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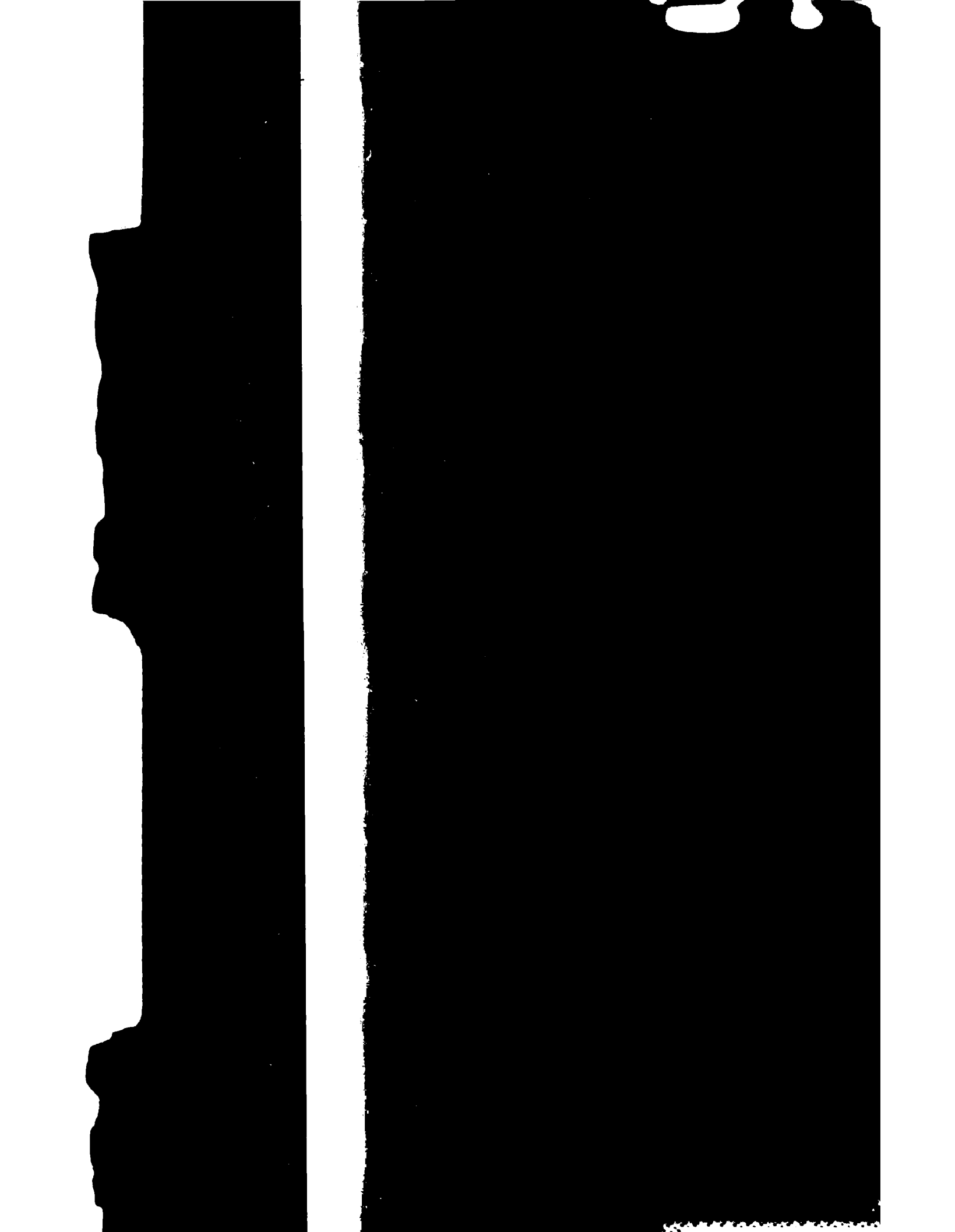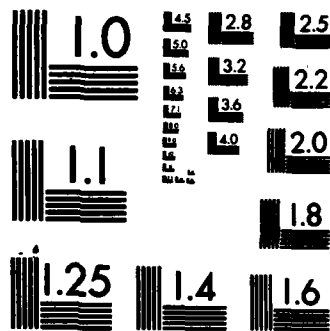# FINITE ELEMENT METHODS FOR THE SOLUTION
OF PROBLEMS WITH ROUGH INPUT DATA

by

I. Babuška

and

J. E. Osborn

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number) This report analyzes a new version of the finite element method when the trial space reflects the structure of the solution and the test spaces guarantee the stability of the method. It is shown on model problems that this approach leads to high accuracy of the approximate solution.

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FINITE ELEMENT METHODS FOR THE SOLUTION
OF PROBLEMS WITH ROUGH INPUT DATA

I. Babuška[+] and J.E. Osborn[‡]

1. Introduction

The finite element method is based on the variational or weak
formulation of the boundary value problem under consideration. The
approximate solution is obtained by restriction of the variational
formulation to finite dimensional trial and test spaces.

Accuracy of the approximate solution is achieved by the choice of
a trial space with good approximation properties and the choice of a
test space which guarantees that the finite element solution is an
approximation of the same quality as is the best approximation of the
exact solution by the trial functions.

The success of the finite element method as a practical computational
tool is related to the special construction of the trial and test func-
tions in terms of "element" trial and test functions (defined on the
finite elements) satisfying appropriate constraints at the nodes. This
element based structure is the basis for the architecture of existing
finite element codes. Usually the "element" trial and test functions
are polynomials.

It is well known that when the solution is rough, which will in
general be the case when the data is rough, then the use of polynomials
as "element" trial functions does not lead to good accuracy. In this
paper we will construct "element" trial functions which reflect the
properties of the problem under consideration. This will lead to
increased accuracy - the maximum possible accuracy - while not changing
the structure of the code. In fact, the approach can be implemented
by changing only the "element" trial functions employed in the method.
The questions of optimal trial functions relative to a class of problems
is also addressed.

The abstract framework of the approach is given in Section 2. In
Section 3, five examples of various types are presented and in terms
of these examples the ideas presented in Section 2 are developed. In
Section 4, we make   some general comments on the design of finite
element methods for problems with rough coefficients.

Detailed proofs of the results in the paper will be presented
elsewhere. The ideas in the paper can be generalized in various
directions.

Throughout the paper we will use the usual L_p spaces and Sobolev spaces H_1, $\mathring{H}_1$.

2. Variational Approximation Methods

In this section we discuss variational approximation methods and state a basic error estimate.

Let $H_1, \|\cdot\|_{H_1}$ and $H_2, \|\cdot\|_{H_2}$ be two Hilbert spaces and let $H_{1,h}, \|\cdot\|_{H_{1,h}}$, $0 < h \leq 1$, $H_{2,h}, \|\cdot\|_{H_{2,h}}$, $0 < h \leq 1$, be two one parameter families of Hilbert spaces satisfying

$$H_1 \subset H_{1,h}, \|u\|_{H_{1,h}} = \|u\|_{H_1} \forall h \text{ and } \forall u \in H_1 \tag{2.1}$$

and

$$H_2 \subset H_{2,h}, \|u\|_{H_{2,h}} = \|u\|_{H_2} \forall h \text{ and } \forall u \in H_2. \tag{2.2}$$

Let $B_h(u,v)$ be a bilinear form on $H_{1,h} \times H_{2,h}$ satisfying

$$|B_h(u,v)| \leq M\|u\|_{H_{1,h}} \|v\|_{H_{2,h}} \forall u \in H_{1,h}, \forall v \in H_{2,h}, \text{ and } \forall h \tag{2.3}$$

(M is independent of h).

Let f be a bounded linear functional on H_2. Corresponding to f we assume that there is a unique $u_0 \in H_1$ satisfying

$$B_h(u_0,v) = f(v) \quad \forall v \in H_2 \text{ and } \forall h. \tag{2.4}$$

u_0 can be thought of as the exact solution to a boundary value problem under consideration and is unknown. B_h and f are given input date and are known.

We are interested in approximating u_0 and toward this end we assume we are given or have chosen finite dimensional spaces $S_{1,h} \subset H_{1,h}$ and $S_{2,h} \subset H_{2,h}$ with $\dim S_{1,h} = \dim S_{2,h}$ and

$$\inf_{\substack{u \in S_{1,h} \\ \|u\|_{H_{1,h}}=1}} \sup_{\substack{u \in S_{2,h} \\ \|v\|_{H_{2,h}}=1}} |B_h(u,v)| \geq \gamma > 0 \quad \forall h \tag{2.5}$$

(γ is independent of h). Then we define u_h by

$$u_h \in S_{1,h}$$
$$B_h(u_h,v) = B_h(u_0,v) \quad \forall v \in S_{2,h} \tag{2.6}$$

and consider u_h to be an approximation to u_0. (2.6) is uniquely solvable. We often denote u_0 be u. Note that u_h is defined for every $u \in H_{1,h}$. Given bases for $S_{1,h}$ and $S_{2,h}$, (2.6) is reduced to a system of linear equations. But (2.6) does not define an algorithm for finding u_h since it depends on the unknown u_0. We note from (2.4), however, that $B_h(u_0,v) = f(v) \, \forall v \in S_{2,h}$ provided

$$S_{2,h} \subset H_2 \; \forall h. \tag{2.7}$$

We now assume that (2.7) holds. (2.6) can then be written

$$u_h \in S_{1,h}$$
$$B_h(u_h,v) = f(v) \quad \forall v \in S_{2,h}. \tag{2.8}$$

(2.8) is called a variational approximation method.

Having defined u_h we are interested in an estimate for $\|u_0 - u_h\|_{H_{1,h}}$. This is provided by the standard

Theorem. The error $u_0 - u_h$ satisfies

$$\|u_0 - u_h\|_{H_{1,h}} \leq (1 + \gamma^{-1}M) \inf_{\chi \in S_{1,h}} \|u_0 - \chi\|_{H_{1,h}} \tag{2.9}$$

where M and γ are the constants in (2.3) and (2.5), respectively.

For a complete discussion of variational approximation methods see [1,2].

The spaces $S_{1,h}$ are called trial spaces and the functions in $S_{1,h}$ are called trial or approximating functions. The spaces $S_{2,h}$ are called test spaces and the functions in $S_{2,h}$ are called test functions.

Remarks:1) In our applications there will be a bilinear form $B(u,v)$ defined on $H_1 \times H_2$ and satisfying $B_h(u,v) = B(u,v) \; \forall u \in H_1, \, v \in H_2$. From (2.4) we see that

$$B(u_0,v) = f(v) \quad \forall v \in H_2. \tag{2.10}$$

(2.10) is the variational formulation of our boundary value problem and H_1, H_2, and B are the spaces and the bilinear form in this formulation.

2) (2.9) suggests we choose $S_{1,h}$ so that $\inf\limits_{\chi \in S_{1,h}} \| u_0 - \chi \|_{H_{1,h}}$ is small, i.e., so that the trial functions have good approximation properties, and, with $S_{1,h}$ so chosen, $S_{2,h}$ can be selected so that (2.5) holds with as large a constant γ as possible.

3) In many applications we can choose $S_{1,h} \subset H_1$, but in others the requirement that $\inf\limits_{\chi \in S_{1,h}} \| u_0 - \chi \|_{H_{1,h}}$ be small leads one to choose $S_{1,h} \not\subset H_1$. The trial space is then nonconforming in the sense that $S_{1,h}$ does not lie in the basic variational space H_1. This fact leads to a use of the family of spaces $H_{1,h}$ and forms B_h.

In certain situations one has $H_1 = H_2$ and one wishes to choose $S_{2,h} = S_{1,h}$. Then, if $S_{1,h}$ is nonconforming, $S_{2,h}$ will be also, and we are led to the use of the family $H_{2,h}$. Note that in this circumstance,

$$B_h(u_0, v) = f(v) \quad \forall v \in S_{2,h} \tag{2.11}$$

is not valid. Nonetheless, an approximation u_h can still be defined as in (2.8). This, in fact, is what is done in the class of methods commonly referred to as nonconforming in the finite element literature (see, e.g., Section 4.2 in [4]). The error analysis for such problems does not follow directly from (2.9) and the additional complications in the analysis are due to the fact that (2.11) does not hold.

In the methods discussed in this paper we will always have $S_{2,h} \subset H_2$, i.e., our test spaces will be conforming. We emphasize that the choice of nonconforming trial spaces causes no difficulty in the analysis, provided the test spaces are conforming.

3. Examples

In the section we will discuss the approximate solution of five specific boundary value problems with rough input data. In each example we will use a variational approximation method employing trial functions which reflect the properties of the underlying problem in that they provide an accurate approximation to the unknown solution.

a. A Two Point Boundary Value Problem with a Rough Coefficient

Consider the problem

$$Lu_0 = -(a(x)u_0')' = f, \quad 0 < x < 1$$
$$u_0(0) = u_0(1) = 0 \tag{3.1}$$

where $a(x)$ is a rather arbitrary function satisfying $0 < \alpha \le a(x) \le \beta$ and $f \in L_2(0,1)$. This simple model problem arises in the analysis of the displacements in a tapered elastic bar. f represents the load, $a(x)$ the elastic and geometric properties of the bar, and u_0 the displacement. If the bar has smoothly but rapidly or abruptly varying (as in the case of composite materials) material properties, then $a(x)$ will be a smoothly but rapidly or abruptly varying function, i.e., a rough function.

The variational formulation of (3.1) is

$$u_0 \in \overset{\circ}{H}_1(0,1)$$
$$\int_0^1 au_0'v'dx = \int_0^1 fvdx \quad \forall v \in \overset{\circ}{H}_1(0,1). \tag{3.2}$$

This has the form (2.10) with $H_1 = H_2 = \overset{\circ}{H}_1$, $B(u,v) = \int_0^1 au'v'dx$, and $f(v) = \int_0^1 fdx$. It is known that the standard finite element method employing C^0 piecewise linear trial and test functions (i.e., the variational approximation method determined by the form B and C^0 piecewise linear trial and test spaces) does not yield accurate approximations to the solution of (3.2) (or (3.1)) when $a(x)$ is rough. We thus consider an alternate method.

Let $T_h = \{0 = x_0 < x_1 < ... < x_n = 1\}$ be an arbitrary mesh on $[0,1]$ and set $I_j = (x_{j-1}, x_j)$, $h_j = x_j - x_{j-1}$, and $h = \max h_j$. The points x_j are called nodes. For the trial space we choose

$$S_{1,h} = \{u: \text{For each } j, \ u\big|_{I_j} = \text{a linear combination of 1 and}$$
$$\int^x \frac{dt}{a(t)}, \text{ i.e., } u\big|_{I_j} \text{ is a solution of } (au')' = 0, \tag{3.3}$$
$$u \text{ is continuous at the nodes, and } u(0) = u(1) = 0\}$$

and for the test space

$$S_{2,h} = \{v: \text{For each } j, \ v\big|_{I_j} = \text{a linear combination of 1 and } x,$$
$$u \text{ is continuous at the nodes, and } u(0) = u(1) = 0\}. \tag{3.4}$$

Now $S_{1,h} \subset H_1$ and $S_{2,h} \subset H_2$ and we may thus choose

$$H_{1,h} = H_1, \ H_{2,h} = H_2, \ B_h = B. \tag{3.5}$$

However, we could also choose

$$H_{1,h} = H_{2,h} = \overset{\circ}{H}_{1,h} \equiv \{u : u|_{I_j} \in H^1(I_j) \text{ for each } j, \ u(0) = u(1) = 0\},$$

$$\|u\|_{H_{1,h}} = \|u\|_{H_{2,h}} = \|u\|_{\overset{\circ}{H}_{1,h}} = [\int_0^1 u^2 dx + \sum_{j=1}^n \int_{I_j} (u')^2 dx]^{1/2}, \quad (3.6)$$

$$B_h(u,v) = \sum_j \int_{I_j} u'v' dx.$$

With either the choice (3.5) or (3.6), we see that (3.2) has the form (2.4) and that (2.3) is satisfied with $M = B$. We will use the choice (3.6) since it leads more naturally to the choice dictated to us in Example b below.

We next consider the variational approximation method (2.8) determined by B_h, $S_{1,h}$, and $S_{2,h}$, i.e.,

$$u_h \in S_{1,h}$$
$$B_h(u_h, v) = \int_0^1 a u_h' v' dx = \int_0^1 fv dx \ \forall v \in S_{2,h}. \quad (3.7)$$

Regarding (3.7) one can prove the inf-sup condition (2.5), namely

$$\inf_{\substack{u \in S_{1,h} \\ \|u\|_{\overset{\circ}{H}_1} = 1}} \sup_{\substack{v \in S_{2,h} \\ \|u\|_{\overset{\circ}{H}_1} = 1}} \left| \int_0^1 a u' v' dx \right| = \gamma(\alpha, \beta) > 0 \ \forall h \quad (3.8)$$

and the approximation result

$$\inf_{\chi \in S_{1,h}} \|u_0 - \chi\|_{\overset{\circ}{H}_1} \le N(\alpha, \beta) h \|(a u_0')'\|_{L_2} = N(\alpha, \beta) h \|f\|_{L_2} \quad (3.9)$$

where $\gamma(\alpha, \beta)$ and $N(\alpha, \beta)$ depend on α and β but not otherwise on $a(x)$ nor on h. From (3.8), (3.9) and the basic estimate (2.9) we immediately have

<u>Theorem a.</u> The error $u_0 - u_h$ satisfies

$$\|u_0 - u_h\|_{\overset{\circ}{H}_1} \le C(\alpha, \beta) h \|f\|_{L_2} \quad (3.10)$$

where $C(\alpha, \beta) = [1 + \gamma^{-1}(\alpha, \beta) M(\alpha, \beta)] N(\alpha, \beta)$ depends only on α and β. Thus we have a $1^{\underline{st}}$ order estimate for $\|u_0 - u_h\|_{\overset{\circ}{H}_1}$ which is uniform

over all $a(x)$ satisfying $\alpha \leq a(x) \leq \beta$. For a proof of (3.8) and
(3.9) see [3]. The estimate (3.10) is the best possible estimate
for $f \in L_2$. This can be seen by an application of the theory of
N-widths (see, e.g., [2, 8]).

(3.9) shows that $S_{1,h}$, as defined in (3.3), yields accurate
approximations to the solutions u_0 of (3.1). (3.8) shows that (2.5)
holds with our choice of $S_{1,h}$ and $S_{2,h}$, as defined in (3.4). Thus
our trial and test spaces have been chosen in accordance with the
suggestions in Remark 2 in Section 2. We further note that if
$\phi_1,\ldots,\phi_N$ is the usual basis for $S_{2,h}$ defined by $\phi_i(x_j) = \delta_{ij}$ and
$\widetilde{\phi}_1,\ldots,\widetilde{\phi}_N$ is the basis for $S_{1,h}$ defined by $\widetilde{\phi}_i(x_j) = \delta_{ij}$, then the
matrix of (3.7) (the stiffness matrix) is given by

$$\int_0^1 a\widetilde{\phi}_j'\phi_i'dx = \int_0^1 \underline{a}_h\phi_j'\phi_i'dx \tag{3.11}$$

where $\underline{a}_h$ is the piecewise harmonic average of $a(x)$, i.e., the step
function defined by

$$\underline{a}_h\Big|_{I_j} = \left(\frac{\int_{I_j} \frac{dx}{a}}{h_j}\right)^{-1}. \tag{3.12}$$

Thus the stiffness matrix is symmetric (which is not immediate since
we are using different trial and test spaces) and is as easily computed
as is the stiffness matrix for the standard method employing $S_{2,h}$ for
both trial and test space, namely $\int_0^1 \bar{a}_h\phi_j'\phi_i'dx$, where $\bar{a}_h$ is the piece-
wise average of $a(x)$.

The accuracy and robustness of the approximation u_h is further
shown by the estimate

$$|u(x_j) - u_h(x_j)| \leq C(\alpha,\beta)V_0^1(a)h^2\|f\|_{L_\infty}, \quad j=1,\ldots,n-1 \tag{3.13}$$

where $V_0^1(a)$ denotes the total variation of $a(x)$. Thus we have
second order convergence at the nodes even if $a(x)$ has several jumps.
The proof of (3.13), which does not follow the lines suggested by
(2.9), can be found in [3].

b. **A Special Class of Two Dimensional Boundary Value Problems with Rough Coefficients**

Consider the problem

$$Lu = -(a(x)u_x)_x - (b(y)u_y)_y = f(x,y), (x,y) \in \Omega = [0,1]^2$$

$$u = 0 \quad \text{on} \quad \Gamma = \partial\Omega \tag{3.14}$$

where $0 < \alpha \leq a(x), b(y) \leq \beta$ and $f \in L_2$. This problem generalizes the model problem considered in a. The variational formulation of (3.14) is

$$u \in \overset{\circ}{H}_1(\Omega)$$

$$B(u,v) = f(v) \quad \forall v \in \overset{\circ}{H}_1(\Omega) \tag{3.15}$$

where

$$B(u,v) = \int_\Omega (au_x v_x + bu_y v_y)\,dxdy$$

and

$$f(v) = \int_\Omega fv\,dxdy.$$

This is of the form (2.10) with $H_1 = H_2 = \overset{\circ}{H}_1$.

Let T_h be a uniform triangulation or mesh on Ω with triangles of size h, as shown in Fig. 1. The vertices of the triangles $T \in T_h$ are
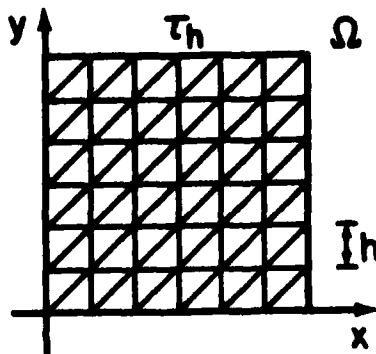

Fig. 1

called the nodes of T_h. By analogy with the definition of $S_{1,h}$ in a , we choose as trial space

$$S_{1,h} = \{u: \text{ For each } T \in T_h, \ u\big|_T = \text{ linear combination of } 1,$$

$$\int^x \frac{dt}{a(t)}, \text{ and } \int^y \frac{ds}{b(s)}, \text{ u continuous at the nodes, and}$$

$$u = 0 \text{ at the boundary nodes}\}. \tag{3.16}$$

For the test space we choose

$$S_{2,h} = \{v: \text{For each } T \in T_h, \left. v \right|_T = \text{linear combination of } 1, x, \text{ and}$$

$$y, \text{ u continuous at the nodes, and } u = 0 \text{ at the}$$

$$\text{boundary nodes}\}. \tag{3.17}$$

In contrast to the situation in Example a, the nodal constraints imposed on the functions in $S_{1,h}$ do not imply the functions are continuous and we have $S_{1,h} \nsubseteq H_1$, i.e., the trial space is nonforming. $S_{2,h} \subset H_2$ in both examples.

We now define

$$H_{1,h} = H_{2,h} = H_{1,h} = \{u: u \in L_2(\Omega), \left. u \right|_T \in H_1(T) \text{ for all } T \in T_h\},$$

$$\|u\|_{H_{1,h}} = \|u\|_{H_{2,h}} = \|u\|_{H_{1,h}} = [\int_\Omega u^2 dxdy + \sum_{T \in T_h} \int_T |\nabla u|^2 dxdy]^{1/2},$$

and

$$B_h(u,v) = \sum_{T \in Th} \int_T (ay_x v_x + bu_y v_y) dxdy.$$

B_h is defined on $H_{1,h} \times H_{1,h}$ and (2.3) holds with $M = \beta$.
The approximate solution is then defined by

$$u_h \in S_{1,h}$$
$$B_h(u_h,v) = \int_\Omega fv dxdy, \forall v \in S_{2,h}. \tag{3.18}$$

It is possible to show that (2.5) holds, i.e., that

$$\inf_{\substack{u \in S_{1,h} \\ \|u\|_{H_{1,h}}=1}} \sup_{\substack{v \in S_{2,h} \\ \|v\|_{H_{1,h}}=1}} |B_h(u,v)| \geq \gamma(\alpha,\beta) > 0 \quad \forall h \tag{3.19}$$

where $\gamma(\alpha,\beta)$ depends on α and β but not otherwise on $a(x)$ and $b(y)$. We have also shown that the functions in $S_{1,h}$ approximate the unknown solution well. In fact

$$\inf_{\chi \in S_{1,h}} \|u - \chi\|_{H_{1,h}} \leq C(\alpha,\beta) h \|u\|_{H_{L,h}} \tag{3.20}$$

where

$$H_{L,h} = \{u: u \in L_2(\Omega), \; u \in H_1(T), \; au_x, \; bu_y \in H_1(T) \; \forall T \in T_h, \; u \text{ is}$$
$$\text{continuous at the nodes of } T_h, \text{ and } u = 0 \text{ at the}$$
$$\text{boundary nodes}\}$$

and

$$\|u\|_{H_{L,h}}^2 = \|u\|_{H_{1,h}}^2 + \sum_T \int_T \{|(au_x)_x|^2 + 2ab|u_{xy}|^2 + |(bu_y)_y|^2\}dxdy.$$

It is easy to see that u, au_x, $bu_y \in H_1(T)$ implies u is continuous on T and so the requirement of continuity at the nodes makes sense. Now, combining (3.19), (3.20), and (2.9) we have

$$\|u - u_h\|_{H_{1,h}} \le C(\alpha,\beta)h\|u\|_{H_{L.h}} \quad \text{for all } h. \tag{3.21}$$

We have been able to prove the following regularity result for (3.14) (or (3.15)): If $f \in L_2$, then $u \in H_L$ and

$$\|u\|_{H_L} \le C(\alpha,\beta)\|f\|_{L_2} \tag{3.22}$$

where

$$H_L = \{u: u \in \overset{\circ}{H}_1(\Omega), \; au_x, \; bu_y \in H_1(\Omega)\}$$

and

$$\|u\|_{H_L}^2 = \|u\|_{H_1}^2 + \int_\Omega \{|(au_x)_x|^2 + 2ab|u_{xy}|^2 + |(bu_y)_y|^2\}dxdy.$$

Because it is immediate that $\|u\|_{H_{L,h}} = \|u\|_{H_L}$ for any $u \in H_L$, from (3.21) and (3.22) we get

Theorem b. The error $u - u_h$ satisfies

$$\|u - u_h\|_{H_{1,h}} \le C(\alpha,\beta)\|f\|_{L_2} \quad \forall h. \tag{3.23}$$

(3.23) shows that we have first order convergence in the "energy norm," with an estimate that is uniform with respect to the class of coefficients satisfying $0 < \alpha \le a(x), b(y) \le \beta$.

If $\phi_1, \ldots, \phi_N$ form the usual basis for $S_{2,h}$ defined by $\phi_i(z_j) = \delta_{ij}$, for all interior nodes $z_1, \ldots, z_n$ of T_h, and $\phi_1, \ldots, \phi_N$

form the basis for $S_{1,h}$ defined by $\phi_i(z_j) = \delta_{ij}$, for al
nodes z_j, then the stiffness matrix of (3.18) is given by

$$B_h(\tilde{\phi}_j,\phi_i) = \int_\Omega (\underline{a}_h\phi_{j,x}\phi_{i,x} + \underline{b}_h\phi_{j,y}\phi_{i,x})dxdy$$

where $\underline{a}_h$ and $\underline{b}_h$ are the (one dimensional) piecewise har
averages of $a(x)$ and $b(y)$, respectively. The result in
completely analogous to that expressed in (3.11) and (3.12)
a . We exphasize that although we are treating a two dimer
problem, the harmonic averages are one dimensional.

As we have seen, trial spaces with good approximation μ
are required in an accurate method. It is thus natural to
trial functions. We have, in fact, been able to show that
defined in (3.16), is an optimal approximation space in the

$$\sup_{\substack{u \in H_{L,h} \\ \|u\|_{H_{L,h}}=1}} \inf_{\chi \in S_{1,h}} \|u - \chi\|_{H_{1,h}} \leq \sup_{\substack{u \in H_{L,h} \\ \|u\|_{H_{L,h}}=1}} \inf_{\chi \in S_N} \|u - \chi\|_{H_1}$$

for all $S_N \subset H_{L,h}$ with $\dim S_N = \dim S_{1,h} = N = $ the numbe
nodes of T_h. Therefore $S_{1,h}$ is optimal in the sence of
(cf. [8]) relative to the norms $\| \cdot \|_{H_{1,h}}$ (the norm we ar
the error in) and $H_{L,h}$ (a $2^{\underline{nd}}$ derivative norm in which ι
bounded by $\|f\|_{L_2}$). We have thus chosen an "ideal" approxi

subspace for our problem. We have chosen the test space $\pounds$
defined in (3.17), so that $B_h(u,v)$ can be calculated fror
data for $v \in S_{2,h}$ (cf. (2.7)) and so that the inf-sup cc
(2.5) holds. These features of the choices for the trial ϵ
space lead to the accuracy of the method. We further note
choice of $S_{2,h}$ led to an easily computed stiffness matri $\rceil$
the precise statement of the accuracy and robustness of the
We note that (3.25) shows (in the case $a = b = 1$) that
$S_{2,h}$ of continuous piecewise linear functions is an optim ϵ
mating subspace in the case when the differential operator
Laplacian. Furthermore, in an asymptotic sense, $S_{2,h}$ is ϵ
problems with smooth coefficients.

c. A Singular Two Point Boundary Value Problem
    Consider the problem

$$-(\sqrt{x}\, u')' = f(x), \quad 0 < x < 1$$
$$u(0) = u(1) = 0. \tag{3.26}$$

Let

$$H_1 = H_2 = H = \{u: \int_0^1 \sqrt{x}\, (u')^2 dx < \infty, \; u(0) = u(1) = 0\}$$

and

$$\|u\|_{H_1}^2 = \|u\|_{H_2}^2 = \|u\|_H^2 = \int_0^1 \sqrt{x}\, (u')^2 dx.$$

The variational formulation of (3.26) is given by

$$u \in H$$
$$\int_0^1 \sqrt{x}\, u'v'dx = \int_0^1 fvdx \quad \forall v \in H.$$

Given a mesh T_h (as in Example a), let

$$H_{1,h} = H_{2,h} = \{u: \int_{I_j} \sqrt{x}\, (u')^2 dx < \infty \quad \text{for} \quad j = 1,\ldots,n,$$
$$u(0) = u(1) = 0\}$$

and

$$\|u\|_{H_{1,h}}^2 = \|u\|_{H_{2,h}}^2 = \|u\|_{L_2}^2 + \sum_{j=1}^n \int_{I_j} \sqrt{x}\, (u')^2 dx.$$

We define the trial and test spaces by

$$S_{1,h} = \{u: \text{For each } j, \; u\big|_{I_j} = \text{a linear combination of } 1 \text{ and }$$
$$\sqrt{x}, \; u \text{ continuous at the nodes, and } u(0) = u(1) = 0\}$$

and $S_{2,h} = C^0$ piecewise linear functions, as in Example a .

The approximate solution $u_h \in S_{1,h}$ is then characterized by

$$B_h(u_h,v) = \int_0^1 fvdx \quad \forall v \in S_{2,h}$$

where

$$B_h(u,v) = \sum_j \int_{I_j} \sqrt{x}\, u'v'dx.$$

One can show that (2.3) and (2.5) hold with M and γ independent of h. Furthermore, one can show that

$$\inf_{\chi \in S_{1,h}} \|u - \chi\|_H \leq Ch \int_0^1 \frac{\left| \left(\sqrt{x}\, u'\right)'\right|^2}{\sqrt{x}} \, dx.$$

Combining these facts with (2.9) leads immediately to

Theorem c. The error $u - u_h$ satisfies

$$\|u - u_h\|_H \leq Ch \left(\int_0^1 \frac{|f|^2}{\sqrt{x}} \, dx \right)^{1/2} \forall h. \tag{3.27}$$

We emphasize that (3.27) holds for an arbitrary mesh. It is not necessary to refine the mesh in the neighborhood of the singular point 0. The rate of convergence in (3.27) is the highest possible. This follows from the theory of N-widths. Problems similar to (3.26) have recently been considered in [6,11].

d. A Boundary Value Problem in a Domain with a Corner

Consider the model problem

$$-\Delta u = f(x,y), \quad (x,y) \in \Omega$$
$$u = 0, \quad (x,y) \in \Gamma = \partial\Omega \tag{3.28}$$

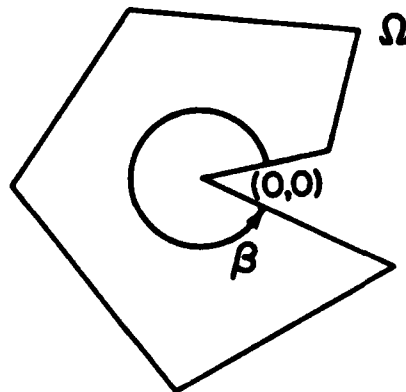where Ω is a polygonal domain with a convex angle (see Fig. 2), which

Fig. 2

we assume is placed at the origin, and $f \in C_0^\infty(\Omega)$. The solution u of (3.28) is singular at $(0,0)$ and as a consequence the standard finite element method with piecewise linear approximating functions

and a quasiuniform mesh gives an inaccurate approximate solution. In
fact

$$\|u - u_h\|_{H_1(\Omega)} \geq Ch^\rho$$

with $\rho < 1$ depending on the angle β.   (If $\beta \leq \pi$, then $\rho = 1$.)
Appropriate refinement of the mesh in the neighborhood of the concave
angle leads to the optimal convergence rate $O(N^{-1/2})$, where N is
the number of degrees of freedom (the dimension of the trial and test
space). A different way to achieve accuracy for problems with corners
is to augment the trial space with singular functions (which will not
have local supports). See, for example [5,7]. We will outline an
approach that preserves the local nature of the finite element method
by selecting special trial functions. The resulting method will have
the highest possible rate of convergence, namely $O(h)$.

Toward this end suppose T_h is a quasi-uniform triangulation of
Ω, let

$$S_{1,h} = \{u: \text{For each } T \in T_h, u\big|_T \text{ is a linear combination of } 1,$$

$r^\alpha \cos \alpha\theta$, and $r^\alpha \sin \alpha\theta$, where $\alpha = {}^\pi/\beta$ and (r,θ)
are polar coordinates of (x,y), u is continuous at
the nodes, and $u = 0$ at the boundary nodes$\}$,

and let $S_{2,h}$ be the usual C° piecewise linear functions relative
to T_h that vanish on Γ. The triangulation will be required to
satisfy certain technical restrictions in addition to being quasi-
uniform. We describe these now. Let $0 < \tau$ and $0 < h << 1$ be
given. Choose K sufficiently large, to be precise, choose $K > K(\beta,\gamma)$,
where $K(\alpha,\tau)$ is an explicitly known expression. Triangulate the
portion of Ω near the concave angle with several isoceles triangles
such as shown in Fig. 3. Triangulate the rest of Ω with triangles
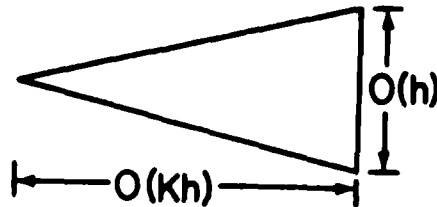

Fig. 3

of size h and minimal angle $\geq \tau$ as in Fig. 4. The bilinear form
B_h, the approximate solution u_h, and the norm $\| \cdot \|_{H_{1,h}}$ are defined
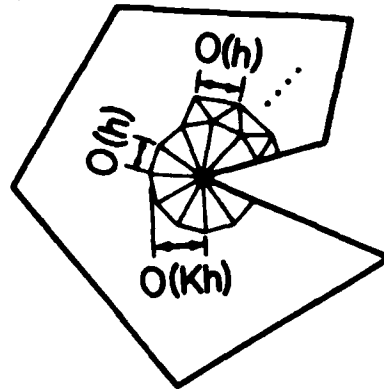
Fig. 4

in the obvious way. Then we can verify (2.3) and (2.5) and use (2.9) to prove

Theorem d. The error $u - u_h$ satisfies

$$\|u - u_h\|_{H_{1,h}} \leq \begin{cases} Ch|\log h|^{1/2}, & \beta = 2 \\ Ch, & \beta < 2 . \end{cases} \tag{3.29}$$

We thus have a first order estimate for the "energy" norm with a nearly uniform mesh.

e. **An Interface Problem**
    Consider the problem

$$-\text{div (a grad u)} = f(x,y), \quad (x,y) \in \Omega$$
$$u = 0 \quad \text{on} \quad \Gamma = \partial\Omega \tag{3.30}$$

where Ω is a convex polygon and

$$a = \begin{cases} a_1 & \text{on} \quad \Omega_1 \\ a_2 & \text{on} \quad \Omega_2 \end{cases}$$

where a_1 and a_2 are positive constants and $\bar{\Omega}_1 \subset \Omega$, $\partial\Omega_1$ is smooth, and $\Omega_2 = \Omega - \bar{\Omega}_1$. The first derivatives of u have jumps across the interface $\partial\Omega_1$ and thus the standard finite element method is inaccurate unless the mesh is chosen so that the interface lies on edges of triangles in the trinagulation. We will now show that maximal accuracy can be achieved without aligning the mesh with the interface if we modify the trial functions properly.

Let T_h be a quasi-uniform triangulation or mesh on Ω. We

describe a function $u \in S_{1,h}$.

    i) For $T \in T_h$,

        if $T \cap \partial\Omega_1 = \phi$, $u\big|_T$ is linear

        if $T \cap \partial\Omega_1 \neq \phi$, $u\big|_T$ is as follows:

            Choose $Q \in \partial\Omega_1 \cap T$. Denote the tangent and normal to $\partial\Omega_1$ by t and n, respectively. Then

            $\alpha)$ $u_1 = u\big|_{T \cap \Omega_1}$ and $u_2 = u\big|_{T \cap \Omega_2}$ are linear,

            $\beta)$ $u_1(Q) = u_2(Q)$,

            $\gamma)$ $\dfrac{\partial u_1}{\partial t}(Q) = \dfrac{\partial u_2}{\partial t}(Q)$.

            $\delta)$ $a_1\dfrac{\partial u_1}{\partial n}(Q) = a_2\dfrac{\partial u_2}{\partial n}(Q)$.

    ii) u is continuous at the nodes of T_h.

Note that i)$\delta)$ is a requirement that the trial functions approximately model the interface condition $a_1 \dfrac{\partial u\big|_{\Omega_1}}{\partial n} = a_2 \dfrac{\partial u\big|_{\Omega_2}}{\partial n}$ satisfied by the exact solution. For test space we choose the usual C° piecewise linear functions. Having defined these spaces we then define the approximate solution by

$$u_h \in S_{1,h}$$
$$B_h(u,v) \equiv \sum_T \left(\int_{T \cap \Omega_1} a\nabla u \cdot \nabla v\, dxdy + \int_{T \cap \Omega_2} a\nabla u \cdot \nabla u\, dxdy \right) =$$

$$\int_\Omega fv\, dxdy \quad \forall v \in S_{2,h}. \tag{3.31}$$

For the method defined in (3.31) we have proved, with the aid of (2.9), the estimate

$$\|u - u_h\|_{\overline{H}_{1,h}} \leq Ch\|f\|_{L_2} \quad \forall h \tag{3.32}$$

where

$$\|v\|_{\overline{H}_{1,h}}^2 = \int_\Omega v^2\, dxdy + \sum_T \left(\int_{T \cap \Omega_1} |\nabla v|^2\, dxdy + \int_{T \cap \Omega_2} |\nabla v|^2\, dxdy \right).$$

(3.32) is a first order estimate for the "energy" norm of the error. We emphasize that there is no relationship between the mesh and the

interface. The method proposed here would provide an alternative to methods in which the interface is modeled by a mesh line, as in [9].

4. An Approach to the Design of Finite Element Methods for Problems with Rough Input Data

Our treatment of the Examples in Section 3 suggests an approach to the design of finite element methods for problems with rough input data. In this section we outline the steps in this approach.

a) Choose the space of right hand sides or source terms f to be considered. In all of our example we chose $f \in L_2$ except in Example 3.c where we considered f's satisfying $\int_0^1 \frac{f^2}{\sqrt{x}} dx < \infty$.

b) Find the space of solutions corresponding to the space of right-hand sides under consideration. This will involve a regularity result. For example, in Example 3.b , the space H_L is the space of solutions corresponding to f's in L_2. Often regularity results are available for problems with smooth data, but are not available in sufficient generality for problems with rough data.

c) Choose the mesh dependent bilinear forms and spaces (norms) B_h, $H_{1,h}(\| \cdot \|_{H_{1,h}})$, $H_{2,h}(\| \cdot \|_{H_{2,h}})$. This choice is usually very natural, following directly from the basic variational formulation, considered triangle by triangle.

d) Select trial spaces which have good (optimal) approximation properties. This is the major problem. Usually such spaces are closely related to local solutions of the equation under consideration. In many situations the proper choice leads to non-conforming functions. Inter-triangle continuity is enforces only at the nodes. The approximation properties of the trial functions is directly tied to the space of solutions (see b)). The problem of selecting optimal trial functions is often not simple. In practice one would like to find a trial space that performs almost as well as the optimal trial space but which is easily implemented.

e) Select a test space so as to ensure the inf-sup condition is satisfied with constant γ that is not too small and so that the stiffness matrix can be calculated. In contrast to the trial space, the test space is chosen to be conforming.

We have illustrated these steps on the relatively simple examples discussed in Section 3. We restricted our attention to first order methods in which the maximal rate of convergence was $O(h)$. As mentioned in the introduction, the ideas in the paper can be generalized in

various directions.

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