*-biset* $S(c)$ *(see Definition 6.20) by dividing out the left* $\mathrm{aut}(x)$*-action and* $R(\mathrm{aut}(x)\backslash S(c))$ *is the associated right* $R[y]$*-module.*

(ii) *The functorial Euler characteristic* $\chi_f(\Gamma; R) \in U(\Gamma)$ *has at* $\overline{y}$ *the value*

$$\sum_{l \geq 0} (-1)^l \cdot \sum_{\overline{x} \in \mathrm{iso}(\Gamma)} \sum_{c \in \mathrm{ch}_l(y,x)} \bigl|\mathrm{aut}(x)\backslash S(c)/\,\mathrm{aut}(y)\bigr|,$$

where $\bigl|\mathrm{aut}(x)\backslash S(c)/\,\mathrm{aut}(y)\bigr|$ *is the order of the set obtained from* $S(c)$ *by dividing out the* $\mathrm{aut}(x)$*-action and the* $\mathrm{aut}(y)$*-action;*

(iii) *The Euler characteristic* $\chi(\Gamma, R)$ *is given by the integer*

$$\sum_{l \geq 0} (-1)^l \cdot \sum_{\overline{x}, \overline{y} \in \mathrm{iso}(\Gamma)} \sum_{c \in \mathrm{ch}_l(y,x)} \bigl|\mathrm{aut}(x)\backslash S(c)/\,\mathrm{aut}(y)\bigr|;$$

(iv) *The functorial* L^2*-Euler characteristic* $\chi_f^{(2)}(\Gamma) \in U^{(1)}(\Gamma)$ *has at* $\overline{y}$ *the value*

$$\sum_{l \geq 0} (-1)^l \cdot \sum_{\overline{x} \in \mathrm{iso}(\Gamma)} \sum_{c \in \mathrm{ch}_l(y,x)} \dim_{\mathcal{N}(y)}\bigl(\mathbb{C}(\mathrm{aut}(x)\backslash S(c)) \otimes_{\mathbb{C}[y]} \mathcal{N}(y)\bigr);$$

where $\dim_{\mathcal{N}(y)}\bigl(\mathbb{C}(\mathrm{aut}(x)\backslash S(c)) \otimes_{\mathbb{C}[y]} \mathcal{N}(y)\bigr)$ *is* $\sum_{i \in I, |L_i| < \infty} 1/|L_i|$ *if the cofinite right* $\mathrm{aut}(y)$*-set* $\mathrm{aut}(x)\backslash S(c)$ *is the disjoint union of homogeneous* $\mathrm{aut}(y)$*-spaces* $\coprod_{i \in I} L_i \backslash \mathrm{aut}(y)$*;*

(v) *The L^2-Euler characteristic $\chi^{(2)}(\Gamma)$ is given by*

$$\sum_{l\geq 0}(-1)^l\cdot \sum_{\overline{x},\overline{y}\in \mathrm{iso}(\Gamma)}\ \sum_{c\in \mathrm{ch}_l(y,x)} \dim_{\mathcal{N}(y)}\bigl(\mathbb{C}(\mathrm{aut}(x)\backslash S(c))\otimes_{\mathbb{C}[y]}\mathcal{N}(y)\bigr).$$

Proof. The category Γ is of type (FP) by Lemma 6.15 (v).

(i) This follows from Theorem 6.22 (ii) since the the $R[y]$-modules $R\otimes_{R\,\mathrm{aut}(x)}RS(c)$ and $R(\mathrm{aut}(x)\backslash S(c))$ are isomorphic.

(ii) and (iii) follow now from Corollary 4.19 and assertion (i).

(iv) and (v) follow from Theorem 5.22, Example 5.4 (ii) and assertion (i). ☐

Example 6.24 (Möbius inversion for a finite partially ordered set)**.** Let $(I,\leq)$ be a partially ordered set. It defines an EI-category $\Gamma(I)$ whose set of objects is I and for which $\mathrm{mor}(x,y)$ consists of precisely one element if $x\leq y$ and is empty otherwise.

Suppose that I is finite. Take $R=\mathbb{Q}$. Then

$$\mathrm{Split}\,K_0(\mathbb{Q}\Gamma(I))=\mathbb{Z}I=\bigoplus_I\mathbb{Z}$$

and the homomorphism ω is given by the matrix $A=\bigl(a_{i,j}\bigr)_{i,j\in I}$ with $a_{i,j}=1$ if $j\leq i$ and $w_{i,j}=0$ otherwise. Let $B=\bigl(b_{i,j}\bigr)_{i,j\in I}$ be the matrix given by

$$b_{i,j}=\sum_{l\geq 0}(-1)^l\cdot|\,\mathrm{ch}_l(j,i)|,$$

where $|\,\mathrm{ch}_0(j,i)|$ is 0 if $j\neq i$ and 1 otherwise, and for $l\geq 1$, $\mathrm{ch}_l(j,i)$ is the set of chains $j=k_0<k_1<\ldots<k_{l-1}<k_l=i$. Then we conclude from Theorem 6.22 that the matrices A and B are inverse to one another. This is the classical Möbius inversion in combinatorics (see for instance [1, IV.2]).

We get from Theorem 6.23 (iii) and (v)

$$\chi(\Gamma;\mathbb{Q})=\chi^{(2)}(\Gamma)=\sum_{i,j\in I}b_{i,j}.$$

Example 6.25 (Möbius inversion for a finite skeletal category with trivial endomorphisms)**.** Generalizing Example 6.24, let Γ be a finite skeletal category in which every endomorphism is an identity, and take $R=\mathbb{Q}$. Then

$$\mathrm{Split}\,K_0(\mathbb{Q}\Gamma)=\mathbb{Z}\,\mathrm{ob}(\Gamma)=\bigoplus_{\mathrm{ob}(\Gamma)}\mathbb{Z}$$

and the homomorphism ω is given by the matrix $A=\bigl(a_{x,y}\bigr)_{x,y\in\mathrm{ob}(\Gamma)}$ with $a_{x,y}=|\,\mathrm{mor}(y,x)|$.

The (bi)set $S(c)$ in Definition 6.20 is simply the set of non-degenerate paths $x_0\to x_1\to\cdots\to x_l$, and $\mu_{x,y}(c)=|S(c)|$. Let $B=\bigl(b_{x,y}\bigr)_{x,y\in\mathrm{ob}(\Gamma)}$ be the matrix given by

$$b_{x,y} = \sum_{l \geq 0} (-1)^l \cdot \sum_{c \in \mathrm{ch}_l(y,x)} |S(c)| = \sum_{l \geq 0} (-1)^l \cdot |\{\text{non-degenerate } l\text{-paths from } y \text{ to } x\}|.$$

Then we conclude from Theorem 6.22 that the matrices A and B are inverse to one another. That is to say, in the terminology of [13], the category Γ has Möbius inversion given by B. Thus Corollary 1.5 in [13] is a special case of the K-theoretic Möbius inversion of Theorem 6.22 (i). See also Example 6.33, which illustrates rational Möbius inversion for a finite, skeletal, free EI-category. See also the related proof of Lemma 7.3, which shows that the L^2-Euler characteristic coincides with Leinster's Euler characteristic in the case of a finite, skeletal, free EI-category.

6.3. **The K-theoretic Möbius inversion and the L^2-rank.** In this subsection we investigate when the homomorphisms ω and μ factorize over the homomorphism given by the L^2-rank.

Condition 6.26 (Condition (I))**.** A group G *satisfies condition (I)* if the map induced by the various inclusions of finite subgroups

$$\bigoplus_{H \subseteq G, |H| < \infty} K_0(\mathbb{Q}H) \otimes_{\mathbb{Z}} \mathbb{Q} \to K_0(\mathbb{Q}G) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is surjective. A category Γ satisfies condition (I) if for every object its automorphism group satisfies condition (I).

Obviously any finite group and any finite category satisfy condition (I).

Remark 6.27 (Condition (I) and the Farrell-Jones Conjecture)**.** Let $\mathcal{FJ}(\mathbb{Q})$ be the class of groups for which the K-theoretic Farrell-Jones Conjecture with coefficients in $\mathbb{Q}$ holds. By [5, Theorem 0.5] every group in $\mathcal{FJ}(\mathbb{Q})$ satisfies condition (I). This class $\mathcal{FJ}(\mathbb{Q})$ is analyzed for instance in [3], [4], and [5]. It contains for instance subgroups of finite products of hyperbolic groups or CAT(0)-groups, directed colimits of hyperbolic groups or CAT(0)-groups, and all elementary amenable groups. For a survey article on the Farrell-Jones Conjecture we refer for instance to [23].

Lemma 6.28. *Let G and H be groups. Suppose that H satisfies condition (I) defined in* (6.26)*. Let S be an H-G-biset which is cofinite proper as a right G-set and free as a left H-set.*

(i) *The image of*

$$\mathrm{rk}_H^{(2)} \colon K_0(\mathbb{Q}H) \to \mathbb{R}, \quad [P] \mapsto \dim_{\mathcal{N}(H)}\big(P \otimes_{\mathbb{Q}H} \mathcal{N}(H)\big)$$

lies in $\mathbb{Q}$;

(ii) *The following diagram commutes*

$$\begin{array}{ccc} K_0(\mathbb{Q}H) & \xrightarrow{\omega_S} & K_0(\mathbb{Q}G) \\ {\scriptstyle \mathrm{rk}_H^{(2)}}\downarrow & & \downarrow{\scriptstyle \mathrm{rk}_G^{(2)}} \\ \mathbb{R} & \xrightarrow[\overline{\omega}_S]{} & \mathbb{R} \end{array}$$

where ω_S sends $[P]$ to $[P \otimes_{\mathbb{Q}H} \mathbb{Q}S]$, and $\overline{\omega}_S$ is multiplication with the rational number $\dim_{\mathcal{N}(G)}\bigl(\mathbb{Q}S \otimes_{\mathbb{Q}G} \mathcal{N}(G)\bigr)$.

Proof. (i) Because H satisfies condition (I), this follows from Lemma 5.3 (ii) and Example 5.4 (i) .

(ii) For a finite group H every element in $K_0(\mathbb{Q}H) \otimes_{\mathbb{Z}} \mathbb{Q}$ can be written as a $\mathbb{Q}$-linear combination of elements of the form $\bigl[\mathbb{Q}[K\backslash H]\bigr]$ (see [25, Theorem 30 in Chapter 13 on page 103]). Since H satisfies condition (I), we can find for every element $\eta \in K_0(\mathbb{Q}H)$ a natural number $k \geq 1$, finitely many finite subgroups K_1, K_2, ..., K_r of H, and integers n_1, n_2, ..., n_r such that we get in $K_0(\mathbb{Q}H)$

$$k \cdot \eta = \sum_{i=1}^{r} n_i \cdot \bigl[\mathbb{Q}[K_i\backslash H]\bigr].$$

Hence it suffices to show for any finite subgroup $K \subseteq H$

$$\begin{aligned} &\dim_{\mathcal{N}(G)}\bigl(\mathbb{Q}[K\backslash H] \otimes_{\mathbb{Q}H} \mathbb{Q}S \otimes_{\mathbb{Q}G} \mathcal{N}(G)\bigr) \\ &\qquad = \dim_{\mathcal{N}(H)}\bigl(\mathbb{Q}[K\backslash H] \otimes_{\mathbb{Q}H} \mathcal{N}(H)\bigr) \cdot \dim_{\mathcal{N}(G)}\bigl(\mathbb{Q}S \otimes_{\mathbb{Q}G} \mathcal{N}(G)\bigr) \end{aligned}$$

We get from Example 5.4 (ii)

$$\begin{aligned} \dim_{\mathcal{N}(H)}\bigl(\mathbb{Q}[K\backslash H] \otimes_{\mathbb{Q}H} \mathcal{N}(H)\bigr) &= \frac{1}{|K|}; \\ \dim_{\mathcal{N}(G)}\bigl(\mathbb{Q}[K\backslash H] \otimes_{\mathbb{Q}H} \mathbb{Q}S \otimes_{\mathbb{Q}G} \mathcal{N}(G)\bigr) &= \dim_{\mathcal{N}(G)}\bigl(\mathbb{Q}[K\backslash S] \otimes_{\mathbb{Q}G} \mathcal{N}(G)\bigr). \end{aligned}$$

Hence it suffices to show for a K-G-biset T which is proper and cofinite as a G-set and free as a left K-set

$$|K| \cdot \dim_{\mathcal{N}(G)}\bigl(\mathbb{Q}[K\backslash T] \otimes_{\mathbb{Q}G} \mathcal{N}(G)\bigr) = \dim_{\mathcal{N}(G)}\bigl(\mathbb{Q}T \otimes_{\mathbb{Q}G} \mathcal{N}(G)\bigr).$$

We can interpret the K-G-biset T as a right $(K \times G)$-set by putting $t \cdot (k, g) = k^{-1}tg$ for $k \in K$, $g \in G$ and $t \in T$, and vice versa. Since K is finite, T is free as left K-set, and T is cofinite and proper as a right G-set, the $(K \times G)$-set T is a finite union of homogeneous spaces of the form $L\backslash(K \times G)$, where L is a finite subgroup of $K \times G$ with $K \times \{1\} \cap L = \{1\}$. Hence we can assume without loss of generality that T is of the form $L\backslash(K \times G)$ for finite $L \subseteq K \times G$ with $K \times \{1\} \cap L = \{1\}$

The projection $\operatorname{pr}\colon K \times G \to G$ induces a bijection $L \xrightarrow{\cong} \operatorname{pr}(L)$. Since the G-sets $K\backslash\big(L\backslash(K \times G)\big)$ and $\operatorname{pr}(L)\backslash G$ are G-isomorphic, we conclude from Example 5.4 (ii)

$$|K| \cdot \dim_{\mathcal{N}(G)}\big(\mathbb{Q}\big[K\backslash\big(L\backslash(H \times G)\big)\big] \otimes_{\mathbb{Q}G} \mathcal{N}(G)\big) \quad = \quad \frac{|K|}{|L|}.$$

We conclude from Lemma 5.3 and Example 5.4 (ii)

$$\begin{aligned}
&\dim_{\mathcal{N}(G)}\big(\mathbb{Q}\big[L\backslash(K \times G)\big] \otimes_{\mathbb{Q}G} \mathcal{N}(G)\big)\\
&\quad = \dim_{\mathcal{N}(G)}\big(\mathbb{Q}\big[L\backslash(K \times G)\big] \otimes_{\mathbb{Q}[K\times G]} \mathbb{Q}[K \times G] \otimes_{\mathbb{Q}G} \mathcal{N}(G)\big)\\
&\quad = \dim_{\mathcal{N}(G)}\big(\mathbb{Q}\big[L\backslash(K \times G)\big] \otimes_{\mathbb{Q}[K\times G]} \mathcal{N}(K \times G)\big)\\
&\quad = |K| \cdot \dim_{\mathcal{N}(K\times G)}\big(\mathbb{Q}\big[L\backslash(K \times G)\big] \otimes_{\mathbb{Q}[K\times G]} \mathcal{N}(K \times G)\big)\\
&\quad = \frac{|K|}{|L|}.
\end{aligned}$$

This finishes the proof of Lemma 6.28. □

Let Γ be a quasi-finite free EI-category. Define the $\mathbb{Q}$-homomorphism

$$\overline{\omega}^{(2)}\colon U(\Gamma) \otimes_{\mathbb{Z}} \mathbb{Q} \to U(\Gamma) \otimes_{\mathbb{Z}} \mathbb{Q} \tag{6.29}$$

by the matrix over the rational numbers

$$\Big(\dim_{\mathcal{N}(y)}\big(\mathbb{Q}\operatorname{mor}(y,x) \otimes_{\mathbb{Q}[y]} \mathcal{N}(y)\big)\Big)_{\overline{x},\overline{y}\in\operatorname{iso}(\Gamma)}.$$

Define the $\mathbb{Q}$-homomorphism

$$\overline{\mu}^{(2)}\colon U(\Gamma) \otimes_{\mathbb{Z}} \mathbb{Q} \to U(\Gamma) \otimes_{\mathbb{Z}} \mathbb{Q} \tag{6.30}$$

by the matrix over the rational numbers

$$\Big(\sum_{l\geq 0}(-1)^l \cdot \sum_{c\in\operatorname{ch}_l(y,x)} \dim_{\mathcal{N}(y)}\big(\mathbb{Q}S(c) \otimes_{\mathbb{Q}[y]} \mathcal{N}(y)\big)\Big)_{\overline{x},\overline{y}\in\operatorname{iso}(\Gamma)}.$$

Notice that these homomorphisms are well-defined because of Example 5.4 (ii) since the right $\operatorname{aut}(y)$-sets $\operatorname{mor}(y,x)$ and $S(c)$ are proper cofinite and for given $\overline{x} \in \operatorname{iso}(\Gamma)$ there are only finitely many $\overline{y} \in \operatorname{iso}(\Gamma)$ for which the sets $\operatorname{mor}(y,x)$ and $S(c)$ are non-empty.

Theorem 6.31 (Rational Möbius inversion)**.** *Let Γ be a quasi-finite free EI-category. Then the homomorphisms $\overline{\omega}^{(2)}$ of* (6.29) *and $\overline{\mu}^{(2)}$ of* (6.30) *are isomorphisms and inverse to one another.*

Proof. Let

$$\overline{\imath}\colon U(\Gamma) = \bigoplus_{\overline{x}\in\operatorname{iso}(\Gamma)} \mathbb{Z} \;\to\; \operatorname{Split} K_0(\mathbb{Q}\Gamma) = \bigoplus_{\overline{x}\in\operatorname{iso}(\Gamma)} K_0(\mathbb{Q}[x])$$

be the homomorphism that sends $\{n_{\overline{x}} \mid \overline{x} \in \operatorname{iso}(\Gamma)\}$ to $\bigl\{n_x \cdot [\mathbb{Q}[x]] \bigm| \overline{x} \in \operatorname{iso}(\Gamma)\bigr\}$. A direct computation shows that

$$\operatorname{rk}_\Gamma^{(2)} \circ \omega \circ \overline{\iota} = \overline{\omega}^{(2)}.$$

The image of $\omega \circ \overline{\iota}$ in Split $K_0(\mathbb{Q}\Gamma)$ has the property that its value at any $\overline{x} \in \operatorname{iso}(\Gamma)$ is an element in $K_0(\mathbb{Q}[x])$ given by a $\mathbb{Z}$-linear combination of classes of the form $\bigl[\mathbb{Q}[K\backslash \operatorname{aut}(x)]\bigr]$ for finite subgroups $K \subseteq \operatorname{aut}(x)$. Hence the argument in the proof of Lemma 6.28 (ii) shows (without using condition (I)) that $\operatorname{rk}_\Gamma^{(2)} \circ \mu = \overline{\mu}^{(2)} \circ \operatorname{rk}_\Gamma^{(2)}$ is true on the image of $\omega \circ \overline{\iota}$. This implies

$$\overline{\mu}^{(2)} \circ \overline{\omega}^{(2)} = \overline{\mu}^{(2)} \circ \operatorname{rk}_\Gamma^{(2)} \circ \omega \circ \overline{\iota} = \operatorname{rk}_\Gamma^{(2)} \circ \mu \circ \omega \circ \overline{\iota}.$$

We conclude $\mu \circ \omega = \operatorname{id}$ from Theorem 6.22. A direct computation shows $\operatorname{rk}_\Gamma^{(2)} \circ \overline{\iota} = \operatorname{id}$. Hence

$$\overline{\mu}^{(2)} \circ \overline{\omega}^{(2)} = \operatorname{id}.$$

Since the matrix defining $\overline{\omega}^{(2)}$ is a triangular matrix whose entries on the diagonal are all 1, $\overline{\omega}^{(2)}$ is an isomorphism. Hence $\overline{\omega}^{(2)}$ of (6.29) and $\overline{\mu}^{(2)}$ of (6.30) are isomorphisms and inverse to one another. □

Remark 6.32. Notice that the condition free is not needed when we want to define the finiteness obstruction or to compute it as long as we stay on the K-theory level. It does enter, when we want to consider the rank or L^2-rank of the finiteness obstruction to ensure that certain comparisons can be done on the level of the Euler characteristics, or, equivalently certain maps on the K_0-level factorize over the rank or L^2-rank homomorphism from $K_0(R\Gamma)$ to $U(\Gamma)$.

Example 6.33 (Rational Möbius inversion for a finite, skeletal, free EI-category)**.** Generalizing Example 6.24, let Γ be a finite skeletal EI-category which is free in the sense of Definition 6.6 , and take $R = \mathbb{Q}$. Then

$$U(\Gamma) \otimes_{\mathbb{Z}} \mathbb{Q} = \bigoplus_{\operatorname{ob}(\Gamma)} \mathbb{Q}$$

and the homomorphism $\overline{\omega}^{(2)}$ is given by the matrix

$$\Bigl(\dim_{\mathcal{N}(y)}\bigl(\mathbb{Q}\operatorname{mor}(y,x) \otimes_{\mathbb{Q}[y]} \mathcal{N}(y)\bigr)\Bigr)_{x,y \in \operatorname{ob}(\Gamma)} = \left(\frac{|\operatorname{mor}(y,x)|}{|\operatorname{aut}(y)|}\right)_{x,y \in \operatorname{ob}(\Gamma)}.$$

The last equality follows from Example 5.4 (ii). If we let ω_L be the matrix

$$(|\operatorname{mor}_\Gamma(y,x)|)_{x,y \in \operatorname{ob}(\Gamma)}$$

and D is the diagonal matrix with entry $|\operatorname{aut}(y)|$ at (y,y) for $y \in \operatorname{ob}(\Gamma)$, then $D \circ \overline{\omega}^{(2)} = \omega_L$.

Then by Theorem 6.31, the homomorphism $\overline{\omega}^{(2)}$ is invertible and its inverse is $\overline{\mu}^{(2)}$. Hence ω_L admits an inverse $\mu_L := (D \circ \overline{\omega}^{(2)})^{-1} = \overline{\mu}^{(2)} \circ D^{-1}$. We calculate

μ_L by way of the matrix for $\overline{\mu}^{(2)}$ using the formula just after equation (6.30). For any l-chain $c \in \operatorname{ch}_l(y,x)$ with $c = x_0 < x_1 < \cdots < x_l$ we have

$$|S(c)| = \frac{|\operatorname{mor}(x_{l-1},x_l)| \cdot |\operatorname{mor}(x_{l-2},x_{l-1})| \cdot \dots \cdot |\operatorname{mor}(x_0,x_1)|}{|\operatorname{aut}(x_{l-1})| \cdot |\operatorname{aut}(x_{l-2})| \cdot \dots \cdot |\operatorname{aut}(x_1)|}$$

by freeness. Then,

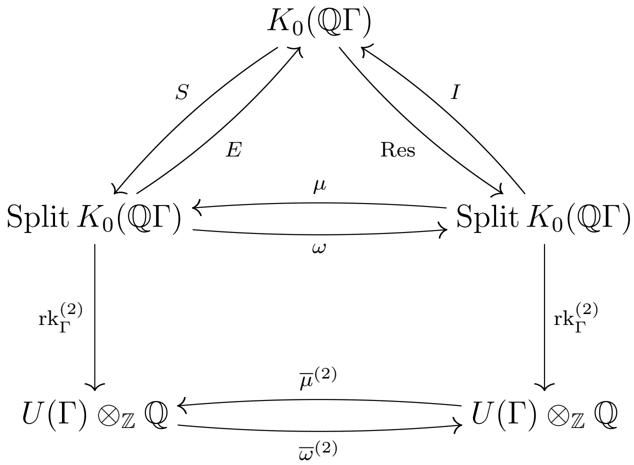
$$\begin{aligned}\dim_{\mathcal{N}(y)}&\big(\mathbb{Q}S(c) \otimes_{\mathbb{Q}[y]} \mathcal{N}(y)\big)\\ &= \frac{|S(c)|}{|\operatorname{aut}(y)|} = \frac{|\operatorname{mor}(x_{l-1},x_l)| \cdot |\operatorname{mor}(x_{l-2},x_{l-1})| \cdot \dots \cdot |\operatorname{mor}(x_0,x_1)|}{|\operatorname{aut}(x_{l-1})| \cdot |\operatorname{aut}(x_{l-2})| \cdot \dots \cdot |\operatorname{aut}(x_1)| \cdot |\operatorname{aut}(x_0)|}\end{aligned}$$

by Example 5.4 (ii). Summing up, we have

$$\begin{aligned}\mu_L &= \overline{\mu}^{(2)} \circ D^{-1}\\ &= \Bigg(\sum_{l \geq 0} (-1)^l \cdot \sum_{c \in \operatorname{ch}_l(y,x)} \dim_{\mathcal{N}(y)}\big(\mathbb{Q}S(c) \otimes_{\mathbb{Q}[y]} \mathcal{N}(y)\big)\Bigg)_{x,y \in \operatorname{ob}(\Gamma)} \circ D^{-1}\\ &= \Bigg(\sum_{l \geq 0} (-1)^l \cdot \sum_{c \in \operatorname{ch}_l(y,x)} \frac{|\operatorname{mor}(x_{l-1},x_l)| \cdot |\operatorname{mor}(x_{l-2},x_{l-1})| \cdot \dots \cdot |\operatorname{mor}(x_0,x_1)|}{|\operatorname{aut}(x_{l-1})| \cdot |\operatorname{aut}(x_{l-2})| \cdot \dots \cdot |\operatorname{aut}(x_1)| \cdot |\operatorname{aut}(x_0)|}\Bigg)_{x,y \in \operatorname{ob}(\Gamma)} \circ D^{-1}\\ &= \Bigg(\sum_{l \geq 0} (-1)^l \cdot \sum_{c \in \operatorname{ch}_l(y,x)} \frac{|\operatorname{mor}(x_{l-1},x_l)| \cdot |\operatorname{mor}(x_{l-2},x_{l-1})| \cdot \dots \cdot |\operatorname{mor}(x_0,x_1)|}{|\operatorname{aut}(x_l)| \cdot |\operatorname{aut}(x_{l-1})| \cdot |\operatorname{aut}(x_{l-2})| \cdot \dots \cdot |\operatorname{aut}(x_1)| \cdot |\operatorname{aut}(x_0)|}\Bigg)_{x,y \in \operatorname{ob}(\Gamma)}\\ &= \Bigg(\sum_{l \geq 0} (-1)^l \cdot \sum \frac{1}{|\operatorname{aut}(x_l)| \cdot |\operatorname{aut}(x_{l-1})| \cdot |\operatorname{aut}(x_{l-2})| \cdot \dots \cdot |\operatorname{aut}(x_1)| \cdot |\operatorname{aut}(x_0)|}\Bigg)_{x,y \in \operatorname{ob}(\Gamma)}.\end{aligned}$$

The final sum is over all l-paths $x_0 \to x_1 \to \cdots \to x_l$ from y to x such that $x_0, \ldots, x_l$ are all distinct. Thus, in the terminology of [13], the category Γ has Möbius inversion given by μ_L. The free case of Theorem 1.4 of [13] is now a special case of rational Möbius inversion (Theorem 6.31). See also the related proof of Lemma 7.3, which shows that the L^2-Euler characteristic coincides with Leinster's Euler characteristic in the case of a finite, skeletal, free EI-category.

Theorem 6.34 (The K-theoretic Möbius inversion and the L^2-rank)**.** *Let Γ be a quasi-finite free EI-category satisfying condition (I) defined in 6.27. Then the* *following diagram commutes*



Here the pairs (S, E) *(see Theorem 3.14),* (Res, I) *(see Theorem 6.16),* (ω, μ) *(see Theorem 6.22), and* $(\overline{\omega}^{(2)}, \overline{\mu}^{(2)})$ *(see Theorem 6.31) are pairs of isomorphisms inverse to one another, and the map* $\mathrm{rk}_\Gamma^{(2)}$ *comes from the map defined in* (5.19)*.*

Proof. The maps $\mathrm{rk}_\Gamma^{(2)}$ takes value in $U(\Gamma) \otimes_{\mathbb{Z}} \mathbb{Q}$ by Lemma 6.28 (i). The other claims follow from Theorem 6.22 and Lemma 6.28 (ii) and Theorem 6.31. □

Theorem 6.35 (The finiteness obstruction and the (functorial) L^2-Euler characteristic)**.**

(i) *Let* Γ *be a quasi-finite EI-category of type (FP). Then the image of the finiteness obstruction* $o(\Gamma; \mathbb{Q})$ *under the homomorphism*

$$\mathrm{Res}\colon K_0(\mathbb{Q}\Gamma) \to \mathrm{Split}\, K_0(\mathbb{Q}\Gamma)$$

defined in Theorem 6.16 has as entry at $\overline{x} \in \mathrm{iso}(\Gamma)$ *the finiteness obstruction* $o\big(\widehat{\mathrm{aut}(x)}; \mathbb{Q}\big)$ *of the category* $\widehat{\mathrm{aut}(x)}$*, i.e., the finiteness obstruction* $o(\mathbb{Q})$ *of the* $\mathbb{Q}[x]$*-module* $\mathbb{Q}$ *with the trivial* $\mathrm{aut}(x)$*-action. This possesses a finite projective* $\mathbb{Q}[x]$*-resolution by Lemma 6.15 (i). As usual, we will write* $[\mathbb{Q}]$ *for* $o\big(\widehat{\mathrm{aut}(x)}; \mathbb{Q}\big)$*.*

(ii) *Suppose that* Γ *is a quasi-finite free EI-category of type (FP) satisfying condition (I) or that* Γ *is a quasi-finite free EI-category of type (FF).*

Then for every object x *the* L^2*-Euler characteristic* $\chi^{(2)}(\mathrm{aut}(x))$ *is a rational number and is non-trivial for only finitely many* $\overline{x} \in \mathrm{iso}(\Gamma)$*. The collection* $\big(\chi^{(2)}(\mathrm{aut}(x))\big)_{\overline{x} \in \mathrm{iso}(\Gamma)}$ *defines an element* $\eta \in U(\Gamma) \otimes_{\mathbb{Z}} \mathbb{Q}$*. The functorial* L^2*-Euler characteristic* $\chi_f^{(2)}(\Gamma)$ *lies in* $U(\Gamma) \otimes_{\mathbb{Z}} \mathbb{Q}$*. We get*

$$\begin{aligned} \overline{\omega}^{(2)}\big(\chi_f^{(2)}(\Gamma)\big) &= \eta; \\ \overline{\mu}^{(2)}(\eta) &= \chi_f^{(2)}(\Gamma), \end{aligned}$$

where $\overline{\omega}^{(2)}$ *and* $\overline{\mu}^{(2)}$ *are the homomorphisms defined in* (6.29) *and* (6.30)*.*

Proof. (i) Since Γ is of type (FP), we conclude from Lemma 6.15 (i) that the $\mathbb{Q}[x]$-module $\mathbb{Q}$ with the trivial $\operatorname{aut}(x)$-action possesses a finite projective $\mathbb{Q}[x]$-resolution and hence defines an element in $K_0(\mathbb{Q}[x])$. Since $\operatorname{Res}_x \colon \mathsf{MOD}\text{-}\mathbb{Q}\Gamma \to \mathsf{MOD}\text{-}\mathbb{Q}[x]$ is exact, the claim follows from Lemma 6.15 (i).

(ii) We begin with the case where Γ is a quasi-finite free EI-category of type (FP) satisfying condition (I). The map $\operatorname{rk}_\Gamma^{(2)} \colon \operatorname{Split} K_0(\mathbb{Q}\Gamma) \to \prod_{\overline{x} \in \operatorname{iso}(\Gamma)} \mathbb{R}$ takes value in $U(\Gamma) \otimes_{\mathbb{Z}} \mathbb{Q}$ by Lemma 6.28 (i). The image of $o(\Gamma;\mathbb{Q})$ under the composite

$$K_0(\mathbb{Q}\Gamma) \xrightarrow{S} \operatorname{Split} K_0(\mathbb{Q}\Gamma) \xrightarrow{\operatorname{rk}_\Gamma^{(2)}} \prod_{\overline{x} \in \operatorname{iso}(\Gamma)} \mathbb{R}$$

is by definition $\chi_f^{(2)}(\Gamma)$. The image of $o(\Gamma;\mathbb{Q})$ under the composite

$$K_0(\mathbb{Q}\Gamma) \xrightarrow{\operatorname{Res}} \operatorname{Split} K_0(\mathbb{Q}\Gamma) \xrightarrow{\operatorname{rk}_\Gamma^{(2)}} \prod_{\overline{x} \in \operatorname{iso}(\Gamma)} \mathbb{R}$$

is by definition η. Now the claim follows from Theorem 6.34.

Next we deal with the case, where Γ is a quasi-finite free EI-category of type (FF) Since Γ is of type (FF), the image of $o(\Gamma;\mathbb{Q})$ under the isomorphism $S \colon K_0(\mathbb{Q}\Gamma) \xrightarrow{\cong} \operatorname{Split} K_0(\mathbb{Q}\Gamma)$ is the image of $\chi_f^2(\Gamma) \in U(\Gamma)$ under the map $\iota \colon U(\Gamma) \to \operatorname{Split} K_0(\mathbb{Q}\Gamma)$ defined in (4.8) since $\operatorname{rk}_\Gamma^{(2)} \circ \iota$ is the identity on $U(\Gamma)$. A direct computation shows that $\overline{\omega}^{(2)} = \operatorname{rk}_\Gamma^{(2)} \circ \omega \circ \iota$. This implies

$$\overline{\omega}^{(2)}(\chi^{(2)}(\Gamma)) = \eta.$$

We get

$$\overline{\mu}^{(2)}(\eta) \quad = \quad \chi_f^{(2)}(\Gamma),$$

from Theorem 6.31. □

6.4. **The example of a biset.** Let H and G be groups and let S be a G-H-biset. They define an EI-category $\Gamma(S)$ with two objects x and y, where the automorphism group of x is H, the automorphism group of y is G, the set of morphisms from x to y is S, the set of morphisms from y to x is empty and the composition in $\Gamma(S)$ comes from the group structure on H and G and the G-H-biset structure on S. Any EI-category with precisely two objects which are not isomorphic arises as $\Gamma(S)$ for some S. The category $\Gamma(S)$ is free if and only if S is free as a left G-set. The category $\Gamma(S)$ is quasi-finite if and only if S is proper and cofinite as a right H-set. The set of isomorphism classes of objects contains precisely two elements, namely x and y.

Suppose that $\Gamma(S)$ is quasi-finite. Then $\Gamma(S)$ is of type (FP) if and only if the trivial $\mathbb{Q}H$-module $\mathbb{Q}$ has a finite projective $\mathbb{Q}H$-resolution and the trivial $\mathbb{Q}G$-module $\mathbb{Q}$ has a finite projective $\mathbb{Q}G$-resolution (see Lemma 6.15 (iv)).

Suppose that $\Gamma(S)$ is quasi-finite and of type (FP). Then the image of the finiteness obstruction under the isomorphism

$$S\colon K_0(\mathbb{Q}\Gamma(S)) \xrightarrow{\cong} K_0(\mathbb{Q}H) \oplus K_0(\mathbb{Q}G)$$

is the element $\mu\big([\mathbb{Q}], [\mathbb{Q}]\big)$ by Theorem 6.22 (ii), where $[\mathbb{Q}]$ stands, of course, for the finiteness obstruction of the trivial $\mathbb{C}H$-module and trivial $\mathbb{Q}G$-module $\mathbb{Q}$, respectively. That is, $[\mathbb{Q}]$ means $o(\widehat{H};\mathbb{Q})$ or $o(\widehat{G};\mathbb{Q})$ respectively.

Suppose that $\Gamma(S)$ is quasi-finite, free, and of type (FP). Then the $\mathbb{Q}H$-module $\mathbb{Q}G\backslash S$ has a finite projective $\mathbb{Q}H$-resolution and the image of the finiteness obstruction under the isomorphism

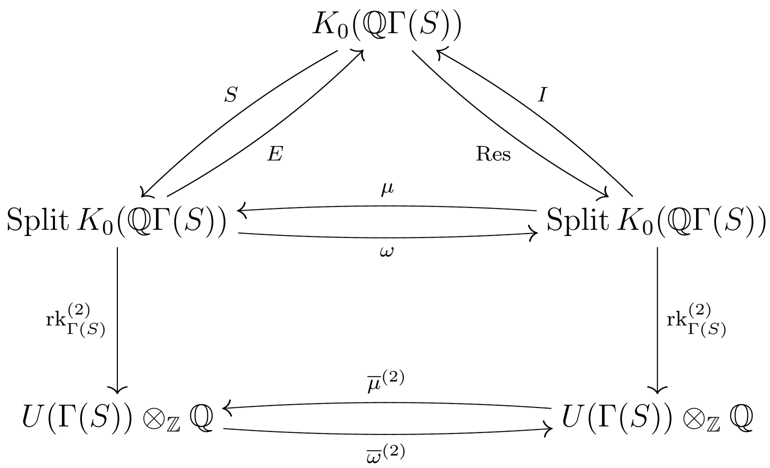
$$S\colon K_0(\mathbb{Q}\Gamma(S)) \xrightarrow{\cong} K_0(\mathbb{Q}H) \oplus K_0(\mathbb{Q}G)$$

is the element

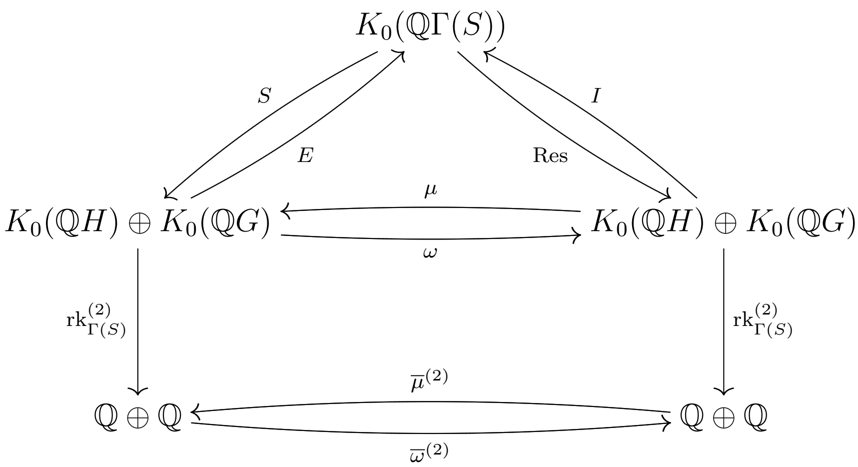
$$\mu([\mathbb{Q}], [\mathbb{Q}]) = ([\mathbb{Q}] - [\mathbb{Q} \otimes_{\mathbb{Q}G} \mathbb{Q}S], [\mathbb{Q}]) = \big([\mathbb{Q}] - [\mathbb{Q}G\backslash S], [\mathbb{Q}]\big)$$

by Theorem 6.22 (ii).

Suppose that $\Gamma(S)$ is quasi-finite, free, and of type (FP), and that H and G satisfy Condition (I) (see 6.26). Then $\Gamma(S)$ satisfies Condition (I) by definition. The commutative diagram appearing in Theorem 6.34



becomes



where ω sends $\big([P],[Q]\big)$ to $\big([P]+[Q\otimes_{\mathbb{Q}G}\mathbb{Q}S],[Q]\big)$ and μ sends $\big([P],[Q]\big)$ to $\big([P]-[Q\otimes_{\mathbb{Q}G}\mathbb{Q}S],[Q]\big)$. If the proper cofinite right H-set S is the disjoint union $\coprod_{i=1}^{r} L_i\backslash H$ and $d := \sum_{i=1}^{r} 1/|L_i|$, then the matrices for $\overline{\omega}^{(2)}$ and $\overline{\mu}^{(2)}$ are respectively $\begin{pmatrix} 1 & 0 \\ d & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ -d & 1 \end{pmatrix}$ by Example 5.4 (ii) and Lemma 6.28 (ii). We conclude from Theorem 6.35, the definition of $\chi_f(\Gamma(S))$, and Corollary 4.19 that

$$\begin{aligned}
\chi_f^{(2)}(\Gamma(S)) &= \big(\chi^{(2)}(H) - d\cdot\chi^{(2)}(G), \chi^{(2)}(G)\big);\\
\chi^{(2)}(\Gamma(S)) &= \chi^{(2)}(H) + (1-d)\cdot\chi^{(2)}(G);\\
\chi_f(\Gamma(S)) &= \big(1-|G\backslash S/H|, 1\big);\\
\chi(\Gamma(S)) &= 2-|G\backslash S/H|.
\end{aligned}$$

The situation above simplifies considerably in the finite case.

Example 6.36 (Finite G-H-biset for finite groups H and G)**.** Let H and G be finite groups and S a finite G-H-biset. Then the category $\Gamma(S)$ is a finite EI-category. We conclude from Theorem 6.23 that $\Gamma(S)$ is of type (FP). The image of the finiteness obstruction under the isomorphism

$$S\colon K_0(\mathbb{Q}\Gamma(S)) \xrightarrow{\cong} K_0(\mathbb{Q}H) \oplus K_0(\mathbb{Q}G)$$

is the element $\mu([\mathbb{Q}],[\mathbb{Q}]) = \big([\mathbb{Q}] - [\mathbb{Q}G\backslash S], [\mathbb{Q}]\big)$, and

$$\begin{aligned}
\chi_f^{(2)}(\Gamma(S)) &= \left(\frac{1}{|H|} - \frac{|G\backslash S|}{|H|}, \frac{1}{|G|}\right);\\
\chi^{(2)}(\Gamma(S)) &= \frac{1}{|H|} + \frac{1}{|G|} - \frac{|G\backslash S|}{|H|};\\
\chi_f(\Gamma(S);\mathbb{Q}) &= (1-|G\backslash S/H|, 1);\\
\chi(\Gamma(S);\mathbb{Q}) &= 2-|G\backslash S/H|,
\end{aligned}$$

since $\dim_{\mathcal{N}(H)}\bigl(\mathbb{C}(G\backslash S) \otimes_{\mathbb{C}H} \mathcal{N}(H)\bigr) = \frac{|G\backslash S|}{|H|}$ by Example 5.4 (ii). If S is free as a left G-set, or, equivalently, if $\Gamma(S)$ is free, we obtain

$$\begin{aligned} \chi_f^{(2)}(\Gamma(S)) &= \left(\frac{1}{|H|} - \frac{|S|}{|G| \cdot |H|}, \frac{1}{|G|}\right); \\ \chi^{(2)}(\Gamma(S)) &= \frac{1}{|H|} + \frac{1}{|G|} - \frac{|S|}{|G| \cdot |H|}, \end{aligned}$$

since in this case $\frac{|G\backslash S|}{|H|} = \frac{|S|}{|G|\cdot|H|}$.

6.5. **The passage to the opposite category.** In this subsection we want to compare the finiteness obstruction of Γ with the finiteness obstruction of the opposite category Γ^{op}.

In general Γ and Γ^{op} behave very differently. It may happen that Γ is of type (FP) but Γ^{op} is not of type (FP) or that both Γ and Γ^{op} are of type (FP), but their finiteness obstructions and functorial Euler characteristics are very different. This is illustrated by the following example.

Example 6.37. Let G be a group. Let S be the G-$\{1\}$ biset consisting of precisely one element. Let $\Gamma(S)$ be the associated EI-category of Subsection 6.4. It has two objects x and y. The sets $\mathrm{mor}_{\Gamma(S)}(x,x)$ and $\mathrm{mor}_{\Gamma(S)}(x,y)$ each contain precisely one element, the set $\mathrm{mor}_{\Gamma(S)}(y,y)$ is equal to G, and the set $\mathrm{mor}_{\Gamma(S)}(y,x)$ is empty. The category $\Gamma(S)$ is quasi-finite in the sense of Definition 6.6 and also directly finite in the sense of Definition 3.1. We conclude from Lemma 6.15 (iv) that $\Gamma(S)$ is of type $(\mathrm{FP}_{\mathbb{Q}})$ if and only if the group G is of type $(\mathrm{FP}_{\mathbb{Q}})$, i.e., the trivial $\mathbb{Q}G$-module $\mathbb{Q}$ possesses a finite projective $\mathbb{Q}G$-resolution.

Now suppose that G is of type $(\mathrm{FP}_{\mathbb{Q}})$. Then the trivial $\mathbb{Q}G$-module $\mathbb{Q}$ has a finite projective $\mathbb{Q}G$-resolution and defines an element $[\mathbb{Q}] = o(G;\mathbb{Q}) \in K_0(\mathbb{Q}G)$. Let $\alpha\colon K_0(\mathbb{Q}G) \to K_0(\mathbb{Q})$ be the homomorphism which sends $[P]$ to $[P \otimes_{\mathbb{Q}G} \mathbb{Q}]$. We conclude from Theorem 6.22 (ii) that the finiteness obstruction $o(\Gamma;\mathbb{Q})$ is sent under the isomorphism of (3.7)

$$S_{\mathbb{Q}\Gamma(S)}\colon K_0(\mathbb{Q}\Gamma(S)) \xrightarrow{\cong} K_0(\mathbb{Q}) \oplus K_0(\mathbb{Q}G)$$

to $\mu([\mathbb{Q}],[\mathbb{Q}]) = ([\mathbb{Q}] - \alpha([\mathbb{Q}]), [\mathbb{Q}])$,

This implies

$$\begin{array}{lcll} \chi_f^{(2)}(\Gamma(S)) &=& \bigl(1 - \chi(BG), \chi^{(2)}(G)\bigr) & \in U(\Gamma(S)) \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q} \oplus \mathbb{Q}; \\ \chi^{(2)}(\Gamma(S)) &=& 1 - \chi(BG) + \chi^{(2)}(G) & \in \mathbb{Q}; \\ \chi_f(\Gamma(S);R) &=& \bigl(1 - \chi(BG), \chi(BG)\bigr) & \in U(\Gamma(S)) = \mathbb{Z} \oplus \mathbb{Z}; \\ \chi(\Gamma(S);R) &=& 1 & \in \mathbb{Z}. \end{array}$$

If G satisfies condition (I) of (6.26) or G is of type (FF), then we conclude from Lemma 5.23 (i)

$$\chi^{(2)}(\Gamma(S)) = 1.$$

The opposite category $\Gamma(S)^{\mathrm{op}} = \Gamma(S^{\mathrm{op}})$ has a terminal object, namely x. Hence it is always of type (FP) and its finiteness obstruction $o(\Gamma(S)^{\mathrm{op}};\mathbb{Q})$ is sent under the isomorphism of (3.7)

$$S_{\mathbb{Q}\Gamma(S)^{\mathrm{op}}} \colon K_0(\mathbb{Q}\Gamma(S)^{\mathrm{op}}) \xrightarrow{\cong} K_0(\mathbb{Q}) \oplus K_0(\mathbb{Q}G)$$

to $\mu([\mathbb{Q}],[\mathbb{Q}]) = ([\mathbb{Q}],0)$.

This implies

$$\begin{array}{lcll} \chi_f^{(2)}(\Gamma(S)^{\mathrm{op}}) & = & \big(1,0\big) & \in U(\Gamma(S)^{\mathrm{op}}) \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q} \oplus \mathbb{Q}; \\ \chi^{(2)}(\Gamma(S)^{\mathrm{op}}) & = & 1 & \in \mathbb{Q}; \\ \chi_f(\Gamma(S)^{\mathrm{op}};R) & = & (1,0) & \in U(\Gamma(S)^{\mathrm{op}}) = \mathbb{Z} \oplus \mathbb{Z}; \\ \chi(\Gamma(S)^{\mathrm{op}};R)) & = & 1 & \in \mathbb{Z}. \end{array}$$

Notice that all the results for $\Gamma(S)$ depend on G, whereas the results for $\Gamma(S)^{\mathrm{op}}$ are all independent of G. So for example, if G is not of type $(\mathrm{FP}_{\mathbb{Q}})$, then $\Gamma(S)$ is not of type $(\mathrm{FP}_{\mathbb{Q}})$, while $\Gamma(S)^{\mathrm{op}}$ is of type $(\mathrm{FP}_{\mathbb{Q}})$.

6.6. **The passage to the opposite category for finite EI-categories.** One can say more about the passage from Γ to Γ^{op} in the special case where Γ is a finite EI-category. Let R be a commutative ring. Given an R-module M, denote by $M^* := \hom_R(M,R)$ its dual R-module. Notice that M^* is again an R-module since R is commutative. This defines a contravariant functor

$$*_R \colon \mathsf{MOD}\text{-}R \to \mathsf{MOD}\text{-}R.$$

There is a natural R-homomorphism $I(M) \colon M \to (M^*)^*$ which sends $m \in M$ to $M^* \to R$, $\phi \mapsto \phi(m)$. It is an isomorphism if M is a finitely generated projective R-module.

We obtain a functor

$$*_{R\Gamma} \colon \mathsf{MOD}\text{-}R\Gamma \to \mathsf{MOD}\text{-}R\Gamma^{\mathrm{op}}$$

which sends a contravariant $R\Gamma$-module P to the contravariant $R\Gamma^{\mathrm{op}}$-module, or equivalently, covariant $R\Gamma$-module P^* given by the composite $\Gamma \xrightarrow{P} \mathsf{MOD}\text{-}R \xrightarrow{*} \mathsf{MOD}\text{-}R$. The functor $*_{R\Gamma}$ is exact when restricted to $R\Gamma$-modules M for which $M(x)$ is a finitely generated projective R-module for every object $x \in \mathrm{ob}(\Gamma)$. Let M be an $R\Gamma$-module such that $M(x)$ is a finitely generated projective R-module for every object $x \in \mathrm{ob}(\Gamma)$. Then M^* is an $R\Gamma^{\mathrm{op}}$-module such that $M(x)$ is a finitely generated projective R-module for every object $x \in \mathrm{ob}(\Gamma^{\mathrm{op}})$ and there is a natural isomorphism of $R\Gamma$-modules $M \xrightarrow{\cong} (M^*)^*$.

Now assume that the order of the automorphism group of every object in Γ is invertible in R. Then an $R\Gamma$-module M, for which the R-module $M(x)$ possesses a finite projective R-resolution for every object $x \in \mathrm{ob}(\Gamma)$, possesses a finite projective $R\Gamma$-resolution by Lemma 6.15 (v). Hence we obtain a well-defined homomorphism

$$*_{R\Gamma} \colon K_0(R\Gamma) \to K_0(R\Gamma^{\mathrm{op}}), \quad [P] \mapsto [P^*] \tag{6.38}$$

The functor $*_{R\Gamma}$ sends the constant $R\Gamma$-module $\underline{R}$ to the the constant $R\Gamma^{\mathrm{op}}$-module $\underline{R}$. We conclude:

Lemma 6.39. *Let Γ be a finite EI-category. Let R be a commutative ring such that the order of the automorphism group of every object in Γ is invertible in R.*

(i) *The map of* (6.38)
$$*_{R\Gamma}\colon K_0(R\Gamma) \to K_0(R\Gamma^{\mathrm{op}})$$
is bijective, an inverse is
$$*_{R\Gamma^{\mathrm{op}}}\colon K_0(R\Gamma^{\mathrm{op}}) \to K_0(R\Gamma);$$

(ii) *Both Γ and Γ^{op} are of type (FP) and*
$$*_{R\Gamma}\big(o(\Gamma;R)\big) = o(\Gamma^{\mathrm{op}};R).$$

The map $*_{R\Gamma}$ is rather complicated as the next result shows.

Lemma 6.40. *Let Γ be a finite EI-category. Let R be a commutative ring such that the order of the automorphism group of every object in Γ is invertible in R. Then the following diagram commutes*

$$\begin{array}{ccc}
K_0(R\Gamma) & \xrightarrow[\cong]{*_{R\Gamma}} & K_0(R\Gamma^{\mathrm{op}}) \\
{\scriptstyle S_{R\Gamma}}\big\downarrow{\scriptstyle\cong} & & {\scriptstyle\cong}\big\downarrow{\scriptstyle S_{R\Gamma^{\mathrm{op}}}} \\
\operatorname{Split} K_0(R\Gamma) & \xrightarrow[\nu]{\cong} & \operatorname{Split} K_0(R\Gamma^{\mathrm{op}})
\end{array}$$

Here $S_{R\Gamma}$ and $S_{R\Gamma^{\mathrm{op}}}$ are the homomorphisms defined in (3.7) *which are isomorphisms by Theorem 3.14, the isomorphism $*_{R\Gamma}$ has been defined in* (6.38) *and the isomorphism ν is the composite*

$$\nu\colon \operatorname{Split} K_0(R\Gamma) \xrightarrow{\omega_{R\Gamma}} \operatorname{Split} K_0(R\Gamma) \xrightarrow{D} \operatorname{Split} K_0(R\Gamma^{\mathrm{op}}) \xrightarrow{\mu_{R\Gamma^{\mathrm{op}}}} \operatorname{Split} K_0(R\Gamma^{\mathrm{op}}),$$

where $\omega_{R\Gamma}$ is the isomorphism defined in (6.18) *for Γ, $\mu_{R\Gamma^{\mathrm{op}}}$ is the isomorphism defined in* (6.21) *for Γ^{op} and D is given by the direct sum of the isomorphisms $K_0(R\operatorname{aut}_\Gamma(x)) \xrightarrow{\cong} K_0(R\operatorname{aut}_{\Gamma^{\mathrm{op}}}(x))$ sending the class of the finitely generated projective $R\operatorname{aut}_\Gamma(x)$-module P to the class of the finitely generated projective $R\operatorname{aut}_{\Gamma^{\mathrm{op}}}(x)$-module P^*.*

Proof. Consider the following diagram.

$$\begin{array}{ccccccc}
 & & K_0(R\Gamma) & \xrightarrow{*_{R\Gamma}} & K_0(R\Gamma^{\mathrm{op}}) & & \\
 & {\scriptstyle S_{R\Gamma}}\swarrow & \big\downarrow{\scriptstyle \operatorname{Res}_{R\Gamma}} & & {\scriptstyle \operatorname{Res}_{R\Gamma^{\mathrm{op}}}}\big\downarrow & \searrow{\scriptstyle S_{R\Gamma^{\mathrm{op}}}} & \\
\operatorname{Split} K_0(R\Gamma) & \xrightarrow[\omega_{R\Gamma}]{} & \operatorname{Split} K_0(R\Gamma) & \xrightarrow[D]{} & \operatorname{Split} K_0(R\Gamma^{\mathrm{op}}) & \xrightarrow[\mu_{R\Gamma^{\mathrm{op}}}]{} & \operatorname{Split} K_0(R\Gamma^{\mathrm{op}})
\end{array}$$

The left and right triangles commute by Theorem 6.22 and the middle square commutes from the definitions, so the entire diagram commutes. □

Lemma 6.41. *Let Γ be a finite EI-category. Suppose that both Γ and Γ^{op} are free in the sense of Definition 6.6. Then the following diagram commutes.*

$$\begin{array}{ccc}
K_0(\mathbb{Q}\Gamma) & \xrightarrow[\cong]{*_{\mathbb{Q}\Gamma}} & K_0(\mathbb{Q}\Gamma^{\mathrm{op}}) \\
{\scriptstyle S_{\mathbb{Q}\Gamma}}\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\downarrow{\scriptstyle S_{\mathbb{Q}\Gamma^{\mathrm{op}}}} \\
\operatorname{Split} K_0(\mathbb{Q}\Gamma) & \xrightarrow[\cong]{\nu} & \operatorname{Split} K_0(\mathbb{Q}\Gamma^{\mathrm{op}}) \\
{\scriptstyle \mathrm{rk}^{(2)}_{\Gamma}}\downarrow & & \downarrow{\scriptstyle \mathrm{rk}^{(2)}_{\Gamma^{\mathrm{op}}}} \\
U(\Gamma)\otimes_{\mathbb{Z}}\mathbb{Q} & \xrightarrow[\cong]{\overline{\nu}^{(2)}} & U(\Gamma)\otimes_{\mathbb{Z}}\mathbb{Q}
\end{array}$$

Here the upper square is taken from Lemma 6.40, the maps $\mathrm{rk}^{(2)}_{\Gamma}$ and $\mathrm{rk}^{(2)}_{\Gamma^{\mathrm{op}}}$ have been defined in (5.19)*, and the isomorphism $\overline{\nu}^{(2)}$ is defined to be $\overline{\mu}^{(2)}_{\Gamma^{\mathrm{op}}}\circ\overline{\omega}^{(2)}_{\Gamma}$, where $\overline{\omega}^{(2)}_{\Gamma}$ is the isomorphism defined in* (6.29) *for Γ and $\overline{\mu}^{(2)}_{\Gamma^{\mathrm{op}}}$ is the isomorphism defined in* (6.30) *for Γ^{op}.*

Proof. This follows from Theorem 6.34 and Lemma 6.40 and the easy to verify fact that the following diagram commutes for the homomorphism D appearing in Lemma 6.40.

$$\begin{array}{ccc}
\operatorname{Split} K_0(\mathbb{Q}\Gamma) & \xrightarrow{D} & \operatorname{Split} K_0(\mathbb{Q}\Gamma^{\mathrm{op}}) \\
{\scriptstyle \mathrm{rk}^{(2)}_{\Gamma}}\downarrow & & \downarrow{\scriptstyle \mathrm{rk}^{(2)}_{\Gamma^{\mathrm{op}}}} \\
U(\Gamma)\otimes_{\mathbb{Z}}\mathbb{Q} & \xrightarrow[\mathrm{id}]{} & U(\Gamma)\otimes_{\mathbb{Z}}\mathbb{Q}
\end{array}$$

□

Example 6.42 (The isomorphism $*$ for a finite G-H-biset for finite groups H and G)**.** Let H and G be finite groups and S a finite G-H-biset. We have defined a finite EI-category $\Gamma(S)$ in Subsection 6.4 and Example 6.36. We conclude from Subsection 6.4 that the commutative diagram appearing in Lemma 6.40 can be identified for $\Gamma(S)$ with

$$\begin{array}{ccc}
K_0(\mathbb{Q}\Gamma(S)) & \xrightarrow[\cong]{*_{\mathbb{Q}\Gamma(S)}} & K_0(\mathbb{Q}\Gamma(S)^{\mathrm{op}}) \\
{\scriptstyle S_{\mathbb{Q}\Gamma(S)}}\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\downarrow{\scriptstyle S_{\mathbb{Q}\Gamma(S)^{\mathrm{op}}}} \\
K_0(\mathbb{Q}H)\oplus K_0(\mathbb{Q}G) & \xrightarrow[\nu]{\cong} & K_0(\mathbb{Q}H^{\mathrm{op}})\oplus K_0(\mathbb{Q}G^{\mathrm{op}}).
\end{array}$$

By the calculation for ω and μ in Subsection 6.4, the homomorphism ν sends $\big([P],[Q]\big)$ to

$$\big([P^*]+[(Q\otimes_{\mathbb{Q}G}\mathbb{Q}S)^*],[Q^*]-[P^*\otimes_{\mathbb{Q}H^{\mathrm{op}}}\mathbb{Q}S^{\mathrm{op}}]-[(Q\otimes_{\mathbb{Q}G}\mathbb{Q}S)^*\otimes_{\mathbb{Q}H^{\mathrm{op}}}\mathbb{Q}S^{\mathrm{op}}]\big)$$

(recall that the roles of G^{op} and H^{op} are switched in the formula for $\mu_{\mathbb{Q}\Gamma^{\mathrm{op}}}$).

Now suppose that both $\Gamma(S)$ and $\Gamma(S)^{\mathrm{op}}$ are free, or, equivalently, that G acts freely from the left on S and H acts freely from the right on S. Then the commutative diagram appearing in Lemma 6.41 can be identified with

$$\begin{array}{ccc}
K_0(\mathbb{Q}\Gamma(S)) & \xrightarrow[\cong]{*_{\mathbb{Q}\Gamma(S)}} & K_0(\mathbb{Q}\Gamma(S)^{\mathrm{op}}) \\
{\scriptstyle S_{\mathbb{Q}\Gamma(S)}} \downarrow \cong & & \cong \downarrow {\scriptstyle S_{\mathbb{Q}\Gamma(S)^{\mathrm{op}}}} \\
K_0(\mathbb{Q}H) \oplus K_0(\mathbb{Q}G) & \xrightarrow[\cong]{\nu} & K_0(\mathbb{Q}H^{\mathrm{op}}) \oplus K_0(\mathbb{Q}G^{\mathrm{op}}) \\
{\scriptstyle \mathrm{rk}^{(2)}_{\Gamma(S)}} \downarrow & & \downarrow {\scriptstyle \mathrm{rk}^{(2)}_{\Gamma(S)^{\mathrm{op}}}} \\
\mathbb{Q} \oplus \mathbb{Q} & \xrightarrow[\overline{\nu}^{(2)}]{} & \mathbb{Q} \oplus \mathbb{Q}
\end{array}$$

where $\overline{\nu}^{(2)}$ is given by the matrix $\begin{pmatrix} 1 & \frac{|S|}{|H|} \\ -\frac{|S|}{|G|} & 1 - \frac{|S|^2}{|H|\cdot|G|} \end{pmatrix}$.

7. Comparison with the invariants of Baez-Dolan and Leinster

In this section we compare our invariants with the groupoid cardinality of Baez-Dolan [2] and the Euler characteristic of Leinster [13]. If Γ is a skeletal, finite, free EI-category, then Γ is of type (FP) and of type (L^2), and Leinster's Euler characteristic coincides with the L^2-Euler characteristic. However, if we leave out the freeness hypothesis, then Leinster's Euler characteristic can very well be different from the L^2-Euler characteristic, see Remark 7.4.

7.1. **Comparison with the groupoid cardinality of Baez-Dolan.** Baez-Dolan define in [2] the *groupoid cardinality* of a groupoid Γ to be

$$\sum_{\overline{x}\in\mathrm{iso}(\Gamma)} \frac{1}{|\operatorname{aut}(x)|},$$

provided this sum converges. In other words, the groupoid cardinality is the count of the isomorphism classes of objects inversely weighted by the size of their symmetry groups. This agrees with the L^2-Euler characteristic of such groupoids as seen in Example 5.12.

7.2. **Review of Leinster's Euler characteristic.** We briefly review the Euler characteristic due to Leinster [13]. Let Γ be a finite category (see Definition 6.6). A *weighting* on Γ is a function $k^\bullet\colon \mathrm{ob}(\Gamma) \to \mathbb{Q}$ such that for all objects $x \in \mathrm{iso}(\Gamma)$ we have $\sum_{y\in\mathrm{ob}(\Gamma)} |\operatorname{mor}(x,y)| \cdot k^y = 1$. A *coweighting* $k_\bullet$ on Γ is a weighting on Γ^{op}.

Definition 7.1. A finite category Γ has an *Euler characteristic in the sense of Leinster* if it has a weighting and a coweighting. Its Euler characteristic in the sense of Leinster is then defined as

$$\chi_L(\Gamma) := \sum_{x\in\mathrm{ob}(\Gamma)} k^x = \sum_{x\in\mathrm{ob}(\Gamma)} k_x$$

for any choice of weighting $k^\bullet$ or coweighting $k_\bullet$.

This is indeed independent of the choice of the weighting and the coweighting. In particular we get $\chi_L(\Gamma) = \chi_L(\Gamma^{\mathrm{op}})$.

Remark 7.2. Leinster's Euler characteristic can only be defined if the category Γ is finite and depends only on the set of objects $\mathrm{ob}(\Gamma)$ and the orders $|\operatorname{mor}(x,y)|$ for $x, y \in \mathrm{ob}(\Gamma)$. This is different from the other invariants such as the finiteness obstruction. For instance χ_L does not distinguish between the category Γ appearing in Example 2.18 and the groupoid $\widehat{\mathbb{Z}/2}$, whereas the finiteness obstructions and the L^2-Euler characteristic do.

7.3. **Finite free skeletal EI-categories.**

Lemma 7.3. *Let Γ be a finite free EI-category which is skeletal, i.e., two isomorphic objects are already equal.*

Then Γ is of type (FP) and of type (L^2), and has an Euler characteristic in the sense of Leinster. We get for the L^2-Euler characteristic $\chi^{(2)}(\Gamma;\mathbb{Q})$ of Definition 5.10 and Leinster's Euler characteristic $\chi_L(\Gamma)$ of Definition 7.1

$$\chi^{(2)}(\Gamma;\mathbb{Q}) = \chi_L(\Gamma).$$

Proof. By [13, Lemma 1.3 and Theorem 1.4] the category Γ^{op} has a Möbius inversion, i.e., the homomorphism

$$\omega_L \colon U(\Gamma) \otimes_{\mathbb{Z}} \mathbb{Q} \to U(\Gamma) \otimes_{\mathbb{Z}} \mathbb{Q}$$

given by the matrix

$$(|\operatorname{mor}_\Gamma(y,x)|)_{x,y\in\mathrm{ob}(\Gamma)}$$

is bijective, and has an Euler characteristic in the sense of Leinster. Then by definition

$$\chi_L(\Gamma) = \chi_L(\Gamma^{\mathrm{op}}) = \sum_{x\in\mathrm{ob}(\Gamma)} k_x$$

for any element $k_\bullet \in U(\Gamma) \otimes_{\mathbb{Z}} \mathbb{Q}$ such that $\omega_L(k_\bullet)$ is the element $\overline{1} \in U(\Gamma)$ which assigns 1 to every element in $\mathrm{ob}(\Gamma)$.

We conclude from Theorem 6.23 that Γ is of type (FP) and hence of type (L^2). Hence it remains to show

$$\omega_L\big(\chi_f^{(2)}(\Gamma)\big) = \overline{1} \quad \in U(\Gamma).$$

since by definition $\chi^{(2)}(\Gamma) = \sum_{x\in\mathrm{ob}(\Gamma)} \chi_f^{(2)}(\Gamma)(x)$.

Since $\mathrm{aut}(y)$ is finite, Example 5.4 (ii) implies

$$\dim_{\mathcal{N}(y)}\big(\mathbb{Q}\operatorname{mor}(y,x) \otimes_{\mathbb{Q}[y]} \mathcal{N}(y)\big) = \frac{|\operatorname{mor}(y,x)|}{|\operatorname{aut}(y)|}$$

for every $x, y \in \mathrm{ob}(\Gamma)$. Hence the homomorphism ω_L agrees with the composite $D \circ \overline{\omega}^{(2)}$, where $\overline{\omega}^{(2)}$ is defined in (6.29) and D is the isomorphism given by the diagonal matrix with entry $|\operatorname{aut}(y)|$ at (y,y) for $y \in \mathrm{ob}(\Gamma)$. Since $D \circ \overline{\omega}^{(2)}$ maps $\chi_f^{(2)}(\Gamma)$ to $\overline{1}$ because of Theorem 6.35 (ii) and because of $\chi^{(2)}(\mathrm{aut}(x)) =$

$1/|\operatorname{aut}(x)|$, Lemma 7.3 follows. We need Γ to be free in the sense of Definition 6.6 in order to apply Theorem 6.35 (ii). $\square$

Remark 7.4. The condition in Lemma 7.3 that Γ is free is necessary as the following example shows. Let H and G be finite groups and S be a finite G-H-biset. Let $\Gamma(S)$ be the associated finite EI-category of Example 6.36. We conclude from Example 6.36 and the definition of $\chi_L(\Gamma(S))$ that

$$\begin{aligned} \chi^{(2)}(\Gamma(S)) &= \frac{1}{|H|} + \frac{1}{|G|} - \frac{|G\backslash S|}{|H|}; \\ \chi(\Gamma(S)) &= 2 - |G\backslash S/H|, \\ \chi_L(\Gamma(S)) &= \frac{1}{|H|} + \frac{1}{|G|} - \frac{|S|}{|G|\cdot|H|}. \end{aligned}$$

Hence $\chi^{(2)}(\Gamma(S)) = \chi_L(\Gamma(S))$ holds if and only if $|G\backslash S| = \frac{|S|}{|G|}$. The latter is equivalent to the condition that $\Gamma(S)$ is free.

Notice that $\chi(\Gamma(S))$ is always an integer and is in general different from both $\chi^{(2)}(\Gamma(S))$ and $\chi_L(\Gamma(S))$.

Remark 7.5 (Homotopy colimit formula)**.** In [12] we prove the compatability of various Euler characteristics of categories with homotopy colimits. There we compare our homotopy colimit results with Leinster's results on Grothendieck fibrations.

7.4. **Passage to the opposite category and initial and terminal objects.** Leinster's Euler characteristic χ_L and the Euler characteristic $\chi(\Gamma)$ do not see a difference between Γ and Γ^{op}. We have discussed in detail in Subsection 6.5 that Γ and Γ^{op} can be distinguished by the finiteness obstruction $o(\Gamma;R)$, the functorial Euler characteristic $\chi_f(\Gamma;R)$, the functorial L^2-Euler characteristic $\chi_f^{(2)}(\Gamma)$, and the L^2-Euler characteristic $\chi^{(2)}(\Gamma)$.

Suppose that Γ has a terminal object x. Let $i\colon \{*\} \to \Gamma$ be the inclusion of the trivial category with value x. Then the finiteness obstruction is the image of $[R]$ under $i_*\colon K_0(R) \to K_0(R\Gamma)$ by Example 2.11. The functorial Euler characteristic $\chi_f(\Gamma;R) \in U(\Gamma)$ and the functorial L^2-Euler characteristic $\chi_f^{(2)}(\Gamma) \in U^{(1)}(\Gamma)$ agree and are given by the element $1\cdot\overline{x}$. The Euler characteristic $\chi(\Gamma;R)$ and the L^2-Euler characteristic $\chi^{(2)}(\Gamma) \in U^{(1)}(\Gamma)$ are both equal to 1. Since Γ has a terminal object, it admits a weighting [13, Example 1.11.c]. If Γ additionally admits a coweighting, then Leinster's Euler characteristic $\chi_L(\Gamma)$ is equal to 1.

If Γ has an initial object, we cannot predict the values of $o(\Gamma;R)$, $\chi_f(\Gamma;R)$, and $\chi^{(2)}(\Gamma)$ in general, as the results in Subsections 6.4 and 6.5 illustrate. In particular, $\chi^{(2)}(\Gamma)$ is not necessarily 1 if Γ has an initial object. For instance, Example 6.36 yields for $H=1$, $S=\{*\}$, and G any finite group $\chi^{(2)}(\Gamma(S)) = 1/|G|$. If Γ has an initial object, then Γ admits a coweighting. If Γ additionally admits a weighting, then Leinster's Euler characteristic $\chi_L(\Gamma)$ is equal to 1.

The naive Euler characteristic $\chi(\Gamma) = \chi(B\Gamma; R)$ is equal to 1 if Γ has an initial or a terminal object.

7.5. **Relationship Between Weightings and Free Resolutions.**

Theorem 7.6 (Weighting from a free resolution). *Let Γ be a small category. Suppose that the constant $R\Gamma$-module $\underline{R}$ admits a finite free resolution P_*. If P_n is free on the finite* $\operatorname{ob}(\Gamma)$*-set C_n, that is*

$$P_n = B(C_n) = \bigoplus_{y \in \operatorname{ob}(\Gamma)} \bigoplus_{C_n^y} R \operatorname{mor}(?, y), \tag{7.7}$$

then the function $k^\bullet \colon \operatorname{ob}(\Gamma) \to \mathbb{Q}$ defined by

$$k^y := \sum_{n \geq 0} (-1)^n \cdot |C_n^y|$$

is a weighting on Γ.

Proof. At each object x of Γ, the R-chain complex $P_*(x)$ has Euler characteristic 1, since it is a resolution of R. Further, calculating the Euler characteristic of $P_*(x)$ using equation (7.7) yields

$$\begin{aligned} 1 = \chi(P_*(x)) &= \sum_{n \geq 0} (-1)^n \operatorname{rk}_R P_n(x) \\ &= \sum_{n \geq 0} (-1)^n \left(\sum_{y \in \operatorname{ob}(\Gamma)} |C_n^y| \cdot |\operatorname{mor}(x, y)| \right) \\ &= \sum_{y \in \operatorname{ob}(\Gamma)} |\operatorname{mor}(x, y)| \left(\sum_{n \geq 0} (-1)^n |C_n^y| \right) \\ &= \sum_{y \in \operatorname{ob}(\Gamma)} |\operatorname{mor}(x, y)| k^y. \end{aligned}$$

□

In [12] we recall the Γ-CW-complexes of [11] in the context of Euler characteristics and homotopy colimits.

Corollary 7.8 (Construction of a weighting from a finite Γ-CW-model for the classifying Γ-space). *Let Γ be a small category. Suppose that Γ admits a finite Γ-CW-model X for the classifying Γ-space $E\Gamma$. Then the function $k^\bullet \colon \operatorname{ob}(\Gamma) \to \mathbb{Q}$ defined by*

$$k^y := \sum_{n \geq 0} (-1)^n (\textit{number of } n\textit{-cells of } X \textit{ based at } y)$$

is a weighting on Γ.

Proof. The composite of the cellular R-chain complex functor with X is a finite free resolution of the constant $R\Gamma$-module $\underline{R}$. The number of n-cells of X based at y is $|C_n^y|$. □

Remark 7.9. We may think of $k^\bullet$ in Corollary 7.8 as the Γ-*Euler characteristic of the* Γ-*CW-space* X. If $R = \mathbb{C}$ and Γ is skeletal and directly finite, then the function $k^\bullet$ is just $\chi_f(\Gamma;\mathbb{C}) = \chi_f^{(2)}(\Gamma)$ by Lemma 5.23 (iii)) and Lemma 4.10 (ii). The role of direct finiteness is to guarantee that the splitting functors S_x are defined.

Example 7.10. Let $\Gamma = \{1 \leftarrow 0 \rightarrow 2\}$ be the category with objects 0, 1, and 2 and only two nontrivial morphisms, one from 0 to 1 and one from 0 to 2. A finite Γ-CW-model for $E\Gamma$ has two zero-cells $\mathrm{mor}(?,1)$ and $\mathrm{mor}(?,2)$ and one 1-cell $\mathrm{mor}(?,0) \times D^1$ whose attaching map $\mathrm{mor}(?,0) \times S^0 \to \mathrm{mor}(?,1) \amalg \mathrm{mor}(?,2)$ is the disjoint union of the canonical maps $\mathrm{mor}(?,0) \to \mathrm{mor}(?,1)$ and $\mathrm{mor}(?,0) \to \mathrm{mor}(?,2)$. This finite model produces the weighting $(k^0, k^1, k^2) = (-1,1,1)$ by Corollary 7.8. This is the same weighting as in 1.11.a of Leinster's article [13].

Example 7.11. Let $\Gamma = \{a \rightrightarrows b\}$ be the category consisting of two objects and a single pair of parallel arrows between them. A finite Γ-CW-model for $E\Gamma$ has a single 0-cell based at b and a single 1-cell based at a. The gluing map $\mathrm{mor}(-,a) \times S^0 \to \mathrm{mor}(-,b)$ is induced by the the two parallel arrows $a \rightrightarrows b$. Corollary 7.8 then produces the weighting $(k^a, k^b) = (-1,1)$, the same weighting as in 3.4.b of Leinster's article [13]

Example 7.12. Let Γ be the category with objects the non-empty subsets of $[q] = \{0,1,\dots,q\}$ and a unique arrow $J \to K$ if and only if $K \subseteq J$. In [12] we construct a finite Γ-CW-model with precisely one $|J|-1$ cell based at J for each nonempty $J \subseteq [q]$. By Corollary 7.8, we obtain a weighting $k^\bullet$ on Γ by defining $k^J := (-1)^{|J|-1}$. This is the same weighting as in 3.4.d of Leinster's article [13].

Remark 7.13. For a finite group G, there is no finite model. So it appears the above method of finding the weighting does not work. However, if we use the L^2-rank, something similar does. Every finite group G has a finite projective resolution of $\underline{\mathbb{Q}}$, namely $\underline{\mathbb{Q}}$ itself. Then we obtain for the weighting

$$k^* = \sum_{n \geq 0} (-1)^n \dim_{\mathcal{N}(G)} \underline{\mathbb{Q}}_* = \dim_{\mathcal{N}(G)} \underline{\mathbb{Q}} = 1/|G|,$$

precisely as by Leinster.

8. The proper orbit category

The principal virtue of the finiteness-obstruction approach to Euler characteristics is the wide variety of examples and familiar notions it encompasses. We have already seen the naive Euler characteristic and the classical L^2-Euler characteristic of a group [19, Chapter 7] as special cases. We turn now to another special case: the equivariant Euler characteristic of the classifying space $\underline{E}G$ for proper G-actions. Recall from Definition 6.7 that the *proper orbit category* $\underline{\mathtt{Or}}(G)$ has as objects the homogeneous spaces G/H with H a finite subgroup of

G, and as morphisms the G-equivariant maps. We have shown in Lemma 6.11 that $\underline{\mathsf{Or}}(G)$ is a quasi-finite and free EI-category. We will explain in this section that the finiteness obstructions and Euler characteristic notions for $\Gamma = \underline{\mathsf{Or}}(G)$ correspond to established notions in equivariant topology for the classifying space $\underline{E}G$ for proper G-actions. This gives in particular the possibility to compute and relate the invariants for $\underline{\mathsf{Or}}(G)$ to more geometric notions.

In Subsection 8.1 we recall G-CW-complexes, the classifying space for proper G-actions, and the relationship between equivariant invariants of $\underline{E}G$ and our category-theoretic invariants of $\underline{\mathsf{Or}}(G)$. In Subsection 8.2 we discuss Möbius inversion for $\underline{\mathsf{Or}}(G)$ in the case where $\underline{E}G$ admits a finite model. If G_0 is a subgroup of G_1 and G_2, then the Euler characteristics of $\underline{\mathsf{Or}}(G_1 *_{G_0} G_2)$ are computed additively from those of $\underline{\mathsf{Or}}(G_0)$, $\underline{\mathsf{Or}}(G_1)$, and $\underline{\mathsf{Or}}(G_2)$ in Subsection 8.3. In Subsection 8.4 we derive the Burnside congruences from an integrality condition involving $(\overline{\mu}^{(2)}, \overline{\omega}^{(2)})$. We work everything out explicitly for G the infinite dihedral group in Subsection 8.5. Fundamental groupoids are considered in Subsection 8.6.

8.1. **The classifying space for proper G-actions.**

Definition 8.1 (G-CW-complex)**.** A *G-CW-complex* X is a G-space X together with a filtration by G-spaces $X_{-1} = \emptyset \subseteq X_0 \subseteq X_1 \subseteq \ldots \subseteq X = \bigcup_{n\geq 0} X_n$ such that $X = \operatorname{colim}_{n\to\infty} X_n$ and for each n there is a G-pushout, that is, a pushout in the category of G-spaces

$$\begin{array}{ccc} \coprod_{i\in I_n} G/H_i \times S^{n-1} & \xrightarrow{\coprod_{i\in I_n} q_i^n} & X_{n-1} \\ \downarrow & & \downarrow \\ \coprod_{i\in I_n} G/H_i \times D^n & \xrightarrow{\coprod_{i\in I_n} Q_i^n} & X_n. \end{array}$$

For more information about G-CW-complexes we refer to [15, Chapters 1 and 2]. A G-CW-complex is *proper* if and only if all its isotropy groups are finite (see [15, Theorem 1.23 on page 18]).

A G-CW-complex is *finite*, i.e., is built out of finitely many equivariant cells $G/H_i \times D^n$ if and only if it is *cocompact*, i.e. $G\backslash X$ is compact. A G-CW-complex X is *finitely dominated* if and only if there exists a finite G-CW-complex Y and G maps $i\colon X \to Y$ and $r\colon Y \to X$ with $r \circ i \simeq_G \mathrm{id}_X$.

Definition 8.2 (Classifying space for proper G-actions)**.** A model for the *classifying space for proper G-actions* is a G-CW-complex $\underline{E}G$ such that the subspace of H-fixed points $\underline{E}G^H$ is contractible for every finite subgroup $H \subseteq G$ and is empty for every infinite subgroup $H \subseteq G$.

For much more information about $\underline{E}G$ than presented here we refer the reader to the survey article [21]. We have $EG = \underline{E}G$ if and only if G is torsion-free. We can choose G/G as a model for $\underline{E}G$ if and only if G is finite.

Remark 8.3. The classifying space for proper G-actions has the following universal property. If X is a proper G-CW-complex, then there is up to G-homotopy precisely one G-map from X to $\underline{E}G$. In other words, a model for $\underline{E}G$ is a terminal object in the G-homotopy category of proper G-CW-complexes. In particular, two models for $\underline{E}G$ are G-homotopy equivalent.

Recall from Notation 4.4 that $U(\Gamma) := \mathbb{Z}\operatorname{iso}(\Gamma)$ for any category Γ.

Definition 8.4 (Equivariant Euler characteristic)**.** Let X be a finite G-CW-complex (see Definition 8.1). Define its equivariant Euler characteristic

$$\chi^G(X) \in U(\underline{\mathtt{Or}}(G))$$

by

$$\chi^G(X) := \sum_{n \geq 0} (-1)^n \cdot \sum_{i \in I_n} \overline{G/H_i}$$

for any choice of G-pushout appearing in Definition 8.1.

Theorem 8.5 (The relation between $\underline{E}G$ and $\underline{\mathtt{Or}}(G)$)**.**

(i) *If there exists a finite G-CW-model for $\underline{E}G$, then the EI-category $\underline{\mathtt{Or}}(G)$ is of type (FF_R) for any ring R;*

(ii) *If there exists a finitely dominated G-CW-model for $\underline{E}G$, then $\underline{\mathtt{Or}}(G)$ is of type (FP_R) for any ring R;*

(iii) *Suppose that G contains only finitely many conjugacy classes of finite subgroups and for every finite subgroup $H \subset G$ its Weyl group $W_GH := N_GH/H$ is finitely presented. Suppose that $R = \mathbb{Z}$. Then the converses of assertions (i) and (ii) are true;*

(iv) *If $\underline{E}G$ is a finitely dominated G-CW-complex, then the equivariant finiteness obstruction of* [15, Definition 14.4 on page 278] *agrees with the finiteness obstruction $o(\underline{\mathtt{Or}}(G);\mathbb{Z})$ of Definition 2.7;*

(v) *Suppose that there is a finite G-CW-complex model for $\underline{E}G$. Then its equivariant Euler characteristic $\chi^G(\underline{E}G) \in U(\underline{\mathtt{Or}}(G))$ agrees with the functorial Euler characteristic $\chi_f(\underline{\mathtt{Or}}(G);\mathbb{Z})$ and the functorial $L^{(2)}$-Euler characteristic $\chi_f^{(2)}(\underline{\mathtt{Or}}(G))$. Moreover, its finiteness obstruction $o(\underline{\mathtt{Or}}(G);R)$ is the image of $\chi_f(\underline{\mathtt{Or}}(G);\mathbb{Z})$ under the composite*

$$U(\underline{\mathtt{Or}}(G)) \xrightarrow{\iota} K_0(\mathbb{Z}\underline{\mathtt{Or}}(G)) \xrightarrow{c} K_0(R\underline{\mathtt{Or}}(G))$$

where ι has been defined in (4.8) *and c is the obvious change of coefficients homomorphism.*

Proof. (i) The cellular $\mathbb{Z}\underline{\mathtt{Or}}(G)$-chain complex $C_*(X)$ of a proper G-CW-complex X sends G/H to the cellular chain complex of the CW-complex $\operatorname{map}_G(G/H, X) = X^H$. It is always free, and it is finite free if and only if X is finite (see [15, Section 18A].

Since $\underline{E}G^H$ is contractible, the cellular $\mathbb{Z}\underline{\mathtt{Or}}(G)$-chain complex $C_*(\underline{E}G)$ is a free and hence projective resolution of the constant $\mathbb{Z}\underline{\mathtt{Or}}(G)$-module $\underline{R}$.

(ii) This follows from [15, Proposition 11.11 on page 222].

(iii) This follows from [22, Theorem 0.1].

(iv) This follows now from the definitions.

(v) This follows for $\chi_f(\underline{\mathsf{Or}}(G);\mathbb{Z})$ from the definitions. For $\chi_f^{(2)}(\underline{\mathsf{Or}}(G))$ apply Lemma 5.23 (iii). $\square$

Remark 8.6. The classifying spaces for proper G-actions $\underline{E}G$ play a prominent role in the Baum-Connes Conjecture (see [6, Conjecture 3.15 on page 254]) and they have been intensively studied in their own right.

Given a group G, there are often nice geometric models for $\underline{E}G$ which are finite. If there is a finitely dominated model for BG, then G must be torsion-free. This is not the case for $\underline{E}G$.

Example 8.7 (Groups with finite $\underline{E}G$)**.** If G is a hyperbolic group in the sense of Gromov, then its Rips complex (for an appropriate parameter) is a finite model for $\underline{E}G$ (see [24]).

If the group G acts simplicially cocompactly and properly by isometries on a CAT(0)-space X, i.e., a complete Riemannian manifold with non-positive sectional curvature or a tree, then X is a finite G-CW-model for $\underline{E}G$. This follows from [8, Corollary II.2.8 on page 179].

Further groups admitting finite models for $\underline{E}G$ are mapping class groups, the group of outer automorphisms of a finitely generated free group, finitely generated one-relator groups, and cocompact lattices in connected Lie groups.

8.2. **The Möbius inversion for the proper orbit category.** Next we take a closer look at Theorem 6.34 in the case of $\Gamma = \underline{\mathsf{Or}}(G)$ for a group G with a finite model for $\underline{E}G$.

Given an object G/H, we obtain by Lemma 6.8 an isomorphism of groups

$$W_GH := N_GH/H \xrightarrow{\cong} \operatorname{aut}(G/H) \tag{8.8}$$

by sending the class $gH \in N_GH/H$ to the G-automorphism $G/H \to G/H, g'H \mapsto g'g^{-1}H$.

We obtain a bijection

$$\{(H) \mid H \subseteq G, |H| < \infty\} \xrightarrow{\cong} \operatorname{iso}(\underline{\mathsf{Or}}(G)), \quad (H) \mapsto \overline{G/H} \tag{8.9}$$

where (H) denotes the conjugacy class of the subgroup H. Define a partial ordering on $\{(H) \mid H \subseteq G, |H| < \infty\}$ by

$$(H) \leq (K) \quad \Leftrightarrow \quad H \text{ is conjugate to a subgroup of } K \tag{8.10}$$

Then the bijection (8.9) is compatible with the partial orderings of (6.4) and (8.10).

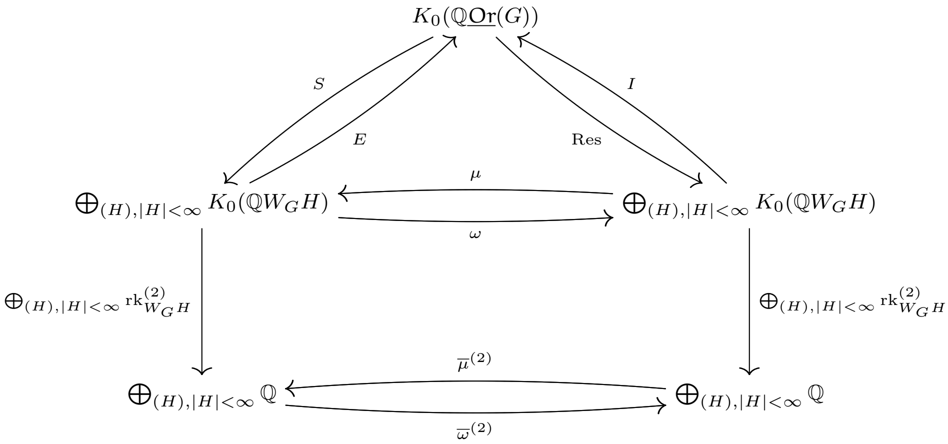
Given two elements $\overline{G/H}, \overline{G/K} \in \operatorname{iso}(\underline{\mathsf{Or}}(G))$, an l-chain $c \in \operatorname{ch}_l(\overline{G/K}, \overline{G/H})$ in the sense of Definition 6.20 is, under the bijection (8.9), the same as a sequence of conjugacy classes of subgroups $(H_0) < (H_1) < \ldots < (H_l)$ with $(H_0) = (K)$

and $(H_l) = (H)$. The $\operatorname{aut}(G/H)$-$\operatorname{aut}(G/K)$-biset $S(c)$ becomes under this identification and the identification (8.8) the W_GH-W_GK-biset

$$\begin{aligned} S(c) &= \operatorname{map}_G(G/H_{l-1}, G/H) \times_{W_GH_{l-1}} \operatorname{map}_G(G/H_{l-2}, G/H_{l-1}) \times_{W_GH_{l-2}} \\ &\qquad \ldots \times_{W_GH_1} \operatorname{map}_G(G/K, G/H_1) \\ &= (G/H)^{H_{l-1}} \times_{W_GH_{l-1}} (G/H_{l-1})^{H_{l-2}} \times_{W_GH_{l-2}} \ldots \times_{W_GH_1} (G/H_1)^K \end{aligned}$$

where we can arrange $K \subsetneq H_1 \subsetneq H_2 \subsetneq \ldots \subsetneq H_{l-1} \subsetneq H$.

The commutative diagram appearing in Theorem 6.34 becomes the following diagram



where $\operatorname{rk}^{(2)}_{W_GH} \colon K_0(\mathbb{Q}W_GH) \to \mathbb{Q}$ sends $[P]$ to $\dim_{\mathcal{N}(W_GH)}\bigl(P \otimes_{\mathbb{Q}W_GH} \mathcal{N}(W_GH)\bigr)$, the map ω is given by the collection of homomorphisms

$$\omega_{(H),(K)} \colon K_0(\mathbb{Q}W_GH) \to K_0(\mathbb{Q}W_GK), \quad [P] \mapsto \bigl[P \otimes_{\mathbb{Q}W_GH} \mathbb{Q}\operatorname{map}_G(G/K, G/H)\bigr],$$

the map μ is given by the collection of homomorphisms

$$\begin{aligned} &\mu_{(H),(K)} \colon K_0(\mathbb{Q}W_GH) \to K_0(\mathbb{Q}W_GK), \\ &\qquad\qquad [P] \mapsto \sum_{l \geq 0} (-1)^l \cdot \sum_{c \in \operatorname{ch}_l((K),(H))} [P \otimes_{\mathbb{Q}W_GH} \mathbb{Q}S(c)], \end{aligned}$$

the map $\overline{\omega}^{(2)}$ is given by the matrix $\left(\overline{\omega}^{(2)}_{(H),(K)}\right)$ over $\mathbb{Q}$, where

$$\overline{\omega}^{(2)}_{(H),(K)} = \sum_{i=1}^{r} \frac{1}{|L_i|}$$

if the right W_GK-set $\operatorname{map}_G(G/K, G/H) = G/H^K$ is the disjoint union $\sum_{i=1}^{r} L_i \backslash W_GK$, and the map $\overline{\mu}^{(2)}$ is given by the matrix $\left(\overline{\mu}^{(2)}_{(H),(K)}\right)$ over $\mathbb{Q}$, where

$$\overline{\mu}^{(2)}_{(H),(K)} = \sum_{l \geq 0} (-1)^l \cdot \sum_{c \in \operatorname{ch}_l((K),(H))} \sum_{i=1}^{r} \frac{1}{|L_i(c)|}$$

if the right set W_GK-set

$$S(c) = (G/K)^{H_{l-1}} \times_{W_GH_{l-1}} (G/H_{l-1})^{H_{l-2}} \times_{W_GH_{l-2}} \dots \times_{W_GH_1} (G/H_1)^H$$

is the disjoint union $\sum_{i=1}^r L_i(c)\backslash W_GK$.

8.3. Additivity of the finiteness obstruction and the Euler characteristic for the proper orbit category.

Theorem 8.11 (Additivity of the finiteness obstruction and the Euler characteristic for the proper orbit category)**.** *Consider two groups G_1 and G_2 with a common subgroup G_0. Let G be the amalgamated product $G = G_1 *_{G_0} G_2$. Then:*

(i) *We obtain a G-pushout of G-CW-complexes*

$$\begin{array}{ccc} G \times_{G_0} \underline{E}G_0 & \xrightarrow{j_1} & G \times_{G_1} \underline{E}G_1 \\ {\scriptstyle j_2}\downarrow & & \downarrow \\ G \times_{G_2} \underline{E}G_2 & \longrightarrow & \underline{E}G \end{array}$$

where j_1 and j_2 are inclusions of G-CW-complexes;

(ii) *If $\underline{\mathrm{Or}}(G_k)$ is of type (FP) for $k = 0, 1, 2$, then $\underline{\mathrm{Or}}(G)$ is of type (FP) and we get for the finiteness obstruction*

$$o(\underline{\mathrm{Or}}(G); R) = (i_1)_*\big(o(\underline{\mathrm{Or}}(G_1); R)\big) + (i_2)_*\big(o(\underline{\mathrm{Or}}(G_2); R)\big) - (i_0)_*\big(o(\underline{\mathrm{Or}}(G_0); R)\big) \quad \in K_0(R\underline{\mathrm{Or}}(G)),$$

where $(i_k)_*\colon K_0(R\underline{\mathrm{Or}}(G_k)) \to K_0(R\underline{\mathrm{Or}}(G))$ is the homomorphism induced by the functor $(i_k)_*\colon \underline{\mathrm{Or}}(G_k) \to \underline{\mathrm{Or}}(G)$ coming from induction associated to the inclusion $i_k\colon G_k \to G$ for $k = 0, 1, 2$;

(iii) *If $\underline{\mathrm{Or}}(G_k)$ is of type (FP) for $k = 0, 1, 2$, then $\underline{\mathrm{Or}}(G)$ is of type (FP) and we get for the functorial Euler characteristic*

$$\chi_f(\underline{\mathrm{Or}}(G)) = (i_1)_*\big(\chi_f(\underline{\mathrm{Or}}(G_1))\big) + (i_2)_*\big(\chi_f(\underline{\mathrm{Or}}(G_2))\big) - (i_0)_*\big(\chi_f(\underline{\mathrm{Or}}(G_0))\big) \quad \in U(\underline{\mathrm{Or}}(G)),$$

where $(i_k)_*\colon U(\underline{\mathrm{Or}}(G_i)) \to U(\underline{\mathrm{Or}}(G))$ is the homomorphism induced by the functor $(i_k)_*\colon \underline{\mathrm{Or}}(G_k) \to \underline{\mathrm{Or}}(G)$ coming from induction associated to the inclusion $i_k\colon G_k \to G$ for $k = 0, 1, 2$, and we get for the Euler characteristic

$$\chi(\underline{\mathrm{Or}}(G)) = \chi(\underline{\mathrm{Or}}(G_1)) + \chi(\underline{\mathrm{Or}}(G_2)) - \chi(\underline{\mathrm{Or}}(G_0)) \quad \in \mathbb{Z}.$$

(iv) *If $\underline{\mathrm{Or}}(G_k)$ is of type (L^2) for $k = 0, 1, 2$, then $\underline{\mathrm{Or}}(G)$ is of type (L^2) and we get for the functorial L^2-Euler characteristic*

$$\chi_f^{(2)}(\underline{\mathrm{Or}}(G)) = (i_1)_*\big(\chi_f^{(2)}(\underline{\mathrm{Or}}(G_1))\big) + (i_2)_*\big(\chi_f^{(2)}(\underline{\mathrm{Or}}(G_2))\big) - (i_0)_*\big(\chi_f^{(2)}(\underline{\mathrm{Or}}(G_0))\big) \quad \in U^{(1)}(\underline{\mathrm{Or}}(G)),$$

where $(i_k)_*\colon U^{(1)}(\underline{\mathtt{Or}}(G_k)) \to U^{(1)}(\underline{\mathtt{Or}}(G))$ *is the homomorphism induced by the functor* $(i_k)_*\colon \underline{\mathtt{Or}}(G_k) \to \underline{\mathtt{Or}}(G)$ *coming from induction associated to the inclusion* $i_k\colon G_k \to G$ *for* $k = 0, 1, 2$*, and we get for the* L^2*-Euler characteristic*

$$\chi^{(2)}(\underline{\mathtt{Or}}(G)) = \chi^{(2)}(\underline{\mathtt{Or}}(G_1)) + \chi^{(2)}(\underline{\mathtt{Or}}(G_2)) - \chi^{(2)}(\underline{\mathtt{Or}}(G_0)) \quad \in \mathbb{R}.$$

Proof. (i) Associated to $G = G_1 *_{G_0} G_2$ there is a 1-dimensional contractible G-CW-complex T which is obtained as a G-pushout

$$\begin{array}{ccc} G/G_0 \times S^0 & \xrightarrow{\mathrm{pr}_1 \amalg \mathrm{pr}_2} & G/G_1 \amalg G/G_2 \\ \downarrow & & \downarrow \\ G/G_0 \times D^1 & \longrightarrow & T \end{array}$$

where $\mathrm{pr}_k\colon G/G_0 \to G/G_k$ is the projection (see [26, Theorem 7 in I.4 on page 32]).

Since the H-fixed point set T^H is a non-empty subtree [26, Proposition 19 in I.4 on p. 36], thus contractible, for every finite subgroup $H \subseteq G$, the product with the diagonal G-action $T \times \underline{E}G$ is again a model for $\underline{E}G$. Note that $\mathrm{res}_G^{G_k} \underline{E}G$ is a model for $\underline{E}G_k$ and

$$G/G_k \times \underline{E}G \xrightarrow{\cong_G} G \times_{G_k} \mathrm{res}_G^{G_k} \underline{E}G, \ (gG_k, x) \mapsto (g, g^{-1}x)$$

is a G-equivariant homeomorphism. Combining everything, we obtain the following G-pushout by crossing the G-pushout for T above with $\underline{E}G$

$$\begin{array}{ccc} G \times_{G_0} \underline{E}G_0 \times S^0 & \longrightarrow & G \times_{G_1} \underline{E}G_1 \amalg G \times_{G_2} \underline{E}G_2 \\ \downarrow & & \downarrow \\ G \times_{G_0} \underline{E}G_0 \times D^1 & \longrightarrow & \underline{E}G. \end{array}$$

We can write the preceding G-pushout equivalently as

$$\begin{array}{ccc} G \times_{G_0} \underline{E}G_0 \times D^1 & \xrightarrow{j_1} & G \times_{G_1} \underline{E}G_1 \\ {\scriptstyle j_2}\downarrow & & \downarrow \\ G \times_{G_2} \underline{E}G_2 & \longrightarrow & \underline{E}G \end{array}$$

where j_1 and j_2 are inclusions of G-CW-complexes. Furthermore, $\underline{E}G_0 \times D^1$ is just another model of $\underline{E}G_0$.

(ii) For $k = 0, 1, 2$ we get

$$\mathrm{ind}_{i_k} C_*(\underline{E}G_k) \cong C_*\big(G \times_{G_k} \underline{E}G_k\big)$$

where $C_*(\underline{E}G_k)$ is the cellular $\mathbb{Z}\underline{\mathtt{Or}}(G_k)$-chain complex of the G_k-CW-complex $\underline{E}G_k$, $C_*\big(G \times_{G_k} \underline{E}G_k\big)$ is the cellular $\mathbb{Z}\underline{\mathtt{Or}}(G)$-chain complex of the G-CW-complex

$G \times_{G_k} \underline{E}G_k$. From the G-pushout of assertion (i) we obtain a short exact sequence of $\mathbb{Z}\underline{\mathtt{Or}}(G)$-chain complexes

$$0 \to \operatorname{ind}_{i_0} C_*(\underline{E}G_0) \to \operatorname{ind}_{i_1} C_*(\underline{E}G_1) \oplus \operatorname{ind}_{i_2} C_*(\underline{E}G_2) \to C_*(\underline{E}G) \to 0.$$

Now apply [15, Theorem 11.2 on page 212], Theorem 4.15, and Theorem 8.5.

(iii) This follows from the definition of $\chi_f(\underline{\mathtt{Or}}(G))$ since $\operatorname{rk}_{R\Gamma} \colon K_0(R\underline{\mathtt{Or}}(G)) \to U(\underline{\mathtt{Or}}(G))$ is compatible with induction homomorphisms induced from group homomorphisms.

(iv) We obtain for any object G/H in $\underline{\mathtt{Or}}(G)$ a short exact sequence of $\mathbb{Z}\underline{\mathtt{Or}}(G)$-chain complexes

$$0 \to S_{G/H}\big(\operatorname{ind}_{i_0} C_*(\underline{E}G_0)\big) \to S_{G/H}\big(\operatorname{ind}_{i_1} C_*(\underline{E}G_1)\big) \oplus S_{G/H}\big(\operatorname{ind}_{i_2} C_*(\underline{E}G_2)\big) \to S_{G/H}\big(C_*(\underline{E}G)\big) \to 0.$$

For every finite subgroup $H \subset G_k$ and $k = 0, 1, 2$ the inclusion $G_k \to G$ induces an injection $W_{G_k}H \to W_G H$. The splitting functor is compatible with induction. Now apply Theorem 5.7. □

8.4. **The Burnside integrality relations and the classical Burnside congruences.** Let G be a group and let X be a finite proper G-CW-complex. We have defined its equivariant Euler characteristic $\chi^G(X) \in U(\underline{\mathtt{Or}}(G))$ in Definition 8.4. The map

$$\overline{\omega}^{(2)} \colon \bigoplus_{(H),|H|<\infty} \mathbb{Q} \to \bigoplus_{(H),|H|<\infty} \mathbb{Q}$$

defined in Subsection 8.2 sends

$$\chi^G(X) \in U(\underline{\mathtt{Or}}(G)) \subseteq U(\underline{\mathtt{Or}}(G)) \otimes_{\mathbb{Z}} \mathbb{Q} = \bigoplus_{(H),|H|<\infty} \mathbb{Q}$$

to the collection $\big(\chi^{(2)}(X^H; \mathcal{N}(W_G H))\big)_{(H),|H|<\infty}$ of the L^2-Euler characteristics of the $\mathcal{N}(W_G H)$-chain complexes $C_*(X^H) \otimes_{\mathbb{Z}W_G H} \mathcal{N}(W_G H)$. If $X = \underline{E}G$, then $\chi^{(2)}(X^H; \mathcal{N}(W_G H)) = \chi^{(2)}(W_G H)$. Notice that we get for the map

$$\overline{\mu}^{(2)} \colon U(\underline{\mathtt{Or}}(G)) \otimes_{\mathbb{Z}} \mathbb{Q} \to U(\underline{\mathtt{Or}}(G)) \otimes_{\mathbb{Z}} \mathbb{Q}$$

defined in Subsection 8.2

$$\overline{\mu}^{(2)}\Big(\big(\chi^{(2)}(X^H; \mathcal{N}(W_G H))\big)_{(H),|H|<\infty}\Big) = \chi^G(X).$$

This implies

Lemma 8.12. *Consider $\eta = \big(\eta_{(H)}\big)_{(H),|H|<\infty} \in \prod_{(H),|H|<\infty} \mathbb{R}$. Then there is a finite proper G-CW-complex X with $\chi^{(2)}(X^H; \mathcal{N}(W_G H)) = \eta_{(H)}$ for every finite subgroup $H \subseteq G$ if and only if $\eta \in U(\underline{\mathtt{Or}}(G)) \otimes_{\mathbb{Z}} \mathbb{Q} = \bigoplus_{(H),|H|<\infty} \mathbb{Q}$ and $\overline{\mu}^{(2)}(\eta)$ lies in $U(\underline{\mathtt{Or}}(G))$.*

Lemma 8.13. *Let G be a group such that $\underline{\mathtt{Or}}(G)$ is of type (FP).*

(i) *If* $\underline{\mathrm{Or}}(G)$ *satisfies condition (I), then*

$$\chi_f^{(2)}(\underline{\mathrm{Or}}(G)) = \overline{\mu}^{(2)}\Big(\big(\chi^{(2)}(W_GH)\big)_{(H),|H|<\infty}\Big);$$

(ii) *If there is a finite model for* $\underline{E}G$*, then the following integrality condition is satisfied*

$$\overline{\mu}^{(2)}\Big(\big(\chi^{(2)}(W_GH)\big)_{(H),|H|<\infty}\Big) \in U(\underline{\mathrm{Or}}(G)).$$

Proof. This follows from Theorem 6.35, Theorem 8.5, and Lemma 8.12. □

Example 8.14 (Burnside congruence)**.** These considerations are already interesting in the case of a finite group G. For every finite G-CW-complex X, the map

$$\overline{\omega}^{(2)}\colon \bigoplus_{(H),|H|<\infty} \mathbb{Q} \to \bigoplus_{(H),|H|<\infty} \mathbb{Q}$$

sends the equivariant Euler characteristic $\chi^G(X)$ to the collection $\big(\chi(X^H)/|W_GH|\big)_{(H)}$, where $\chi(X^H)$ is the classical Euler characteristic of the H-fixed point set. We conclude from Lemma 8.12 that for an element $\eta = (\eta_{(H)})_{(H)} \in \bigoplus_{(H),|H|<\infty} \mathbb{Q}$ there exists a finite G-CW-complex X such that $\chi(X^H)/|W_GH| = \chi^{(2)}(X^H;\mathcal{N}(W_GH))$ agrees with $\eta_{(H)}$ for any subgroup $H \subseteq G$, if and only if $\overline{\mu}^{(2)}(\eta) \in U(\underline{\mathrm{Or}}(G))$. The latter is a kind of integrality condition. In the case of a finite group G it can be transformed into equivalent congruence conditions for integers.

Let

$$\mathrm{ch} = \mathrm{ch}^G\colon U(\underline{\mathrm{Or}}(G)) \to \bigoplus_{(H)} \mathbb{Z}$$

be the map uniquely determined by the property that it sends $\chi^G(X)$ to the collection $\big(\chi(X^H))\big)_{(H)}$ for every finite G-CW-complex X. Under the obvious identification of $U(\underline{\mathrm{Or}}(G))$ with the Burnside ring $A(G)$ the map ch corresponds to the character map which sends a finite G-set S to the collection $\big(|S^H|\big)_{(H)}$. We have

$$i \circ \mathrm{ch} = D \circ \overline{\omega}^{(2)} \circ i,$$

if $i\colon U(G) \to U(G) \otimes_{\mathbb{Z}} \mathbb{Q}$ is the obvious inclusion and the map $D\colon U(G) \otimes_{\mathbb{Z}} \mathbb{Q} \to U(G) \otimes_{\mathbb{Z}} \mathbb{Q}$ is given by the diagonal matrix whose entry at (H) is $|W_GH|$. Let

$$\nu\colon \bigoplus_{(H)} \mathbb{Z} \to \bigoplus_{(H)} \mathbb{Z}$$

be the map uniquely determined by $i \circ \nu = D \circ \overline{\mu}^{(2)} \circ D^{-1} \circ i$. One easily checks that it is given by the integer matrix whose entry at $\big((H), (K)\big)$ is

$$\sum_{l \geq 0} (-1)^l \cdot \sum_{(H_0) < \cdots < (H_l) \in \operatorname{ch}_l\big((K),(H)\big)} \prod_{i=1}^{l} \big|W_G H_{i+1} \backslash \operatorname{map}_G(G/H_i, G/H_{i+1})\big|.$$

Notice that $i \circ \nu \circ \chi = D \circ i$. We conclude that an element $\xi \in \bigoplus_{(H)} \mathbb{Z}$ lies in the image of ch if and only if for every conjugacy class (H) of subgroups the following congruence of integers holds

$$\nu(\xi)_{(H)} \equiv 0 \mod |W_G H|.$$

These are the *Burnside ring congruences*. For more information about the Burnside ring we refer for instance to [28, Chapter 1].

If G is the cyclic group $\mathbb{Z}/p$ of order p for a prime p, then $U(\underline{\mathsf{Or}}(\mathbb{Z}/p)) = \mathbb{Z}^2$,

$$\operatorname{ch} = \begin{pmatrix} p & 1 \\ 0 & 1 \end{pmatrix} : U(\underline{\mathsf{Or}}(\mathbb{Z}/p)) = \mathbb{Z}^2 \to U(\underline{\mathsf{Or}}(\mathbb{Z}/p)) = \mathbb{Z}^2$$

and

$$\nu = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} : U(\underline{\mathsf{Or}}(\mathbb{Z}/p)) = \mathbb{Z}^2 \to U(\underline{\mathsf{Or}}(\mathbb{Z}/p)) = \mathbb{Z}^2.$$

The Burnside ring congruences reduce to one congruence, namely

$$\eta_{(\mathbb{Z}/p)/\{1\}} - \eta_{(\mathbb{Z}/p)/(\mathbb{Z}/p)} \equiv 0 \mod p.$$

The latter reflects the fact that the cardinality of $S - S^{\mathbb{Z}/p}$ is a multiple of p for a finite $\mathbb{Z}/p$-set S.

Example 8.15 (Amenable G)**.** Let G be an amenable group. Suppose that $\underline{\mathsf{Or}}(G)$ is of type (FP). Then $\chi^{(2)}(\underline{\mathsf{Or}}(G))$ is the image of $\eta = \big(\eta_{(H)}\big)_{(H),|H|<\infty}$ under

$$\overline{\mu}^{(2)} : U(\underline{\mathsf{Or}}(G)) \otimes_{\mathbb{Z}} \mathbb{Q} \to U(\underline{\mathsf{Or}}(G)) \otimes_{\mathbb{Z}} \mathbb{Q}$$

where $\eta_{(H)} = 0$ if $W_G H$ is infinite and $\eta_{(H)} = 1/|W_G H|$ if $W_G H$ is finite.

In particular, if $W_G H$ is infinite for every finite subgroup $H \subseteq G$, then $\chi^{(2)}(\underline{\mathsf{Or}}(G))$ vanishes.

This follows from Theorem 6.35, Lemma 8.13, and the result of Cheeger and Gromov that all the L^2-Betti numbers of any infinite amenable group G vanish (see [10] and [19, Theorem 7.2 on page 294]).

8.5. **The infinite dihedral group.** Consider the infinite dihedral group

$$D_\infty = \langle t, s \mid s^2 = 1, sts = t^{-1} \rangle \cong \mathbb{Z} \rtimes \mathbb{Z}/2 \cong \mathbb{Z}/2 * \mathbb{Z}/2.$$

As an illustration we want to make all the material of this section explicit for this easy special case.

The infinite dihedral group D_∞ has three conjugacy classes of finite subgroups (C_1), (C_2), and (T), where $C_1 = \langle s \rangle$ and $C_2 = \langle ts \rangle$ have order two and T is the trivial group.

One easily checks that $W_{D_\infty} C_i$ is trivial for $i = 1, 2$ and $W_{D_\infty} T = D_\infty$. Hence we get

$$\operatorname{Split} K_0(\mathbb{Q}\underline{\mathsf{Or}}(D_\infty)) = K_0(\mathbb{Q}D_\infty) \oplus K_0(\mathbb{Q}) \oplus K_0(\mathbb{Q}) = K_0(\mathbb{Q}D_\infty) \oplus \mathbb{Z} \oplus \mathbb{Z}$$

by the discussion in Subsection 8.2.

The $W_{D_\infty} C_i$-$W_{D_\infty} T$-biset $\operatorname{map}_{D_\infty}(D_\infty/T, D_\infty/C_i)$ is given by the right D_∞-set $C_i \backslash D_\infty$ for $i = 1, 2$. The $W_{D_\infty} T$-$W_{D_\infty} T$-biset $\operatorname{map}_{D_\infty}(D_\infty/T, D_\infty/T)$ is D_∞ regarded as D_∞ -D_∞ biset. The $W_{D_\infty} C_j$-$W_{D_\infty} C_i$-biset $\operatorname{map}_{D_\infty}(D_\infty/C_i, D_\infty/C_j)$ is empty for $i \neq j$ and is the $\{1\}$-$\{1\}$-biset consisting of one point for $i = j$. The $W_{D_\infty} T$-$W_{D_\infty} C_i$-biset $\operatorname{map}_{D_\infty}(D_\infty/C_i, D_\infty/T)$ is empty for $i = 1, 2$. There are exactly two 1-chains in $\underline{\mathsf{Or}}(D_\infty)$, namely $(T) < (C_1)$ and $(T) < (C_2)$.

Hence we get

$$\begin{aligned} \omega \colon K_0(\mathbb{Q}D_\infty) \oplus \mathbb{Z} \oplus \mathbb{Z} &\to K_0(\mathbb{Q}D_\infty) \oplus \mathbb{Z} \oplus \mathbb{Z}, \\ (x, n_1, n_2) &\mapsto \bigl(x + n_1 \cdot [\mathbb{Q}C_1 \backslash D_\infty] + n_2 \cdot [\mathbb{Q}C_2 \backslash D_\infty], n_1, n_2\bigr), \end{aligned}$$

$$\begin{aligned} \mu \colon K_0(\mathbb{Q}D_\infty) \oplus \mathbb{Z} \oplus \mathbb{Z} &\to K_0(\mathbb{Q}D_\infty) \oplus \mathbb{Z} \oplus \mathbb{Z}, \\ (x, n_1, n_2) &\mapsto \bigl(x - n_1 \cdot [\mathbb{Q}C_1 \backslash D_\infty] - n_2 \cdot [\mathbb{Q}C_2 \backslash D_\infty], n_1, n_2\bigr), \end{aligned}$$

$$\overline{\omega}^{(2)} = \begin{pmatrix} 1 & 1/2 & 1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \colon \mathbb{Z}^3 \to \mathbb{Z}^3, \quad (n_0, n_1, n_2) \mapsto (n_0 + n_1/2 + n_2/2, n_1, n_2),$$

and

$$\overline{\mu}^{(2)} = \begin{pmatrix} 1 & -1/2 & -1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \colon \mathbb{Z}^3 \to \mathbb{Z}^3, \quad (n_0, n_1, n_2) \mapsto (n_0 - n_1/2 - n_2/2, n_1, n_2).$$

The map

$$\operatorname{rk}^{(2)}_{\underline{\mathsf{Or}}(D_\infty)} \colon K_0(\mathbb{Q}D_\infty) \oplus \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$$

sends $\bigl([P], n_1, n_2\bigr)$ to $\bigl(\dim_{\mathcal{N}(D_\infty)}(P \otimes_{\mathbb{Q}D_\infty} \mathcal{N}(D_\infty)), n_1, n_2\bigr)$.

There is the isomorphism

$$\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\cong} K_0(\mathbb{Q}D_\infty), \quad \bigl(n_0, n_1, n_2\bigr) \mapsto n_0 \cdot [\mathbb{Q}D_\infty] + n_1 \cdot [\mathbb{Q}C_1 \backslash D_\infty] + n_2 \cdot [\mathbb{Q}C_2 \backslash D_\infty]$$

(see for example the Mayer-Vietoris sequence for amalgated products in [30, Corollary 2.15 on page 216] and the subsequent remarks there). Under this identification

$$\operatorname{rk}^{(2)}_{\underline{\mathsf{Or}}(D_\infty)} = \begin{pmatrix} 1 & 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \colon \mathbb{Z}^5 \to \mathbb{Z}^3,$$

$$\omega = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} : \mathbb{Z}^5 \to \mathbb{Z}^5,$$

and

$$\mu = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} : \mathbb{Z}^5 \to \mathbb{Z}^5.$$

The infinite dihedral group $D_\infty = \mathbb{Z} \rtimes \mathbb{Z}/2$ acts on $\mathbb{R}$ by the action of $\mathbb{Z}$ on $\mathbb{R}$ given by addition and the action of $\mathbb{Z}/2$ in $\mathbb{R}$ given by multiplication with (-1). There is a D_∞-CW-structure on $\mathbb{R}$ such that there are three equivariant cells of the type $D_\infty/C_1 \times D^0$, $D_\infty/C_2 \times D^0$, and $D_\infty/T \times D^1$. One easily checks that this is a model for $\underline{E}D_\infty$. Hence we get for the equivariant Euler characteristic of $\underline{E}D_\infty$

$$\chi^{D_\infty}(\underline{E}D_\infty) = \overline{D_\infty/C_1} + \overline{D_\infty/C_2} - \overline{D_\infty/T} \in U(\underline{\mathtt{Or}}(D_\infty)).$$

By Theorem 6.22 (ii) and Theorem 8.5 (v) the image of the finiteness obstruction $o(\underline{\mathtt{Or}}(D_\infty))$ under the isomorphism

$$S\colon K_0(\mathbb{Q}\underline{\mathtt{Or}}(D_\infty)) \xrightarrow{\cong} \operatorname{Split} K_0(\mathbb{Q}\underline{\mathtt{Or}}(D_\infty)) = K_0(\mathbb{Q}D_\infty) \oplus \mathbb{Z} \oplus \mathbb{Z} = \mathbb{Z}^5$$

is $(-1,0,0,1,1)$. The image of this element under ω is $(-1,1,1,1,1)$. All this is consistent with Theorem 8.11 applied to $D_\infty = \mathbb{Z}/2 * \mathbb{Z}/2$.

The trivial $\mathbb{Q}D_\infty$-module $\mathbb{Q}$ has a finite projective $\mathbb{Q}D_\infty$-resolution of the form $0 \to \mathbb{Q}D_\infty \to \mathbb{Q}D_\infty/C_1 \oplus \mathbb{Q}D_\infty/C_1 \to \mathbb{Q}$ coming from the $\mathbb{Q}D_\infty$-chain complex of $\mathbb{R}$. This implies that the homomorphism

$$\operatorname{Res}\colon K_0(\mathbb{Q}\underline{\mathtt{Or}}(D_\infty)) \xrightarrow{\cong} \operatorname{Split} K_0(\mathbb{Q}\underline{\mathtt{Or}}(D_\infty)) = K_0(\mathbb{Q}D_\infty) \oplus \mathbb{Z} \oplus \mathbb{Z} = \mathbb{Z}^5$$

sends $o(\underline{\mathtt{Or}}(D_\infty);\mathbb{Q})$ to $(-1,1,1,1,1)$ (see Theorem 6.35 (i)). This is consistent with the fact that ω sends the image of the finiteness obstruction $o(\underline{\mathtt{Or}}(D_\infty))$ under S, which is given by $(-1,0,0,1,1) \in \mathbb{Z}^5$, to the element $(-1,1,1,1,1) \in \mathbb{Z}^5$ (see Theorem 6.22).

We have $\chi_f^{(2)}(\underline{\mathtt{Or}}(D_\infty);\mathbb{Q}) = (-1,1,1) \in U(\underline{\mathtt{Or}}(D_\infty)) = \mathbb{Z}^3$. The composite

$$\operatorname{rk}^{(2)}_{\underline{\mathtt{Or}}(D_\infty)} \circ \operatorname{Res}\colon K_0(\mathbb{Q}\underline{\mathtt{Or}}(D_\infty)) \to U(\underline{\mathtt{Or}}(D_\infty)) = \mathbb{Z}^3$$

sends $o(\underline{\mathtt{Or}}(D_\infty);\mathbb{Q})$ to $\big(\chi^{(2)}(D_\infty), \chi^{(2)}(\{1\}), \chi^{(2)}(\{1\})\big)$. Since the L^2-Euler characteristic of an infinite amenable group vanishes (see [10]) and the L^2-Euler characteristic of the trivial group is 1, we get $\big(\chi^{(2)}(D_\infty), \chi^{(2)}(\{1\}), \chi^{(2)}(\{1\})\big) =$

$(0,1,1)$. This is consistent with the fact that $\overline{\omega}^{(2)}$ sends $(-1,1,1)$ to $(0,1,1)$ and with Example 8.15.

8.6. **The fundamental category.** Let X be a G-space. Consider the functor

$$F\colon \mathsf{Or}(G) \to \mathsf{GROUPOIDS}, \quad G/H \mapsto \Pi\big(\mathrm{map}_G(G/H,X)\big),$$

which sends G/H to the fundamental groupoid of $X^H = \mathrm{map}_G(G/H,X)$. Its homotopy colimit is by definition the *fundamental groupoid* $\Pi(G,X)$ which plays an important role in transformation groups (see [15, Definition 8.13 on page 144]).

Denote by $\underline{\Pi}(G,X)$ the homotopy colimit of the functor F above restricted to $\underline{\mathsf{Or}}(G)$. If all isotropy groups of X are finite, then $\Pi(G,X)$ and $\underline{\Pi}(G,X)$ agree.

Suppose that there is a finite G-CW-model for $\underline{E}G$. Let I_n be the set of equivariant n-cells $c = G/H_c \times (D^n - S^{n-1})$. Consider a G-CW-complex X. Suppose that for every finite subgroup $H \subseteq G$ each groupoid $\Pi(X^H)$ is of type $(\mathrm{FP}_\mathbb{Q})$. This is equivalent to requiring that for every finite subgroup $H \subseteq G$ the set $\pi_0(X^H)$ is finite and at each base point $x \in X^H$ the fundamental group $\pi_1(X^H,x)$ is of type $(\mathrm{FP}_\mathbb{Q})$. This follows from [9, Exercise 8 in VIII.6 on page 205] using the facts that W_GH is of type $(\mathrm{FP}_\mathbb{Q})$ because $\underline{\mathsf{Or}}(G)$ is of type $(\mathrm{FP}_\mathbb{Q})$ (see Theorem 8.5 (i) and Lemma 6.15 (i)) and for every object $x\colon G/H \to X$ in $\underline{\Pi}(G,X)$ there exists an exact sequence

$$1 \to \pi_1(X^H,x) \to \mathrm{aut}(x\colon G/H \to X) \to W_GH(x) \to 1 \tag{8.16}$$

for the subgroup $W_GH(x) \subseteq W_GH$ of finite index which is the isotropy group of the component in X^H determined by x under the W_GH-action on $\pi_0(X^H)$ (see [15, Proposition 8.33 on page 150]). Hence the homotopy colimit formula of [12] applies. For instance we get

$$\begin{aligned}\chi^{(2)}(\underline{\Pi}(G;X)) &= \sum_{n\geq 0}(-1)^n \cdot \sum_{c\in I_n}\ \sum_{C\in\pi_0(X^{H_c})/W_GH_c} \chi^{(2)}\big(\mathrm{aut}(x(C))\big);\\ \chi(\underline{\Pi}(G;X)) &= \sum_{n\geq 0}(-1)^n \cdot \sum_{c\in I_n}\ \sum_{C\in\pi_0(X^{H_c})/W_GH_c} \chi\big(B\,\mathrm{aut}(x(C));\mathbb{Q}\big),\end{aligned}$$

where for a component $C \in \pi_0(X^{H_c})$ we denote by $x(C)\colon G/H_c \to X$ an object in $\underline{\Pi}(G,X)$ such that $x(C)(eH_c)$ lies in the component C and $\mathrm{aut}(x(C))$ is its automorphism group in $\underline{\Pi}(G,X)$ which fits into the exact sequence (8.16).

If we take $X = \{\bullet\}$ itself, we get back Theorem 8.5 (v).

One can define for a functor $\mu\colon \mathsf{Or}(G) \to \mathsf{GROUPOIDS}$ its *equivariant Eilenberg Mac Lane space* $E(\mu,1)$ which is a G-CW-complex such that μ can be identified with the functor $\mathsf{Or}(G) \to \mathsf{GROUPOIDS}$ sending G/H to $\Pi(E(\mu,1)^H)$ and we have $\pi_n(E(\mu,1)^H,x)$ is trivial for all $n \geq 2$, $H \subseteq G$ and $x \in E(\mu,1)^H$ (see [14]). There is a natural equivalence $\mathrm{hocolim}_{\mathsf{Or}(G)}\,\mu \to \Pi(G;E(\mu,1))$ which induces an isomorphism

$$K_0\big(\mathbb{Z}\,\mathrm{hocolim}_{\mathsf{Or}(G)}\,\mu\big) \to K_0\big(\mathbb{Z}\Pi(G;E(\mu,1))\big).$$

Under this isomorphism the finiteness obstruction of $\operatorname{hocolim}_{\mathsf{Or}(G)} \mu$ in the sense of Definition 2.7 corresponds to the finiteness obstruction of $E(\mu, 1)$ in the sense of [15, Definition 14.4 on page 278].

9. An example of a finite category without property EI

For the remainder of this section we will consider the following category Γ. It has precisely two objects x and y. There is precisely one morphism $u\colon x \to y$ and precisely one morphism $v\colon y \to x$. There are precisely two endomorphisms of x, namely, $v \circ u$ and id_x. There are precisely two endomorphisms of y, namely, $u \circ v$ and id_x. We have $vuv = v$ and $uvu = u$. Obviously Γ is a free finite category. It has two idempotents which are not the identity, namely, vu and uv. It is directly finite but it is not Cauchy complete and not an EI-category. In this section we compute the homomorphisms S, E, Res for $K_0(R\Gamma)$ and determine the finiteness obstruction.

Given an R-module M, we define three $R\Gamma$-modules I_xM, I_yM, and I_cM as follows. The contravariant functor I_xM sends x to M and y to $\{0\}$ and every morphism except id_x to the zero homomorphism. The contravariant functor I_yM sends y to M and x to $\{0\}$ and every morphism except id_y in Γ to the zero homomorphism. The contravariant functor I_cM sends both x and y to M and every morphism in Γ to the identity id_M.

Lemma 9.1. *Let M be an $R\Gamma$-module. Then there is an isomorphism of $R\Gamma$-modules, natural in M*

$$f\colon I_x\big(\ker(M(vu))\big) \oplus I_y\big(\ker(M(uv))\big) \oplus I_c(\operatorname{im}(vu)) \xrightarrow{\cong} M.$$

Proof. The transformation f is given at the object x by the direct sum of the obvious inclusions

$$i_x \oplus j_x\colon \ \ker(M(vu)) \oplus \operatorname{im}(M(vu)) \xrightarrow{\cong} M(x).$$

This is an isomorphism since $M(vu)^2 = M((vu)^2) = M(vu)$. The transformation f is given at the object y by the direct sum of the inclusion i_y and the map induced by $M(v)$

$$i_y \oplus M(v)|_{\operatorname{im}(M(uv))}\colon \ \ker(M(uv)) \oplus \operatorname{im}(M(vu)) \xrightarrow{\cong} M(y).$$

This is an isomorphism of R-modules, an inverse is given by

$$(\mathrm{id} - M(uv)) \times M(u)\colon M(y) \to \ker(M(uv)) \oplus \operatorname{im}(M(vu)).$$

It remains to check that f is a transformation. We check this for the morphism v, the proof for u is analogous. We have to show that the following diagram is commutative

$$\begin{array}{ccc}
\ker(M(vu)) \oplus \operatorname{im}(M(vu)) & \xrightarrow{0 \oplus \mathrm{id}} & \ker(M(uv)) \oplus \operatorname{im}(M(vu)) \\
\downarrow{\scriptstyle i_x \oplus j_x} & & \downarrow{\scriptstyle i_y \oplus M(v)|_{\operatorname{im}(M(uv))}} \\
M(x) & \xrightarrow[M(v)]{} & M(y)
\end{array}$$

This is equivalent to showing that $M(v)|_{\ker(M(vu))} = 0$. This follows from $M(v) = M(vuv) = M(v) \circ M(vu)$. $\square$

Lemma 9.2. *Let M be an R-module.*

(i) *The functors* Res_x *and* Res_y *respectively from* $\mathsf{MOD}\text{-}R\Gamma$ *to* $\mathsf{MOD}\text{-}R$*, which are given by evaluation at x and y respectively, are exact and send finitely generated projective $R\Gamma$-modules to finitely generated projective R-modules;*

(ii) *The following assertions are equivalent:*
  (a) *M is a finitely generated projective R-module;*
  (b) *I_xM is a finitely generated projective $R\Gamma$-module;*
  (c) *I_yM is a finitely generated projective $R\Gamma$-module;*
  (d) *I_cM is a finitely generated projective $R\Gamma$-module.*

Proof. (i) Obviously Res_x and Res_y are exact. Hence it remains to show that they send both $R\operatorname{mor}(?,x)$ and $R\operatorname{mor}(?,y)$ to a finitely generated projective R-module. This is obviously true.

(ii) Suppose that I_xM is a finitely generated projective $R\Gamma$-module. Then M is a finitely generated R-module because of assertion (i) since $I_x(M)(x) = M$. Analogously one shows that M is finitely generated projective if I_yM or I_cM is a finitely generated projective $R\Gamma$-module.

Suppose that M is a finitely generated projective R-module. We want to show that I_xM, I_yM, and I_cM are finitely generated projective $R\Gamma$-modules. Since the functors I_x, I_y, and I_c are exact, it suffices to check this in the special case $M = R$. This follows from Lemma 9.1 since $R\operatorname{mor}(?,x)$ and $R\operatorname{mor}(?,y)$ are free $R\Gamma$-modules and I_xR, I_yR, and I_cR are direct summands in $R\operatorname{mor}(?,x)$ or $R\operatorname{mor}(?,y)$. $\square$

Corollary 9.3. *The constant functor $\underline{R}\colon \Gamma^{\mathrm{op}} \to R\text{-}\mathsf{MOD}$ with value R defines a projective $R\Gamma$-module. In particular, $\underline{R}$ admits a finite projective resolution and Γ is of type (FP).*

Lemma 9.4. *We obtain isomorphisms, inverse to one another,*

$$\alpha\colon K_0(R) \oplus K_0(R) \oplus K_0(R) \xrightarrow{\cong} K_0(R\Gamma),$$
$$\big([P_1],[P_2],[P_3]\big) \mapsto [I_x(P_1)] + [I_y(P_2)] + [I_c(P_3)]$$

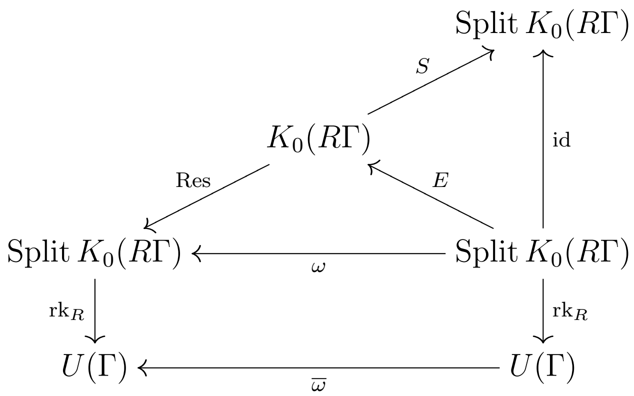
and

$$\beta\colon K_0(R\Gamma) \xrightarrow{\cong} K_0(R) \oplus K_0(R) \oplus K_0(R), \quad [P] \mapsto \big([S_xP],[S_yP],[\operatorname{Res}_x P] - [S_xP]\big),$$

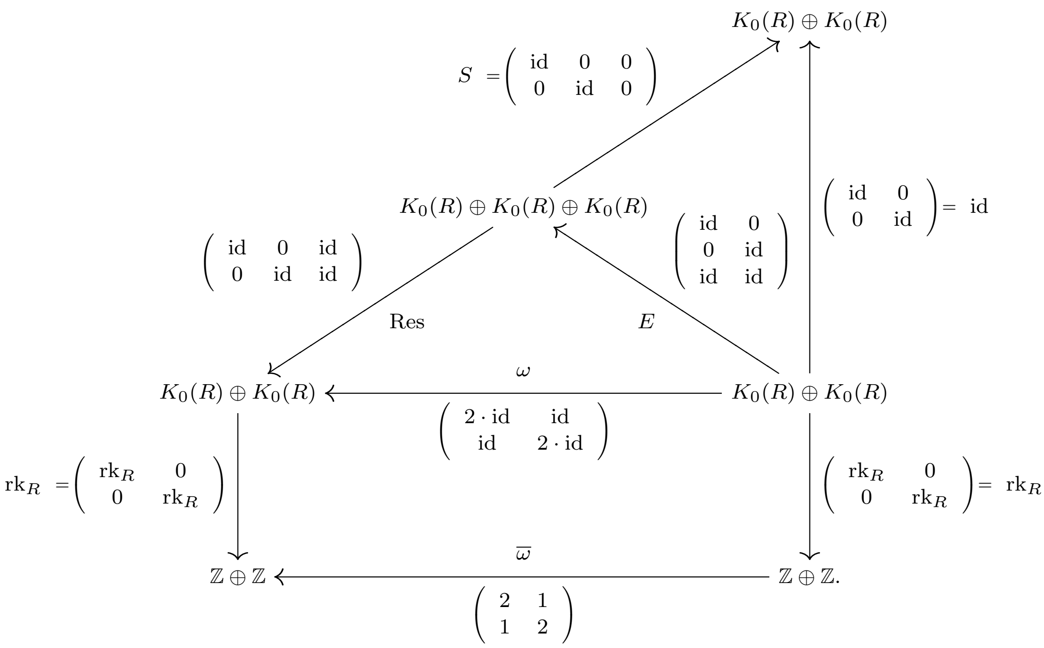
where the functors S_x and S_y are the splitting functors defined in (3.3).

Proof. This follows from Lemma 9.1 and Lemma 9.2. $\square$

Consider the following commutative diagram



where the homomorphisms S and E have been defined in (3.7) and in (3.8) and satisfy $S \circ E = \mathrm{id}$ by Lemma 3.9, the homomorphism Res sends $[P]$ to $\big([\mathrm{Res}_x P], [\mathrm{Res}_y P]\big)$, the homomorphism ω has been defined in (6.18), the map rk_R is given by the direct sum of the homomorphisms $K_0(R) \to \mathbb{Z}$ sending $[P]$ to $\mathrm{rk}_R(P)$ and $\overline{\omega}$ is given by the matrix $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. Under the identification α of Lemma 9.4 and the definitions $\mathrm{Split}\, K_0(R\Gamma) := K_0(R) \oplus K_0(R)$ and $U(\Gamma) = \mathbb{Z} \oplus \mathbb{Z}$, where the first summand corresponds to x and the second to y, this diagram becomes



The finiteness obstruction $o(\Gamma; R) \in K_0(R\Gamma)$ of Definition 2.7 corresponds under the identification α of Lemma 9.4 to the element $(0, 0, [R]) \in K_0(R) \oplus$

$K_0(R) \oplus K_0(R)$. Its image under $S\colon K_0(R\Gamma) \to \operatorname{Split} K_0(R\Gamma) = K_0(R) \oplus K_0(R)$ is $(0,0)$. Its image under $\operatorname{Res}\colon K_0(R\Gamma) \to \operatorname{Split} K_0(R\Gamma) = K_0(R) \oplus K_0(R)$ is $([R],[R])$. Its image under the composite $\operatorname{rk}_R \circ \operatorname{Res}\colon K_0(R\Gamma) \to U(\Gamma) = \mathbb{Z} \oplus \mathbb{Z}$ is $(1,1)$. An inverse $\overline{\mu}$ of the isomorphism induced by $\overline{\omega}\colon U(\Gamma) \otimes_{\mathbb{Z}} \mathbb{Q} \to U(\Gamma) \otimes_{\mathbb{Z}} \mathbb{Q}$ is given by

$$\begin{pmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{pmatrix} : \mathbb{Q} \oplus \mathbb{Q} \to \mathbb{Q} \oplus \mathbb{Q}.$$

The Euler characteristic in the sense of Leinster [13] is $2/3 + (-1/3) + (-1/3) + 2/3 = 2/3$. We see that the Euler characteristic in the sense of Leinster [13] is the image of the finiteness obstruction under the composite

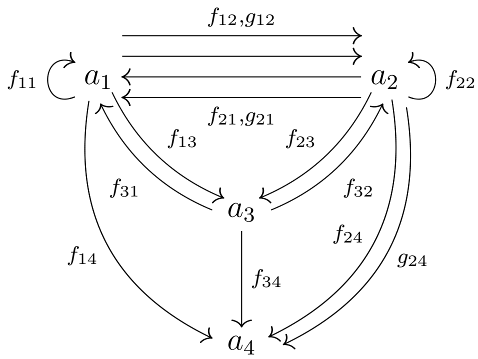
$$K_0(R\Gamma) \xrightarrow{\operatorname{Res}} \operatorname{Split} K_0(R\Gamma) \xrightarrow{\operatorname{rk}_R} U(\Gamma) \xrightarrow{i} U(\Gamma) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\overline{\mu}} U(\Gamma) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\epsilon} \mathbb{Q}$$

where i is the obvious inclusion and ϵ is the augmentation homomorphism.

10. A finite category without property (FP)

In this section we investigate the finite category $\mathbb{A}$ appearing in [13, Example 1.11.d], recalled below. Leinster showed that $\mathbb{A}$ has no weighting. Obviously $\mathbb{A}$ is Cauchy complete but not directly-finite and in particular not an EI-category. We will show that it is not of type (FP), give a full classification of the finitely generated projective $R\mathbb{A}$-modules, and compute $K_0(R\mathbb{A})$, $G_0(R\mathbb{A})$, and $H_n(B\mathbb{A}; R) = H_n(\mathbb{A}; R)$.

The nontrivial morphisms of Leinster's example $\mathbb{A}$ are drawn in the diagram below.



He also defines $f_{33} := \mathrm{id}_{a_3}$ and $f_{44} := \mathrm{id}_{a_4}$. Composition in the category $\mathbb{A}$ is: for any composable pair $a_i \xrightarrow{p} a_j \xrightarrow{q} a_k$ in $\mathbb{A}$ for which neither p nor q is an identity we have $q \circ p = f_{ik}$.

Lemma 10.1. *The space $|N\mathbb{A}|$ is homotopy equivalent to a point.*

Proof. We consider the subcategory $\mathbb{U}$ of $\mathbb{A}$ which does not contain g_{24}, but otherwise is the same as $\mathbb{A}$. The object a_4 is a terminal object for $\mathbb{U}$, so $|N\mathbb{U}| \simeq *$.

But $|N\mathbb{U}| \simeq |N\mathbb{A}|$. We have the inclusion $i\colon \mathbb{U} \to \mathbb{A}$. The functor $r\colon \mathbb{A} \to \mathbb{U}$ is the identity functor, except on g_{24}, which r maps to f_{24}. Then $ri = \mathrm{id}_{\mathbb{U}}$ and we also have a natural transformation $\alpha\colon ir \Rightarrow \mathrm{id}_{\mathbb{A}}$ defined by

$$\begin{aligned}\alpha(a_1) &= \mathrm{id}_{a_1}\\ \alpha(a_2) &= f_{22}\\ \alpha(a_3) &= \mathrm{id}_{a_3}\\ \alpha(a_4) &= \mathrm{id}_{a_4}\,.\end{aligned}$$

The continuous maps $|Nr|$ and $|Ni|$ are homotopy inverses. □

Although $\mathbb{A}$ has the homotopy type of a point, $\mathbb{A}$ is not equivalent to the trivial category, for the unique functor $\mathbb{A} \to *$ is not fully faithful. Alternatively, we note that the trivial category is of type (FP) while $\mathbb{A}$ is not of type (FP), as we now show.

10.1. **Property (FP).**

Theorem 10.2. *The above finite category* $\mathbb{A}$ *appearing in* [13, Examples 1.11.d] *is not of type* (FP_R) *for any associative, commutative ring* R *with unit.*

Proof. In the sequel we use the notation in $\mathbb{A}$ appearing in [13, Examples 1.11.d], recalled above. Let M be the $R\mathbb{A}$-module M which is uniquely determined by $M(a_i) = \{0\}$ for $i = 1, 3, 4$, $M(a_2) = R$, and $M(f_{22}) = 0$. Such an $R\mathbb{A}$-module M exists since $\mathrm{id}_{a_2} = a \circ b$ implies $a = b = \mathrm{id}_{a_2}$. Let $u_0\colon R\,\mathrm{mor}(?, a_4) \to \underline{R}$ be the $R\mathbb{A}$-homomorphism uniquely defined by the property that it sends id_{a_4} to $1 \in R$. Let $u_1\colon M \to R\,\mathrm{mor}(?, a_4)$ be the $R\mathbb{A}$-homomorphism uniquely determined by the property that its evaluation at a_2 sends $1 \in R = M(a_2)$ to $f_{24} - g_{24}$. Let $v_1\colon R\,\mathrm{mor}(?, a_2) \to M$ be the $R\mathbb{A}$-homomorphism uniquely determined by the property that it sends id_{a_2} to $1 \in R = M(a_2)$. Let $v_2\colon R\,\mathrm{mor}(?, a_1) \to R\,\mathrm{mor}(?, a_2)$ be the $R\mathbb{A}$-homomorphism uniquely determined by the property that it sends id_{a_1} to $g_{12} \in R\,\mathrm{mor}(a_1, a_2)$. Let $v_3\colon M \to R\,\mathrm{mor}(?, a_1)$ be the $R\mathbb{A}$-homomorphism uniquely determined by the property that its evaluation at a_2 sends $1 \in R = M(a_2)$ to $f_{21} - g_{21}$. Then we obtain exact sequences of $R\mathbb{A}$-modules

$$0 \to M \xrightarrow{u_1} R\,\mathrm{mor}(?, a_4) \xrightarrow{u_0} \underline{R} \to 0, \tag{10.3}$$

and

$$0 \to M \xrightarrow{v_3} R\,\mathrm{mor}(?, a_1) \xrightarrow{v_2} R\,\mathrm{mor}(?, a_2) \xrightarrow{v_1} M \to 0. \tag{10.4}$$

The first exact sequence and [15, Lemma 11.6 on page 216] imply that $\underline{R}$ has a finite-dimensional projective $R\mathbb{A}$-resolution if and only if M has. By concatenating copies of 10.4 we obtain an exact sequence

$$0 \to M \to F_n \to \cdots \to F_0 \to M \to 0$$

with free $R\mathbb{A}$-modules F_i of arbitrarily long length n. Thus, using [9, Lemma (2.1) on p. 184], M has a finite-dimensional projective $R\mathbb{A}$-resolution if and only if M is projective. Hence $\underline{R}$ has a finite-dimensional projective $R\mathbb{A}$-resolution if and only if M is projective. Since v_1 is surjective, M is projective only if v_1 has a section. Hence it suffices to show that v_1 has no section.

Let $s\colon M \to R\operatorname{mor}(?, a_2)$ be any $R\mathbb{A}$-homomorphism. Consider the homomorphism $g_{12}^*\colon R\operatorname{mor}(a_2, a_2) \to R\operatorname{mor}(a_1, a_2)$ given by composition with g_{12}. It sends the R-basis $\{\mathrm{id}_{a_2}, f_{22}\}$ bijectively to the R-basis $\{g_{12}, f_{12}\}$ and is hence an isomorphism. The composite $g_{12}^* \circ s(a_2)\colon M(a_2) \to R\operatorname{mor}(a_1, a_2)$ factorizes through $M(a_1)$ and hence is trivial since $M(a_1) = \{0\}$. Hence the $R\mathbb{A}$-morphism $s\colon M \to R\operatorname{mor}(?, a_2)$ is trivial and cannot be a section of v_1. □

10.2. **Finitely generated projective modules.** We want to classify all finitely generated projective $R\mathbb{A}$-modules. Let P be a finitely generated projective R-module. For $i = 1, 2$ let $K_1(P)$ be the $R\mathbb{A}$-module whose evaluation at both a_1 and a_2 is P and whose evaluation at both a_3 and a_4 is $\{0\}$. We require that g_{21} for $i = 1$ and that g_{12} for $i = 2$ induces the identity $\mathrm{id}\colon P \to P$, whereas all other morphisms in $\mathbb{A}$ besides the identity morphisms of the objects a_1 and a_2 induce the zero homomorphism. Then

Theorem 10.5. *Let P be an $R\mathbb{A}$-module.*

(i) *P is finitely generated projective if and only if there exists finitely generated projective R-modules P_1, P_2, P_3, and P_4 such that*
$$P \cong K_1(P_1) \oplus K_2(P_2) \oplus E_{a_3}(P_3) \oplus E_{a_4}(P_4),$$
where E_{a_3} and E_{a_4} denote the extension functors defined in (3.4)*;*

(ii) *Suppose that there exists finitely generated projective R-modules P_1, P_2, P_3, and p_4 such that*
$$P \cong K_1(P_1) \oplus K_2(P_2) \oplus E_{a_3}(P_3) \oplus E_{a_4}(P_4).$$
Then
$$\begin{aligned} P_1 &\cong S_{a_1}P; \\ P_2 &\cong S_{a_1}P; \\ P_3 &\cong \operatorname{coker}\bigl(P(f_{34})\colon P(a_4) \to P(a_3)\bigr); \\ P_4 &\cong P(a_4), \end{aligned}$$
where S_{a_i} is the splitting functor defined in (3.3)*;*

(iii) *P is finitely generated free if and only if there exists finitely generated free R-modules F_1, F_2, F_3, and F_4 such that*
$$P \cong K_1(F_1) \oplus K_2(F_2) \oplus E_{a_3}(F_1 \oplus F_2 \oplus F_3) \oplus E_{a_4}(F_4).$$

Proof. (i) Recall that the extension functor E_{a_j} satisfies $E_{a_j}(R) = R\operatorname{mor}(?, a_j)$, is compatible with direct sums, and sends finitely generated projective modules to finitely generated projective modules (see Lemma 3.5 (i)). In particular $E_{a_3}(P_3)$

and $E_{a_4}(P_4)$ are finitely generated projective $R\mathbb{A}$-modules if P is a finitely generated projective R-module.

Given a category Γ and an endomorphism $u\colon x \to x$ of an object in Γ and an $R[x]$-module Q, we obtain a morphism of $R\Gamma$-modules $u_*\colon E_xQ \to E_xQ$ as follows. Its evaluation at an object y is given by

$$Q \otimes_{R[x]} R\operatorname{mor}(y,x) \to Q \otimes_{R[x]} R\operatorname{mor}(y,x), \quad q \otimes v \mapsto q \otimes uv.$$

Obviously $(\mathrm{id}_x)_* = \mathrm{id}_{E_xQ}$ and $(u_1)_* \circ (u_2)_* = (u_1 \circ u_2)_*$ for two endomorphisms u_1 and u_2.

Consider a finitely generated projective R-module P. Consider $i \in \{1,2\}$. The construction above applied to the idempotent $f_{ii}\colon a_i \to a_i$ yields an idempotent endomorphism of $R\mathbb{A}$-modules $(f_{ii})_*\colon E_{a_i}P \to E_{a_i}P$. We obtain a direct sum decomposition of finitely generated projective $R\mathbb{A}$-modules

$$E_{a_i}P \quad \cong \quad \operatorname{im}\big((f_{ii})_*\big) \oplus \ker\big((f_{ii})_*\big). \tag{10.6}$$

Next we show for $i = 1,2$

$$\operatorname{im}\big((f_{ii})_*\big) \quad \cong \quad E_{a_3}P; \tag{10.7}$$

$$\ker\big((f_{ii})_*\big) \quad \cong \quad K_i(P). \tag{10.8}$$

We only treat the case $i = 1$, the case $i = 2$ is completely analogous. Let

$$\alpha\colon E_{a_3}P \to E_{a_1}P \tag{10.9}$$

be the $R\Gamma$-homomorphism which is the adjoint under the adjunction of [15, Lemma 9.31 a) on page 171] of the R-homomorphism $P \to E_{a_1}P(a_3) = P \otimes_R R\operatorname{mor}(a_3,a_1)$ sending p to $p \otimes f_{31}$. Explicitly the evaluation of α at an object a_j is given by

$$P \otimes_R R\operatorname{mor}(a_j,a_3) \to P \otimes_R R\operatorname{mor}(a_j,a_1), \quad p \otimes u \mapsto p \otimes (f_{31} \circ u).$$

One easily checks that α is injective. The image of $\alpha(a_j)$ is $\{0\}$ for $j = 4$ and is $\{p \otimes f_{j1} \mid p \in P\}$ for $j = 1,2,3$. This is the same as the image of $(f_{11})_*\colon E_{a_1}P \to E_{a_1}P$ and (10.7) follows. The cokernel of α is isomorphic to $\ker\big((f_{11})_*\big)$ since $(f_{11})_*$ is an idempotent. Obviously the cokernel evaluated at a_4 and a_3 is $\{0\}$. The cokernel evaluated at the objects a_1 and a_2 is isomorphic to R. The element id_{a_1} projects down to a generator in $\operatorname{coker}(\alpha)(a_1)$ and the element g_{21} projects down to a generator in $\operatorname{coker}(\alpha)(a_2)$. Hence the morphism g_{21} induces a map $\operatorname{coker}(\alpha)(a_1)$ to $\operatorname{coker}(\alpha)(a_2)$ that respects these generators. The morphisms f_{11}, f_{12}, f_{22} and g_{12} induce the trivial homomorphism on the cokernel of α. Now (10.8) follows.

In particular we see that $K_i(P)$ is a finitely generated projective $R\mathbb{A}$-module if P is a finitely generated projective R-module.

Now consider a finitely generated projective $R\mathbb{A}$-module P. Choose a finitely generated free $R\Gamma$-module F together with $R\Gamma$-maps $i\colon P \to F$ and $r\colon F \to P$. Let

$$\sigma_{a_4}(P)\colon E_{a_4}P(a_4) \to P$$

be the adjoint of the adjunction of [15, Lemma 9.31 on page 171] of the R-homomorphism $\mathrm{id}_{a_4}\colon P(a_4) \to P(a_4)$. Explicitly its evaluation at a_j is given by

$$P(a_4) \otimes_R R\operatorname{mor}(a_j, a_4) \to P(a_j), \quad p \otimes u \mapsto P(u)(p).$$

The map $\sigma_{a_4}(P)$ is natural in P. Let $\overline{P}$ and $\overline{F}$ respectively be the cokernel of $\sigma_{a_4}(P)$ and $\sigma_{a_4}(F)$ respectively. Denote by $\mathrm{pr}(P)\colon P \to \overline{P}$ and $\mathrm{pr}(F)\colon P \to \overline{F}$ the canonical projections.

Choose non-negative integers m_1, m_2, m_3, and m_4 such that

$$F \cong \bigoplus_{j=1}^{4} R\operatorname{mor}(?, a_j)^{m_j}.$$

Since the are no morphisms from a_4 to the other objects a_1, a_2 and a_3, one easily checks that the sequence

$$E_{a_4}F(a_4) \xrightarrow{\sigma_{a_4}} F \xrightarrow{\mathrm{pr}(F)} \overline{F}$$

can be identified with the obvious split exact sequence

$$R\operatorname{mor}(?, a_4)^{m_4} \to \bigoplus_{j=1}^{4} R\operatorname{mor}(?, a_j)^{m_j} \to \bigoplus_{j=1}^{3} R\operatorname{mor}(?, a_j)^{m_j}.$$

We obtain a commutative diagram

$$\begin{array}{ccccc}
0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow \\
E_{a_4}(P(a_4)) & \xrightarrow{E_{a_4}(i(a_4))} & E_{a_4}(F(a_4)) & \xrightarrow{E_{a_4}(r(a_4))} & E_{a_4}(P(a_4)) \\
\downarrow \sigma_{a_4}(P) & & \downarrow \sigma_{a_4}(F) & & \downarrow \sigma_{a_4}(P) \\
P & \xrightarrow{i} & F & \xrightarrow{r} & P \\
\downarrow \mathrm{pr}(P) & & \downarrow \mathrm{pr}(F) & & \downarrow \mathrm{pr}(P) \\
\overline{P} & \xrightarrow{\overline{i}} & \overline{F} & \xrightarrow{\overline{r}} & P \\
\downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0
\end{array}$$

where $\overline{i}$ and $\overline{r}$ are the maps induced by i and r. We know already that the middle row is exact. We conclude $E_{a_4}(r(a_4)) \circ E_{a_4}(i(a_4)) = \mathrm{id}$ and $\overline{r} \circ \overline{i} = \mathrm{id}$ from $r \circ i = \mathrm{id}$. An easy diagram shows that all rows are exact.

Hence $\overline{P}$ is a finitely generated projective $R\mathbb{A}$-module, we have the isomorphisms

$$\begin{aligned} P &\cong E_{a_4}(P(a_4)) \oplus \overline{P}; \\ \overline{F} &\cong \bigoplus_{j=1}^{3} R\operatorname{mor}(?, a_j)^{m_j}, \end{aligned}$$

and $R\mathbb{A}$-homomorphisms $\overline{i} \colon \overline{P} \to \overline{F}$ and $\overline{r} \colon \overline{F} \to \overline{P}$ with $\overline{r} \circ \overline{i} = \mathrm{id}$. The R-module $P(a_4)$ is a finitely generated projective R-module since it is a direct summand in the finitely generated free R-module $F(a_4) = R^{m_4}$. Hence it suffices to proof the claim for $\overline{P}$.

Now we more or less repeat the argument above, but nor replacing a_4 by a_3. So we define

$$\begin{aligned} \sigma_{a_3}(\overline{P}) &\colon E_{a_3}\overline{P}(a_3) \to \overline{P} \\ \sigma_{a_3}(\overline{P}) &\colon E_{a_3}\overline{F}(a_3) \to \overline{F} \end{aligned}$$

as above. Denote by $\overline{\overline{P}}$ and $\overline{\overline{F}}$ respectively the cokernel of $\sigma_{a_3}(\overline{P})$ and $\sigma_{a_3}(\overline{F})$ respectively. Let $\operatorname{pr}(\overline{P}) \colon \overline{P} \to \overline{\overline{P}}$ and $\operatorname{pr}(\overline{F}) \colon \overline{F} \to \overline{\overline{F}}$ be the canonical projections. Denote by $\overline{\overline{i}} \colon \overline{\overline{P}} \to \overline{\overline{F}}$ and $\overline{\overline{r}} \colon \overline{\overline{F}} \to \overline{\overline{P}}$ the maps induced by $\overline{i}$ and $\overline{r}$. The maps $\sigma_{a_3}(P)$ are natural in P and compatible with direct sums. One easily checks that the $R\mathbb{A}$-homomorphism $\sigma_{a_3}(R\operatorname{mor}(?, a_3))$ is an isomorphism. Hence also the $R\mathbb{A}$-homomorphism

$$\sigma_{a_3}(R\operatorname{mor}(?, a_3)^{m_3}) \colon E_{a_3}R\operatorname{mor}(a_3, a_3)^{m_3} \to R\operatorname{mor}(?, a_3)^{m_3}$$

is an isomorphism. The map $\sigma_{a_3}(R\operatorname{mor}(?, a_1)) \colon E_{a_3}R\operatorname{mor}(a_3, a_1) \to R\operatorname{mor}(?, a_1)$ is the same as the map α defined in (10.9). Hence it is injective and its cokernel is $K_1(R)$. This implies that

$$\sigma_{a_3}(R\operatorname{mor}(?, a_1)^{m_1}) \colon E_{a_3}R\operatorname{mor}(a_3, a_1)^{m_1} \to R\operatorname{mor}(?, a_1)^{m_1}$$

is injective with the finitely generated projective $R\mathbb{A}$-module $K_1(R^{m_1})$ as cokernel. Analogously one shows that

$$\sigma_{a_3}(R\operatorname{mor}(?, a_2)^{m_2}) \colon E_{a_3}R\operatorname{mor}(a_3, a_2)^{m_2} \to R\operatorname{mor}(?, a_2)^{m_2}$$

is injective with the finitely generated projective $R\mathbb{A}$-module $K_2(R^{m_2})$ as kernel. This implies

$$\overline{F} \cong K_1(R^{m_1}) \oplus K_2(R^{m_2}).$$

As above we obtain a commutative diagram with exact rows

$$\begin{array}{ccccc}
0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow \\
E_{a_3}(\overline{P}(a_3)) & \xrightarrow{E_{a_3}(\overline{i}(a_3))} & E_{a_3}(\overline{F}(a_3)) & \xrightarrow{E_{a_3}(\overline{r}(a_3))} & E_{a_3}(\overline{P}(a_3)) \\
\downarrow{\scriptstyle \sigma_{a_3}(\overline{P})} & & \downarrow{\scriptstyle \sigma_{a_3}(\overline{F})} & & \downarrow{\scriptstyle \sigma_{a_3}(\overline{P})} \\
\overline{P} & \xrightarrow{\overline{i}} & \overline{F} & \xrightarrow{\overline{r}} & \overline{P} \\
\downarrow{\scriptstyle \operatorname{pr}(\overline{P})} & & \downarrow{\scriptstyle \operatorname{pr}(\overline{F})} & & \downarrow{\scriptstyle \operatorname{pr}(\overline{P})} \\
\overline{\overline{P}} & \xrightarrow{\overline{\overline{i}}} & \overline{\overline{F}} & \xrightarrow{\overline{\overline{r}}} & \overline{\overline{P}} \\
\downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0
\end{array}$$

Hence $\overline{\overline{P}}$ is a finitely generated projective $R\mathbb{A}$-module with is a direct summand in $\overline{F} \cong K_1(R^{m_1}) \oplus K_2(R^{m_2})$ and we obtain an isomorphism

$$\overline{P} \cong E_{a_3}(\overline{P}(a_3)) \oplus \overline{\overline{P}}.$$

Since $\overline{P}(a_3)$ is a direct summand in the finitely generated free R-module $\overline{F}(a_3) \cong R^{m_1+m_2+m_3}$, it is finitely generated projective R-module. Hence it remains to prove the claim for $\overline{\overline{P}}$.

Since $\overline{\overline{P}}$ is a direct summand in $K_1(R^{m_1}) \oplus K_2(R^{m_2})$, one easily checks that we have exact sequences of finitely generated projective R-modules

$$0 \to \operatorname{im}\bigl(\overline{\overline{P}}(g_{12})\bigr) \xrightarrow{i_1} \overline{\overline{P}}(a_1) \xrightarrow{\overline{\overline{P}}(g_{21})} \operatorname{im}\bigl(\overline{\overline{P}}(g_{21})\bigr) \to 0,$$

and

$$0 \to \operatorname{im}\bigl(\overline{\overline{P}}(g_{21})\bigr) \xrightarrow{i_2} \overline{\overline{P}}(a_2) \xrightarrow{\overline{\overline{P}}(g_{12})} \operatorname{im}\bigl(\overline{\overline{P}}(g_{12})\bigr) \to 0,$$

where i_1 and i_2 are the inclusions. Choose R-maps

$$\begin{array}{rcl} r_1\colon \overline{\overline{P}}(a_1) & \to & \operatorname{im}\bigl(\overline{\overline{P}}(g_{12})\bigr), \\ r_2\colon \overline{\overline{P}}(a_2) & \to & \operatorname{im}\bigl(\overline{\overline{P}}(g_{21})\bigr), \end{array}$$

satisfying $r_1 \circ i_1 = \operatorname{id}$ and $r_2 \circ i_2 = \operatorname{id}$. Next we define an $R\mathbb{A}$-isomorphism

$$\beta\colon \overline{\overline{P}} \xrightarrow{\cong} K_1\left(\operatorname{im}\bigl(\overline{\overline{P}}(g_{21})\bigr)\right) \oplus K_2\left(\operatorname{im}\bigl(\overline{\overline{P}}(g_{12})\bigr)\right).$$

Its evaluation at a_1 is given by the R-isomorphism

$$\overline{\overline{P}}(g_{21}) \oplus r_1\colon \overline{\overline{P}}(a_1) \xrightarrow{\cong} \operatorname{im}\bigl(\overline{\overline{P}}(g_{21})\bigr) \oplus \operatorname{im}\bigl(\overline{\overline{P}}(g_{12})\bigr)$$

and its evaluation at a_2 by the R-isomorphism

$$r_2 \oplus \overline{\overline{P}}(g_{12})\colon \overline{\overline{P}}(a_2) \xrightarrow{\cong} \operatorname{im}\bigl(\overline{\overline{P}}(g_{21})\bigr) \oplus \operatorname{im}\bigl(\overline{\overline{P}}(g_{12})\bigr).$$

This finishes the proof of assertion (i) of Theorem 10.5.

(ii) Recall that $K_i(P_i)$ is a direct summand in $E_{a_i}(P_i)$ for $i = 1, 2$ (see (10.8)). Using Lemma 3.5 (ii) one easily checks

$$\begin{array}{lllll} S_{a_i}(P) & \cong & S_{a_i}(K_i(P_i)) & \cong & P_i \quad \text{for } i = 1, 2; \\ P(a_4) & \cong & P_4. & & \end{array}$$

A direct computation shows

$$\begin{aligned} & \operatorname{coker}\big(P(f_{34})\colon P(a_4) \to P(a_3)\big) \\ \cong\ & \bigoplus_{i=1}^{2} \operatorname{coker}\big(K_i(p_i)(f_{34})\big) \oplus \operatorname{coker}\big(E_{a_3}(P_3)(f_{34})\big) \oplus \operatorname{coker}\big(E_{a_4}(P_4)(f_{34})\big) \\ \cong\ & \operatorname{coker}\big(E_{a_3}(P_3)(f_{34})\big) \\ \cong\ & P_3. \end{aligned}$$

This finishes the proof of assertion (ii).

(iii) This follows from assertions (i) and (ii) and the isomorphism for $i = 1, 2$ (see (10.6), (10.7) and (10.8))

$$R\operatorname{mor}(?, a_i) \cong R\operatorname{mor}(?, a_3) \oplus K_1(R).$$

This finishes the proof of Theorem 10.5. □

Remark 10.10. Notice that the decomposition of Theorem 10.5 (i) is not natural in P. However, the cofiltration by epimorphisms

$$P \to \overline{P} \to \overline{\overline{P}}$$

and the identifications

$$\begin{array}{rcl} \overline{\overline{P}} & \cong & K_1(S_{a_1}(P)) \oplus K_2(S_{a_2}(P)); \\ \ker\big(\overline{P} \to \overline{P}/\overline{\overline{P}}\big) & \cong & E_{a_3}\left(\operatorname{coker}\big(P(f_{34})\colon P(a_4) \to P(a_3)\big)\right); \\ \ker\big(P \to \overline{P}\big) & \cong & E_{a_4}(P(a_4)), \end{array}$$

are natural in P.

Let $K_0^f(R\mathbb{A})$ be the Grothendieck group of finitely generated free $R\mathbb{A}$-modules. Let

$$\iota\colon U(\Gamma) \to K_0^f(R\mathbb{A})$$

be the homomorphism which sends a basis element $\overline{x} \in \operatorname{iso}(\mathbb{A})$ to the class of $R\operatorname{mor}(?, x)$.

Theorem 10.11 ($K_0(R\mathbb{A})$)**.** (i) *The maps*

$$\begin{array}{rcl} \xi\colon K_0(R)^4 & \xrightarrow{\cong} & K_0(R\mathbb{A}), \\ \eta\colon K_0(R\mathbb{A}) & \xrightarrow{\cong} & K_0(R)^4, \end{array}$$

defined by

$$\begin{aligned}\xi\big([P_1],[P_2],[P_3],[P_4]\big) &= [K_1(P_1)]+[K_2(P_2)]+[E_{a_3}(P_3)]+[E_{a_4}(P_4)],\\ \eta([P]) &= \big([S_{a_1}P],[S_{a_2}P],\big[\operatorname{coker}\big(P(f_{34})\colon P(a_4)\to P(a_3)\big)\big],[S_{a_4}P]\big)\,,\end{aligned}$$

are isomorphisms, inverse to another.

(ii) *The map*

$$\iota\colon U(\mathbb{A}) \xrightarrow{\cong} K_0^f(R\mathbb{A})$$

is bijective. If R is a principal domain, then the forgetful map

$$F^f\colon K_0^f(R\mathbb{A}) \xrightarrow{\cong} K_0(R\mathbb{A})$$

is bijective.

Proof. (i) This follows from Theorem 10.5 (i) and (ii).

(ii) The map ι is obviously surjective. The composite

$$U(\Gamma) \xrightarrow{\iota} K_0^f(R\mathbb{A}) \xrightarrow{F^f} K_0(R\mathbb{A}) \xrightarrow{\eta} K_0(R)^4 \xrightarrow{\mathrm{rk}_R} \mathbb{Z}^4$$

can be identified with the injection

$$\mathbb{Z}^4 \xrightarrow{\cong} \mathbb{Z}^4, \quad \big(m_1,m_2,m_3,m_4\big) \mapsto \big(m_1,m_2,m_1+m_2+m_3,m_4\big)$$

by Theorem 10.5 (iii). The forgetful map $F^f\colon K_0^f(R\mathbb{A}) \to K_0(R\mathbb{A})$ is surjective by Theorem 10.5 (iii) provided that R is an integral domain and hence $\mathbb{Z} \to K_0(R)$, $n \mapsto [R^n]$ is an isomorphism. This finishes the proof of Theorem 10.11. ☐

10.3. K_0 **versus** G_0**.** Let R be a commutative Noetherian ring and let Γ be a finite category (see Definition 6.6). Denote by $G_0(\mathbb{Q}\Gamma)$ the Grothendieck group of finitely generated $\mathbb{Q}\Gamma$-modules. Since Γ is finite, an $R\Gamma$-module is finitely generated if and only if for every object x the $\mathbb{Q}$-module $M(x)$ is finitely generated as an R-module. In particular the category of $R\Gamma$-modules is Noetherian, i.e., a submodule of a finitely generated $R\Gamma$-module is finitely generated (see [15, Lemma 16.10 on page 327]).

Remark 10.12. Notice that the constant R-module $\underline{R}$ defines an element $[\underline{R}]$ in $G_0(R\Gamma)$ which may be viewed as a kind of analogue of the finiteness obstruction. Only if Γ is of type (FP), then we get also an element $o(\Gamma;R) := [\underline{R}]$ in $K_0(R\Gamma)$ which is mapped under the forgetful homomorphism

$$F_{R\Gamma}\colon K_0(R\Gamma) \;\to\; G_0(R\Gamma).$$

to $[\underline{R}] \in G_0(R\Gamma)$.

Notice that $F_{R\Gamma}$ is bijective if Γ is a finite EI-category and the order $\operatorname{aut}(x)$ is invertible in R for every object x in Γ (see [15, Proposition 16.28 on page 332]). This is not true in general as the following example shows.

Example 10.13. We conclude from (10.4) that

$$[R\,\mathrm{mor}(?,a_1)] = [R\,\mathrm{mor}(?,a_2)] \qquad \in G_0(R\mathbb{A}). \tag{10.14}$$

This together with Theorem 10.11 (ii) implies that

$$F\colon K_0(R\mathbb{A}) \to G_0(R\mathbb{A})$$

is not injective.

Define a map

$$\mathrm{Res}\colon G_0(R\Gamma) \to \bigoplus_{\overline{x}\in\mathrm{iso}(\Gamma)} G_0(R[x]), \quad [P] \mapsto \{[P(x)] \mid \overline{x} \in \mathrm{iso}(\Gamma)\}. \tag{10.15}$$

Provided that the order $\mathrm{aut}(x)$ is invertible in R for every object x in Γ, we also obtain a map

$$\mathrm{Res}\colon K_0(R\Gamma) \to \bigoplus_{\overline{x}\in\mathrm{iso}(\Gamma)} K_0(R[x]), \quad [P] \mapsto \{[P(x)] \mid \overline{x} \in \mathrm{iso}(\Gamma)\}, \tag{10.16}$$

and we get a commutative diagram

$$\begin{array}{ccc} K_0(R\Gamma) & \xrightarrow{F_{R\Gamma}} & G_0(R\Gamma) \\ \downarrow{\scriptstyle \mathrm{Res}} & & \downarrow{\scriptstyle \mathrm{Res}} \\ \bigoplus_{\overline{x}\in\mathrm{iso}(\Gamma)} K_0(R[x]) & \xrightarrow[\cong]{\bigoplus_{\overline{x}\in\mathrm{iso}(\Gamma)} F_{R[x]}} & \bigoplus_{\overline{x}\in\mathrm{iso}(\Gamma)} G_0(R[x]) \end{array}$$

whose lower horizontal arrow is an isomorphism.

Now we consider the special case $\Gamma = \mathbb{A}$ and $R = \mathbb{Q}$. For a $\mathbb{Q}$-module P and $k \in \{1,2,4\}$ denote by $I_k(P)$ the $\mathbb{Q}\mathbb{A}$-module for which $I_k(\mathbb{Q})(a_k) = \mathbb{Q}$, $I_k(\mathbb{Q})(a_j) = \{0\}$ for $j \neq k$ and all morphisms except id_{a_k} induce the trivial homomorphism. One easily checks that this is a well-defined $\mathbb{Q}$-module. (Notice that this definition does not make sense for the object a_3).

Theorem 10.17 ($G_0(\mathbb{Q}\mathbb{A})$)**.** *The homomorphisms*

$$\begin{aligned} &\omega\colon \mathbb{Z}^4 \to G_0(R\mathbb{A}), \\ &\quad (n_1,n_2,n_3.n_4) \mapsto n_1\cdot[I_1(\mathbb{Q})] + n_2\cdot[I_2(\mathbb{Q})] + n_3\cdot[R\,\mathrm{mor}(?,a_3)] + n_4\cdot[I_4(\mathbb{Q})] \end{aligned}$$

and the composite

$$G_0(\mathbb{Q}\mathbb{A}) \xrightarrow{\mathrm{Res}} \bigoplus_{i=1}^{4} G_0(\mathbb{Q}) \xrightarrow{\bigoplus_{i=1}^4 \mathrm{rk}_{\mathbb{Q}}} \mathbb{Z}^4$$

are isomorphisms.

Proof. The composite

$$\mathbb{Z}^4 \xrightarrow{\omega} G_0(R\mathbb{A}) \xrightarrow{\mathrm{Res}} \bigoplus_{i=1}^{4} G_0(\mathbb{Q}) \xrightarrow{\bigoplus_{i=1}^4 \mathrm{rk}_{\mathbb{Q}}} \mathbb{Z}^4$$

sends (m_1, m_2, m_3, m_4) to $(m_1 + m_3, m_2 + m_3, m_3, m_4)$ and is hence an isomorphism. Therefore it suffices to show that ω is surjective.

Consider a finitely generated $\mathbb{QA}$-module M. There is the epimorphism of $\mathbb{QA}$-modules $M \to I_4(M(a_4))$ whose evaluation at a_4 is the identity. Let N be its kernel. Then we get $[M] = [N] + [I_4(M(a_4)]$ in $G_0(\mathbb{QA})$ and $N(a_4) = \{0\}$. Hence it suffices to prove that $[N]$ lies in the image of ω.

Consider the $\mathbb{QA}$-homomorphism $f\colon E_3(N(a_3)) \to N$ uniquely determined by the property that its evaluation at a_3 is the isomorphism $N(a_3) \otimes_{\mathbb{Q}} \mathbb{Q}\operatorname{mor}(a_3, a_3) \xrightarrow{\cong} N(a_3)$ sending $x \otimes \mathrm{id}_{a_3}$ to x. Let K be its kernel and L be its cokernel. We get in $[N] = [E_3(N(a_3))] + [L] - [K]$ in $G_0(\mathbb{QA})$ and $K(a_3) = K(a_4) = L(a_3) = L(a_4) = \{0\}$. Hence it suffices to show that K lies in the image of ω if K is a finitely generated $\mathbb{QA}$-module with $K(a_3) = K(a_4) = 0$.

Notice that the all morphisms in $\mathbb{A}$ possibly except g_{12} and g_{21} and the identity morphisms for a_1 and a_2 induce the trivial homomorphism on K since they factor through the object a_3 or a_4 and $K(a_3) = K(a_4) = 0$. Consider the $\mathbb{QA}$-homomorphism

$$g\colon I_1\big(\ker(N(g_{21})\big) \to K$$

given by the inclusion $\ker(N(g_{21})) \to N(a_1)$. Let P be its cokernel. By construction the map $P(g_{21})\colon P(a_1) \to P(a_2)$ is injective. Since $P(a_3) = 0$, we get

$$P(g_{21}) \circ P(g_{12}) = P(g_{12} \circ g_{21}) = P(f_{11}) = P(f_{31} \circ f_{13}) = P(f_{13}) \circ P(f_{31}) = 0.$$

Since $P(g_{21})$ is injective, $P(g_{12}) = 0$. Hence the identity on $P(a_2)$ induces an injection $I_{a_2}(P(a_2)) \to P$. Let Q be its cokernel. Then $Q(a_2) = Q(a_3) = Q(a_4)$. This implies $Q = I_{a_1}(Q(a_1))$. Hence we get in $G_0(\mathbb{QA})$

$$[K] = [I_{a_1}(\ker(N(g_{21}))] + [I_{a_2}(P(a_2))] + [I_{a_1}(Q(a_1))].$$

This finishes the proof of Theorem 10.17. $\square$

Example 10.18. Put $R = \mathbb{Q}$ and $\Gamma = \mathbb{A}$. Then the following diagram commutes

$$\begin{array}{ccccc}
U(\mathbb{A}) = \mathbb{Z}^4 & \xrightarrow[\cong]{\iota} & K_0(\mathbb{QA}) & \xrightarrow{F_{\mathbb{QA}}} & G_0(\mathbb{QA}) \\
 & \searrow^{A} & \downarrow{\scriptstyle \mathrm{Res}} & & \cong \downarrow{\scriptstyle \mathrm{Res}} \\
 & & \bigoplus_{i=1}^4 K_0(\mathbb{Q}) & \xrightarrow[\cong]{\bigoplus_{\overline{x} \in \mathrm{iso}(\mathbb{A})} F_{\mathbb{Q}}} & \bigoplus_{i=1}^4 G_0(\mathbb{Q}) \\
 & & \cong \downarrow{\scriptstyle \bigoplus_{i=1}^4 \mathrm{rk}_{\mathbb{Q}}} & & \cong \downarrow{\scriptstyle \bigoplus_{i=1}^4 \mathrm{rk}_{\mathbb{Q}}} \\
 & & \mathbb{Z}^4 & \xrightarrow[\cong]{\mathrm{id}} & \mathbb{Z}^4
\end{array}$$

where A is given by the matrix

$$\begin{pmatrix} 2 & 2 & 1 & 1 \\ 2 & 2 & 1 & 2 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Notice that $(1,1,1,1)$ is not in the image of $A\colon \mathbb{Z}^4 \to \mathbb{Z}^4$. Obviously $[\underline{\mathbb{Q}}] \in G_0(\mathbb{Q}\mathbb{A})$ is sent under the composite

$$\bigoplus_{i=1}^{4} \mathrm{rk}_{\mathbb{Q}} \circ \mathrm{Res}\colon G_0(\mathbb{Q}\mathbb{A}) \to \mathbb{Z}^4$$

to $(1,1,1,1)$. Hence we see again that $\mathbb{A}$ is not of type (FP) since otherwise $[\underline{R}] \in G_0(\mathbb{Q}\mathbb{A})$ lies in the image of $F_{\mathbb{Q}\mathbb{A}}$ and hence $(1,1,1,1)$ lies in the image of $A\colon \mathbb{Z}^4 \to \mathbb{Z}1^4$.

10.4. **Homology of $\mathbb{A}$.** We obtain from the short exact sequence (10.4) the following periodic projective resolution P_* of the $R\mathbb{A}$-module M

$$\cdots \xrightarrow{v_3 \circ v_1} R\,\mathrm{mor}(?,a_1) \xrightarrow{v_2} R\,\mathrm{mor}(?,a_2) \xrightarrow{v_3 \circ v_1} R\,\mathrm{mor}(?,a_1) \xrightarrow{v_2} R\,\mathrm{mor}(?,a_2) \xrightarrow{v_1} M.$$

Recall that v_2 sends id_{a_1} to g_{12} and $v_3 \circ v_1$ sends id_{a_2} to $f_{21} - g_{21}$. The R-chain complex $P_* \otimes_{R\mathbb{A}} \underline{R}$ looks like

$$\cdots \xrightarrow{0} R \xrightarrow{\mathrm{id}} R \xrightarrow{0} R \xrightarrow{\mathrm{id}} R$$

Hence we get for $n \geq 0$

$$H_n^R(\mathbb{A};M) := H_n\big(P_* \otimes_{R\mathbb{A}} M\big) = \{0\}. \tag{10.19}$$

We conclude from $\underline{R} \otimes_{R\mathbb{A}} \underline{R} \cong R$, from (10.19), and the short exact sequence (10.3) that

$$H_n(B\mathbb{A};R) = H_n(\mathbb{A};R) = H_n^R(\mathbb{A};\underline{R}) = \begin{cases} R & \text{if } n = 0 \\ \{0\} & \text{if } n > 0, \end{cases}$$

as we may expect from the contractibility of $B\mathbb{A}$.

References

[1] M. Aigner. *Combinatorial theory.* Springer-Verlag, Berlin, 1979.

[2] J. Baez and J. Dolan. From finite sets to Feynman diagrams. In *Mathematics Unlimited - 2001 and Beyond*, vol. 1, eds. Björn Engquist and Wilfried Schmid, Springer, Berlin, 2001, pp. 29-50.

[3] A. Bartels and W. Lück. The Borel conjecture for hyperbolic and CAT(0)-groups. Preprintreihe SFB 478 — Geometrische Strukturen in der Mathematik, Münster, `arXiv:0901.0442v1` [math.GT], 2009.

[4] A. Bartels, W. Lück, and H. Reich. The K-theoretic Farrell-Jones conjecture for hyperbolic groups. *Invent. Math.*, **172(1)** (2008), 29–70.

[5] A. Bartels, W. Lück, and H. Reich. On the Farrell-Jones Conjecture and its applications. *Journal of Topology*, **1** (2008), 57–86.

[6] P. Baum, A. Connes, and N. Higson. Classifying space for proper actions and K-theory of group C^*-algebras. In *C^*-algebras: 1943–1993 (San Antonio, TX, 1993)*, pages 240–291. Amer. Math. Soc., Providence, RI, 1994.

[7] C. Berger and T. Leinster. The Euler characteristic of a category as the sum of a divergent series. *Homology, Homotopy Appl.*, **10(1)** (2008), 41–51.

[8] M. R. Bridson and A. Haefliger. *Metric spaces of non-positive curvature.* Springer-Verlag, Berlin, 1999. Die Grundlehren der mathematischen Wissenschaften, Band 319.

[9] K. S. Brown. *Cohomology of groups*, vol. 87 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1982.
[10] J. Cheeger and M. Gromov. L_2-cohomology and group cohomology. *Topology*, **25(2)** (1986), 189–215.
[11] J. F. Davis and W. Lück. Spaces over a category and assembly maps in isomorphism conjectures in K- and L-theory. *K-Theory*, **15(3)** (1998), 201–252.
[12] T. M. Fiore, W. Lück, and R. Sauer. Homotopy colimits and Euler characteristics of categories.
[13] T. Leinster. The Euler characteristic of a category. *Doc. Math.*, **13** (2008), 21–49.
[14] W. Lück. Equivariant Eilenberg-Mac Lane spaces $K(G, \mu, 1)$ for possibly non-connected or empty fixed point sets. *Manuscripta Math.*, **58** (1987), 67–75.
[15] W. Lück. *Transformation groups and algebraic K-theory*. Springer-Verlag, Berlin, 1989. Mathematica Gottingensis.
[16] W. Lück. Dimension theory of arbitrary modules over finite von Neumann algebras and L^2-Betti numbers. I. Foundations. *J. Reine Angew. Math.*, **495** (1998), 135–162.
[17] W. Lück. Dimension theory of arbitrary modules over finite von Neumann algebras and L^2-Betti numbers. II. Applications to Grothendieck groups, L^2-Euler characteristics and Burnside groups. *J. Reine Angew. Math.*, **496** (1998), 213–236.
[18] W. Lück. The universal functorial Lefschetz invariant. *Fund. Math.*, **161(1-2)** (1999), 167–215. Algebraic topology (Kazimierz Dolny, 1997).
[19] W. Lück. *L^2-invariants: theory and applications to geometry and K-theory*, vol. 44 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2002.
[20] W. Lück. L^2-invariants from the algebraic point of view. Preprintreihe SFB 478 — Geometrische Strukturen in der Mathematik, Heft 295, Münster, `arXiv:math.GT/0310489 v1`, 2003.
[21] W. Lück. Survey on classifying spaces for families of subgroups. In *Infinite groups: geometric, combinatorial and dynamical aspects*, vol. 248 of *Progr. Math.*, pages 269–322. Birkhäuser, Basel, 2005.
[22] W. Lück and D. Meintrup. On the universal space for group actions with compact isotropy. In *Geometry and topology: Aarhus (1998)*, pages 293–305. Amer. Math. Soc., Providence, RI, 2000.
[23] W. Lück and H. Reich. The Baum-Connes and the Farrell-Jones conjectures in K- and L-theory. In *Handbook of K-theory. vol. 1, 2*, pages 703–842. Springer, Berlin, 2005.
[24] D. Meintrup and T. Schick. A model for the universal space for proper actions of a hyperbolic group. *New York J. Math.*, **8** (2002), 1–7 (electronic).
[25] J.-P. Serre. *Linear representations of finite groups*. Springer-Verlag, New York, 1977. Translated from the second French edition by Leonard L. Scott, Graduate Texts in Mathematics, vol. 42.
[26] J.-P. Serre. *Trees*. Springer-Verlag, Berlin, 1980. Translated from the French by J. Stillwell.
[27] N. E. Steenrod. A convenient category of topological spaces. *Michigan Math. J.*, **14** (1967), 133–152.
[28] T. tom Dieck. *Transformation groups and representation theory*. Springer-Verlag, Berlin, 1979.
[29] T. tom Dieck. *Transformation groups*. vol. 8 of *de Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, 1987.
[30] F. Waldhausen. Algebraic K-theory of generalized free products. I, II. *Annals of Math. (2)*, **108** (1978), 135–256.

[31] C. T. C. Wall. Finiteness conditions for CW-complexes. *Annals of Math. (2)*, **81** (1965), 56–69.

[32] C. T. C. Wall. Finiteness conditions for CW-complexes. II. *Proc. Roy. Soc. Ser. A*, **295**(1966), 129–139.

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