Uniform Variate Generation

Refs: Chapter 7 in Law,
Pierre Lecuyer Tutorial, Winter Simulation Conference 2015

Peter J. Haas

CS 590M: Simulation
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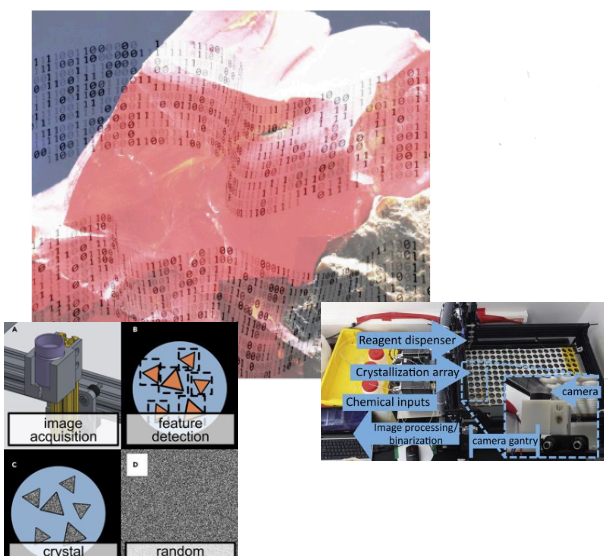
Mad Scientists Are Using Crystals to Generate Random Numbers

It's all to make your computer safer. But it sounds fun, too.

By Courtney Linder Feb 19, 2020 Popular Mechanics



Pseudo-Random Numbers

A deterministic sequence that "looks random"

- Deterministic recurrence relation: sequence of integer seeds
- Each seed converted into a uniform "random number"
- A good generator has desirable theoretical properties, passes statistical tests

Repeatability is a good thing

- Facilitates debugging and verification
- Allows "common random numbers" (efficiency improvement)

Versus "natural" sources of randomness

- Ex: Silicon graphics / Cloudflare "lava lamp" generator
- Ex: HotBits site uses radioactive decay
- Ex: random.org uses atmospheric noise (radio static)
- Non-reproducible and slow
  (OK for lotteries, games, generating encryption keys)

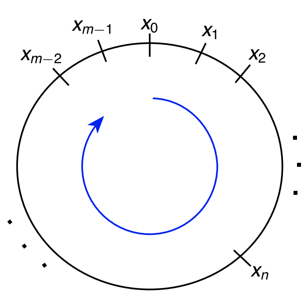
Linear Congruential Generators (LCGs)

Fundamental recurrence:

$x_{n+1} = (ax_n + c) \bmod m$

- a = multiplier, c = increment, m = modulus
- $k \bmod m$ is remainder after dividing k by m (e.g., $14 \bmod 5 = 4$ and $2 \bmod 10 = 2$)
- x_n's take values in $\{0, 1, 2, \ldots, m-1\}$
- Return $U_n = x_n/m$
- In C: `rand()` returns seed between 1 and `RAND_MAX`
- Historically the earliest (effective) rng

Period of an LCG



An LCG is periodic

- Period $\leq m$
- Want full period (every number in $[0..m-1]$ appears once)
  - Maximizes number of available random variates in a simulation
  - Otherwise, gaps may cause statistical anomalies

Multiplicative Congruential Generators (MCGs)

Fundamental recurrence:

$x_{n+1} = ax_n \bmod m$

- Special case of LCG with increment c equal to 0
- Full period: all values in $\{1, 2, \ldots, m-1\}$ visited in cycle

Theorem:

An MCG has full period if m is prime and a is a primitive element modulo m

- I.e., $r(a) \triangleq \min\{k > 0 : a^k \bmod m = 1\} = m - 1$
- Ex: $x_{n+1} = 3x_n \bmod 4$
- Ex: $x_{n+1} = 3x_n \bmod 5$

k	1	2	3	4
$3^k \bmod 5$	3	4	2	1

Classical MCGs

Modulus choices

- $m = 2^b$ for convenience on binary computer
  - mod 2^b is simple: retain b lowest-order bits
  - Ex: IBM RANDU generator with $m = 2^{31}$ and $a = 2^{16} + 3$
  - For $b > 3$, period is at most $m/4$
- $m = 2^{31} - 1$ is, fortuitously, a (Mersenne) prime number
  - Because $2^{31} - 1$ is "almost" 2^{31}, can compute mod quickly [Bratley et al., pp. 212–213]

Lewis-Learmonth Generator ("Minimal Standard Generator")

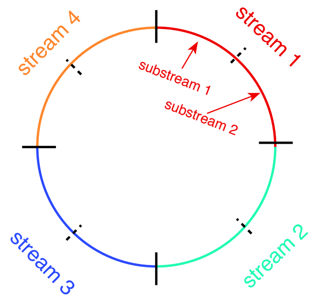
$x_{n+1} = 7^5 x_n \bmod (2^{31} - 1) = 6807 x_n \bmod 2{,}147{,}483{,}647$

Used for many years, but fails modern statistical tests, cycle is too short

Streams and Substreams in an MCG

Jumping ahead

- Goal: quickly compute x_k for k large
- $x_k = (a^k x_0) \bmod m = \big((a^k \bmod m)\, x_0\big) \bmod m$
- Precompute numbers $\alpha_k = a^k \bmod m$ for multiple values of k
- Allows partitioning of cycle into streams and substreams
  - Better than, e.g., setting $y_n = x_{2n}$ and $z_n = x_{2n+1}$
  - Caution: For an MCG, non-overlap is not sufficient (see demo)

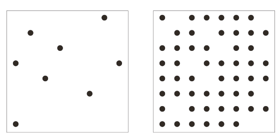



```
unigen.ResetNextSubstream()
unigen.ResetStartStream()
unigen.ResetStartSubstream()
```


Pitfalls of MCGs (and Other Generators)

Short cycles

- MCG numbers fall on a lattice
- Only want to use $O(\sqrt{\text{period}})$ numbers



Early in cycle   Late in cycle

Low-order bits

Claim:

If $x_{n+1} = a x_n \bmod 2^b$ and $r_{n+1} = x_{n+1} \bmod 2^k$, where $3 < k < b$, then $(r_n : n \geq 1)$ has period at most 2^{k-2}

- r_n's are k low-order bits
- Ex: $x_{n+1} = 13 x_n \bmod 2^{31}$ with $k = 4$ [period $= (2^{31}/4) - 1 \approx 537 \times 10^6$]
- So avoid algorithm that sets $X = \lfloor nU \rfloor$ and $V = nU - X$ where $U \overset{D}{\sim} \text{Uniform}(0,1)$

n	x_n	r_n
1	16049371	11
2	208641823	15
3	564860051	3
4	900729719	7
5	972068107	11
6	1899467151	15
7	1070752835	3
8	1034884967	7

Other Pitfalls (Demo)

Starting seeds for Lewis-Learmonth generator

- X: use starting seed $s = 1$
- Y: use starting seed $s' = 2$
- s and s' are over 1.3 billion steps apart in cycle
- Plot (X, Y) pairs
- Plot histogram of $X + Y$

Box-Muller and MCG

1. Generate U, V i.i.d. $U[01,]$
2. Set $X = \sqrt{-2 \log u} \cos(2\pi V)$
3. Set $Y = \sqrt{-2 \log u} \sin(2\pi V)$
4. Return X and Y independent $N(0,1)$

Combined Generators

Example: rngStream (used in Arena and other packages)

$$x_n = (1403580x_{n-2} - 810728x_{n-3}) \bmod 4294967087$$
$$y_n = (527612y_{n-1} - 1370589y_{n-3}) \bmod 4294944443$$
$$z_n = (x_n - y_n) \bmod 4294967087$$
$$u_n = z_n/4294967087$$

- Seed = vector of six 32-bit integers
- Cycle length $\approx 2^{191} \approx 10^{57}$ (1 octodecillion)
- # streams $= 2^{64} \approx 10^{19}$ (10 quintillion)
- Stream length $= 2^{127} \approx 10^{38}$ (100 undecillion)
- # substreams $= 2^{51} = 10^{15}$ (1 trillion)
- Substream length $= 2^{76} \approx 10^{22}$ (10 sextillion)
- Well-behaved up to at least 45 dimensions

Fun Fact
Time to mostly use up a generator with period of 2^{191} with 1 trillion computers generating one seed per nanosecond: $> 10^{38}$ years

Mersenne Twister

General Form (Bit-wise Generator):

$$X_n = (A_1X_{n-1} + \cdots + A_kX_{n-k}) \bmod 2$$
$$Y_n = BX_n \bmod 2 = (y_{n,1}, \ldots, y_{n,W}) \text{ where } W = 32 \text{ or } 64$$
$$u_n = \sum_{j=1}^{W} y_{n,j}2^{-j} \text{ or } u_n = 0.y_{n,1}y_{n,2}\cdots y_{n,W}$$

- $X_n = (x_{n,1}, \ldots, x_{n,L})^\top$ with each $x_{n,i} = 0$ or 1
- Binary matrices $A_1, \ldots, A_k$ ($L \times L$) and B ($W \times L$)
- Fast: XOR operations and bit shifting

Mersenne Twister

- Default generator in Python and many other systems
- Seed = vector of 623 integers (32 bit)
- Period $= 2^{19937} - 1 \approx 10^{6002}$ and well-behaved up to $d = 623$
- Drawbacks: large, slow initialization (demo), some stat. issues
- WELL generator (Lecuyer et al.) scambles bits better

More Generators

Cryptographic generators

- "Impossible" to guess the next number in the sequence
- Ex: RC4 (open source: Arc4Random), Threefish, ChaCha20
- Good for security, slow and statistically shaky for simulation

Counter-based generators

- Trivial seeds: $s_n = n$ for $n \geq 1$ (great for substreams!)
- $u_n = f(n)$ where f is a "weak" but fast encryption function
- Perform well, equidistribution properties not well understood

Permuted congruential generator (PCG)

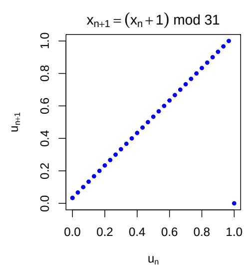
- Use "improving" transformation of fast but shaky LCG
- Under evaluation

Testing Uniform Random Number Generators

A simple generator:
$x_{n+1} = (x_n + 1) \bmod m$

Properties

- Full period
- Values uniformly spread out over $[0, 1]$
- Yet: this is a *terrible* generator



Two ways of showing poor quality

- Compare to expected statistical behavior of uniform sequence $U_1 = \frac{1}{m-1}, U_2 = \frac{2}{m-1}, U_3 = \frac{3}{m-1}, \ldots$ (not very random)
- Look at possible values in higher dimensions (see plot)

Structural (Theoretical) Tests

Possible values in d dimensions

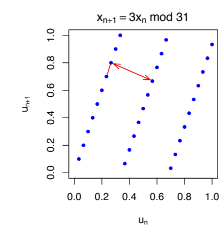
- Group the cycle into d-vectors: $V_i = (U_i, \ldots, U_{i+d-1})$
- Want equidistribution over $[0, 1]^d$

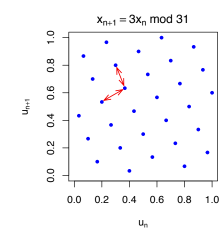
Structural tests for MCGs

- Points of MCG lie on lattice: how even? (spectral test)
- For modulus m, points lie on at most $(d!m)^{1/d}$ hyperplanes
- RANDU (demo)

For other generators:

- Not a lattice; 32/64-bit U values can appear multiple times in cycle
- Want same # of points in each "grid cell"





d	Upper bound
1	2^{31}
2	2^{16}
3	2344
4	476
5	192
6	108

Statistical (Empirical) Tests

How does generator jump from point to point?

- Do U_i numbers look i.i.d. uniform to a statistician?

Many kinds of statistical tests

- General tests for goodness of fit (e.g., χ^2 test)
  1. Divide $[0, 1]$ into k (> 100) equal intervals
  2. Generate $U_1, \ldots, U_n$ (where $n \approx 10k$)
  3. Count number $f_1, \ldots, f_k$ that fall into each interval
  4. Compute likelihood under i.i.d. uniform hypothesis

$$\chi^2 = \sum_{j=1}^{k} \frac{(f_j - \frac{n}{k})^2}{n/k}$$

- Serial test: essentially d-dimensional version of χ^2 test
- Runs-up test (see homework)

Test suites (the PRNG arms race)

- Gold standard: TestU01 suite (incl. "SmallCrush", "Crush") [Lecuyer et al., simul.iro.umontreal.ca/testu01/tu01.html]