# Undamped Vibration of a Beam

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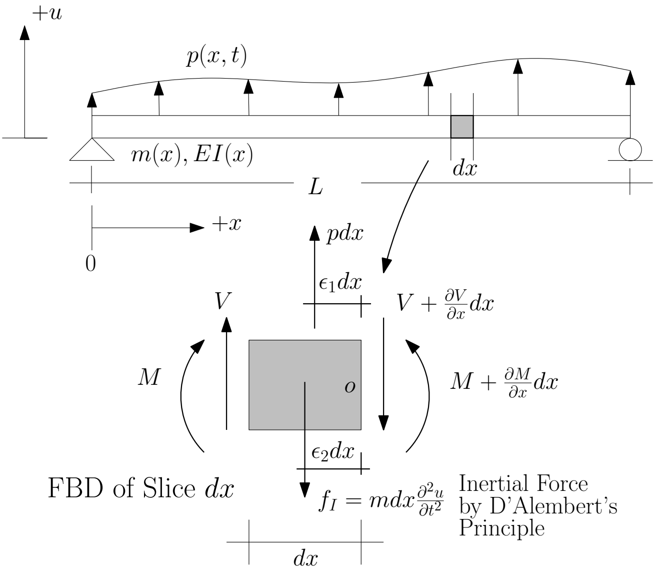
# Problem - Undamped Transverse Beam Vibration



# Derivation of PDE

- Sum Forces Vertically, choosing + up

$$V - \left(V + \frac{\partial V}{\partial x}dx\right) + p(x,t)dx - m(x)dx\frac{\partial^2 u}{\partial t^2} = 0 \qquad (1)$$

- Sum Moments about o, choosing CCW as + rotation

$$-M - Vdx + p(x,t)\epsilon_1 dx^2 + m(x)\epsilon_2 dx^2\frac{\partial^2 u}{\partial t^2} + M + \frac{\partial M}{\partial x}dx = 0 \qquad (2)$$

  Simplifying (1), and in (2) ignoring higher order terms in the limit as $dx \longrightarrow 0$ gives

$$\frac{\partial V}{\partial x} = p(x,t) - m(x)\frac{\partial^2 u}{\partial t^2}, \quad and \quad V = \frac{\partial M}{\partial x} \qquad (3)$$

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# Derivation of PDE

- From mechanics of materials class, moment curvature relation (given here to save time)

$$M = EI(x)\frac{\partial^2 u}{\partial x^2} \qquad (4)$$

- Substituting equation two of (3) and equation (4) into equation one of (3) and rearranging yields

$$m(x)\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 u}{\partial x^2}\right] = p(x,t) \qquad (5)$$

- Equation (5) is the PDE governing the motion $u(x,t)$, subject to the external forcing function $p(x,t)$.

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# Solving the PDE

- Analytical solution difficult or impossible to obtain due to $m(x)$ and $I(x)$.
- Numerical methods such as Finite Element Method or Finite Differences can solve the PDE.
- Can simplify the PDE to demonstrate analytical methods by the following assumptions:
  - $m(x) = m =$constant along the beam length
  - $I(x) = I =$constant along the beam length
  - $p(x, t) = 0$, ie, no forcing function

# Solving the PDE

- After simplifying assumptions the governing PDE (5) becomes

$$m\frac{\partial^2 u}{\partial t^2} + EI\frac{\partial^4 u}{\partial x^4} = 0 \tag{6}$$

- To make things pretty at the end, define $a^2 = EI/m$, so that

$$\frac{\partial^2 u}{\partial t^2} + a^2\frac{\partial^4 u}{\partial x^4} = 0 \tag{7}$$

# Solving the PDE

- Assume a solution of the following form

$$u(x,t) = \phi(x)q(t) \tag{8}$$

- Substitute (8) into the PDE (7) to get

$$\phi\frac{\partial^2 q}{\partial t^2} + a^2 q\frac{\partial^4 \phi}{\partial x^4} = 0 \tag{9}$$

- By separation of variables, observe that l.h.s and r.h.s must equal a constant, $\beta^4$

$$-\frac{1}{a^2 q}\frac{\partial^2 q}{\partial t^2} = \frac{1}{\phi}\frac{\partial^4 \phi}{\partial x^4} = \beta^4 \tag{10}$$

# Solving the PDE

- From (10), two ODE's are obtained

$$\frac{\partial^4 \phi}{\partial x^4} - \beta^4 \phi = 0 \qquad (11)$$

$$\frac{\partial^2 q}{\partial t^2} + \beta^4 a^2 q = 0 \qquad (12)$$

- The respective solutions are

$$\phi(x) = A \sinh \beta x + B \cosh \beta x + C \sin \beta x + D \cos \beta x \qquad (13)$$

$$q(t) = E \sin a\beta^2 t + F \cos a\beta^2 t \qquad (14)$$

- $\therefore$ solution of the PDE (6) is $u(x,t) = \phi(x)q(t)$.

# Solving a Boundary Value Problem (BVP)

- To solve a realistic problem, boundary conditions must be specified
- The six boundary conditions (BC's) are
  1. @ $x = 0,\ u(0, t) = \phi(0)q(t) = 0$
  2. @ $x = L,\ u(L, t) = \phi(L)q(t) = 0$
  3. @ $x = 0,\ u''(0, t) = \phi''(0)q(t) = 0$
  4. @ $x = L,\ u''(L, t) = \phi''(L)q(t) = 0$
  5. @ $t = 0,\ \dot{u}(x, 0) = \phi(x)\dot{q}(0) = 0$
  6. @ $t = 0,\ u(x, 0) = Gx(L - x)$, G specified constant

# Applying the boundary conditions to $\phi(x)$

- Applying the first four boundary conditions yield the following results
  1. $\phi(0) = B + D = 0$
  2. $\phi(L) = A\sinh\beta L + B\cosh\beta L + C\sin\beta L + D\cos\beta L = 0$
  3. $\phi''(0) = B\beta^2 - D\beta^2 = 0 \quad \Rightarrow \quad B - D = 0$
  4. $$\phi''(L) = A\beta^2\sinh\beta L + B\beta^2\cosh\beta L \\ -C\beta^2\sin\beta L - D\beta^2\cos\beta L = 0$$
- From BC's (1) and (3), $B = 0$ and hence $D = 0$

# Applying the boundary conditions to $\phi(x)$

- From BC's (2) and (4)
  $A \sinh \beta L + C \sin \beta L = 0$
  $A \sinh \beta L - C \sin \beta L = 0$
- The above results imply

$$A \sinh \beta L = 0 \quad \textit{and} \quad C \sin \beta L = 0 \tag{15}$$

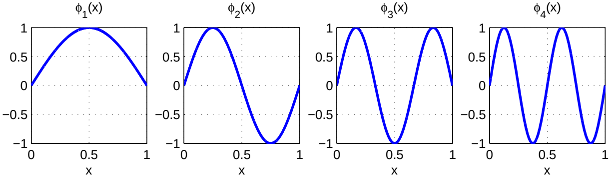
- From the first expression of (15), $A = 0$. If $A = 0$ is not chosen, $\beta = 0$ is required and this leads to $\phi(x) = 0$ for all $x$ which is the at rest condition (not very interesting).
- Using the remaining case (since $A = 0$), either $C = 0$ or $\sin \beta L = 0$. Choosing $C = 0$ isn't an option since that leads to $\phi(x) = 0$ for all $x$ which is the uninteresting at rest condition.

# Applying the boundary conditions to $\phi(x)$

- Therefore, must have $\sin\beta L = 0$, which implies $\beta L = n\pi$.
- After solving for $\beta$, the $n$ solutions (which satisfy the B.C's) for $\phi(x)$ are

$$\phi_n(x) = C_n \sin\frac{n\pi x}{L} \tag{16}$$

- This implies that the beam vibrates in the following natural mode shapes for $n = 1, 2, 3, 4...$



# Applying boundary condition (5)

- Applying BC (5) yields

$$\dot{q}(0) = -a\beta^2 E \sin a\beta^2 0 + a\beta^2 F \cos a\beta^2 0 = 0 \qquad (17)$$

- The sine term equals zero and hence $F = 0$. As a result

$$q(t) = E \cos a\beta^2 t \qquad (18)$$

- In light of the fact that $\beta = n\pi/L$

$$q_n(t) = E_n \cos \frac{an^2\pi^2 t}{L^2} \qquad (19)$$

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# Applying boundary condition (6)

- Combining (16) and (19) and defining $b_n = C_n E_n$ yields

$$u_n(x,t) = \phi_n(x) q_n(t) = b_n \sin \frac{n\pi x}{L} \cos \frac{an^2\pi^2 t}{L^2} \qquad (20)$$

- Equation (20) satisfies the PDE and the first 5 BC's for any value of $n$ and arbitrary constants $b_n$. As a result, any linear combination of (20) also satisfies the requirements so that

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} \cos \frac{an^2\pi^2 t}{L^2} \qquad (21)$$

# Applying the boundary condition (6)

- To satisfy BC (6) the following must be true

$$u(x,0) = Gx(L-x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} \qquad (22)$$

- Hence, the $b_n$ are the sine Fourier coefficients for $Gx(L-x)$. That is

$$b_n = \frac{2}{L}\int_0^L Gx(L-x)\sin\frac{n\pi x}{L}dx \qquad (23)$$

$$= \frac{8GL^2}{n^3\pi^3} \quad \textit{for } n \textit{ odd} \qquad (24)$$

$$= 0 \quad \textit{for } n \textit{ even} \qquad (25)$$

# Final solution of the BVP

- Using the results of (21) and (23) gives the final solution of the BVP.

$$u(x,t) = \sum_{n=1,3,5,\ldots}^{\infty} \frac{8GL^2}{n^3\pi^3} \sin\frac{n\pi x}{L} \cos\frac{an^2\pi^2 t}{L^2} \tag{26}$$

- Comments:
  - Recall $a^2 = EI/m$ which is known
  - $G$ specifies initial amplitude at $t = 0$, hence is known
  - By observing the cosine term of (26) it is concluded that the natural frequencies for the beam are

$$\omega_n = \frac{n^2\pi^2}{L^2}\sqrt{\frac{EI}{m}}$$

  - Reference: Miller, Kenneth S., "Partial Differential Equations in Engineering Problems", Prentice-Hall, Englewood Cliffs, NJ, 1953.