Unfaithful complex hyperbolic triangle groups II: Higher order reflections

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Abstract

We consider symmetric complex hyperbolic triangle groups generated by three complex reflections with angle $2\pi/p$. We restrict our attention to those groups where certain words are elliptic. Our goal is to find necessary conditions for such a group to be discrete. The main application we have in mind is that such groups are candidates for non-arithmetic lattices in SU(2,1).

1 Introduction

In [11] Mostow constructed the first examples of non-arithmetic complex hyperbolic lattices. These lattices were generated by three complex reflections R_1, R_2 and R_3 with the property that there exists a complex hyperbolic isometry J of order 3 so that $R_{j+1} = JR_jR^{-1}$ (here and throughout the paper the indices will be taken mod 3). In Mostow's examples the generators R_j have order $p = 3$, 4 or 5. Subsequently Deligne–Mostow and Mostow constructed further non-arithmetic lattices as monodromy groups of certain hypergeometric functions, [3] and [12] (the lattices from [3] in dimension 2 were known to Picard who did not consider their arithmetic nature). These lattices are (commensurable with) groups generated by complex reflections R_j with other values of p; see Mostow [12] and Sauter [19]. Subsequently no new non-arithmetic lattices have been constructed.

In [15] Parker considered the case $p = 2$. That is, he considered complex involutions I_1, I_2 and I_3 with the property that there is a J of order 3 so that $I_{j+1} = JI_jJ^{-1}$. In particular, he used a theorem of Conway and Jones [1] to classify all such groups where I_1I_2 and $I_1I_2I_3$ are elliptic.

Remarkably, when $p \geqslant 3$ finding groups for which R_1R_2 and $R_1R_2R_3$ are elliptic involves solving the same equation as in the case $p = 2$. In this paper we use the solutions to this equation found using [1] in [15] in the general case.

We describe the configuration space of all groups generated by a complex reflection R_1 of order p and a regular elliptic motion J of order 3. This configuration space is parametrised by the conjugacy class of the product R_1J, which we represent geometrically in two different manners. The first, following Goldman, Parker, is to consider the trace of R_1J; this determines the conjugacy class of R_1J when it is loxodromic, but there is a threefold indetermination when it is elliptic or parabolic. The second manner, following Paupert, is to use the geometric invariants of the conjugacy class, i.e. an angle pair for elliptic isometries and a pair (angle, length) for loxodromic isometries. We will use both parameter spaces in this paper, where we focus on the elliptic case.

Our first result is the direct analogue of the main theorem of [15], and can be roughly stated as follows:

Theorem 1.1 *Let R_1 be a complex reflection of order p and J a regular elliptic of order 3 in* $\mathrm{PU}(2,1)$. *Suppose that R_1J and $R_1R_2 = R_1JR_1J^{-1}$ are elliptic. If the group $\Gamma = \langle R_1, J\rangle$ is discrete then one of the following is true:*

- Γ *is one of Mostow's lattices.*
- Γ *is a subgroup of one of Mostow's lattices.*
- Γ *is one of the sporadic groups described below.*

The sporadic groups correspond to the 18 exceptional solutions from [1], which do not depend on p (the groups do change with p of course). We determine for each $p \geqslant 3$ which of these points lie inside our configuration space. One must then analyse each of these groups to decide whether or not it is discrete, if so whether or not it is a lattice and if so whether or not it is arithmetic. We illustrate ways to attack this problem by showing that certain solutions are arithmetic and certain other solutions are non-discrete. We analyse in detail the situation for $p = 3$, which can be summarised as follows:

Theorem 1.2 *There are 16 sporadic groups for $p = 3$, with the following properties:*

- *Four of them fix a point in $H^2_{\mathbb{C}}$.*
- *One stabilises a complex line.*
- *One is contained in an arithmetic lattice.*
- *The ten remaining groups are none of the above.*

The crucial question is then to determine whether or not the ten remaining groups are discrete. We give a negative answer for four of them, by finding elliptic elements of infinite order in the group.

2 Parameters, traces and angles

In this section we show how to use a single complex number τ to parametrise symmetric complex hyperbolic triangle groups generated by three complex reflections thorough angle ψ. This generalises the construction in [15] where involutions (that is $\psi = \pi$) were considered.

We give an alternative description following Paupert [17] where the same space is parametrised by a pair of angles. It is fairly easy to pass between then two parametrisations and we shall use whichever one is best adapted to a particular problem.

We then describe how the properties of the group (for example the type of R_1R_2 and $R_1R_2R_3$) vary with τ. Much of this section follows the relevant parts of [15] rather closely. Otherwise we will try to keep this account as self contained as possible. However, we shall assume a certain amount of background knowledge of complex hyperbolic geometry. For such background material on complex hyperbolic space see [6] and for material on complex hyperbolic triangle groups see [21] or [18]. Moreover, many of the ideas we use may be traced back to Mostow's paper [11].

2.1 Complex reflections through angle ψ

Let L_1 be a complex line in complex hyperbolic 2-space $\mathbf{H}^2_{\mathbb{C}}$ and write R_1 for the complex reflection through angle ψ and fixing L_1. In our applications we shall only consider the case where $\psi = 2\pi/p$ for an integer p at least 2. Points of L_1 are fixed by R_1 and $\mathbf{n}_1$, the polar vector of L_1, is sent by R_1 to $e^{i\psi}\mathbf{n}_1$. Hence the matrix in $\mathrm{U}(2,1)$ associated to R_1 has determinant $e^{i\psi}$. Since we shall be dealing with traces, we want to lift R_1 to a matrix in $\mathrm{SU}(2,1)$. Hence we multiply the $\mathrm{U}(2,1)$ matrix by $e^{-i\psi/3}$, and take R_1 to be the map given by:

$$R_1(\mathbf{z}) = e^{-i\psi/3}\,\mathbf{z} + (e^{2i\psi/3} - e^{-i\psi/3})\frac{\langle \mathbf{z}, \mathbf{n}_1\rangle}{\langle \mathbf{n}_1, \mathbf{n}_1\rangle}\,\mathbf{n}_1. \tag{1}$$

We remark that we can add arbitrary multiples of 2π to the angle ψ. Equivalently, since $\mathrm{SU}(2,1)$ is a triple cover of $\mathrm{PU}(2,1)$ we could have multiplied R_1 by any power of $e^{2\pi i/3}$. Thus in the case where R_1 has order 2, the in the analogous formula (1) in [15], the angle ψ was chosen to be 3π and so $e^{-i\psi/3}$ and $e^{2i\psi/3}$ become -1 and $+1$ respectively. Equivalently, to obtain this we multiply (1) with $\psi = \pi$ by $e^{-2\pi i/3}$.

2.2 Traces

We define $\tau = \mathrm{tr}(R_1 J)$ to be the parameter in our space. We now show how to express τ in terms of the polar vectors and Hermitian form. We go on to find the trace of R_1R_2 in the same way.

Our first lemma and its corollary generalise Lemma 2.4 and Corollary 2.5 of [15] and are proven in a similar manner.

Lemma 2.1 *Let A be any element of $\mathrm{SU}(2,1)$. Then*

$$\mathrm{tr}(R_1 A) = e^{-i\psi/3}\mathrm{tr}\,(A) + (e^{2i\psi/3} - e^{-i\psi/3})\frac{\langle A(\mathbf{n}_1), \mathbf{n}_1\rangle}{\langle \mathbf{n}_1, \mathbf{n}_1\rangle}.$$

Corollary 2.2 *Let R_1 be a complex involution fixing a complex line L_1 with polar vector $\mathbf{n}_1$. Let $J \in \mathrm{SU}(2,1)$ be a regular elliptic map of order 3. Then*

$$\tau := \mathrm{tr}(R_1 J) = (e^{2i\psi/3} - e^{-i\psi/3})\frac{\langle J(\mathbf{n}_1), \mathbf{n}_1\rangle}{\langle \mathbf{n}_1, \mathbf{n}_1\rangle} = 2\sin(\psi/2)ie^{i\psi/6}\frac{\langle J(\mathbf{n}_1), \mathbf{n}_1\rangle}{\langle \mathbf{n}_1, \mathbf{n}_1\rangle}. \tag{2}$$

The following theorem is just a restatement of Theorem 6.2.4 of [6]

Proposition 2.3 *Let τ be given by (2). Then R_1J is regular elliptic if and only if τ lies in the interior of the deltoid*

$$\mathbf{\Delta} = \left\{z \in \mathbb{C} \;:\; |z|^4 - 8\,\mathrm{Re}\,(z^3) + 18|z|^2 - 27 \leq 0\right\}. \tag{3}$$

The curve given by the equality in Proposition 2.3 is a deltoid. Groups for which R_1J is regular elliptic correspond to points in the interior of this deltoid. Points on the deltoid correspond to points where R_1J has repeated eigenvalues, and so is either a complex reflection or is parabolic. Since $R_1R_2R_3 = (R_1J)^3$ we can determine the type of $R_1R_2R_3$ from R_1J. (Note that it may be that R_1J is regular elliptic and that $R_1R_2R_3$ is a complex reflection.)

We now consider R_1R_2, and hence by symmetry R_2R_3 and R_3R_1 as well. This generalises Lemma 2.7 of [15] and its proof is analogous.

Proposition 2.4 *Let L_1 be a complex line in $\mathbf{H}^2_{\mathbb{C}}$ with polar vector $\mathbf{n}_1$. Let $J \in \mathrm{SU}(2,1)$ be a regular elliptic map of order 3 and write $L_2 = J(L_1)$. Let R_1 and $R_2 = JR_1J^{-1}$ denote the complex reflections through with angle ψ fixing L_1 and L_2 respectively. Then*

$$\mathrm{tr}(R_1R_2) = e^{i\psi/3}\left(2 - |\tau|^2\right) + e^{-2i\psi/3}.$$

where τ is given by (2).

Corollary 2.5 *If $|\tau| > 2$ then R_1R_2 is loxodromic. If $|\tau| = 2\cos(\theta)$ then R_1R_2 has eigenvalues $e^{-2i\psi/3}$, $-e^{i\psi/3+2i\theta}$ and $-e^{i\psi/3-2i\theta}$.*

PROOF: Each point on L_1 is an $e^{-i\psi/3}$ eigenvector for R_1 and each point on L_2 is an $e^{-i\psi/3}$ eigenvector for R_2. Therefore if $\mathbf{z} \in \mathbb{C}^{2,1}$ lies on the intersection of L_1 and L_2 we have

$$R_1R_2(\mathbf{z}) = e^{-i\psi/3}R_1(\mathbf{z}) = e^{-2i\psi/3}\mathbf{z}.$$

Hence $\mathbf{z}$ is an $e^{-2i\psi/3}$ eigenvector for R_1R_2.

If $|\tau| > 2$ then R_1R_2 has an eigenvector whose modulus is greater than one. Hence R_1R_2 is loxodromic.

So we assume that $|\tau| = 2\cos(\theta)$. Since R_1R_2 has an eigenvalue $e^{-2i\psi/3}$ and determinant 1, we see that its other eigenvalues must be $-e^{i\psi/3+2i\theta}$ and $-e^{i\psi/3-2i\theta}$ as claimed. □

2.3 Angles

Conjugacy classes of elliptic isometries in $\mathrm{PU}(2,1)$ are characterised by an unordered angle pair $\{\theta_1, \theta_2\}$ with $\theta_i \in \mathbb{R}/2\pi\mathbb{Z}$. These angles can be defined by noting that an elliptic isometry g belongs to a maximal compact subgroup of $\mathrm{PU}(2,1)$, which is a group conjugate to $\mathrm{U}(2)$. Then all elements of $\mathrm{U}(2)$ which are conjugate to g have the same eigenvalues of norm 1, which we define to be $e^{i\theta_1}$ and $e^{i\theta_2}$. In concrete terms, a matrix $A \in \mathrm{U}(2,1)$ whose associated isometry $g \in \mathrm{PU}(2,1)$ is elliptic is semisimple with eigenvalues of norm 1, say $e^{i\alpha_1}$, $e^{i\alpha_2}$ and $e^{i\alpha_3}$. One of these is *of negative type* in the sense that the associated eigenspace intersects the negative cone in $\mathbb{C}^3$ of the ambient Hermitian form (of signature $(2,1)$). Supposing for instance that $e^{i\alpha_3}$ is of negative type, the angle pair of g is then $\{\alpha_1 - \alpha_3, \alpha_2 - \alpha_3\}$. It is thus in general not sufficient to know the eigenvalues of an elliptic matrix to obtain the angle pair of the corresponding isometry; one must determine which of them is of negative type.

This can also be seen in terms of the trace of matrix representatives in $\mathrm{SU}(2,1)$. Recall from [6] that elliptic isometries have matrix representatives in $\mathrm{SU}(2,1)$ whose trace lies in the deltoid $\mathbf{\Delta}$ given in (3). Multiplication of A, and hence of its trace, by a cube root of unity corresponds to the same complex hyperbolic isometry in $\mathrm{PU}(2,1)$. Up to this ambigiuty, the map from angle pairs to traces is given by

$$\{\theta_1,\, \theta_2\} \longmapsto e^{2i\theta_1/3 - i\theta_2/3} + e^{2i\theta_2/3 - i\theta_1/3} + e^{-i\theta_1/3 - i\theta_2/3}.$$

This map is three-to-one, except at the exceptional central point $\{4\pi/3, 2\pi/3\}$ which is the only preimage of 0. The three preimages in this case correspond to the fact that one of the three eigenvalues corresponds to negative eigenvectors and the other two to positive ones. There are three possible choices and the trace is the same for each of them. Conversely, given a trace $\tau = e^{i\alpha} + e^{i\beta} + e^{-i\alpha - i\beta}$ the three preimages of this trace are the three angle pairs

$$\{2\alpha + \beta,\, \alpha + 2\beta\}, \quad \{\alpha - \beta,\, -\alpha - 2\beta\}, \quad \{\beta - \alpha,\, -2\alpha - \beta\}.$$

In order to get these into the parameter space where $0 \le \theta_1 \le 2\pi$ and $0 \le \theta_2 \le \theta_1$ one may have to add an integer multiple of 2π to either or both angles and one may have to change their order. The reader can refer to [16], pp. 29–30, for more details on angle pairs and their relation to traces.

2.4 The trace parameter space

Suppose we are given a symmetrical configuration of three complex lines L_1, L_2 and L_3 with polar vectors $\mathbf{n}_1$, $\mathbf{n}_2$ and $\mathbf{n}_3$. Because they are symmetrical $J(\mathbf{n}_j) = \mathbf{n}_{j+1}$ where $j = 1, 2, 3$ taken mod 3. Because J preserves the Hermitian form $\langle \mathbf{n}_j, \mathbf{n}_j \rangle$ is the same positive real number for each j. We normalise $\mathbf{n}_j$ so that this number is $2\sin(\psi/2)$. Likewise $\langle \mathbf{n}_{j+1}, \mathbf{n}_j \rangle = \langle J(\mathbf{n}_j), \mathbf{n}_j \rangle$ is the same complex number for each j which we define to be σ. Using Corollary 2.2 we see that

$$\langle J(\mathbf{n}_j), \mathbf{n}_j \rangle = \frac{\tau \langle \mathbf{n}_j, \mathbf{n}_j \rangle}{e^{2i\psi/3} - e^{-i\psi/3}} = -ie^{-i\psi/6}\tau.$$

That is

$$\langle \mathbf{n}_1, \mathbf{n}_1 \rangle = \langle \mathbf{n}_2, \mathbf{n}_2 \rangle = \langle \mathbf{n}_3, \mathbf{n}_3 \rangle = 2\sin(\psi/2), \qquad \langle \mathbf{n}_2, \mathbf{n}_1 \rangle = \langle \mathbf{n}_3, \mathbf{n}_2 \rangle = \langle \mathbf{n}_1, \mathbf{n}_3 \rangle = -ie^{-i\psi/6}\tau. \quad (4)$$

All of this works for any Hermitian form of signature $(2, 1)$. We now make a choice of vectors $\mathbf{n}_1$, $\mathbf{n}_2$ and $\mathbf{n}_3$. This determines a Hermitian form. We choose

$$\mathbf{n}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbf{n}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad \mathbf{n}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Together with (2), this means that, with this choice, the Hermitian form must be $\langle \mathbf{z}, \mathbf{w} \rangle = \mathbf{w}^* H_\tau \mathbf{z}$ where

$$H_\tau = \begin{bmatrix} 2\sin(\psi/2) & -ie^{-i\psi/6}\tau & ie^{i\psi/6}\overline{\tau} \\ ie^{i\psi/6}\overline{\tau} & 2\sin(\psi/2) & -ie^{-i\psi/6}\tau \\ -ie^{-i\psi/6}\tau & ie^{i\psi/6}\overline{\tau} & 2\sin(\psi/2) \end{bmatrix}. \quad (5)$$

We can immediately write down J and, using (1), the reflections R_j. They are

$$J = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad (6)$$

$$R_1 = \begin{bmatrix} e^{2i\psi/3} & \tau & -e^{i\psi/3}\overline{\tau} \\ 0 & e^{-i\psi/3} & 0 \\ 0 & 0 & e^{-i\psi/3} \end{bmatrix}, \quad (7)$$

$$R_2 = \begin{bmatrix} e^{-i\psi/3} & 0 & 0 \\ -e^{i\psi/3}\overline{\tau} & e^{2i\psi/3} & \tau \\ 0 & 0 & e^{-i\psi/3} \end{bmatrix}, \quad (8)$$

$$R_3 = \begin{bmatrix} e^{-i\psi/3} & 0 & 0 \\ 0 & e^{-i\psi/3} & 0 \\ \tau & -e^{i\psi/3}\overline{\tau} & e^{2i\psi/3} \end{bmatrix}. \quad (9)$$

From this it is clear that the groups $\Gamma = \langle R_1, J \rangle$ and $\Delta = \langle R_1, R_2, R_3 \rangle$ are completely determined up to conjugacy by the parameter τ. However, not all values of τ correspond to complex hyperbolic triangle groups. It may be that the Hermitian matrix H_τ does not have signature $(2, 1)$. We now determine this by finding the eigenvalues of H_τ. In this lemma and throughout the paper we write $\omega = e^{2\pi i/3} = (-1 + i\sqrt{3})/2$.

In what follows we will be interested in the case where $R_1 J$ is elliptic. In this case its eigenvalues are unit modulus complex numbers whose product is 1. We write then as $e^{i\alpha}$, $e^{i\beta}$ and $e^{-i\alpha - i\beta}$. This means that $\tau = e^{i\alpha} + e^{i\beta} + e^{-i\alpha - i\beta}$.

Lemma 2.6 *Let H_τ be given by (5) where $\tau = e^{i\alpha} + e^{i\beta} + e^{-i\alpha - i\beta}$. Then the eigenvalues of H_τ are:*

$$-8\sin(\alpha/2 + \psi/6 + 2k\pi/3)\sin(\beta/2 + \psi/6 + 2k\pi/3)\sin(-\alpha/2 - \beta/2 + \psi/6 + 2k\pi/3)$$

for $k = 0,\ 1,\ 2$.

PROOF: We could solve the characteristic polynomial of H_τ. However, it is easier to observe that eigenvectors for H_τ are

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \qquad \begin{bmatrix} 1 \\ \overline{\omega} \\ \omega \end{bmatrix}, \qquad \begin{bmatrix} 1 \\ \omega \\ \overline{\omega} \end{bmatrix}.$$

We can immediately write down the eigenvalues as:

$$\lambda_k = 2\sin(\psi/2) - ie^{-i\psi/6 - 2k\pi i/3}\tau + ie^{i\psi/6 + 2k\pi i/3}\overline{\tau}$$

for $k = 0,\ 1,\ 2$. Subsitituting for τ gives

$$\begin{aligned} \lambda_k \quad = \quad & 2\sin(\psi/2) + 2\sin(-\alpha - \beta - \psi/6 - 2k\pi/3) \\ & + 2\sin(\alpha - \psi/6 - 2k\pi/3) + 2\sin(\beta - \psi/6 - 2k\pi/3). \end{aligned}$$

Using elementary trigonometry (and $-2k\pi/3 = 4k\pi/3 - 2k\pi$), we see that

$$\lambda_k = -8\sin(\alpha/2 + \psi/6 + 2k\pi/3)\sin(\beta/2 + \psi/6 + 2k\pi/3)\sin(-\alpha/2 - \beta/2 + \psi/6 + 2k\pi/3).$$

□

Corollary 2.7 *When $\tau = e^{i\alpha} + e^{i\beta} + e^{-i\alpha - i\beta}$ the matrix H_τ has signature $(2,1)$ if and only if*

$$8\sin(3\alpha/2 + \psi/2)\sin(3\beta/2 + \psi/2)\sin\bigl(-3(\alpha + \beta)/2 + \psi/2\bigr) < 0. \tag{10}$$

PROOF: It is easy to check (for example by adding them) that all three eigenvalues cannot be negative. Thus H_τ has signature $(2,1)$ if and only if its determinant is negative. Using the trigonometric identity $\sin(3\theta) = -4\sin(\theta)\sin(\theta + 2\pi/3)\sin(\theta + 4\pi/3)$ we see that the product of the three eigenvalues is.

$$8\sin(3\alpha/2 + \psi/2)\sin(3\beta/2 + \psi/2)\sin\bigl(-3(\alpha + \beta)/2 + \psi/2\bigr).$$

Hence H_τ has signature $(2,1)$ if and only if this expression is negative. □

The locus where each eigenvalue is zero corresponds to a line in $\mathbb{C}$. They divide $\mathbb{C}$ into seven components which fall into three types:

- The central triangle. Here all three eigenvalues are positive and H_τ is positive definite.
- Three infinite components that each share a common edge with the central triangle. Here two eigenvalues are positive and one negative. Hence H_τ has signature $(2,1)$. This is our parameter space.
- Three infinite components that each only abut the central triangle in a point. Here one eigenvalue is positive and two are negative. These correspond to groups of complex hyperbolic isometries generated by three complex involutions that each fix a point.

Therefore the parameter space we are interested in, that is those values of τ satisfying (10), has three components related by multiplication by powers of $\omega = e^{2\pi i/3}$. This corresponds to that fact that $J \in \mathrm{SU}(2,1)$ is only defined up to multiplication by a cube root of unity. Hence τ is only defined up to a cube root of unity. If we factor out by this equivalence, our parameter space becomes one of these components.

2.5 Arithmetic

When we are discussing the discreteness of $\langle R_1, R_2, R_3\rangle$ below, we shall analyse whether or not the group is arithmetic. In order to do so one must find a suitable ring containing the matrix entries (after possibly conjugating).

Proposition 2.8 *The maps R_1, R_2 and R_3 may be conjugated within* $\mathrm{SU}(2,1)$ *and scaled so that their matrix entries lie in the ring $\mathbb{Z}[\tau, \overline{\tau}, e^{i\psi}, e^{-i\psi}]$.*

PROOF: Conjugating the matrices R_1, R_2 and R_3 given above by $C = \mathrm{diag}(e^{-i\psi/3},\, 1,\, e^{i\psi/3})$ these matrices become

$$\begin{aligned}
CR_1C^{-1} &= e^{-i\psi/3}\begin{bmatrix} e^{i\psi} & \tau & -\overline{\tau} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \\
CR_2C^{-1} &= e^{-i\psi/3}\begin{bmatrix} 1 & 0 & 0 \\ -e^{i\psi}\overline{\tau} & e^{i\psi} & \tau \\ 0 & 0 & 1 \end{bmatrix}, \\
CR_3C^{-1} &= e^{-i\psi/3}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ e^{i\psi}\tau & -e^{i\psi}\overline{\tau} & e^{i\psi} \end{bmatrix}.
\end{aligned}$$

Hence the group generated by $e^{i\psi/3}CR_jC^{-1}$ for $j = 1,\, 2,\, 3$ consists of matrices whose entries lie in the ring $\mathbb{Z}[\tau, \overline{\tau}, e^{i\psi}, e^{-i\psi}]$. □

It is very important to keep track of what happens to H_τ when performing this conjugation. The matrices above preserve the Hermitian form given by (any real multiple of) $CH_\tau C^{-1}$. So we may take the Hermitian form to be given by:

$$2\sin(\psi/2)CH_\tau C^{-1} = \begin{bmatrix} 2 - e^{i\psi} - e^{-i\psi} & (e^{-i\psi} - 1)\tau & (1 - e^{-i\psi})\overline{\tau} \\ (e^{i\psi} - 1)\overline{\tau} & 2 - e^{i\psi} - e^{-i\psi} & (e^{-i\psi} - 1)\tau \\ (1 - e^{i\psi})\tau & (e^{i\psi} - 1)\overline{\tau} & 2 - e^{i\psi} - e^{-i\psi} \end{bmatrix}. \tag{11}$$

Hence the entries of $2\sin(\psi/2)CH_\tau C^{-1}$ also lie in the ring $\mathbb{Z}[\tau, \overline{\tau}, e^{i\psi}, e^{-i\psi}]$.

2.6 The angle parameter space

In [17], Paupert characterised which angle pairs can arise for the product AB (when it is elliptic), as A and B each vary inside a fixed elliptic conjugacy class (i.e. each have a fixed angle pair). In the present case, the R_1 (the first generator) has angles $\{0, 2\pi/p\}$, and J (the second generator) has angles $\{4\pi/3,\, 2\pi/3\}$. The allowable region in the surface $\mathbb{T}^2/\mathfrak{S}_2$ is then a convex quadrilateral (degenerating to a triangle for $p = 3$) with the following properties, which we quote from [17]. The two "totally reducible vertices" V_1 and V_2 (points representing an Abelian group) have respective coordinates $\{4\pi/3, 2\pi/3 + 2\pi/p\}$ and $\{4\pi/3 + 2\pi/p, 2\pi/3\}$. These two vertices are joined by a line segment of slope -1 corresponding to the reducible groups which fix a point inside $\mathbf{H}^2_{\mathbb{C}}$. Each of these vertices is also the endpoint of a line segment corresponding to the reducible groups which stabilise a complex line. The first of these segments starts upward at V_1 with slope 2, bounces off the diagonal and goes on (with slope $1/2$) until it reaches the boundary of the square. The second segment starts upward at V_2 with slope $1/2$ and goes off to the boundary. The last side of the quadrilateral is the resulting portion of the boundary segment $\{2\pi\} \times [0, 2\pi]$; in all corresponding groups the product R_1J is parabolic as was proven in [17]. The polygons for $p = 3, \ldots, , 10$ are

illustrated in Figure 2. In what follows we shall assume that R_1R_2 is elliptic. From Corollary 2.5, we see that this implies $|\tau| \leq 2$. This condition does not have a striaghtforward interpretation in the angle coordinates. In Figure 2 we have drawn the curve corresponding to $|\tau| = 2$.

3 When $R_1R_2R_3$ is elliptic and R_1R_2 is non-loxodromic

We restrict our attention to those groups for which $R_1R_2R_3$ is elliptic of finite order and R_1R_2 is either elliptic of finite order or else parabolic. These are groups for which τ lies inside or on the deltoid and inside or on the circle. Since they have finite order, the eigenvalues of $R_1R_2R_3$ and R_1R_2 are all roots of unity. This fact leads to a linear equation in certain cosines of rational multiples of π. We find all solutions to this equation using a theorem of Conway and Jones [1]. We then go on find which of these solutions lie in parameter space.

3.1 The eigenvalue equation

We now investigate when both R_1R_2 and $R_1R_2R_3$ are elliptic of finite order. In fact our proof will be valid when R_1R_2 is parabolic as well. We know that, R_1J (and hence $R_1R_2R_3$) is elliptic of finite order if and only if

$$\tau = \mathrm{tr}(R_1J) = e^{i\alpha} + e^{i\beta} + e^{-i\alpha-i\beta}, \tag{12}$$

where α and β are rational multiples of π. Likewise for R_1R_2. In fact we know slightly more. Since the intersection of L_1 and L_2 is an $e^{-i\psi/3}$-eigenvector for each of R_1 and R_2 it must be a $e^{-2i\psi/3}$-eigenvector for R_1R_2. From Proposition 2.4 we know that

$$\mathrm{tr}(R_1R_2) = e^{i\psi/3}\left(2 - |\tau|^2\right) + e^{-2i\psi/3}.$$

Hence the other two eigenvalues of R_1R_2 are $-e^{i\psi/3+2i\theta}$ and $-e^{i\psi/3-2i\theta}$. (We have taken minus signs in order to make our angles agree with [15].) Thus

$$|\tau|^2 - 2 = 2\cos(2\theta), \tag{13}$$

where θ is a rational multiple of π.

We solve equations (12) and (13) by eliminating τ. That is, we seek θ, α, β rational multiples of π so that

$$2\cos(2\theta) + 2 = |\tau|^2 = 3 + 2\cos(\alpha-\beta) + 2\cos(\alpha+2\beta) + 2\cos(-2\alpha-\beta).$$

Rearranging, this becomes

$$\frac{1}{2} = \cos(2\theta) - \cos(\alpha-\beta) - \cos(\alpha+2\beta) - \cos(-2\alpha-\beta). \tag{14}$$

In [15] Parker used a theorem of Conway and Jones [1] to solve (14). Up to complex conjugating

τ (so changing the sign of α and β) and multiplying by a power of ω, the only solutions are:

	2θ	$\alpha-\beta$	$\alpha+2\beta$	$2\alpha+\beta$	α	β	$\alpha+\beta$
(i)	$2\pi/3$	$\pi-\phi/2$	π	$2\pi-\phi/2$	$\pi-\phi/3$	$\phi/6$	$\pi-\phi/6$
(ii)	ϕ	$\pi/3-\phi$	$\pi/3+\phi$	$2\pi/3$	$\pi/3-\phi/3$	$2\phi/3$	$\pi/3+\phi/3$
(iii)	$\pi/3$	$\pi/4$	$\pi/2$	$3\pi/4$	$\pi/3$	$\pi/12$	$5\pi/12$
(iv)	$\pi/5$	$3\pi/10$	$2\pi/5$	$7\pi/10$	$\pi/3$	$\pi/30$	$11\pi/30$
(v)	$3\pi/5$	$\pi/10$	$4\pi/5$	$9\pi/10$	$\pi/3$	$7\pi/30$	$17\pi/30$
(vi)	$\pi/2$	$2\pi/7$	$4\pi/7$	$6\pi/7$	$8\pi/21$	$2\pi/21$	$10\pi/21$
(vii)	$\pi/2$	$\pi/15$	$11\pi/15$	$4\pi/5$	$13\pi/45$	$2\pi/9$	$23\pi/45$
$(viii)$	$\pi/2$	$2\pi/5$	$7\pi/15$	$13\pi/15$	$19\pi/45$	$\pi/45$	$4\pi/9$
(ix)	$\pi/7$	$\pi/21$	$4\pi/7$	$13\pi/21$	$2\pi/9$	$11\pi/63$	$25\pi/63$
(x)	$5\pi/7$	$5\pi/21$	$19\pi/21$	$8\pi/7$	$29\pi/63$	$2\pi/9$	$43\pi/63$
(xi)	$3\pi/7$	$2\pi/7$	$11\pi/21$	$17\pi/21$	$23\pi/63$	$5\pi/63$	$4\pi/9$

We then write down $\tau=\mathrm{tr}(R_1J)=e^{i\alpha}+e^{i\beta}+e^{-i\alpha-i\beta}$ and $\mathrm{tr}\,(R_1R_2R_3)=e^{3i\alpha}+e^{3i\beta}+e^{-3i\alpha-3i\beta}$ using this table. As indicated earlier, the parameters $\tau\omega^j$ for $j=1,\,2$ correspond to the same group as τ.

Proposition 3.1 (Proposition 3.2 of [15]) *Suppose that R_1R_2 and $R_1R_2R_3$ are both elliptic of finite order (or possibly R_1R_2 is parabolic). Up to complex conjugating τ and multiplying by a power of ω, then one of the following is true:*

(i) $\tau=-e^{-i\phi/3}=:\mathrm{m}(\phi)$ *for some angle ϕ;*

(ii) $\tau=e^{2i\phi/3}+e^{-i\phi/3}=e^{i\phi/6}\,2\cos(\phi/2)=:\mathrm{s}(\phi)$, *for some angle ϕ;*

(iii) $\tau=e^{i\pi/3}+e^{-i\pi/6}\,2\cos(\pi/4)=:\sigma(\pi/6,\pi/4)$;

(iv) $\tau=e^{i\pi/3}+e^{-i\pi/6}\,2\cos(\pi/5)=:\sigma(\pi/6,\pi/5)$;

(v) $\tau=e^{i\pi/3}+e^{-i\pi/6}\,2\cos(2\pi/5)=:\sigma(\pi/6,2\pi/5)$;

(vi) $\tau=e^{2\pi i/7}+e^{4\pi i/7}+e^{8\pi i/7}=:\sigma(2\pi/7)$;

(vii) $\tau=e^{2\pi i/9}+e^{-i\pi/9}\,2\cos(2\pi/5)=:\sigma(\pi/9,2\pi/5)$;

(viii) $\tau=e^{2\pi i/9}+e^{-\pi i/9}\,2\cos(4\pi/5)=:\sigma(\pi/9,4\pi/5)$;

(ix) $\tau=e^{2\pi i/9}+e^{-i\pi/9}\,2\cos(2\pi/7)=:\sigma(\pi/9,2\pi/7)$;

(x) $\tau=e^{2\pi i/9}+e^{-i\pi/9}\,2\cos(4\pi/7)=:\sigma(\pi/9,4\pi/7)$;

(xi) $\tau=e^{2\pi i/9}+e^{-i\pi/9}\,2\cos(6\pi/7)=:\sigma(\pi/9,6\pi/7)$.

Before proceeding we explain the notation. The possible solutions τ either lie on one of two curves or are on a finite list of points. The points on the curve from part (i) (for $p\geqslant 3$) correspond to Mostow's groups $\Gamma(p,t)$ (compare [11]). So we call this the *Mostow curve* and denote points on it by $\mathrm{m}(\phi)$. The points on the curve from part (ii) correspond to subgroups of Mostow's groups, by the results of section 5.1. These results generalise the isomorphisms discovered by Sauter in [19], so we call this the *Sauter curve* and denote points on it by $\mathrm{s}(\phi)$. The remaining parts (iii) to (xi) consist of finitely many isolated points which we call *sporadic*. All the points except for those in part (vi) depend on two angles and so we write them as $\sigma(\phi,\eta)=e^{2i\phi}+e^{-i\phi}2\cos(\eta)$. Note

that complex conjugation sends $\sigma(\phi,\eta)$ to $\sigma(-\phi,\eta)$ and multiplication by $e^{2\pi i/3}$ sends $\sigma(\phi,\eta)$ to $\sigma(\phi-2\pi/3,\eta)$.

We use these various values of $\tau=\mathrm{tr}(R_1J)$ to parametrise groups in $\mathrm{SU}(2,1)$; note that while the values of τ in the above list do not depend on ψ (the angle of rotation of the reflection R_1), the corresponding groups do vary with ψ. We will denote by $\Gamma(\psi,\tau)$ the subgroup of $\mathrm{SU}(2,1)$ generated by R_1 and J, where the rotation angle of R_1 is ψ and the trace of R_1J is τ. Note that there should be no confusion with Mostow's notation $\Gamma(p,t)$, where p is an integer and t is real. Explicitly, by a simple computation using Mostow's generator R_1 (appropriately normalised in $\mathrm{SU}(2,1)$), Mostow's group $\Gamma(p,t)$ is our $\Gamma(2\pi/p, e^{i\pi(3/2+1/p-t/3-2/(3p))})$.

3.2 The points in angle parameters

We now list the corresponding points in the angle parameter space. This involves considering the differences of the eigenvalues and making sure we have a pair $\{\theta_1,\theta_2\}$ with $4\pi/3\leqslant\theta_1\leqslant 2\pi$ and $2\pi/3\leqslant\theta_2\leqslant\theta_1$.

For the groups of Mostow and Sauter type the corresponding points in the angle space are piecewise linear curves. We now write these down.

Lemma 3.2 *(i) The point $\tau=\mathrm{m}(\phi)=-e^{-i\phi/3}$ corresponds to the angle pair $\{2\pi-\phi/2,\pi\}$ if $\phi/2\in[0,\pi]$ and $\{2\pi+\phi/2,\pi+\phi/2\}$ if $\phi/2\in[-\pi,0]$.*

(ii) The point $\tau=\mathrm{s}(\phi)=e^{2i\phi/3}+e^{-i\phi/3}$ corresponds to the angle pair $\{5\pi/3+\phi,\,4\pi/3\}$ if $\phi\in[-\pi/3,\pi/3]$ and $\{7\pi/3-\phi,5\pi/3-\phi\}$ if $\phi\in[\pi/3,\pi]$.

For the sporadic groups, in each case for τ the angle pair is $\{2\pi+\beta-\alpha,\,2\pi-2\alpha-\beta\}$ and for $\overline{\tau}$ it is either $\{2\pi-\alpha-2\beta,\,2\pi-2\alpha-\beta\}$ or $\{2\pi+\beta-\alpha,\alpha+2\beta\}$. These angles may be read off from the table just before Proposition 3.1.

	τ	$\{\theta_1,\theta_2\}$
(iii)	$e^{\pi i/3}+e^{-\pi i/6}2\cos(\pi/4)$	$\{7\pi/4,\,5\pi/4\}$
	$e^{-\pi i/3}+e^{\pi i/6}2\cos(\pi/4)$	$\{3\pi/2,\,5\pi/4\}$
(iv)	$e^{\pi i/3}+e^{-\pi i/6}2\cos(\pi/5)$	$\{17\pi/10,\,13\pi/10\}$
	$e^{-\pi i/3}+e^{\pi i/6}2\cos(\pi/5)$	$\{8\pi/5,\,13\pi/10\}$
(v)	$e^{\pi i/3}+e^{-\pi i/6}2\cos(2\pi/5)$	$\{19\pi/10,\,11\pi/10\}$
	$e^{-\pi i/3}+e^{\pi i/6}2\cos(2\pi/5)$	$\{19\pi/10,\,4\pi/5\}$
(vi)	$e^{2\pi i/7}+e^{4\pi i/7}+e^{8\pi i/7}$	$\{12\pi/7,\,8\pi/7\}$
	$e^{-2\pi i/7}+e^{-4\pi i/7}+e^{-8\pi i/7}$	$\{10\pi/7,\,8\pi/7\}$
(vii)	$e^{2\pi i/9}+e^{-\pi i/9}2\cos(2\pi/5)$	$\{29\pi/15,\,18\pi/15\}$
	$e^{-2\pi i/9}+e^{\pi i/9}2\cos(2\pi/5)$	$\{29\pi/15,\,11\pi/15\}$
$(viii)$	$e^{2\pi i/9}+e^{-\pi i/9}2\cos(4\pi/5)$	$\{8\pi/5,\,17\pi/15\}$
	$e^{-2\pi i/9}+e^{\pi i/9}2\cos(4\pi/5)$	$\{23\pi/15,\,17\pi/15\}$
(ix)	$e^{2\pi i/9}+e^{-\pi i/9}2\cos(2\pi/7)$	$\{41\pi/21,\,29\pi/21\}$
	$e^{-2\pi i/9}+e^{\pi i/9}2\cos(2\pi/7)$	$\{10\pi/7,\,29\pi/21\}$
(x)	$e^{2\pi i/9}+e^{-\pi i/9}2\cos(4\pi/7)$	$\{37\pi/21,\,6\pi/7\}$
	$e^{-2\pi i/9}+e^{\pi i/9}2\cos(4\pi/7)$	$\{37\pi/21,\,19\pi/21\}$
(xi)	$e^{2\pi i/9}+e^{-\pi i/9}2\cos(6\pi/7)$	$\{12\pi/7,\,25\pi/21\}$
	$e^{-2\pi i/9}+e^{\pi i/9}2\cos(2\pi/7)$	$\{31\pi/21,\,25\pi/21\}$

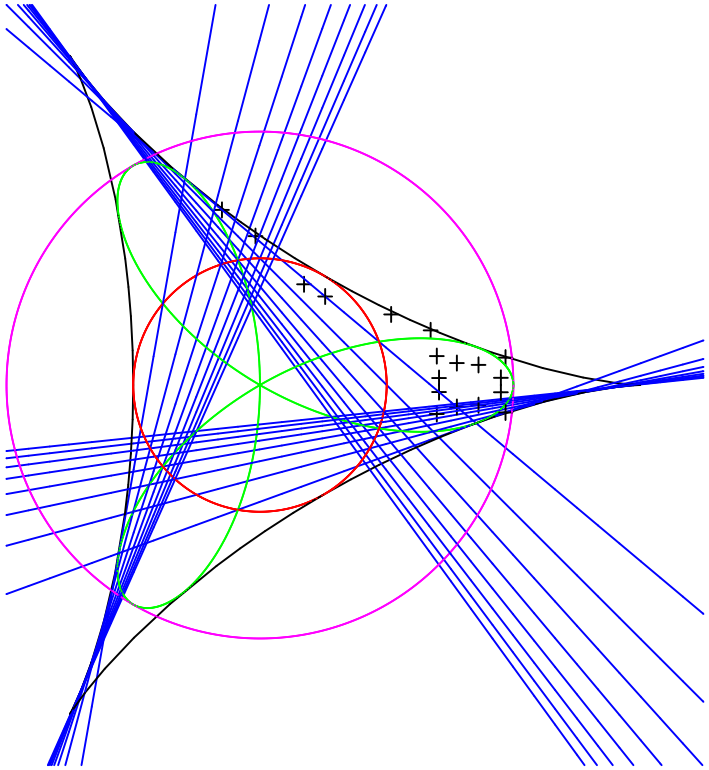

Figure 1: *The points together with the trace parameter spaces for* $p = 3, \ldots, 10$.

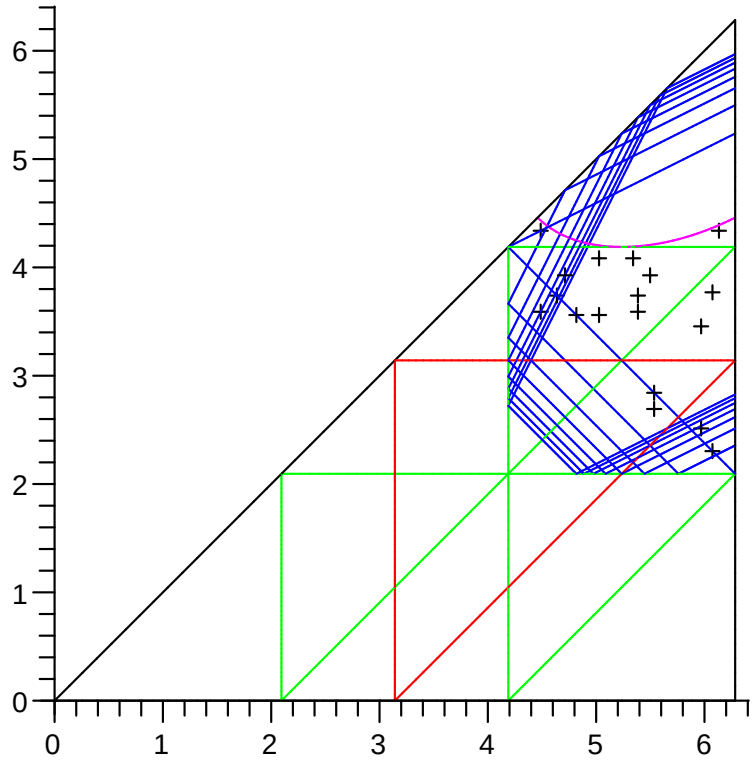

Figure 2: *The points together with the angle parameter spaces for* $p = 3, \ldots, 10$.

3.3 Points in parameter space

In this section we consider the values of τ from Proposition 3.1 or the corresponding angle pairs and we determine which of them is in the parameter space. In each case, we are only interested in the case where $\psi = 2\pi/p$ for some integer $p \geqslant 2$. In each case we know the eigenvalues $e^{i\alpha}$, $e^{i\beta}$ and $e^{-i\alpha-i\beta}$ of $R_1 J$ and we can use the criterion (10) of Corollary 2.7 to determine for which values of p the form H_τ has signature $(2,1)$. Alternatively, we could use the angle pairs and check the linear conditions given in Section 2.6. We have plotted the points and parameter spaces for $p = 3, \ldots, 10$ in the τ-plane in Figure 1 and we have plotted the same thing in the angle parameter space in Figure 2.

Proposition 3.3 *Let R_1 have angle $\psi \in (0, 2\pi)$.*

(i) Suppose that $\tau = \mathrm{m}(\phi) = -e^{-i\phi/3}$. Then H_τ has signature $(2,1)$ if and only if

$$\sin(\psi + \phi/2)\cos(\psi/2 - \phi/2) > 0.$$

(ii) Suppose that $\tau = \mathrm{s}(\phi) = e^{2i\phi/3} + e^{-i\phi/3}$. Then H_τ has signature $(2,1)$ if and only if $\sin(\psi/2 + \phi) > 0$.

Proof: We insert the values of α and β found above into Corollary 2.7 and simplify.

In (i) we have $\alpha = \pi - \phi/3$ and $\beta = \phi/6$. This leads to

$$8\sin(3\pi/2 - \phi/2 + \psi/2)\sin(\phi/4 + \psi/2)\sin(-3\pi/2 + \phi/2 + \psi/2) < 0.$$

In (ii) we have $\alpha = \pi/3 - \phi/3$ and $\beta = 2\phi/3$. This leads to

$$8\sin(\phi + \psi/2)\sin(-\phi/2 + \pi/2 + \psi/2)\sin(-\phi/2 - \pi/2 + \psi/2) < 0$$

Simplifying, this is equivalent to

$$\sin(\psi/2 + \phi)\big(\cos(\psi - \phi) + 1\big) > 0.$$

We can ignore the possibility $\cos(\psi - \phi) = -1$ as this only occurs when $\tau = e^{2i\psi/3} - e^{-i\psi/3}$, in which case H_τ has a repeated eigenvalue of 0. (By inspection this does not occur in the interval where H_τ has signature $(2,1)$). □

For the sporadic groups we simply work by inspection. The results are given in the following table:

	α	β	$-\alpha-\beta$	$(2,1)$	Degenerate	$(3,0)$
(iii)	$\pi/3$	$\pi/12$	$-5\pi/12$	$p \geq 3$	$p = 2$	
	$-\pi/3$	$5\pi/12$	$-\pi/12$	$3 \leq p \leq 7$	$p = 2,\ 8$	$p \geq 9$
(iv)	$\pi/3$	$\pi/30$	$-11\pi/30$	$p \geq 3$	$p = 2$	
	$-\pi/3$	$11\pi/30$	$-\pi/30$	$3 \leq p \leq 19$	$p = 2,\ 20$	$p \geq 21$
(v)	$\pi/3$	$7\pi/30$	$-17\pi/30$	$p \geq 3$	$p = 2$	
	$-\pi/3$	$17\pi/30$	$-7\pi/30$	$3 \leq p \leq 6$	$p = 2$	$p \geq 7$
(vi)	$2\pi/7$	$4\pi/7$	$-6\pi/7$	$4 \leq p \leq 6$	$p = 7$	$p = 2,\ 3,\ p \geq 8$
	$-2\pi/7$	$-4\pi/7$	$6\pi/7$	$p \geq 3$		$p = 2$
(vii)	$2\pi/9$	$13\pi/45$	$-23\pi/45$	$p \geq 2$		
	$-2\pi/9$	$23\pi/45$	$13\pi/45$	$p = 2,\ 4$	$p = 3$	$p \geq 5$
$(viii)$	$2\pi/9$	$31\pi/45$	$-41\pi/45$	$p \geq 3$		$p = 2$
	$-2\pi/9$	$41\pi/45$	$-31\pi/45$	$4 \leq p \leq 29$	$p = 3,\ 30$	$p = 2,\ p \geq 31$
(ix)	$2\pi/9$	$11\pi/63$	$-25\pi/63$	$p \geq 2$		
	$-2\pi/9$	$25\pi/63$	$-11\pi/63$	$p = 2$	$p = 3$	$p \geq 4$
(x)	$2\pi/9$	$29\pi/63$	$-43\pi/36$	$4 \leq p \leq 41$	$p = 42$	$p = 2,\ 3,\ p \geq 43$
	$-2\pi/9$	$43\pi/63$	$-29\pi/63$	$p \geq 4$	$p = 3$	$p = 2$
(xi)	$2\pi/9$	$47\pi/63$	$-61\pi/63$	$p \geq 3$		$p = 2$
	$-2\pi/9$	$61\pi/63$	$-47\pi/63$	$4 \leq p \leq 8$	$p = 3$	$p = 2,\ p \geq 9$

4 Non-discreteness results

There are two simple ways to see that a subgroup of $\mathrm{PU}(2,1)$ is not discrete: finding an elliptic element of infinite order in the group, or finding a subgroup which stabilises a complex line, acting non-discretely on it. Of course, these elementary facts can only be useful if one finds the appropriate element (word) or subgroup. We will present in this section some results of this nature.

4.1 Traces for certain words

We compute the traces of certain short words in our generators R_1 and J (or R_1, R_2, R_3). These words seem relevant to us for experimental and/or historical reasons (see [11] and [21] for instance). These traces can be computed in a straightforward manner from the generators, or by using the formulae from [18].

W	$\mathrm{tr}(W)$
R_1J	τ
$J^{-1}R_1$	$-e^{i\psi/3}\overline{\tau}$
$R_1R_2J^{-1} = (R_1J)^2$	$\tau^2 - 2\overline{\tau}$
$JR_2R_1 = (J^{-1}R_1)^2$	$e^{2i\psi/3}\overline{\tau}^2 + 2e^{-i\psi/3}\tau$
R_1R_2	$e^{-2i\psi/3} + e^{i\psi/3}\left(2 - \lvert\tau\rvert^2\right)$
$R_1R_2R_3R_2^{-1}$	$e^{-2i\psi/3} + e^{i\psi/3}\left(2 - \lvert\tau^2 - \overline{\tau}\rvert^2\right)$
$R_1R_2^{-1}R_3R_2$	$e^{-2i\psi/3} + e^{i\psi/3}\left(2 - \lvert\tau^2 + e^{i\psi}\overline{\tau}\rvert^2\right)$
$R_1R_2^{-1}$	$1 + 2\cos(\psi) + \lvert\tau\rvert^2$
$R_1R_2R_3^{-1}R_2^{-1}$	$1 + 2\cos(\psi) + \lvert\tau^2 - \overline{\tau}\rvert^2$
$R_1R_2^{-1}R_3^{-1}R_2$	$1 + 2\cos(\psi) + \lvert\tau^2 + e^{i\psi}\overline{\tau}\rvert^2$
$R_1R_2R_3 = (R_1J)^3$	$3 - 3\lvert\tau\rvert^2 + \tau^3$
$R_1R_2R_1R_2^{-1}R_1^{-1}R_2^{-1}$	$3 + \left(\lvert\tau\rvert^2 - 1\right)\left(\lvert\tau\rvert^4 + 6\lvert\tau\rvert^2 + 2 - 2\cos(\psi)\right)$

We use this to prove non-discreteness results, some valid for any value of p like the following, and some depending on the value of p (see last section for $p = 3$).

Lemma 4.1 *Let R_1, R_2 and R_3 be given by (7), (8) and (9). If $|\tau^2 - \overline{\tau}| > 2$ then $R_1R_2R_3R_2^{-1}$ is loxodromic. If $|\tau^2 - \overline{\tau}| \leq 2$ then $R_1R_2R_3R_2^{-1}$ is elliptic with eigenvalues $e^{-2i\psi/3}$, $-e^{i\psi/3+2i\theta'}$ and $-e^{i\psi/3-2i\theta'}$ where $|\tau^2 - \overline{\tau}| = 2\cos\theta'$.*

PROOF: A short computation shows that the trace of $R_1R_2R_3R_2^{-1}$ is

$$\mathrm{tr}(R_1R_2R_3R_2^{-1}) = e^{i\psi/3}\big(2 - |\tau^2 - \overline{\tau}|^2\big) + e^{-2i\psi/3}.$$

Since all points of L_1 (the mirror of R_1) and of $R_2(L_3)$ (the mirror of $R_2R_3R_2^{-1}$) are $e^{-i\psi/3}$ eigenvectors for R_1 and $R_2R_3R_2^{-1}$ respectively, we see that their intersection is an $e^{-2i\psi/3}$ eigenvector for $R_1R_2R_3R_2^{-1}$. If this point is outside $\mathbf{H}^2_{\mathbb{C}}$ then $R_1R_2R_3R_2^{-1}$ is loxodromic. On the other hand, if this point is inside complex hyperbolic space then $R_1R_2R_3R_2^{-1}$ is elliptic with eigenvalues $e^{-2i\psi/3}$, $-e^{i\psi/3+2i\theta'}$ and $-e^{i\psi/3-2i\theta'}$ where $|\tau^2 - \overline{\tau}| = 2\cos\theta'$. □

We now list $|\tau^2 - \overline{\tau}|^2$ for the values of τ given in Proposition 3.1. It is clear that complex conjugating τ or multiplying by a cube root of unity does not affect this quantity. We also give the value of θ' or indicate that $R_1R_2R_3R_2^{-1}$ is loxodromic.

	τ	$\lvert\tau\rvert^2$	2θ	$\lvert\tau^2 - \overline{\tau}\rvert^2$	$2\theta'$
(i)	$-e^{-i\phi/3}$	1	$2\pi/3$	$2 + 2\cos(\phi)$	ϕ
(ii)	$e^{2i\phi/3} + e^{-i\phi/3}$	$2 + 2\cos(\phi)$	ϕ	$2 + 2\cos(\phi)$	ϕ
(iii)	$e^{i\pi/3} + e^{-i\pi/6}2\cos(\pi/4)$	3	$\pi/3$	2	$\pi/2$
(iv)	$e^{i\pi/3} + e^{-i\pi/6}2\cos(\pi/5)$	$2 + 2\cos(\pi/5)$	$\pi/5$	3	$\pi/3$
(v)	$e^{i\pi/3} + e^{-i\pi/6}2\cos(2\pi/5)$	$2 + 2\cos(3\pi/5)$	$3\pi/5$	3	$\pi/3$
(vi)	$e^{2\pi i/7} + e^{4\pi i/7} + e^{8\pi i/7}$	2	$\pi/2$	1	$2\pi/3$
(vii)	$e^{2i\pi/9} + e^{-i\pi/9}2\cos(2\pi/5)$	2	$\pi/2$	$2 + 2\cos(2\pi/5)$	$2\pi/5$
$(viii)$	$e^{2i\pi/9} + e^{-i\pi/9}2\cos(4\pi/5)$	2	$\pi/2$	$2 + 2\cos(4\pi/5)$	$4\pi/5$
(ix)	$e^{2i\pi/9} + e^{-i\pi/9}2\cos(2\pi/7)$	$2 + 2\cos(\pi/7)$	$\pi/7$	$3 + 2\cos(2\pi/7)$	loxodromic
(x)	$e^{2i\pi/9} + e^{-i\pi/9}2\cos(4\pi/7)$	$2 + 2\cos(5\pi/7)$	$5\pi/7$	$3 + 2\cos(4\pi/7)$	irrational
(xi)	$e^{2i\pi/9} + e^{-i\pi/9}2\cos(6\pi/7)$	$2 + 2\cos(3\pi/7)$	$3\pi/7$	$3 + 2\cos(6\pi/7)$	irrational

In case (ix) $|\tau^2 - \overline{\tau}|^2 > 4$ and so $R_1R_2R_3R_2^{-1}$ is loxodromic.

In the last two cases $|\tau^2 - \overline{\tau}|^2$ is less than 4. Therefore, using Lemma 4.1 we see that $R_1R_2R_3R_2^{-1}$ is elliptic and we find the angle θ' satisfying $2 + 2\cos(2\theta') = |\tau^2 - \overline{\tau}|^2 = 3 + 2\cos(4\pi/7)$ and $3 + 2\cos(6\pi/7)$, respectively. Using the theorem of Conway and Jones (Theorem 7 of [1]), we see that θ' is not a rational multiple if π and hence $R_1R_2R_3R_2^{-1}$ is elliptic of infinite order. Therefore, for these values of τ, the group $\langle R_1, J\rangle$ is not discrete. This proves:

Corollary 4.2 *Suppose that $\tau = e^{2\pi i/9} + e^{-\pi i/9}2\cos(4\pi/7)$ or $\tau = e^{2\pi i/9} + e^{-\pi i/9}2\cos(6\pi/7)$. Then $R_1R_2R_3R_2^{-1}$ is elliptic of infinite order.*

Hence when examining possible candidates for discrete groups it suffices to consider the values of τ given in parts (i) to (ix) of Proposition 3.1.

4.2 Triangle subgroups

We will systematically analyse triangle subgroups in order to find conditions for discreteness. By triangle subgroups we mean subgroups which fix a point in $\mathbb{C}P^2$; depending on the position of that point, the subgroup will stabilise a complex line (if the point is outside of $\overline{\mathbf{H}^2_{\mathbb{C}}}$), fix a point in $\mathbf{H}^2_{\mathbb{C}}$ or on its boundary. The two-generator group in question then acts as a hyperbolic, spherical, or Euclidean triangle group respectively, and the angles of the triangle action are obtained from the eigenvalues of the generators, see Proposition 4.4 below.

We use the lists which answer the classical question of plane geometry: given three angles α, β and γ, when is the group generated by the reflections in the sides of an (α, β, γ) triangle discrete?

In the case of a spherical triangle, the list is due to Schwarz ([20]), see table 4.2 below; for hyperbolic triangles, it is essentially due to Knapp ([8]), see Proposition 4.3 below. In the Euclidean case there are the three obvious triangles $(\pi/3, \pi/3, \pi/3)$, $(\pi/2, \pi/4, \pi/4)$ and $(\pi/2, \pi/3, \pi/6)$, and only one other, namely $(2\pi/3, \pi/6, \pi/6)$ (see [2]).

We start by listing the Schwarz triangles, i.e. the spherical triangles answering the above question. The possible triangles are arranged as in [2]: colunar triangles appear together on one line, in increasing order of size.

Schwarz triangles
$(\pi/2, \pi/2, p\pi/q)$
$(\pi/2, \pi/3, \pi/3)$, $(\pi/2, 2\pi/3, \pi/3)$, $(\pi/2, 2\pi/3, 2\pi/3)$ $(2\pi/3, \pi/3, \pi/3)$, $(2\pi/3, 2\pi/3, 2\pi/3)$
$(\pi/2, \pi/3, \pi/4)$, $(\pi/2, 2\pi/3, \pi/4)$, $(\pi/2, \pi/3, 3\pi/4)$, $(\pi/2, 2\pi/3, 3\pi/4)$ $(2\pi/3, \pi/4, \pi/4)$, $(\pi/3, 3\pi/4, \pi/4)$, $(2\pi/3, 3\pi/4, 3\pi/4)$
$(\pi/2, \pi/3, \pi/5)$, $(\pi/2, 2\pi/3, \pi/5)$, $(\pi/2, \pi/3, 4\pi/5)$, $(\pi/2, 2\pi/3, 4\pi/5)$ $(2\pi/5, \pi/3, \pi/3)$, $(3\pi/5, 2\pi/3, \pi/3)$, $(2\pi/5, 2\pi/3, 2\pi/3)$ $(2\pi/3, \pi/5, \pi/5)$, $(\pi/3, 4\pi/5, \pi/5)$, $(2\pi/3, 4\pi/5, 4\pi/5)$ $(\pi/2, 2\pi/5, \pi/5)$, $(\pi/2, 3\pi/5, \pi/5)$, $(\pi/2, 2\pi/5, 4\pi/5)$, $(\pi/2, 3\pi/5, 4\pi/5)$ $(3\pi/5, \pi/3, \pi/5)$, $(2\pi/5, 2\pi/3, \pi/5)$, $(2\pi/5, \pi/3, 4\pi/5)$, $(3\pi/5, 2\pi/3, 4\pi/5)$ $(2\pi/5, 2\pi/5, 2\pi/5)$, $(3\pi/5, 3\pi/5, 2\pi/5)$ $(2\pi/3, \pi/3, \pi/5)$, $(\pi/3, \pi/3, 4\pi/5)$, $(2\pi/3, 2\pi/3, 4\pi/5)$ $(4\pi/5, \pi/5, \pi/5)$, $(4\pi/5, 4\pi/5, 4\pi/5)$ $(\pi/2, 2\pi/5, \pi/3)$, $(\pi/2, 3\pi/5, \pi/3)$, $(\pi/2, 2\pi/5, 2\pi/3)$, $(\pi/2, 3\pi/5, 2\pi/3)$ $(3\pi/5, 2\pi/5, \pi/3)$, $(2\pi/5, 2\pi/5, 2\pi/3)$, $(3\pi/5, 3\pi/5, 2\pi/3)$

Proposition 4.3 (Lemma 2.1 of Klimenko and Sakuma [7]) *Suppose that the group generated by reflections in the sides of a hyperbolic triangle is discrete. Then the angles of the triangle appear on the following list:*

(i) π/p, π/q, π/r *where* $1/p + 1/q + 1/r < 1$;

(ii) π/p, π/p, $2\pi/r$ *where* r *is odd and* $1/p + 1/r < 1/2$;

(iii) π/p, $\pi/2$, $2\pi/p$ *where* $p \geq 7$ *is odd;*

(iv) π/p, $\pi/3$, $3\pi/p$ *where* $p \geq 7$ *is not divisible by 3;*

(v) π/p, π/p, $4\pi/p$ *where* $p \geq 7$ *is odd;*

(vi) $2\pi/p$, $2\pi/p$, $2\pi/p$ *where* $p \geq 7$ *is odd;*

(vii) $\pi/7$, $\pi/3$, $2\pi/7$.

The case when two of the angles are π/p and π/q was proved by Knapp in Theorem 2.3 of [8]. This list is the same as that above without case (vi). Knapp's theorem was rediscovered by Mostow and appears as Theorem 3.7 of [13] except that Mostow missed case (vii).

Given a group generated by reflections in the sides of a triangle (hyperbolic, Euclidean or spherical) we can consider the index 2 group of holomorphic motions. The product of a pair of reflections in sides that make an angle θ is an elliptic rotation through 2θ. If we represent this map as a matrix in $\mathrm{SU}(2)$ or $\mathrm{SU}(1,1)$ the eigenvalues are $-e^{\pm i\theta}$. This group of holomorphic motions is generated by A and B where A, B, AB are elliptic rotations corresponding to products of pairs of reflections.

Proposition 4.4 *For* $j = 1, 2$ *suppose that* B_j *is a complex reflection in* $\mathrm{SU}(2,1)$ *with eigenvalues* $e^{2i\psi_j/3}$, $e^{-i\psi_j/3}$, $e^{-i\psi_j/3}$ *and mirror* L_j. *Then* $L_1 \cap L_2$ *corresponds to an* $e^{-i\psi_1/3-i\psi_2/3}$ *eigenvector of* B_1B_2. *Suppose that* B_1B_2 *is elliptic and its other eigenvalues are* $-e^{i\psi_1/6+i\psi_2/6\pm i\psi_3/2}$. *Then on the orthogonal complement of* $L_1 \cap L_2$ *the group* $\langle B_1, B_2\rangle$ *acts as the holomorphic subgroup of a* $(\psi_1/2, \psi_2/2, \psi_3/2)$ *triangle group.*

Proof: We conjugate so that $L_1 \cap L_2$ corresponds to the vector $(1,0,0)^T$. Then B_1 and B_2 are block diagonal matrices in $S\big(\mathrm{U}(1)\times\mathrm{U}(1,1)\big)$ or $S\big(\mathrm{U}(1)\times\mathrm{U}(2)\big)$. Scaling, we may suppose that they lie in $\mathrm{U}(1)\times\mathrm{SU}(1,1)$ or $\mathrm{U}(1)\times\mathrm{SU}(2)$. Then the action on the orthogonal complement of $L_1 \cap L_2$ is given by the 2×2 unimodular matrices in the lower right hand block. These block matrices have eigenvalues $-e^{\pm i\psi_1/2}$, $-e^{\pm i\psi_2/2}$, $-e^{\pm i\psi_3/2}$ for B_1, B_2, B_1B_2 (up to maybe scaling by -1). The result follows from the remarks above. □

For example, consider R_1 and R_2. Their eigenvalues are $e^{2i\psi/3}$, $e^{-i\psi/3}$, $e^{-i\psi/3}$. Scaling, we see that the eigenvalues of $-e^{-i\psi/6}R_1$ and $-e^{-i\psi/6}R_2$ are $-e^{i\psi/2}$, $-e^{-i\psi/2}$, and $-e^{-i\psi/2}$. The eigenvalues of R_1R_2 are $-e^{i\psi/3+2i\theta}$, $-e^{i\psi/3-2i\theta}$, and $e^{-2i\psi/3}$, so that the eigenvalues of $e^{-i\psi/3}R_1R_2$ are $-e^{2i\theta}$, $-e^{-2i\theta}$, and $e^{-i\psi} = (-e^{-i\psi/2})(-e^{-i\psi/2})$. Therefore $\langle R_1, R_2\rangle$ acts on the orthognal complement of $L_1 \cap L_2$ as the holomorphic subgroup of a $(\psi/2, \psi/2, 2\theta)$ triangle group.

Similarly, R_1 and $R_2R_3R_2^{-1}$ have a common eigenvector $L_1 \cap R_2(L_3)$. When $R_1R_2R_3R_2^{-1}$ is elliptic the group $\langle R_1, R_2R_3R_2^{-1}\rangle$ acts on the orthogonal complement of $L_1 \cap R_2(L_3)$ as a $(\psi/2, \psi/2, 2\theta')$ triangle group.

We can apply the result of Klimenko and Sakuma, Proposition 4.3 (in fact we only need the earlier version of Knapp [8]) to eliminate some of the cases, using the values for 2θ and $2\theta'$ given in Table 4.1. This gives the following:

Proposition 4.5 *Suppose that* $\psi = 2\pi/p$.

(i) If $p \geqslant 3$ *and* τ *or* $\overline{\tau} = e^{i\pi/3} + e^{-i\pi/6}2\cos(2\pi/5)$ *then* $\langle R_1, R_2\rangle$ *is not discrete.*

(ii) If $p \neq 3, 5$ *and* τ *or* $\overline{\tau} = e^{2\pi i/9} + e^{-\pi i/9}2\cos(4\pi/5)$ *then* $\langle R_1, R_2R_3R_2^{-1}\rangle$ *is not discrete.*

5 The two curves in parameter space

In this section we determine exactly which points on the two curves from Proposition 3.1 correspond to discrete groups.

The first curve corresponds to Mostow's original groups from [11], where sufficient conditions for discreteness were found; these sufficient conditions were generalised in [12] (to a condition which Mostow calls Σ INT, which also contains the groups from [3]). A necessary condition, using triangle subgroups, was then found in [13], leaving only nine groups in dimension 2 and one in dimension 3 not covered by either of these criteria. Mostow proves in [13] that all but three of these are discrete; the last three cases were shown to be discrete by Sauter in [19]. See [13] or [19] for more details on the history of this question.

Determining which points of the second curve (or Sauter curve) correspond to discrete groups uses the result that every such group is a subgroup of a group on the Mostow curve, see section 5.1 below. It then remains to discard the possibility that a discrete group on the Sauter curve could be a subgroup (of infinite index) of a non-discrete group on the Mostow curve (theorem 5.8). This is done by a careful analysis of triangle subgroups.

5.1 Sauter groups are subgroups of Mostow groups

In this section we prove that each group on the Sauter curve is isomorphic to a subgroup of a group on the Mostow curve, but for a different value of ψ, the rotation angle of R_1:

Proposition 5.1 *$\Gamma(\phi, e^{2i\psi/3} + e^{-i\psi/3})$ is isomorphic to a subgroup of $\Gamma(\psi, -e^{-i\phi/3})$.*

Recall that $\Gamma(\psi, \tau)$ denotes the group generated by R_1 and J, where ψ is the rotation angle of R_1 and τ is the trace of $R_1 J$.

Proof: We begin by defining some elements of $\langle R_1, J\rangle$, with ψ as rotation angle of R_1. These definitions follow Sauter.

$$A_1 = R_2^{-1}R_1^{-1}J = J^{-1}(J^{-1}R_1^{-1})^2 J = \begin{bmatrix} -\tau & e^{i\psi/3}\overline{\tau} & e^{-i\psi/3} \\ e^{-i\psi/3}\left(1-|\tau|^2\right) & \overline{\tau}^2 - \tau & e^{-2i\psi/3}\overline{\tau} \\ 0 & e^{2i\psi/3} & 0 \end{bmatrix},$$

$$A_2 = JR_2^{-1}R_1^{-1} = (J^{-1}R_1^{-1})^2\ ,\ A_3 = J^{-1}R_2^{-1}R_1^{-1}J^{-1} = J(J^{-1}R_1^{-1})^2 J^{-1}$$

The result follows by noting that the subgroup $\langle A_1, J\rangle$ of $\langle R_1, J\rangle$ corresponds to the required parameters. Namely, if $\tau = -e^{-i\phi/3}$ then A_1 is a complex reflection with angle ϕ (as can be seen by its trace), and $A_1 J^{-1}$ has trace $e^{-i\psi/3}(2-|\tau|^2) + e^{2i\psi/3} = e^{-i\psi/3} + e^{2i\psi/3}$. Note that we have replaced J with J^{-1}, which is conjugate to J (as in Sauter's isomorphisms). □

In fact we can say more:

Lemma 5.2 *$\langle A_1, A_2, A_3\rangle$ is a normal subgroup of $\langle R_1, J\rangle$.*

Proof: If $|\tau| = 1$ then $R_i R_j R_i = R_j R_i R_j$. Using this and $R_2 = JR_1J^{-1}$, we find

$$\begin{aligned} R_1 A_1 R_1^{-1} &= R_1 R_2^{-1}R_1^{-1}JR_1^{-1} = R_2^{-1}R_1^{-1}R_2R_2^{-1}J = A_1, \\ R_1 A_2 R_1^{-1} &= R_1 JR_2^{-1}R_1^{-1}R_1^{-1} = R_1R_2J^{-1}J^{-1}R_2^{-1}R_1^{-1}J^{-1}JR_2^{-1}R_1^{-1} = A_2^{-1}A_3A_2, \\ R_1 A_3 R_1^{-1} &= R_1 J^{-1}R_2^{-1}R_1^{-1}J^{-1}R_1^{-1} = JR_2^{-1}R_1^{-1} = A_2. \end{aligned}$$

(compare the identities in Section 7 of [14]). Moreover, from the definition of A_1, A_2, A_3 we have: $JA_iJ^{-1} = A_{i+1}$ (taking the index i mod. 3). The lemma follows. □

When $\tau = -e^{-i\phi/3}$, Sauter considers the map $R_1 \longmapsto A_2$, $R_2 \longmapsto A_1$, $R_3 \longmapsto A_3$. That is, he considers the group for which the trace of the generator is $\overline{\tau}^2 - 2\tau = e^{2i\phi/3} + 2e^{-i\phi/3}$ and the parameter is $e^{2i\psi/3} + e^{-i\psi/3}$. This is a point on the curve of the second type. In fact, as Sauter is only interested in the case of Mostow groups, he only considers this map in the case when $\psi = 2\pi/3$, Theorem 6.1 of [19]. In this case $e^{2i\psi/3} + e^{-i\psi/3} = e^{i\pi/9}$ which is an intersection point of the curves of Mostow and Sauter types (see section 5.5).

The special case of groups on the Mostow curve with $\tau = 1$ (or equivalently ω or $\overline{\omega}$) was considered by Livné in [9]. Such groups have signature $(2, 1)$ when $\psi < \pi/2$. In [14] Parker showed that Livné's groups contain subgroups that are triangle groups generated by involutions. This is another special case of Proposition 5.1.

Conversely, if R_1, R_2 and R_3 have the form (7), (8), (9) then the map

$$Q = \begin{bmatrix} e^{-i\phi/3} & 0 & 0 \\ 0 & 0 & e^{i\psi/3+i\phi/3} \\ 0 & -e^{-i\psi/3} & e^{2i\phi/3} + e^{-i\phi/3} \end{bmatrix}$$

satisfies

$$QR_1Q^{-1} = R_1, \quad QR_2Q^{-1} = R_3, \quad QR_3Q^{-1} = R_3^{-1}R_2R_3.$$

Therefore $\langle R_1, R_2, R_3\rangle$ is a normal subgroup of $\langle Q, R_1, J\rangle$.

5.2 The Mostow curves

Proposition 5.3 *Suppose that $\psi = 2\pi/p$ and $\tau = -e^{-i\phi/3}$. Then the following subgroups of $\langle R_1, J\rangle$ have a common fixed vector and on its orthogonal complement they act as (the holomorphic subgroup of) a triangle group as follows:*

(i) $\langle R_1, R_2\rangle$ *acts as a* $(\psi/2, \psi/2, 2\pi/3)$ *triangle group;*

(ii) $\langle R_1, R_2R_3R_2^{-1}\rangle$ *acts as a* $(\psi/2, \psi/2, \phi)$ *triangle group;*

(iii) $\langle R_1, R_2^{-1}R_3R_2\rangle$ *acts as a* $(\psi/2, \psi/2, \pi - \psi - \phi)$ *triangle group;*

(iv) $\langle A_1, A_2\rangle = \langle R_2^{-1}R_1^{-1}J, JR_2^{-1}R_1^{-1}\rangle$ *acts as a* $(\phi/2, \phi/2, \psi)$ *triangle group;*

(v) $\langle A_1, A_1A_3A_2\rangle = \langle R_2^{-1}R_1^{-1}J, (R_2^{-1}R_1^{-1})^3\rangle$ *acts as a* $(\phi/2, \pi/2-\psi/2-\phi/2, \pi/2-3\psi/2)$ *triangle group.*

PROOF: (i) The eigenvalues of R_1 and R_2 are $e^{2i\psi/3}$, $e^{-i\psi/3}$, $e^{-i\psi/3}$, and those of R_1R_2 are $-e^{i\psi/3+2i\pi/3}$, $-e^{i\psi/3-2i\pi/3}$, $e^{-2i\psi/3}$. The result follows from Proposition 4.4.
(ii) The eigenvalues of R_1 and $R_2R_3R_2^{-1}$ are $e^{2i\psi/3}$, $e^{-i\psi/3}$, $e^{-i\psi/3}$. The trace of $R_1R_2R_3R_1^{-1}$ is $e^{-2i\psi/3} - e^{i\psi/3}2\cos(\phi)$. Therefore its eigenvalues are $-e^{i\psi/3+i\phi}$, $-e^{i\psi/3-i\phi}$, $e^{-2i\psi/3}$. The result follows from Proposition 4.4.
(iii) This is similar to (ii) except that

$$\mathrm{tr}(R_1R_2^{-1}R_3R_2) = e^{-2i\psi/3} + e^{i\psi/3}2\cos(\phi+\psi) = e^{-2i\psi/3} - e^{i\psi/3}2\cos(\pi-\phi-\psi).$$

(iv) The eigenvalues of A_1 and A_2 are $e^{2i\phi/3}$, $e^{-i\phi/3}$, $e^{-i\phi/3}$. The trace of A_1A_2 is $e^{-2i\phi/3} - e^{i\phi/3}2\cos(\psi)$. Hence its eigenvalues are $e^{-2i\phi/3}$, $-e^{i\psi+i\phi/3}$ and $-e^{-i\psi+i\phi/3}$. The result is again similar.
(v) Finally, the eigenvalues of A_1 are $e^{2i\phi/3}$, $e^{-i\phi/3}$, $e^{-i\phi/3}$ and those of $A_1A_3A_2 = (A_1J^{-1})^3 = (R_2^{-1}R_1^{-1})^3$ are $e^{2i\psi}$, $-e^{-i\psi}$, $-e^{-i\psi}$. The eigenvalues of $A_3A_2 = (A_1^{-1})(A_1A_2A_3)$ are $e^{-2i\phi/3}$, $-e^{i\psi+i\phi/3}$, $-e^{-i\psi+i\phi/3}$. The result follows from Proposition 4.4. □

Combining Proposition 5.3 with Proposition 4.3 we obtain

Corollary 5.4 *Suppose that $\psi = 2\pi/p$ and $\tau = -e^{-i\phi/3}$. If $\langle R_1, J\rangle$ is discrete then either*

(i) $\phi = 2\pi/q$, $\pi - \psi - \phi = \pi - 2\pi/p - 2\pi/q = 2\pi/r$ *or*

(ii) $\phi = 2\psi = 4\pi/q$ *and* $\pi - 3\psi = \pi - 6\pi/p = 2\pi/r$ *where p is odd.*

Suppose that p, q and r are integers (or posssibly ∞) in modulus at least 3 so that $1/p+1/q+1/r = 1/2$ then up to permutation p, q and r take one of the values in this table:

p	3	3	3	3	3	3	3	3	3	4	4	4	4	5	6
q	3	4	5	6	7	8	9	10	12	4	5	6	8	5	6
r	-6	-12	-30	∞	42	24	18	15	12	∞	20	12	8	10	6
$2p/(p-6)$	-2	-2	-2	-2	-2	-2	-2	-2	-2	-4	-4	-4	-4	-10	∞
$2q/(q-6)$	-2	-4	-10	∞	14	8	6	5	4	-4	-10	∞	8	-10	∞
$2r/(r-6)$	1	4/3	5/3	2	7/3	8/3	3	10/3	4	2	20/7	4	8	5	∞

The following theorem is a restatement of work of Mostow [13] and Sauter [19].

Theorem 5.5 *Suppose that $\psi = 2\pi/p$ and $\tau = -e^{-i\phi/3}$ where*

$$\max\Big\{-2\psi,\ \psi - \pi\Big\} < \phi < \min\Big\{\pi + \psi,\ 2\pi - 2\psi\Big\}$$

The group $\langle R_1, J\rangle$ corresponding to these parameters is discrete if and only if one of the following is true:

(i) $p = 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10,\ 12,\ 18$ *or* ∞ *and* $\phi = 2\pi/q$, $\pi - \psi - \phi = 2\pi/r$ *for some integers q, r (possibly infinity);*

(ii) $p = 5,\ 7$ *or* 9 *and* $\phi = 4\pi/p$;

(iii) $p = 5,\ 7$ *or* 9 *and* $\phi = \pi - 6\pi/p$;

(iv) $p = 15,\ 24,\ 42,\ -30$ *or* -12 *and* $\phi = 2\pi/3$;

(v) $p = 15,\ 24,\ 42,\ -30$ *or* -12 *and* $\phi = \pi/3 - 2\pi/p$.

Theorem 6.1 of Sauter [19] shows that the group from (v) with $\psi = 2\pi/p$ and $\phi = 2\pi/3$ for $p = 15,\ 24,\ 42,\ -30$ or -12 is isomorphic to the group from (i) with $\psi = 2\pi/3$ and $\phi = 2\pi/p$. This isomorphism is given by identifying A_1 with R_1 and J with J^{-1}. Theorem 6.2 of Sauter [19] shows that the group from (ii) with $\psi = 2\pi/p$, $\phi = 4\pi/p$ is isomorphic to the group from (i) with $\psi = 2\pi/p$ and $\phi = \pi$. This isomorphism is slightly more complicated and involves sending a square root of A_1 to R_1 and J to J^{-1}. The groups from (ii) and (iii) are isomorphic as are the groups from (iv) and (v) via the map that fixes R_1 but sends J to J^{-1}.

5.3 The Sauter curves

Proposition 5.6 *Suppose that $\psi = 2\pi/p$ and $\tau = e^{2i\phi/3} + e^{-i\phi/3}$. Then the following subgroups of $\langle R_1, J\rangle$ have a common fixed vector and on its orthogonal complement they act as (the holomorphic subgroup of) a triangle group as follows:*

(i) $\langle R_1, R_2\rangle$ *acts as a* $(\psi/2, \psi/2, \phi)$ *triangle group;*

(ii) $\langle R_1, R_2R_3\rangle$ *acts as a* $(\psi/2, \pi/2 - \phi/2 - \psi/2, \pi/2 - 3\phi/2)$ *triangle group.*

PROOF: When $\tau = e^{2i\phi/3}+e^{-i\phi/3}$ then $R_1R_2R_3 = (R_1J)^3$ has one eigenvalue $e^{2i\phi}$ and a repeated eigenvalue $-e^{-i\phi}$. Also, R_2R_3 has trace

$$\mathrm{tr}(R_2R_3) = e^{i\psi/3}\big(2-|\tau|^2\big) + e^{-2i\psi/3} = e^{i\psi/3}2\cos(\phi) + e^{-2i\psi/3}.$$

Hence R_2R_3 has eigenvalues $e^{-2i\psi/3}$, $-e^{i\psi/3+i\phi}$ and $e^{i\psi/3-i\phi}$. Then $\mathbf{n} = L_1 \cap L_{123}$ is a common eigenvector of R_1 and $R_1R_2R_3$, and hence of R_2R_3. The eigenvalues are $e^{-i\psi/3}$, $-e^{-i\phi}$ and $-e^{i\psi/3-i\phi}$ respectively. Thus on the orthogonal complement of $\mathbf{n}$ the group $\langle R_1,\, R_2R_3\rangle$ acts as a $(\psi/2, \pi/2-\phi/2-\psi/2, \pi/2-3\phi/2)$ triangle group.

□

We again use Proposition 4.3 to eliminate all but finitely many points.

Corollary 5.7 *Suppose that $\psi = 2\pi/p$ and $\tau = e^{2i\phi/3}+e^{-i\phi/3}$. If $\langle R_1, J\rangle$ is discrete then $\phi = 2\pi/q$ where either*

(i) $\pi-\psi-\phi = \pi-2\pi/p-2\pi/q = 2\pi/r$ *and* $\pi-3\phi = \pi-6\pi/q = 2\pi/s$ *or*

(ii) $p = 3$ *and* $\pi/3-\phi = \pi/3-2\pi/q = 2\pi/s$.

PROOF: Using $\psi = 2\pi/p$ in $\langle R_1, R_2\rangle$ and Proposition 4.3, we see that either $\phi = 2\pi/q$ or $4\pi/p$ and in the latter case p is odd.

Assume that $\phi = 2\pi/q$. Then $\langle R_1, R_2R_3\rangle$ acts as a $(\pi/p, \pi/2-\pi/p-\pi/q, \pi/2-3\pi/q)$ triangle group. From Proposition 4.3 we see that either $\pi/2-\pi/p-\pi/q$ or $\pi/2-3\pi/q$ has the form π/r. The result follows from the table above. Note that in (ii) we have $3\pi/s = \pi/2-3\pi/q = 3(\pi/2-\pi/3-\pi/q)$ and, when s is not divisible by 3 we are in case (iv) of Proposition 4.3.

If $\phi = 4\pi/p$ then $\langle R_1, R_2R_3\rangle$ acts as a $(\pi/p, \pi/2-3\pi/p, \pi/2-6\pi/p)$ triangle group. By inspection we see that the only possible values of p satisfying Proposition 4.3 are $p = 6$, 8, 9, 10, 12, 14, 18. Of these, only $p = 9$ is odd. We now eliminate this case.

Suppose that $\psi = 2\pi/9$ and $\phi = 4\pi/9$. That is:

$$R_1 = \begin{bmatrix} e^{4\pi i/27} & e^{2\pi i/27}2\cos(2\pi/9) & -2\cos(2\pi/9) \\ 0 & e^{-2\pi i/27} & 0 \\ 0 & 0 & e^{-2\pi i/27} \end{bmatrix}.$$

We calculate

$$(R_1R_2)^6 = \begin{bmatrix} e^{4\pi i/9}\big(2\cos(2\pi/9)-1\big) & -e^{16\pi i/27}\big(2\cos(4\pi/9)+1\big) & 0 \\ e^{8\pi i/27}\big(2\cos(4\pi/9)+1\big) & -e^{4\pi i/9}2\cos(2\pi/9) & 0 \\ 0 & 0 & e^{-8\pi i/9} \end{bmatrix}.$$

The eigenvalues of $(R_1R_2)^6$ are $e^{16\pi i/9}$, $e^{-8\pi i/9}$, $e^{-8\pi i/9}$. Thus $(R_1R_2)^6$ is a complex reflection with angle $8\pi/3$, that is $2\pi/3$. Also

$$\mathrm{tr}\Big((R_1R_2)^6J\Big) = -e^{16\pi i/27}\big(2\cos(4\pi/9)+1\big).$$

This trace does not appear on our list of possible values of τ. Therefore $\langle R_1, J\rangle$ is not discrete. Indeed, we may calculate that

$$\mathrm{tr}\Big((R_1R_2)^6(R_2R_3)^6\Big) = e^{8\pi i/9}\Big(2\cos(2\pi/9)-1-2\cos(4\pi/9)\Big) + e^{-16\pi i/9}.$$

Now $2\cos(2\pi/9) - 1 - 2\cos(4\pi/9)$ lies in $(-2, 2)$, but is not equal to twice the cosine of a rational multiple of π (for example this follows from the theorem of Conway and Jones). Hence $(R_1R_2)^6(R_2R_3)^6$ is elliptic of infinite order. □

We now consider the points we have not eliminated by Corollary 5.7. By inspection, we see that each of these groups is a subgroup of one of the discrete groups from Theorem 5.5. Therefore we have proven:

Theorem 5.8 *Suppose that $\psi = 2\pi/p$ and $\tau = e^{2i\phi/3} + e^{-i\phi/3}$ where $-\pi/p < \phi < \pi - \pi/p$. The group $\langle R_1, J\rangle$ is discrete if and only if $\phi = 2\pi/q$ where either*

(i) $q = 3$, 4, 5, 6, 7, 8, 9, 10, 12, 18 or ∞ and $\pi - 2\pi/p - 2\pi/q = 2\pi/r$ for some integer r;

(ii) $p = 3$ and $q = 15$, 24, 42, -30 or -12.

Note that (i) includes the case where $p = 2$ and $r = -q$ (where $q \geqslant 5$), which is Proposition 4.5 of [15], and the case where $p = 3$ and $q = 3$, 4, 5, 6, 7, 8, 9, 10, 12, 18 or ∞.

5.4 An example of a discrete group on the Sauter curve

In §5.4.1 of [16] Paupert considers the group with $p = 3$ and angle parameters $(5\pi/3, 4\pi/3)$. Writing R_1J as a unimodular matrix, these angle parameters translate to eigenvalues 1, $e^{i\pi/3}$ and $e^{-i\pi/3}$. (The fixed point of R_1J corresponds to the $e^{i\pi/3}$-eigenspace.) Hence the trace of R_1J is $\tau = 1 + e^{i\pi/3} + e^{-i\pi/3} = 2$. This appears on the list of discrete groups. In this case:

$$R_1 = \begin{bmatrix} e^{4\pi i/9} & 2 & -2e^{2\pi i/9} \\ 0 & e^{-2\pi i/9} & 0 \\ 0 & 0 & e^{-2\pi i/9} \end{bmatrix},$$

$$R_2 = \begin{bmatrix} e^{-2\pi i/9} & 0 & 0 \\ -2e^{2\pi i/9} & e^{4\pi i/9} & 2 \\ 0 & 0 & e^{-2\pi i/9} \end{bmatrix},$$

$$R_3 = \begin{bmatrix} e^{-2\pi i/9} & 0 & 0 \\ 0 & e^{-2\pi i/9} & 0 \\ 2 & -2e^{2\pi i/9} & e^{4\pi i/9} \end{bmatrix}.$$

We can easily check that an $e^{i\pi/3}$-eigenvector of R_1J is

$$\begin{bmatrix} e^{5\pi i/9} \\ 1 \\ e^{-5\pi i/9} \end{bmatrix}$$

which is negative.

We now give another, more direct, way to see that this group is discrete. Using Proposition 2.8 we see that $\langle R_1, R_2, R_3\rangle$ may be conjugated in $\mathrm{PU}(2,1)$ so that their matrix entries lie in $\mathbb{Z}\big[e^{2\pi i/3}\big]$, the Eisenstein integers. Since this is a discrete subring of $\mathbb{C}$ the group is automatically discrete.

5.5 The two curves are related for $p = 3$

In section 5.1 we have seen that each group on the Sauter curve is a subgroup of a group on the Mostow curve, but for a different value of ψ (the angle of rotation of the generators). In the special case where $\psi = 2\pi/3$, we can apply this twice to see that each group on the Sauter curve is a

subgroup of a group on the Mostow curve, for the same value of ψ. Moreover, Theorem 5.8 tells us that the Sauter subgroup in question is discrete only if the larger Mostow group is discrete. In other words, in terms of discrete groups, the two curves are the same; this is visible on Figure 4. The precise statement is the following:

Proposition 5.9 *$\Gamma(2\pi/3, e^{2i\phi/3}+e^{-i\phi/3})$ is isomorphic to a subgroup of $\Gamma(2\pi/3, -e^{-i\phi/3})$.*

PROOF: We have seen in section 5.1 that $\Gamma(\phi, e^{2i\psi/3}+e^{-i\psi/3})$ is isomorphic to a subgroup of $\Gamma(\psi, -e^{-i\phi/3})$. Applying this with $\psi = 2\pi/3$, we obtain that $\Gamma(\phi, e^{4i\pi/9}+e^{-2i\pi/9})$ is isomorphic to a subgroup of $\Gamma(2\pi/3, -e^{-i\phi/3})$. Now the point is that $e^{4i\pi/9}+e^{-2i\pi/9} = e^{i\pi/9} = -e^{-8i\pi/9}$, so that the former group (of Sauter type) is also of Mostow type, and we can therefore repeat the process. This tells us that $\Gamma(8\pi/3, e^{2i\phi/3}+e^{-i\phi/3})$ is isomorphic to a subgroup of $\Gamma(2\pi/3, -e^{-i\phi/3})$. Note that $8\pi/3 = 2\pi/3$ mod. 2π. □

In fact we can say more and identify the corresponding subgroups in terms of generators: $\langle (R_1R_2)^2, J\rangle$ is a subgroup of $\langle A_1, J\rangle$, which is in turn a subgroup of $\langle R_1, J\rangle$ (see section 5.1), and these various subgroups correspond to the values of the parameters described above.

Indeed, when $p = 3$ and $|\tau| = 1$, we have $\mathrm{tr}((R_1R_2)^2) = e^{16\pi i/9} + 2e^{-8\pi i/9}$. Therefore $(R_1R_2)^2$ is a complex reflection with angle $e^{24\pi i/9} = e^{2\pi i/3}$. We also have $\mathrm{tr}((R_1R_2)^2J) = \overline{\tau}^2 + 2\cos(2\pi/3)\tau$. When $\tau = -e^{-i\phi/3}$ then this trace is $e^{2i\phi/3}+e^{-i\phi/3}$ as required.

6 Sporadic groups generated by complex reflections of order 3

6.1 Which sporadic points are in the parameter space?

This was determined earlier for all values of p (see Table 3.3). For $p = 3$, the situation is the following:

- H_τ has signature $(2,1)$ for the following sporadic values of τ:

$$\begin{array}{lll} e^{i\pi/3}+e^{-i\pi/6}2\cos(\pi/4), & e^{i\pi/3}+e^{-i\pi/6}2\cos(\pi/5), & e^{i\pi/3}+e^{-i\pi/6}2\cos(2\pi/5), \\ e^{-i\pi/3}+e^{i\pi/6}2\cos(\pi/4), & e^{-i\pi/3}+e^{i\pi/6}2\cos(\pi/5), & e^{-i\pi/3}+e^{i\pi/6}2\cos(2\pi/5), \\ e^{2\pi i/9}+e^{-\pi i/9}2\cos(2\pi/5), & e^{2\pi i/9}+e^{-\pi i/9}2\cos(4\pi/5), & e^{2\pi i/9}+e^{-\pi i/9}2\cos(2\pi/7), \\ e^{2\pi i/9}+e^{-\pi i/9}2\cos(6\pi/7), & e^{-2\pi i/7}+e^{-4\pi i/7}+e^{-8\pi i/7}. & \end{array}$$

- H_τ is degenerate (signature (2,0) or (1,1), see below) for $\tau = e^{-2\pi i/9}+e^{\pi i/9}2\cos(\phi)$ where $\phi = 2\pi/5, 4\pi/5, 2\pi/7, 4\pi/7$ or $6\pi/7$. The corresponding five points are on the boundary of our parameter space.

- H_τ is positive definite when $\tau = e^{2\pi i/9}+e^{-\pi i/9}2\cos(4\pi/7)$ and $\tau = e^{2\pi i/7}+e^{4\pi i/7}+e^{8\pi i/7}$. The corresponding two points are outside of our parameter space.

6.2 Reducible sporadic groups

These correspond to $\tau = e^{-2\pi i/9}+e^{\pi i/9}2\cos(\phi)$ where $\phi = 2\pi/5, 4\pi/5, 2\pi/7, 4\pi/7$ or $6\pi/7$. These groups have a common eigenvector $(1,1,1)^T$; as in Proposition 4.4 we analyse their action on the orthogonal complement of this vector. In this case, normalising R_1 for convenience as $-e^{-i\pi/9}R_1$, the eigenvalues on this complement are $-e^{i\pi/3}$ and $-e^{-i\pi/3}$ for R_1, $-e^{i\pi/3}$ and $-e^{-i\pi/3}$ for J, and $-e^{i\phi}$ and $-e^{-i\phi}$ for R_1J. Therefore the group $\langle R_1, J\rangle$ acts as (the holomorphic subgroup of) a $(\pi/3, \pi/3, \phi)$ triangle group. For $\phi = 2\pi/5$ or $4\pi/5$, this is in fact a (finite) $(2,3,5)$ triangle group;

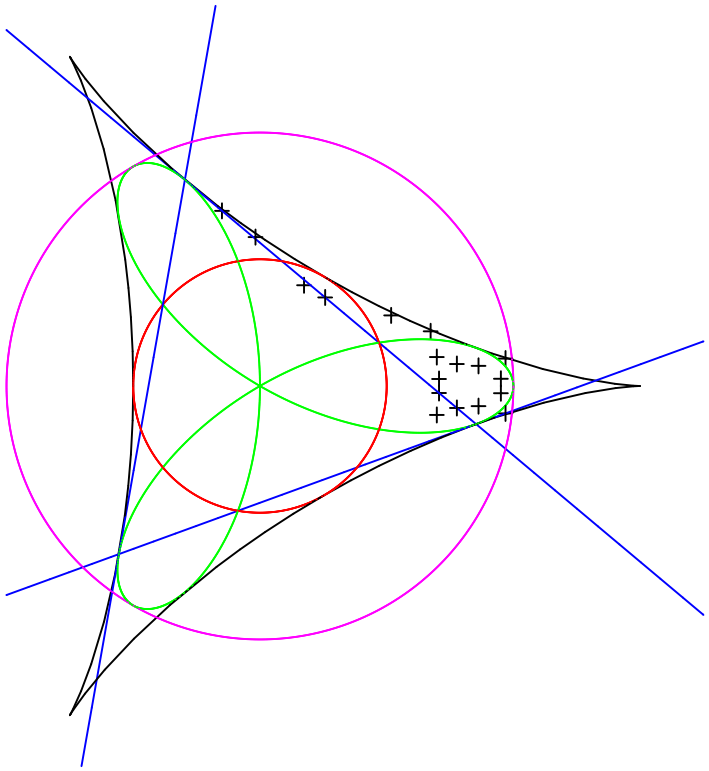

Figure 3: *The sporadic points together with the trace parameter space for $p = 3$.*

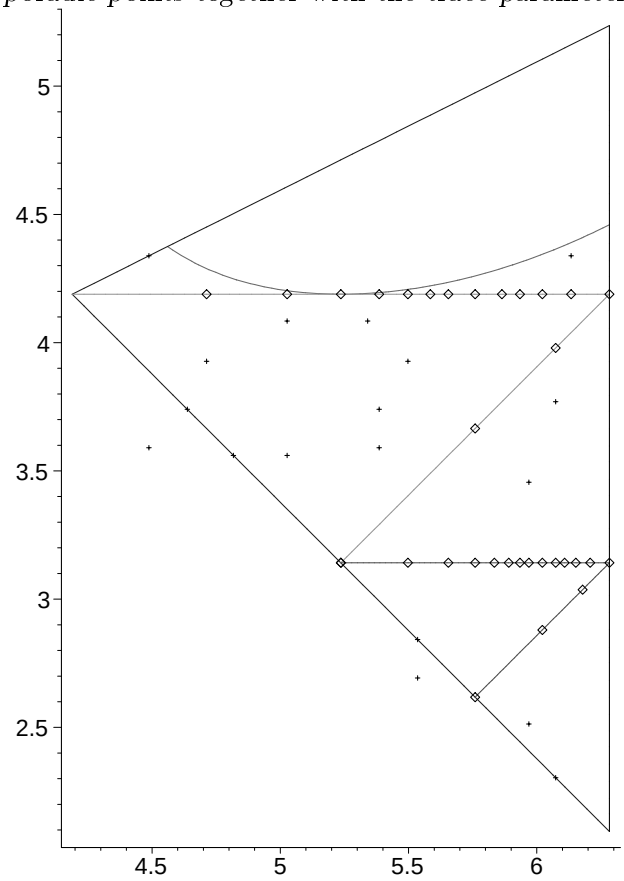

Figure 4: *The sporadic points (crosses) together with the angle parameter space for $p = 3$. We have also included all discrete points (diamonds) on the two curves.*

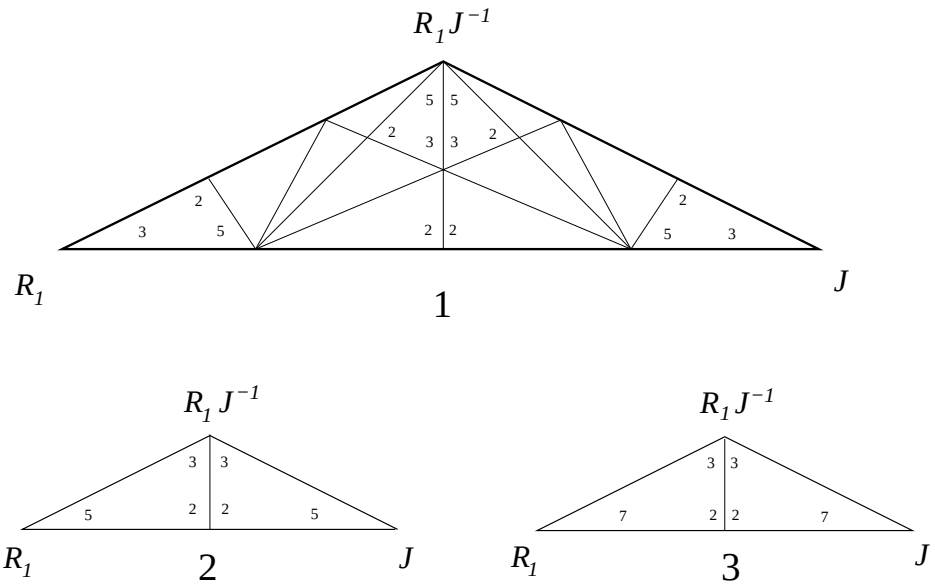

Figure 5: *Schematic picture of triangle group action when* $\tau = e^{-2\pi i/9} + e^{\pi i/9} 2\cos(\phi)$ *for* $\phi = 2\pi/5$ (2), $4\pi/5$ (1), $2\pi/7$ (3).

for $\phi = 2\pi/7$, it is a discrete hyperbolic $(2,3,7)$ triangle group. For $\phi = 4\pi/7$ or $6\pi/7$, it is a non-discrete spherical triangle group; in fact we already know that these two groups are non-discrete from Corollary 4.2, as $R_1R_2R_3R_2^{-1}$ is elliptic of infinite order.

In Figure 5 we draw a schematic picture of the triangle group action induced by the generators in the three discrete cases.

6.3 Non-discrete irreducible sporadic groups

The trace of $R_1R_2^{-1}$ is

$$\mathrm{tr}(R_1R_2^{-1}) = 2\cos(\psi) + |\tau|^2 + 1.$$

When $\psi = 2\pi/3$ we have

$$\mathrm{tr}(R_1R_2^{-1}) = |\tau|^2 = 2 + 2\cos(2\theta)$$

where 2θ is given in the table just before Proposition 3.1. Evaluating in each case we can find whether $R_1R_2^{-1}$ is loxodromic, parabolic or elliptic and, in the latter case, whether or not it has finite order. This enables us to eliminate one pair of complex conjugate τ (see also Proposition 4.5):

Lemma 6.1 *Suppose that* $\psi = 2\pi/3$ *and* $\tau = e^{\pi i/3} + e^{-\pi i/6} 2\cos(2\pi/5)$ *or its complex conjugate. The* $R_1R_2^{-1}$ *is elliptic of infinite order.*

PROOF: We know that

$$\mathrm{tr}(R_1R_2^{-1}) = |\tau|^2 = 1 + 4\cos^2(2\pi/5) = 3 + 2\cos(4\pi/5).$$

Since $R_1R_2^{-1}$ has real trace it must have eigenvalues $e^{i\Theta}$, $e^{-i\Theta}$ and 1. Hence its trace is $1+2\cos(\Theta)$. In other words,

$$2\cos(\Theta) = 2 - 2\cos(\pi/5).$$

Using the theorem of Conway and Jones, Theorem 7 of [1], we see that Θ is not a rational multiple of π. This proves the result. □

The proof of the following lemma is similar to the proof of Lemma 4.1, knowing that:

$$\mathrm{tr}(R_1R_2^{-1}R_3R_2) = e^{i\psi/3}\Big(2 - \big|\tau^2 - e^{i\psi}\overline{\tau}\big|^2\Big) + e^{-2i\psi/3}.$$

Lemma 6.2 *Let R_1, R_2 and R_3 be given by (7), (8) and (9). If $|\tau^2 + \overline{\tau}e^{i\psi}| > 2$ then $R_1R_2^{-1}R_3R_2$ is loxodromic. If $|\tau^2 - \overline{\tau}| \leq 2$ then $R_1R_2^{-1}R_3R_2$ is elliptic with eigenvalues $e^{-2i\psi/3}$, $-e^{i\psi/3+2i\Theta}$ and $-e^{i\psi/3-2i\Theta}$ where $|\tau^2 + \overline{\tau}e^{i\psi}| = 2\cos\Theta$.*

Corollary 6.3 *Suppose that $\psi = 2\pi/3$ and $\tau = e^{2\pi i/9} + e^{-\pi i/9}2\cos(4\pi/5)$. Then $R_1R_2^{-1}R_3R_2$ is elliptic of infinite order.*

Proof: When $\psi = 2\pi/3$ and $\tau = e^{2\pi i/9} + e^{-i\pi/9}2\cos(4\pi/5)$ we see that

$$|\tau^2 + \overline{\tau}e^{i\psi}|^2 = 6 - \sqrt{5}.$$

Hence $|\tau^2 + \overline{\tau}e^{i\psi}| = 2\cos\Theta$ where Θ is not a rational multiple of π. □

6.4 Arithmeticity of sporadic groups

We begin by applying Proposition 2.8 to show that when $\tau = e^{-2\pi i/7} + e^{-4\pi i/7} + e^{-8\pi i/7}$ the group $\langle R_1, R_2, R_3\rangle$ is contained in an arithmetic lattice. In particular, putting $\psi = 2\pi/3$ into the expression in equation (11) for $2\sin(\pi/3)CH_\tau C^{-1}$ where $C = \mathrm{diag}(e^{-2\pi i/9}, 1, e^{2\pi i/9})$ we obtain:

$$\sqrt{3}CH_\tau C^{-1} = \begin{bmatrix} 3 & -(3+i\sqrt{3})\tau/2 & (3+i\sqrt{3})\overline{\tau}/2 \\ -(3-i\sqrt{3})\overline{\tau}/2 & 3 & -(3+i\sqrt{3})\tau/2 \\ (3-i\sqrt{3})\tau/2 & -(3-i\sqrt{3})\overline{\tau}/2 & 3 \end{bmatrix}.$$

Proposition 6.4 *Let R_1, R_2 and R_3 be complex reflections with angle $\psi = 2\pi/3$ so that the group $\langle R_1, R_2, R_3\rangle$ has parameter $\tau = e^{-2\pi i/7} + e^{-4\pi i/7} + e^{-8\pi i/7} = (-1 - i\sqrt{7})/2$. Then $\langle R_1, R_2, R_3\rangle$ is contained in an arithmetic lattice and hence is discrete.*

Proof: We have $e^{i\psi} = (-1 + i\sqrt{3})/2$ and $\tau = (-1 - i\sqrt{7})/2$. Both of these are algebraic integers in the field $\mathbb{Q}\big(\sqrt{21}, i\sqrt{3}\big)$. In particular, using Proposition 2.8, we can ensure that the matrix entries of all elements of $\langle R_1, R_2, R_3\rangle$ are algebraic integers in $\mathbb{Q}\big(\sqrt{21}, i\sqrt{3}\big)$. This field is a totally imaginary quadratic extension of the totally real number field $\mathbb{Q}\big(\sqrt{21}\big)$. The only non-trivial Galois conjugation in $\mathbb{Q}\big(\sqrt{21}\big)$ sends $\sqrt{21}$ to $-\sqrt{21}$. This is compatible with g, the Galois conjugation in $\mathbb{Q}\big(\sqrt{21}, i\sqrt{3}\big)$ fixing $(-1+i\sqrt{3})/2$ and sending $\tau = (-1-i\sqrt{7})/2$ to $\overline{\tau} = (-1+i\sqrt{7})/2$. The matrix $\sqrt{3}CH_\tau C^{-1}$ has entries in the ring of integers of $\mathbb{Q}\big(\sqrt{21}, i\sqrt{3}\big)$ and signature $(2,1)$. The Galois conjugation g sends $\sqrt{3}H_\tau$ to $\sqrt{3}CH_{\overline{\tau}}C^{-1}$. But we know that the point $\overline{\tau}$ lies outside our parameter space and so corresponds to a group which has signature $(3,0)$. Therefore using standard arguments (for example [10] or Proposition 4.3 of [15]) we see that this group is arithmetic and hence discrete. □

There are ten more sporadic groups with signature $(2,1)$. The goal of the rest of this section is to show that none of them are contained in an arithmetic lattice.

Proposition 6.5 *Let R_1, R_2 and R_3 be complex reflections with angle $\psi = 2\pi/3$ so that the group $\langle R_1, R_2, R_3\rangle$ has parameter $\tau \neq e^{\pm 2\pi i/7} + e^{\pm 4\pi i/7} + e^{\pm 8\pi i/7} = (-1 \pm i\sqrt{7})/2$. Then $\langle R_1, R_2, R_3\rangle$ is not contained in an arithmetic lattice.*

The method of proof will be very similar to the proof of Proposition 6.5 and we give an outline. In each case we conjugate (multiples of) H_τ and R_j so that their matrix entries are algebraic integers in a number field E. In each case the number field E will be a purely imaginary quadratic extension of a totally real field F. The following lemma lists the values of τ and the number fields. It is easy to verify and we leave the details to the reader. Note that in parts (i) and (ii) we conjugate by C and in parts (iii) and (iv) we conjugate by C^{-1}.

Lemma 6.6 *Suppose that R_1, R_2 and R_3 are given by (7), (8) and (9) with $\psi = 2\pi/3$ and that $C = \mathrm{diag}(e^{-2\pi i/9}, 1, e^{2\pi i/9})$.*

(i) If $\tau = \sigma(\pi/6, \pi/4)$ or $\sigma(-\pi/6, \pi/4)$ then the entries of $\sqrt{3}CH_\tau C^{-1}$ and $e^{2\pi i/9}CR_jC^{-1}$, for $j = 1, 2, 3$, are algebraic integers in the number field $\mathbb{Q}(\sqrt{6}, i\sqrt{3})$.

(ii) If $\tau = \sigma(\pi/6, \pi/5)$, $\sigma(\pi/6, 2\pi/5)$, $\sigma(-\pi/6, \pi/5)$ or $\sigma(-\pi/6, 2\pi/5)$ then the entries of $\sqrt{3}CH_\tau C^{-1}$ and $e^{2\pi i/9}CR_jC^{-1}$, for $j = 1, 2, 3$, are algebraic integers in the number field $\mathbb{Q}\big(\sqrt{3}, \sqrt{5}, i\big)$.

(iii) If $\tau = \sigma(\pi/9, 2\pi/5)$ or $\sigma(\pi/9, 4\pi/5)$ then the entries of $\sqrt{3}C^{-1}H_\tau C$ and $e^{2\pi i/9}C^{-1}R_jC$, for $j = 1, 2, 3$, are algebraic integers in the number field $\mathbb{Q}\big(2\cos(2\pi/5), e^{2\pi i/3}\big)$.

(iv) If $\tau = \sigma(\pi/9, 2\pi/7)$, $\sigma(\pi/9, 4\pi/7)$ or $\sigma(\pi/9, 6\pi/7)$ then the entries of $\sqrt{3}C^{-1}H_\tau C$ and $e^{2\pi i/9}C^{-1}R_jC$, for $j = 1, 2, 3$, are algebraic integers in $\mathbb{Q}\big(2\cos(2\pi/7), e^{2\pi i/3}\big)$.

We then examine all Galois conjugations in E that are compatible with non-trivial Galois conjugations in F. For the number fields in Lemma 6.6 the Galois conjugations permute the values of τ given in each part.

For example, in part (ii) of Lemma 6.6 E is $\mathbb{Q}\big(\sqrt{3}, \sqrt{5}, i\big)$ and F is $\mathbb{Q}\big(\sqrt{3}, \sqrt{5}\big)$. The Galois conjugation in F fixing $\sqrt{3}$ and changing the sign of $\sqrt{5}$ extends to a Galois conjugation g_1 in E that fixes $e^{2\pi i/3}$ and swaps $\sigma(\pi/6, \pi/5)$ and $\sigma(\pi/6, 2\pi/5)$ and swaps $\sigma(-\pi/6, \pi/5)$ and $\sigma(-\pi/6, 2\pi/5)$. Likewise, the Galois conjugation in F fixing $\sqrt{5}$ and changing the sign of $\sqrt{3}$ extends to a Galois conjugation g_2 in E that fixes $e^{2\pi i/3}$ and swaps $\sigma(\pi/6, \pi/5)$ and $\sigma(-\pi/6 + 2\pi/3, \pi/5)$ and swaps $\sigma(\pi/6, 2\pi/5)$ and $\sigma(-\pi/6 + 2\pi/3, 2\pi/5)$. Note that $\sigma(\phi + 2\pi/3, \eta)$ is equivalent to $\sigma(\phi, \eta)$. The remaining Galois conjugation in E is the product of g_1 and g_2.

In each case all non-trivial Galois conjugations will send our value of τ to one of the other sporadic values of τ. Since we have analysed which of the points τ lie in our parameter space we know the signature of the corresponding Hermitian form H_τ. In each case there will be at least one non-trivial Galois conjugation that sends H_τ to another form of signature $(2, 1)$. This is sufficient to ensure that $\langle R_1, R_2, R_3 \rangle$ is non-arithmetic; compare Proposition 12.6.1 of [4]. This completes our sketch proof of Proposition 6.5.

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