Ultimate Design of Prestressed Concrete Beams

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A method is presented by which prestressed concrete beams can be designed on the basis of strength and ductility. The requirements of strength and ductility are developed in a general form and their influence on the dimensions of the beam is studied. The influence of compression steel on ductility and the required area of the beam is presented. Numerical examples are included to show the practical application of the method in design.

•IN PRESENT design practice, prestressed concrete beams are almost always designed and proportioned by working stress design. The provisions of ultimate design are used to check the flexural strength of a section that has already been designed. It can be shown that the provisions of ultimate design can be used to proportion a section with a rigorous control of both strength and ductility. The provisions of working stress design can then be used to check the stresses at transfer, and at service loads in the section so designed. A rational design of a section is considerably simpler by ultimate design than by service load design.

A simply supported bonded beam is considered, and it is assumed that the strength of the beam is measured by flexure. It is assumed that the only loads acting—in addition to the prestressing force—are the weight of the beam, the superimposed dead load and live load.

NOTATION

The following symbols have been adopted for use in this paper:

a = distance from the neutral axis to the top fiber
A = gross cross-sectional area of the beam
A_s = area of prestressed steel
A_s' = area of non-prestressed compression steel
b = width of compression zone or top flange
b' = web thickness
d = distance from the center of gravity of prestressed steel to the top fiber
d' = distance from the center of gravity of the non-prestressed compression steel to the top fiber
$F(\epsilon_{su})$ = f_{su}, equation of the stress-strain diagram of prestressed steel
$f(\epsilon)$ = stress in the concrete, equation of the stress-strain diagram of concrete
f_c' = cylinder strength of concrete at 28 days
f_{su} = stress in prestressed steel at failure
f_{su}' = stress in non-prestressed compression steel at failure
f_y = yield point of non-prestressed compression steel
$G(\epsilon_{su}')$ = f_{su}' equation of the stress-strain diagram of non-prestressed compression steel
h = overall depth of the beam

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L = span length of a simply supported beam
M_g = moment due to the weight of the beam
M_ℓ = moment due to the live load
M_s = moment due to the superimposed dead load or slab
M_w = moment due to any dead load acting on the roadway slab
M_u = required flexural strength of the beam
M_{cu} = flexural strength of composite section
N_d = load factor for the dead load
N_ℓ = load factor for the live load
p = percentage of prestressed steel, A_s/bd
p' = percentage of non-prestressed compression steel, A_s'/bd
Q = $M_u/bd^2 f_c'$
S = effective width of slab in composite section
t = flange thickness
t_s = thickness of slab
γ = unit weight of concrete
ϵ = strain
ϵ_{ce} = strain in concrete at the level of steel due to effective prestress
ϵ_{se} = strain in the prestressed steel due to effective prestress
$\epsilon_{s\ell}$ = limiting strain in prestressed steel
ϵ_{su} = strain in the prestressed steel at ultimate
ϵ_{su}' = strain in the non-prestressed compression steel at ultimate
ϵ_u = ultimate strain of concrete in flexural compression
ϵ_y = strain at yield of non-prestressed steel
φ = curvature of the section
ψ = a dimensionless shape factor, A/bh

ANALYSIS OF PRESTRESSED CONCRETE BEAMS AT ULTIMATE

Analysis of a prestressed concrete beam at ultimate is discussed for beams having an idealized section as shown in Figure 1. The section considered is flanged, the prestressed steel is assumed to be bonded to concrete, and in addition to prestressed steel the section is assumed to have non-prestressed compression steel. Detailed studies of flexural strength of prestressed concrete beams have been reported previously (1, 2, 3); the presentation here is brief, and is in a form suitable for ultimate design.

The calculation of the ultimate moment is based on the following assumptions:

1. The strain distribution in concrete varies linearly with depth in the compression zone of the beam.
2. The stress-strain diagrams for the prestressed as well as non-prestressed reinforcement are known; the stress-strain diagram for concrete is known and is the same for all fibers in the compression zone.
3. Failure occurs when the strain in concrete at the top fiber reaches a limiting value.
4. The strain in non-prestressed compression steel is equal to the strain in concrete at the level of compression steel.
5. The average strain in steel is not greatly different from the maximum strain, hence the area of steel is concentrated at its centroid.

In addition to the above assumptions, the tension contributed by concrete is usually neglected since it is small at ultimate.

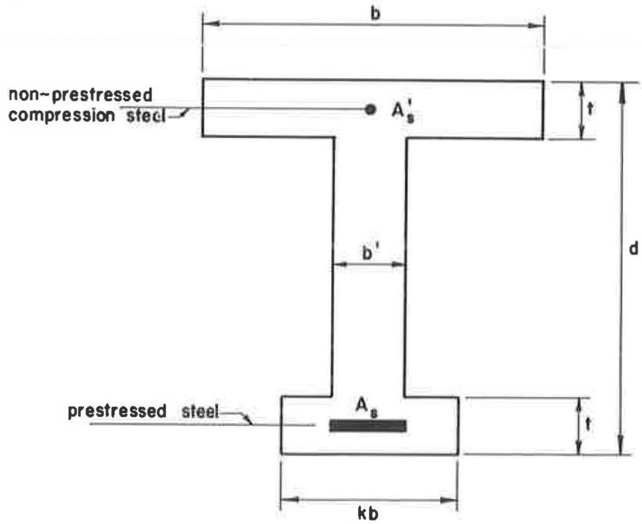

Figure 1. Idealized I-section.

The neutral axis at failure may be either in the flange or below the flange depending on the dimensions of the beam, the amount of steel and the properties of steel and concrete. The case in which the neutral axis falls in the flange is considered first.

Flexural Strength of Section in Which the Neutral Axis at Ultimate Falls in the Flange

In this case the width of the compression zone is constant and is equal to b (Fig. 2). The equation for the stress-strain diagram for concrete is expressed as $f = f(\epsilon)$. Since the width of the compression zone is constant and the strain distribution is assumed to be linear with depth in the compression zone, equations of equilibrium of moments and forces in the section may be written as

$$\frac{a^2 b}{\epsilon_u^2} \int_0^{\epsilon_u} \epsilon f(\epsilon) d\epsilon + A_s f_{su}(d - a) + A'_s f'_{su}(a - d') = M_u \tag{1}$$

$$\frac{ab}{\epsilon_u} \int_0^{\epsilon_u} f(\epsilon) d\epsilon + A'_s f'_{su} = A_s f_{su} \tag{2}$$

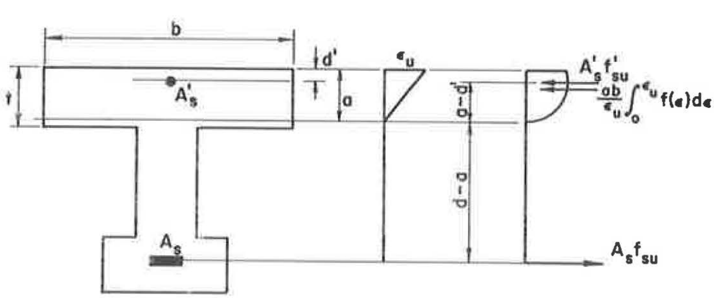

Figure 2. Flanged section, neutral axis in the flange.

where

M_u = ultimate moment,
a = distance from neutral axis to the top fiber,
ϵ_u = limiting strain at the extreme fiber of the beam which defines the condition of flexural failure,
f_{su} = stress in prestressed steel at failure,
f'_{su} = stress in non-prestressed compression steel at failure,
b = width of compression zone or top flange,
d = distance from the center of gravity of prestressed steel to the top fiber,
d′ = distance from the center of gravity of the non-prestressed compression steel to the top fiber,
A_s = area of prestressed steel, and
A'_s = area of non-prestressed compression steel.

The strain in prestressed steel and non-prestressed compression steel is given by

$$\epsilon_{su} = \epsilon_{se} + \epsilon_{ce} + \frac{\epsilon_u}{a}(d - a)\,F \tag{3}$$

and

$$\epsilon'_{su} = \frac{\epsilon_u}{a}(a - d') \tag{4}$$

where

ϵ_{su} = strain in prestressed steel at failure,
ϵ_{se} = strain in prestressed steel due to effective prestress,
ϵ_{ce} = strain in concrete at the level of steel due to effective prestress,
ϵ'_{su} = strain in non-prestressed compression steel at failure, and
F = a strain compatibility factor taken as unity.

The stress-strain relations for prestressed and non-prestressed compression steel are given by

$$f_{su} = F(\epsilon_{su}) \tag{5}$$

$$f'_{su} = G(\epsilon'_{su}) \tag{6}$$

The first term on the left side of Eqs. 1 and 2 is the force and moment contributed by concrete respectively and does not take into account the area of concrete replaced by the compression steel. This effect is small and if necessary can be taken into account.

In order to analyze a beam with given dimensions and a specified value for ϵ_u, Eqs. 1 through 6 can be solved simultaneously for the 6 unknowns, M_u, a, ϵ_{su}, f_{su}, ϵ'_{su} and f'_{su}.

Flexural Strength of Section in Which the Neutral Axis at Ultimate Falls Below the Flange

When the neutral axis at ultimate falls below the flange Eqs. 1 and 2 should be replaced by the following 2 equations:

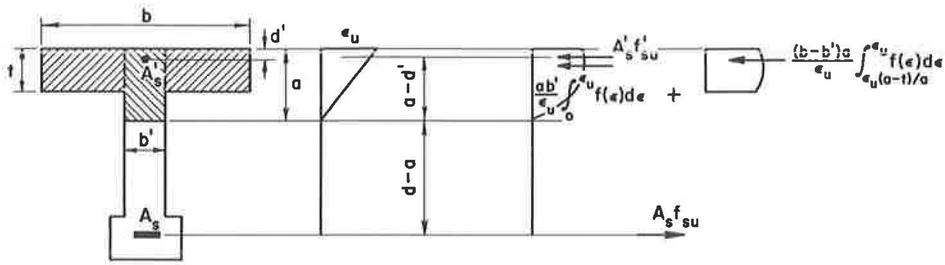

Figure 3. Flanged section, neutral axis below the flange.

$$M_u = \frac{b'a^2}{\epsilon_u^2}\int_0^{\epsilon_u} \epsilon f(\epsilon)d\epsilon + \frac{(b - b')\,a^2}{\epsilon_u^2}\int_{\epsilon_u(a - t)/a}^{\epsilon_u} \epsilon f(\epsilon)d\epsilon$$

$$+ A_s f_{su}(d - a) + A_s' f_{su}'(a - d') \qquad (7)$$

$$\frac{b'a}{\epsilon_u}\int_0^{\epsilon_u} f(\epsilon)d\epsilon + \frac{(b - b')\,a}{\epsilon_u}\int_{\epsilon_u(a - t)/a}^{\epsilon_u} f(\epsilon)d\epsilon + A_s' f_{su}' = A_s f_{su} \qquad (8)$$

where

b' = the web thickness, and
t = the flange thickness.

Equations 7 and 8 describe the equilibrium of moments and horizontal forces respectively. Figure 3 shows the forces in the section.

Expressions for Ultimate Moment in Dimensionless Form

For convenience in design Eqs. 1 and 2 will be expressed in dimensionless form as follows:

$$Q = \frac{M_u}{bd^2 f_c'} = \frac{(a/d)^2}{f_c' \epsilon_u^2}\int_0^{\epsilon_u} \epsilon f(\epsilon)d\epsilon + p\frac{f_{su}}{f_c'}\left(1 - \frac{a}{d}\right) + p'\frac{f_{su}'}{f_c'}\left(\frac{a}{d} - \frac{d'}{d}\right) \qquad (1a)$$

$$\frac{a/d}{\epsilon_u f_c'}\int_0^{\epsilon_u} f(\epsilon)d\epsilon + p'\frac{f_{su}'}{f_c'} = p\frac{f_{su}}{f_c'} \qquad (2a)$$

where $p = \frac{A_s}{bd}$, $p' = \frac{A_s'}{bd}$, and f_c' = cylinder strength of concrete at 28 days.

Similarly Eqs. 7 and 8 may be expressed in dimensionless form:

$$Q = \frac{M_u}{bd^2 f'_c} = \frac{(b'/b)(a/d)^2}{f'_c \epsilon_u^{\ 2}} \int_0^{\epsilon_u} \epsilon f(\epsilon) d\epsilon + \frac{(1 - b'/b)(a/d)^2}{f'_c \epsilon_u^{\ 2}} \int_{\epsilon_u (a-t)/a}^{\epsilon_u} \epsilon f(\epsilon) d\epsilon$$

$$+ p \frac{f_{su}}{f'_c} \left(1 - \frac{a}{d}\right) + p' \frac{f'_{su}}{f'_c} \left(\frac{a}{d} - \frac{d'}{d}\right) \qquad (7a)$$

and

$$\frac{(b'/b)(a/d)}{\epsilon_u f'_c} \int_0^{\epsilon_u} f(\epsilon) d\epsilon + \frac{(1 - b'/b)(a/d)}{\epsilon_u f'_c} \int_{\epsilon_u (a-t)/a}^{\epsilon_u} f(\epsilon) d\epsilon + p' \frac{f'_{su}}{f'_c} = p \frac{f_{su}}{f'_c} \qquad (8a)$$

ULTIMATE DESIGN

Ultimate design of a prestressed concrete beam is based on the ultimate moment and ductility of the section. The section is proportioned in such a way that the ultimate moment is greater than the moment developed under service loads by a prescribed quantity, and that it deforms a certain amount before it fails.

These concepts may be stated in the form

$$M_u \geq N_d (M_g + M_s) + N_\ell M_\ell \qquad (9)$$

and

$$\epsilon_{su} \geq \epsilon_{s\ell} \qquad (10)$$

where

M_u = flexural strength of the beam,

N_d = load factor for the dead load,

M_g = moment due to weight of the beam,

M_s = moment due to the superimposed dead load,

N_ℓ = load factor for the live load,

M_ℓ = moment due to the live load,

ϵ_{su} = strain in steel at ultimate, and

$\epsilon_{s\ell}$ = limiting strain in steel.

Expression 9 states that the required flexural strength of the beam should be at least equal to $N_d(M_g + M_s) + N_\ell M_\ell$, which is a requirement for the strength of the beam.

Expression 10 states that the ductility of the beam should be large enough so that the strain in steel at ultimate is at least equal to a given limiting value designated as $\epsilon_{s\ell}$. Ductility is usually measured by the curvature at ultimate, which may be defined as follows:

$$\varphi = \frac{\epsilon_u}{a} = \frac{\epsilon_{su} - \epsilon_{se} - \epsilon_{ce} + \epsilon_u}{d}$$

where φ is the curvature of the section. For given values of ϵ_{se}, ϵ_{ce}, ϵ_u and d, ϵ_{su} may be used as a measure of ductility.

Determination of the Area of the Beam

Expression 9 can be written as an equation in the following form:

$$Qbd^2f_c' = N_d(M_g + M_s) + N_\ell M_\ell$$

Substituting $A/h\psi$ for b where A is the gross cross-sectional area of the beam, h is the over-all depth, and ψ is a dimensionless shape factor, the following is obtained:

$$M_g = \frac{\gamma AL^2}{8} = Q\frac{A}{h\psi}\, d^2 f_c' \,\frac{1}{N_d} - \frac{N_\ell}{N_d} M_\ell - M_s$$

and

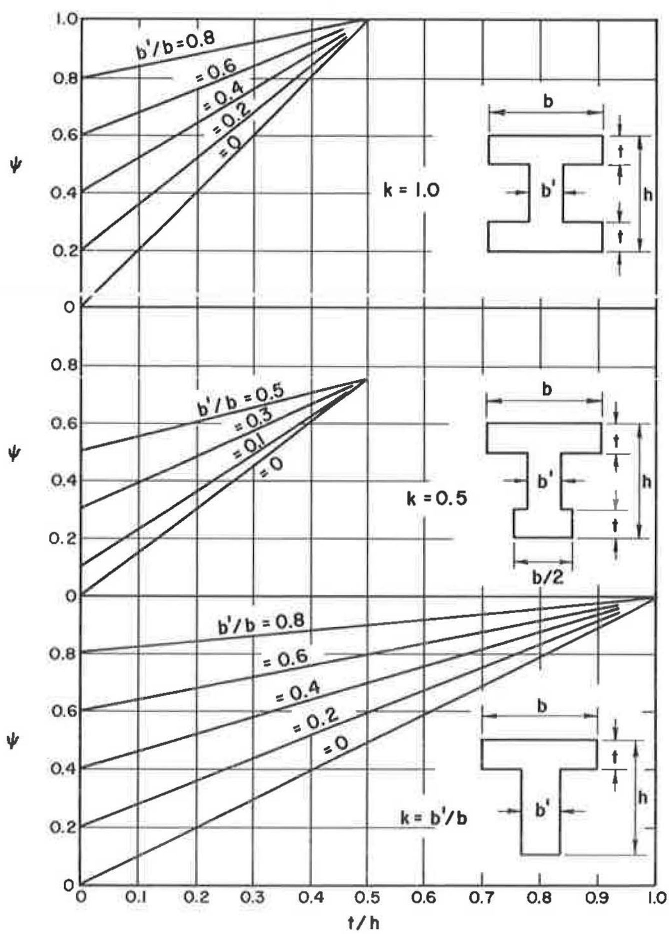

Figure 4. Relationship between ψ and geometric parameters of the section.

$$A = \frac{M_s + \frac{N_\ell}{N_d} M_\ell}{\frac{d^2 f'_c Q}{h\psi\, N_d} - \frac{\gamma L^2}{8}} \tag{11}$$

where γ is the unit weight of concrete.

For the idealized I-section shown in Figure 1, ψ is given by

$$\psi = \frac{t}{h}\,(1 + k) + \frac{b'}{b}\left(1 - 2\frac{t}{h}\right) \tag{12}$$

The quantity k in Eq. 12 is the ratio of the width of bottom flange to that of top flange. Equation 12 is plotted in Figure 4 for typical sections.

A study of Eq. 11 indicates that for a given design problem, in which the depth and type of concrete are specified, A depends on ψ and Q only. It can be seen that A decreases with Q and increases with ψ; i.e., to decrease the area of the beam it is necessary to increase Q and decrease ψ, or to increase the ratio Q/ψ.

The quantities t/d and b′/b usually decrease with increasing Q/ψ; hence they should be made small, without causing the dimensions of the beam to become unreasonably thin.

From Eq. 12 it can be seen that ψ increases and Q/ψ decreases with k. Therefore a small bottom flange is desirable. However, since the bottom flange of the beam should be large enough to permit the placing of steel, k cannot be reduced indefinitely.

From Eqs. 1a and 7a it can be seen that Q increases with a/d and hence it is desirable to make a/d as large as possible; however, Expression 10 for the required ductility sets the upper limit for a/d. Since Expression 10 sets the required minimum ductility of the beam at a strain in steel equal to $\epsilon_{s\ell}$, the required maximum a/d consistent with the required ductility can be computed from Eq. 3 as follows:

$$(a/d)_{max} = \frac{\epsilon_u}{\epsilon_{s\ell} - \epsilon_{se} - \epsilon_{ce} \quad \epsilon_u} \tag{3a}$$

Equation 3a contains the quantity ϵ_{se}, the strain in steel due to effective prestress. It can be seen that since ϵ_{se} increases with the maximum value of a/d, it should be taken as large as practicable. The practical upper limit for ϵ_{se} for the materials used in pretensioned construction is about 0.005.

It should be pointed out that d/h also influences A, the area of the beam, and from Eq. 11 it can be seen that A decreases with d/h. In most practical problems, however, d/h cannot exceed 0.9.

Design Procedure

In the design method presented here it is assumed that the span length, the acting load, the load factors, the strength and unit weight of concrete are given. It is further assumed that the limiting strain in concrete ϵ_u, the requirement of ductility $\epsilon_{s\ell}$, the effective prestrain ϵ_{se}, as well as the stress-strain relations for all materials are given. Hence for a selected value of h, the calculation of A from Eq. 11 means determination of d^2Q/ψ. The quantities d, ψ and Q may be determined as follows:

1. Assign a reasonable value to d as close to h as the arrangement of strands would permit.
2. Assign values to b′/b, t/h and k, and calculate ψ from Eq. 12. These values should be as small as possible.
3. Calculate a/d from Eq. 3a based on the given values of ϵ_u, ϵ_{se} and $\epsilon_{s\ell}$.

4. Calculate $p(f_{su}/f_c')$ from either Eq. 2 a or 8a, whichever applies.
5. Calculate Q from Eq. 1a or 7a, whichever applies.

EXAMPLES OF APPLICATION

Example 1

The following example illustrates the procedure for the ultimate design of a prestressed concrete beam and shows the influence of the required ductility on the dimensions of the beam so designed.

Given a simply supported beam of 54-ft span subjected to a superimposed dead load of 1.0 klf and a live load of 0.6 klf which produce midspan moments of M_s = 4370 in-k and M_ℓ = 2630 in-k respectively. The load factors are N_d = 1.5 and N_ℓ = 1.8. Design the section (a) for a minimum ductility corresponding to $\epsilon_{s\ell}$ = 0.01, and (b) for a mini-

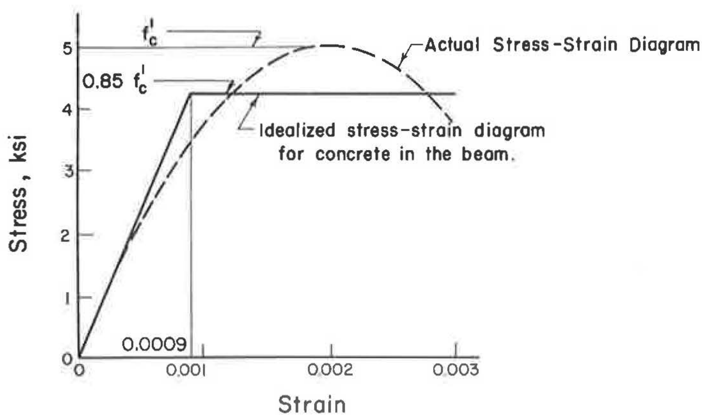

Figure 5. Stress-strain diagram for concrete.

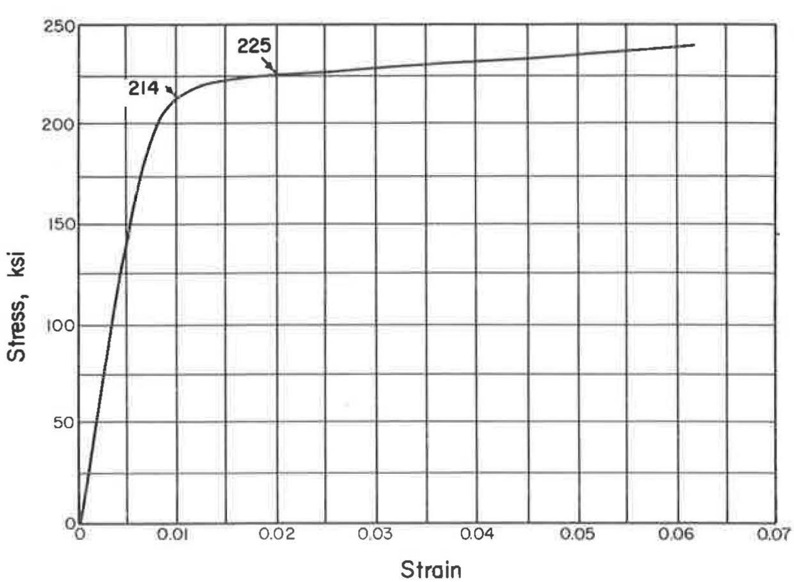

Figure 6. Stress-strain diagram for steel.

mum ductility corresponding to $\epsilon_{s\ell} = 0.02$. Non-prestressed compression steel is not to be used.

The effective prestress or the prestress after losses is given as 145 ksi, which corresponds to a transfer prestress of 170 ksi. The strain due to effective prestress is $\epsilon_{se} = 0.0048$. The quantity ϵ_{ce} is approximated as 0.0005 initially, which may be verified after the section is designed. The limiting strain in concrete is given as $\epsilon_u = 0.003$, unit weight of concrete as $\gamma = 0.15$ kcf, and overall depth as h = 36 in. The strength of concrete f_c' is specified as 5 ksi. The stress-strain diagram for concrete and steel are as shown in Figures 5 and 6 respectively.

1a. Section With Minimum Required Ductility Corresponding to $\epsilon_{su} = 0.01$.—It was shown before that the quantities t/h, b′/b and k increase with A, and thus they should be taken as small as possible. Here they will be taken as t/h = 1/6 (or t = 6 in.), b′/b = 0.3 and k = 1.0. Substitution of these values in Eq. 12 gives $\psi = 0.533$. It is further assumed that d/h = 0.9, which for h = 36 in. yields d = 32.4 in. For the given ductility, $\epsilon_{su} = \epsilon_{s\ell} = 0.01$, Eq. 3a gives depth to the neutral axis as a = 12.64 in. Since in this case the neutral axis is in the web, Eqs. 7a and 8a apply.

From Figure 5, $f(\epsilon) = 4722\epsilon$ when $\epsilon \leq 0.0009$, and $f(\epsilon) = 4.25$ when $\epsilon \geq 0.0009$, the quantity $p(f_{su}/f_c')$ may be calculated from Eq. 8a as follows:

$$p\frac{f_{su}}{f_c'} = \frac{(0.3)(0.390)}{(0.003)\ (5)}\left[\int_0^{0.0009} 4722\epsilon d\epsilon + \int_{0.0009}^{0.003} 4.25\ d\epsilon\right]$$

$$+ \frac{(0.70)(0.390)}{(0.003)\ (5)}\int_{0.00158}^{0.003} 4.25\ d\epsilon = 0.085 + 0.11 = 0.195$$

For the above value of $p(f_{su}/f_c')$, in a similar way Q is obtained from Eq. 7a:

$$Q = \frac{(0.3)(0.390)^2}{(5)\ (0.003)^2}\left[\int_0^{0.0009} 4722\ \epsilon^2 d\epsilon + \int_{0.0009}^{0.003} 4.25\ \epsilon d\epsilon\right]$$

$$+ \frac{(1-0.3)(0.390)^2}{(5)\ (0.003)^2}\int_{0.00158}^{0.003} 4.25\ \epsilon d\epsilon + 0.195\ (1-0.390) = 0.171$$

From Eq. 11 the area of the beam is 284 in^2. In addition, the following quantities are obtained: b = b′ = 14.8 in.; b′ = 0.3 × 14.8 = 4.4 in.

The stress-strain diagram for steel (Fig. 6) yields $f_{su} = 214$ ksi. The amount of prestressing can be found from $p(f_{su}/f_c') = 0.195$ to be p = 0.00455 from which $A_s = 2.18\ in^2$. A total of sixteen ½-in. strands is needed. Each ½-in. strand has an area of 0.1438 in^2. The final dimensions of the section in this solution are shown in Figure 7a. The bottom flange has been widened to properly accommodate the reinforcement, and it is tapered to facilitate construction. The final width of the top flange is taken the same as that of bottom flange to maintain the symmetry of the section originally assumed. The properties of the gross section and the transformed section as well as the stresses at the top and bottom fibers before and after losses for both assumptions are given in Table 1.

1b. Section With Minimum Required Ductility Corresponding to $\epsilon_{su} = 0.02$.—The ultimate strain in the steel required for this example is large, and is not necessarily

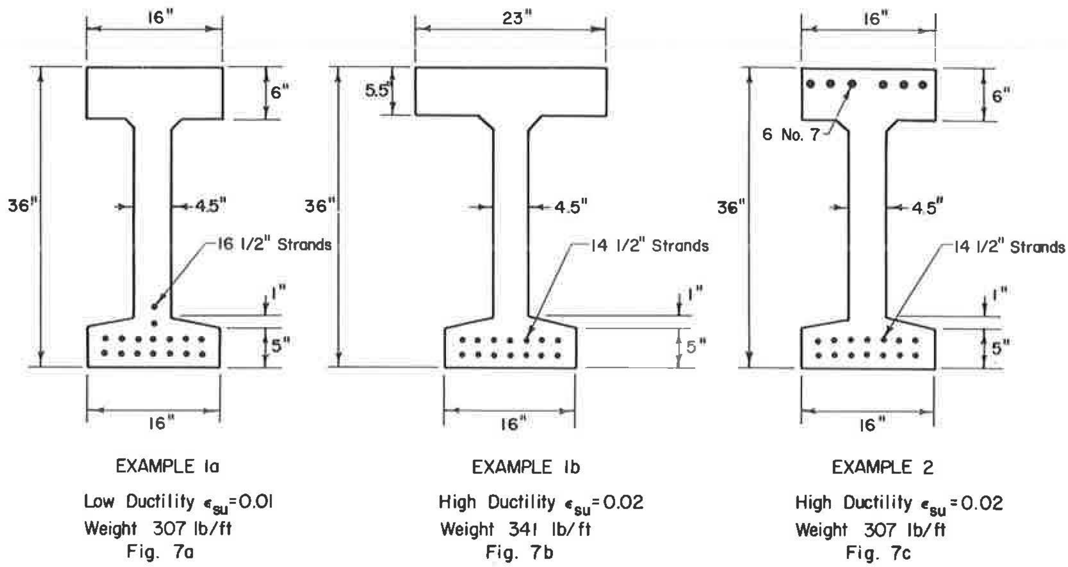

Figure 7. Sections of design examples.

TABLE 1

SUMMARY OF SECTION PROPERTIES AND STRESSES FOR SECTIONS OBTAINED BY ULTIMATE DESIGN
(For each example the results shown are based first on the gross area of the section and second on the transformed section assuming n = 7. Negative stresses are tensile.)

Section	A (in^2)	y_t (in.)	y_b (in.)	I (in^4)	A_s (in^2)	A_s' (in^2)	Weight (lb/ft)	Stress Before Losses (Transfer) (ksi)		Stress After Losses (ksi)	
								Top (Tens.)	Bottom (Comp.)	Top (Comp.)	Bottom (Tens.)
Example 1a	294	17.76	18.24	48,080	2.30	—	307	-0.34	3.04	2.37	-0.13
ϵ_{su} = 0.01	308	18.43	17.57	51,040				-0.27	2.73	2.37	-0.14
Example 1b	327	16.20	19.80	55,230	2.01	—	341	-0.23	2.60	1.92	-0.37
ϵ_{su} = 0.02	339	16.81	19.19	58,610				-0.18	2.36	1.92	-0.35
Example 2	294	17.76	18.24	48,080	2.01	3.60	307	-0.30	2.66	2.40	-0.46
ϵ_{su} = 0.02	328	17.29	18.71	56,270				-0.22	2.41	2.02	-0.34

A = area; y_t = distance from centroidal axis to top fiber; y_b = distance from centroidal axis to bottom fiber; I = moment of inertia; A_s = area of prestressed steel; A_s' = area of non-prestressed compressive steel; n = modular ratio for both types of steel; f_c' = strength of concrete; f_{ci}' = strength of concrete at transfer; prestress at transfer = 170 ksi; effective prestress after losses = 145 ksi. The underlined quantities correspond to the results obtained on the basis of the transformed section.

used in practice. It has been selected to show that direct design for the largest levels of ductility is possible, and to study how it affects the shape of the section.

All the quantities are the same as in Example 1a except that in this case $\epsilon_{su} = 0.02$, which corresponds to a higher ductility. Since a higher required ductility results in a wider top flange, and the bottom flange need only be large enough to accommodate the reinforcement, k is taken as 0.75 in this case. For a similar reason the quantity b′/b is taken as 0.2. Taking t = 5.5 in., t/h = 0.153, Eq. 12 yields $\psi = 0.407$. As before, d = 32.4 in.

From Eq. 3a we obtain a = 5.49 in., which places the neutral axis in the flange. For the stress-strain diagram adopted for concrete from Eq. 2a we have $p(f_{su}/f_c') = 0.122$, and from Eq. 1a we obtain $Q = 0.113$. From Eq. 11, $A = 336$ in^2., and $b = 22.9$ in., $b' = 4.6$ in., $kb = 17.2$ in.

From the stress-strain diagram for steel (Fig. 6), $f_{su} = 225$ ksi. From $p(f_{su}/f'_c) = 0.122$, $p = 0.00271$, from which $A_s = 2.01$ in^2. Fourteen ½-in. are strands needed. Figure 7b shows the final section of the beam. The dimension of the bottom flange is the minimum required to accommodate the prestressing steel at the required depth. As kb turned out to be larger than necessary, only the minimum required was used, because the bottom flange does not contribute to the strength and ductility of the section. Had the adjustment of the dimensions been large, recalculation may have been necessary to improve the shape of the section. The properties of the section and the stresses before and after losses for this part are given in Table 1.

Example 2

In order to show that the non-prestressed compressive reinforcement increases the ductility without increasing the area of the section, the following example is presented. It is required to design the section in Example 1 in such a way that for a ductility corresponding to $\epsilon_{su} = 0.02$, the area of the section will be the same as that for a ductility corresponding to $\epsilon_{su} = 0.01$. The yield point stress of the compressive reinforcement may be assumed as $f_y = 50$ ksi. The section designed in Example 1a has a ductility corresponding to $\epsilon_{su} = 0.01$. The problem is to determine how much compressive steel of the type given should be placed so that the ductility of the section will reach that corresponding to $\epsilon_{su} = 0.02$.

The distance of the neutral axis from the top fiber was determined as $a = 5.49$ in. in Example 1b for the same required ϵ_{su}. Since in this case the neutral axis falls in the flange, Eqs. 1a and 2a may be used with $Q = 0.171$, as in Example 1a, and $d' = 2$ in., to write the following independent relations between $p(f_{su}/f'_c)$ and $p(f'_{su}/f'_c)$:

$$0.171 = 0.012 + p\frac{f_{su}}{f'_c}(1 - 0.170) + p'\frac{f'_{su}}{f'_c}(0.170 - 0.062) \quad (1a)$$

$$0.122 + p'\frac{f'_{su}}{f'_c} = p\frac{f_{su}}{f'_c} \quad (2a)$$

The simultaneous solution of the above equations yields $p(f_{su}/f'_c) = 0.183$ and $p'(f'_{su}/f'_c) = 0.061$.

From Figure 6, $\epsilon_{su} = 0.02$ corresponds approximately to $f_{su} = 225$ ksi. Therefore, $p = 0.00407$ and $A_s = 1.95$ in^2. A total of fourteen ½-in. strands is needed. Since the strain in compression steel is $0.003(3.49/5.49) = 0.0019 > \epsilon_y$ the intermediate grade steel has yielded and the net f'_{su} is $50 - 4.25 = 45.75$ ksi. Therefore, $p' = 0.00666$, $A'_s = 3.20$ in^2. and six No. 7 bars of intermediate grade steel are required. The length of these non-prestressed bars need not be the total span of the beam. Theoretically they are not needed at a section where the required Q is that of the section without the compression reinforcement. The properties of the section and the stresses before and after losses for this example are shown in Table 1; Figure 7c shows the beam section.

A reduction in the amount of non-prestressed compression reinforcement is possible with a section having a wider top flange. The parameter $p(f'_{su}/f'_c)$ is related to $p(f_{su}/f'_c)$ by Eq. 2a. Selection of a smaller value of $p(f'_{su}/f'_c)$ would fix $p(f_{su}/f'_c)$ and permit the determination of the required Q by Eq. 1a. The area of the section and its final shape can be determined as usual from Eq. 11. If the proper values of t/h, b'/b, and k were selected, the new section will present a flange wider than that of Example 1a, but not as large as that of Example 1b. Also, the compressive reinforcement required will be smaller than that of Example 2. This solution would show that to obtain high ductility a compromise section can be obtained if some increment of weight is tolerated with a smaller amount of non-prestressed compression steel.

Comparison of the Three Solutions

It has been shown that ultimate strength design provides a convenient procedure which leads to well-proportioned sections. The desired ductility and strength were used as the fundamental constraints for proportioning the sections, while the stresses at transfer and under service loads were checked.

An examination of Table 1 shows interesting details. The beam of Example 1a with a required ductility corresponding to $\epsilon_{su} = 0.01$ required more prestressing steel (2 strands) than the beams of Examples 1b and 2 with a required ductility corresponding to $\epsilon_{su} = 0.02$.

For the stress-strain diagram of prestressing steel adopted in these examples, any increase in ductility is accompanied by an increase in stress in steel at ultimate. For the larger ductility considered here the stress in steel increases at ultimate from 214 ksi to 225 ksi. This increase in steel stress causes a decrease in the required area of prestressing steel.

The beam of Example 1b shows that by increasing the width of the top flange and thereby adding concrete area to the compression zone, high ductility can be obtained. This, however, increases the weight of the section by 11 percent, but decreases the amount of prestressing steel to 14 strands. The increase in stress in steel at ultimate not only supports the additional weight of the beam, but also permits a reduction in the required area of steel. Under the service loads this beam shows, however, a tendency for a larger tensile stress at the bottom fiber due to the smaller amount of prestressing force.

The beam of Example 2 shows a different way of obtaining high ductility. Six No. 7 intermediate grade bars are added to the top flange of the low ductility section of Example 1a. This increment in compression area raises the neutral axis and increases the lever arm of the resisting couple by approximately 7 percent. In addition the stress in the steel at ultimate is increased from 214 ksi to 225 ksi, approximately 5 percent. These 2 factors combined explain the 12 percent reduction in the number of prestressing strands, from 16 to 14, since the required tensile force at ultimate can be obtained with less area of steel at a higher stress and a larger lever arm. The non-prestressed bars also provide additional tensile strength for the top part of the beam at transfer and during handling operations. Furthermore, they have a tendency to reduce the inelastic deflections due to creep.

SUMMARY AND CONCLUSIONS

A method is presented by which a prestressed concrete beam can be proportioned by ultimate design. Particular emphasis has been placed on the requirement of ductility and its influence on the dimensions of the section. The design examples presented show the actual method of proportioning as well as the influence of ductility on the dimensions of the beam.

The following conclusions may be drawn from the study.

1. A prestressed concrete beam can be proportioned for given required minimum flexural strength and ductility. The stresses at transfer and at service conditions may be checked in a section thus obtained.
2. The dimensions of a section are influenced greatly by the required ductility. An increase in the required ductility results in an increase in the required area of the section, unless compression steel is provided.
3. Compression steel contributes appreciably to the ductility of the section. Example 2 shows that the most expeditious way for increasing the ductility of a section is by placing non-prestressed compression reinforcement as near the top fiber as possible.

For a large required ductility considerable saving in the area of the beam may be effected by use of non-prestressed compression steel. Compression steel has additional advantages such as its contribution to the crack stability of the top fiber, its use as spacer for the web reinforcement and its function in providing more safety for the beam during transportation and erection.

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Appendix

DESIGN OF COMPOSITE SECTIONS

The method discussed in the preceding sections may be used to design composite prestressed concrete beams on the basis of safety and ductility. Composite construction, which is used extensively in highway bridges, consists of a reinforced concrete slab cast in place on top of precast prestressed concrete stringers. It is assumed that the shear connection between slab and stringer is strong enough to develop the flexural strength of the composite section.

Design of stringers in highway bridges is based on the assumption that a known portion of the loads acts on each stringer, and that a strip of the roadway slab contributes to support any load which comes on the bridge after the roadway concrete has set. Thus the problem is reduced to the design of a stringer with a slab of given width, thickness and strength at the top. The slab becomes an integral part of the stringer section for all the loads acting after the slab concrete has hardened. The loads as well as the effective slab section available for each stringer are different for the end and intermediate stringers.

The strength of a composite section, which consists of the stringer and the slab of known width and thickness, is calculated assuming that it is a unit all by itself. Strength calculated in this fashion provides only a measure of safety, and should not be confused with the safety of the entire bridge for the intended loads.

In the discussion that follows it will be assumed that a concrete deck of known thickness is cast on top of parallel stringers at a given spacing, and that a portion of the deck slab behaves compositely with each stringer. The design of the stringer is to be based on the prescribed strength and ductility of the composite section as well as that of the stringer section by itself.

The stringer should be designed such that the following two inequalities are satisfied in the composite section:

$$M_{cu} \geq N_d(M_g + M_s + M_d + M_w) + N_\ell M_\ell \tag{9a}$$

and

$$\epsilon_{csu} \geq \epsilon_{s\ell} \tag{10a}$$

The above inequalities are similar to Inequalities 9 and 10. The quantity M_{cu} is the flexural strength of the composite section; ϵ_{csu} is the strain in the prestressing steel at failure of the composite section; M_s denotes the moment caused by the weight of that portion of the slab which contributes to the composite section; M_d is the moment due to any additional dead load that may be present before the slab concrete has set, and M_w is the moment due to any dead load (such as wearing surface) that may act on the structure after the slab concrete has set.

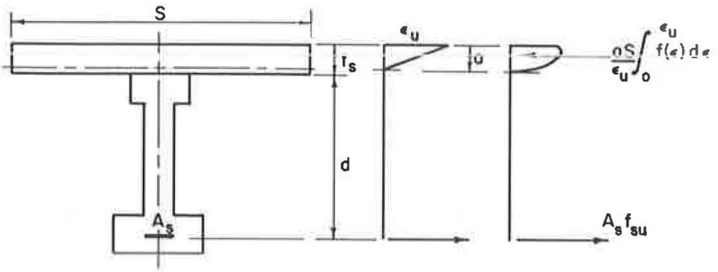

Figure 8. Composite section.

In addition, the following inequality should be satisfied in the stringer section:

$$M_u \geq N_d (M_g + M_s + M_d) \tag{9b}$$

Thus Inequalities 9a, 10a and 9b constitute the basis of design.

The condition expressed by Inequality 10a can easily be satisfied in a composite section, since a comparatively large area of concrete is available in the compression zone. Figure 8 shows a ductile beam in which the neutral axis at failure is in the slab. It can be seen that a considerable compression force may be developed in the slab if the width of slab is large, even though the strength of concrete in the slab is usually in the neighborhood of 3000 psi.

The effective width of slab is usually taken as the center-to-center spacing of stringers, which ranges between 4 and 7 ft. A width of this order of magnitude for the slab provides large compression forces even for high ductilities. On the other hand the higher the ductility, the smaller the available compression force and the required area of prestressing steel. Hence if the ductility is too high, the requirement expressed by Inequality 9a may no longer be satisfied. Thus it can be seen that although high ductility is available, Inequality 9a provides an upper limit for it. In practice the highest ductility compatible with Inequality 9a should be used.

The requirement expressed by Inequality 9b assures that the flexural strength of the stringer section is adequate. Where the available area of slab is large, this requirement is automatically satisfied.

Determination of Area and Prestressing Force

The preceding discussions may be expressed conveniently in algebraic form for use in design. It is assumed that the neutral axis of the composite section at failure will fall in the slab. Ignoring the effect of non-prestressed compression reinforcement, if any, and designating S and t_s respectively as the width and thickness of the slab, from Eqs. 1 and 2 the following can be derived:

$$M_{cu} = \frac{a^2 S}{\epsilon_u^2} \int_0^{\epsilon_u} \epsilon f(\epsilon) d\epsilon + A_s f_{su} (d - a) \tag{1b}$$

and

$$A_s f_{su} = \frac{aS}{\epsilon_u} \int_0^{\epsilon_u} f(\epsilon) d\epsilon \tag{2b}$$

where

$$a = \frac{\epsilon_u}{\epsilon_{s\ell} - \epsilon_{se} - \epsilon_{ce} + \epsilon_u}(d + t_s) \qquad (3b)$$

The substitution of Eq. 1b for M_{cu} and $(A\gamma L^2)/8$ for M_g in Inequality 9a yield the following expression for A, the area of the stringer:

$$A \leq \frac{8}{\gamma L^2}\left[\frac{a^2 S}{N_d \epsilon_u{}^2}\int_0^{\epsilon_u} \epsilon f(\epsilon) d\epsilon + \frac{A_s f_{su}(d - a)}{N_d} - \frac{N_\ell}{N_d} M_\ell - M_s - M_d - M_w\right] \qquad (13)$$

Inequality 13 provides an upper bound for A.

Since Inequality 9b for the stringer section is similar to non-composite design, a lower bound for A may be obtained by expressing 9b in the following form:

$$A \geq \frac{M_s + M_d}{\frac{d^2 f_c' Q}{h\psi N_d} - \frac{\gamma L^2}{8}} \qquad (11a)$$

Since Inequality 11a seldom governs in practical problems, it is more convenient to check the flexural strength of the stringer section directly by Inequality 9b.

The steps to be taken in the determination of A and A_s may be summarized as follows:

1. On the basis of a ductility greater than or equal to the prescribed ductility calculate a from Eq. 3b.
2. Calculate A_s from Eq. 2b using the given stress-strain diagram for concrete.
3. Find the upper bound of A from Inequality 13. If A is too large it means that the ductility may be increased further. On the other hand if A is unreasonably small the ductility should be decreased.
4. Determine the proportions of the section and check Inequality 9b.

Example 3

Given a simply supported bridge of 54-ft span consisting of precast prestressed concrete stringers and cast-in-place slab; the roadway slab is 6½ in. thick and the structure is to be designed for H20-S16-44 loading. It is anticipated that the structure will have to support a 2-in. wearing surface. The bridge will have one diagram at midspan connecting the stringers, whose weight is equivalent to a concentrated load of 1.25 k per intermediate stringer. Design an intermediate stringer assuming a center-to-center spacing of 5 ft 0 in. and overall depth of 36 in.

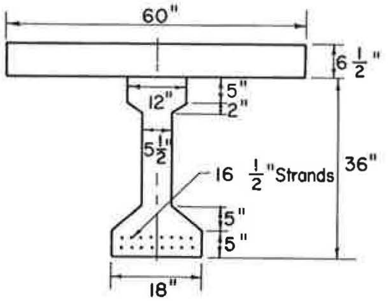

Figure 9. Design Example 3.

The following are given: for slab, f_c' = 3000 psi; for stringer, f_c' = 5000 psi; ϵ_{se} = 0.0048; ϵ_u = 0.003; γ = 0.15 kcf.

The quantity ϵ_{ce} may be taken as 0.0005. The stress-strain diagrams for concrete and prestressing steel are as shown in Figures 5 and 6 respectively.

The moments may be calculated as follows:

TABLE 2

SUMMARY OF SECTION PROPERTIES AND STRESSES IN EXTREME FIBERS OF STRINGER

Section		A (in^2)	y_t (in.)	y_b (in.)	I (in^4)	Stress Before Losses (Transfer) (ksi)		Stress After Losses (ksi)	
						Top (Tens.)	Bottom (Comp.)	Top (Comp.)	Bottom (Tens.)
Gross	Stringer	330.8	20.24	15.76	48050	-0.23	2.16	1.10	-0.03
	Composite	642.8	8.85	27.15	137960				
Transformed	Stringer	344.6	20.78	15.22	50190	-0.17	1.98	1.14	-0.04
	Composite	656.6	9.36	26.64	145800				

In the above calculation the ratio of modulus of elasticity of concrete in the slab to that in the stringer is taken as 0.8. The properties of the transformed section are calculated assuming N = 7 and $A_s = 2.3$ in^2.

$$M_s = 0.406 \times \frac{(54)^2}{8} \times 12 = 1775 \text{ in. k}$$

$$M_d = 1.25 \times \frac{(54)}{4} \times 12 = 203 \text{ in. k}$$

$$M_e = \tfrac{1}{2} \times 699.3 \times \frac{5}{5} \times 1.28 \times 12 = 5370 \text{ in. k}$$

$$M_w = 0.125 \times \frac{(54)^2}{8} \times 12 = 546 \text{ in. k}$$

Assume $\epsilon_{s\ell} = 0.031$, and $d + t_s = 33 + 6.5 = 39.5$ in. From Eq. 3b, a = 4.12 in., from Eq. 2b, $A_s = 2.33$ in^2; from Inequality 13, $A \leq 330$ in^2.

To provide the required area for prestressing steel, sixteen ½-in. strands are used. Figure 9 shows the stringer section and the arrangement of strands. The web is taken as 5½ in. to provide sufficient room for draping. The top flange is made 12 in. wide in order to provide sufficient area to transmit the shearing stresses which occur between slab and stringer in the composite section.

It can be shown that the section of Figure 9 satisfies Inequality 9b. Table 2 shows a summary of the section properties and stresses in the stringer before and after losses.

In the example the effect of non-prestressed reinforcement either in slab or stringer is not taken into account. The non-prestressed compression reinforcement increases the ductility of the stringer section, but is usually under tensile stress at the failure of the composite section. The longitudinal reinforcement in the slab increases the ductility of the composite section, if placed at the top of the slab.