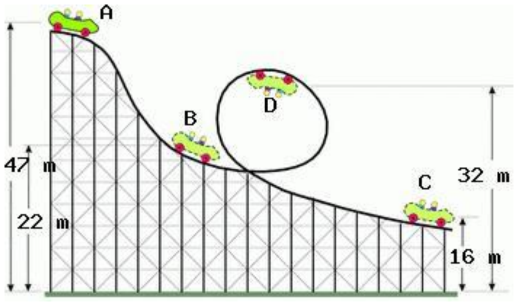
Chapter 6: Polynomials

Objectives

- To add, subtract and multiply polynomials.
- To **divide polynomials**.
- To use the **remainder theorem**, **factor theorem** and **rational-root theorem** to identify the linear factors of cubic and quartic polynomials.
- To solve **equations** and **inequalities** involving cubic and quartic polynomials.
- To recognise and sketch the graphs of **cubic and quartic functions**.
- To find the rules for given cubic graphs.
- To apply cubic functions to solving problems.
- To use the **bisection method** to solve polynomial equations numerically.



6A – The language of polynomials

A **Polynomial function** follows the rule

$$y = a_n x^n + a_{n-1}x^{n-1} + \dots\dots\dots\dots a_1 x + a_0 \qquad n \in N$$

Where $a_0,\ a_1, \dots\dots\dots\dots\ a_n$ are coefficients.

Degree of a polynomial is the highest power of x with a non-zero coefficient.

In summary:

- A **polynomial function** is a function that can be written in the form

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where $n \in \mathbb{N} \cup \{0\}$ and the coefficients $a_0, \dots, a_n$ are real numbers with $a_n \neq 0$.
- The number 0 is called the **zero polynomial**.
- The **leading term**, $a_n x^n$, of a polynomial is the term of highest index among those terms with a non-zero coefficient.
- The **degree of a polynomial** is the index n of the leading term.
- A **monic polynomial** is a polynomial whose leading term has coefficient 1.
- The **constant term** is the term of index 0. (This is the term not involving x.)

Example 1

Let $Q(x) = 2x^6 - x^3 + ax^2 + bx + 20$. If $Q(-1) = 2Q(2) = 0$, find the values of a and b.

The arithmetic of polynomials

The operations of addition, subtraction and multiplication for polynomials are naturally defined. The sum, difference and product of two polynomials is a polynomial.

Example 2

Let $f(x) = x^3 - 2x^2 + x$, $g(x) = 2 - 3x$ and $h(x) = x^2 + x$, simplify the following:

a) $f(x) + h(x)$

b) $g(x)h(x)$

Equating coefficients

Two polynomials P and Q are equal only if their corresponding coefficients are equal.

Example 3

The polynomial $P(x) = x^3 + 3x^2 + 2x + 1$ can be written in the form $(x-2)(x^2 + bx + c) + r$ where $b,\ c$ and r are real numbers. Find the values of $b,\ c$ and r.

The expansion of $(a+b)^n$

We know that $(a+b)^2 = a^2 + 2ab + b^2$

This is called an identity.

If we multiply both sides by $(a+b)$, we obtain

$$(a+b)^3 = (a+b)(a^2 + 2ab + b^2)$$
$$= a^3 + 3a^2b + 3ab^2 + b^3$$

And so on …….

Quadratic function	$(a+b)^2 = ax^2 + bx + b^2,$	$a \neq 0$
Cubic function	$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3,$	$a \neq 0$
Quartic functions	$(a+b)^4 = ax^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4,$	$a \neq 0$

Complete Exercise 6A Questions page 214

6B – Division of polynomials

In order to sketch the graphs of many cubic and quartic functions (as well as higher degree polynomials) it is often necessary to find the x-axis intercepts. As with quadratics, finding x-axis intercepts can be done by factorising and then solving the resulting equation using the null factor theorem.
All cubic functions will have at least one x-axis intercept. Some will have two and others three.

When we divide the polynomial $P(x)$ by the polynomial $D(x)$ we obtain two polynomials, $Q(x)$ the **quotient** and $R(x)$ the **remainder**, such that

$$P(x) = D(x)Q(x) + R(x)$$

and either $R(x) = 0$ or $R(x)$ has degree less than $D(x)$.
Here $P(x)$ is the **dividend** and $D(x)$ is the **divisor**.

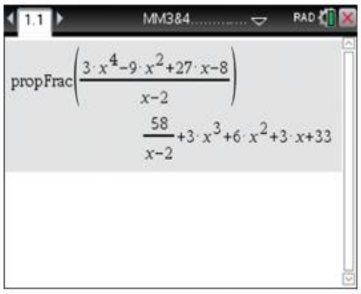
Examples of long division:

1. Divide $x^3 - 4x^2 - 11x + 30$ by $x - 2$.

2. Divide $2x^3 - 6x^2 - 10x + 25$ by $x^2 - 2$

– Using the TI-Nspire

Use **propFrac** from (menu) > **Algebra** > **Fraction Tools** > **Proper Fraction** as shown.



Complete Exercise 6B Questions page 219

6C – Factorisation of polynomials

Remainder theorem and factor theorem

Remainder theorem

When $P(x)$ is divided by $\beta x + \alpha$, the remainder is $P\left(-\frac{\alpha}{\beta}\right)$.

Factor theorem

For a polynomial $P(x)$:

- If $P(\alpha) = 0$, then $x - \alpha$ is a factor of $P(x)$.
- Conversely, if $x - \alpha$ is a factor of $P(x)$, then $P(\alpha) = 0$.

More generally:

- If $\beta x + \alpha$ is a factor of $P(x)$, then $P\left(-\frac{\alpha}{\beta}\right) = 0$.
- Conversely, if $P\left(-\frac{\alpha}{\beta}\right) = 0$, then $\beta x + \alpha$ is a factor of $P(x)$.

Examples :

1. Find the remainder when $P(x) = 3x^4 - 9x^2 + 27x - 8$ is divided by $x - 2$.

2. Factorise $P(x) = x^3 - 4x^2 - 11x + 30$ by $x - 2$ and hence solve for x.
3. Given $x + 1$ and $x - 2$ are factors of $6x^4 - x^3 + ax^2 - 6x + b$, find a and b.

Sums and differences

If $P(x) = x^3 - a^3$, then $x - a$ is a factor and so by division:

$$x^3 - a^3 = (x - a)(x^2 + ax + a^2)$$

If a is replaced by $-a$, then

$$x^3 - (-a)^3 = \big(x - (-a)\big)\big(x^2 + (-a)x + (-a)^2\big)$$

This gives:

$$x^3 + a^3 = (x + a)(x^2 - ax + a^2)$$

Examples:

1. Factorise $27x^3 - 1$	2. Factorise $8a^3 + 125b^3$

The rational-root theorem

Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial of degree n with all the coefficients a_i integers. Let α and β be integers such that the highest common factor of α and β is 1 (i.e. α and β are relatively prime).

If $\beta x + \alpha$ is a factor of $P(x)$, then β divides a_n and α divides a_0.

Examples:

Use the rational-root theorem to factorise $P(x) = 3x^3 + 8x^2 + 2x - 5$.

Alternative method for division of polynomials

Synthetic division is an alternative method to long division that can be used when factorising a polynomial, and the method is not the priority.

A third method, called **synthetic division**, is described in the example below.
Divide $x^3 + 2x^2 - 1$ by $x - 3$ using synthetic division.

Solution / **Explanation**

Step 1

	1	2	0	−1
3				
	1			

Enter the coefficients in the first row:
the coeffcient of x^3 is 1, the coeffcient of x^2 is 2,
the coeffcient of x is 0, the constant is -1.
Then the '3' from $x - 3$ is placed to the left, and the leading coefficient 1 is placed below the line.

Step 2

	1	2	0	−1
3		3		
	1			

Multiply the 1 in the row below the line by 3, and enter the result in the second row.

Step 3

	1	2	0	−1
3		3		
	1	5		

Enter 5 below the line, as it is the sum of the numbers 2 and 3 above.

Step 4

	1	2	0	−1
3		3	15	
	1	5	15	

Multiply the 5 below the line by 3, place the result below the 0, and then add the two values as shown.

Step 5

	1	2	0	−1
3		3	15	45
	1	5	15	44

Multiply the 15 below the line by 3, place the result below the -1, and then add the two values as shown.

Therefore

$$(x^3 + 2x^2 - 1) \div (x - 3) = x^2 + 5x + 15 + \frac{44}{x - 3}$$

Now read off the answer from the bottom row of the table as shown.

Complete Exercise 6C Questions page 227

6D – Solving cubic equations

In order to solve a cubic equation, the first step is often to factorise. Apply the same rule(s) practiced in Exercise 6C to find a factor(s) or by other means, then equate to zero to find solutions.

Examples:

1. Solve each of the following:
   a) $2(x-1)^3 = 32$

   b) $2x^3 - mx^2 - 22x + 11m$

2. Solve $2x^3 - 5x^2 + 5x - 2 = 0$

Complete Exercise 6D Questions page 231

6E – Cubic functions of the form $f(x) = a(x-h)^3 + k$

In Chapter 3 we saw that all quadratic functions can be written in '**turning point form'** and that the graphs of all quadratics have one basic form, the **parabola**. This is not true of cubic functions.

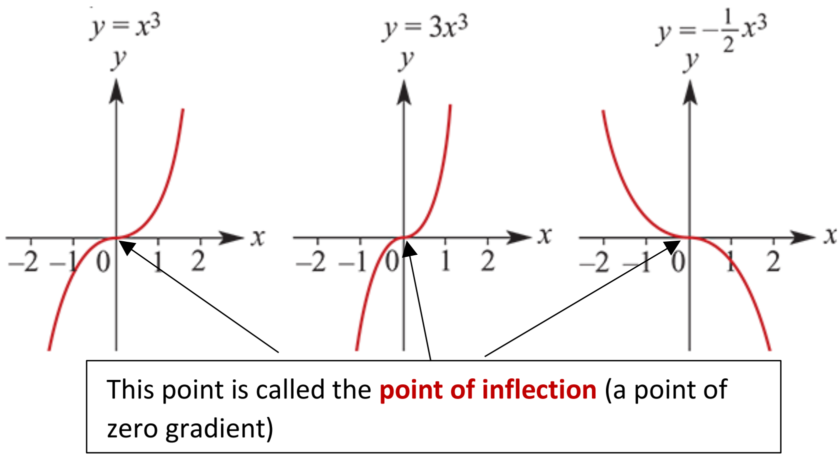
Consider cubic functions in the form $f(x) = a(x-h)^3 + k$.

The graphs of these functions can be formed by simple **transformations** of the graph of $f(x) = x^3$.

For example, the graph of $f(x) = (x-1)^3 + 3$ is obtained from the graph of $f(x) = x^3$ by a translation of 1 unit in the positive direction of the x-axis and 3 units in the positive direction of the y-axis.

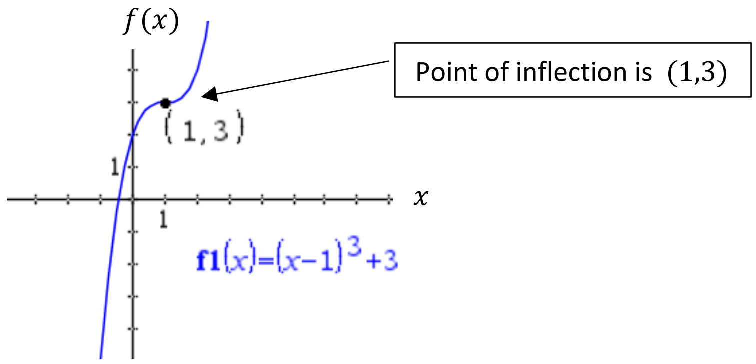
■ Transformations of the graph $f(x) = x^3$

Dilations from an axis and reflections in an axis to cubic functions in the form $f(x) = ax^3$



It should be noted that the implied **domain** of all cubics is $\mathbb{R}$ and the **range** is also $\mathbb{R}$.

The point of inflection can move as described in the above function $f(x) = (x-1)^3 + 3$



General form:

For the graph of a cubic function of the form

$$y = a(x-h)^3 + k$$

the point of inflection is at (h, k).

Vertical translation

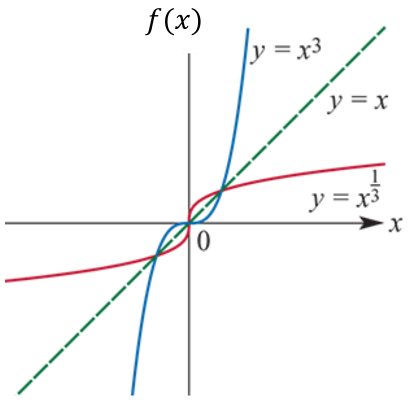
Dilation factor

Horizontal translation

■ The function $\boldsymbol{f}: \mathbb{R} \to \mathbb{R}, \boldsymbol{f(x) = x^{\frac{1}{3}}}$

The functions with rules of the form $f(x) = a(x-h)^3 + k$ are **one-to-one functions**. Hence each of these functions has an **inverse function**.

The inverse function of $f(x) = x^3$ is $f^{-1}(x) = x^{\frac{1}{3}}$.



Examples:

1. Sketch the graph of $f(x) = -2(x+1)^3 + 1$. Show all intercepts andpoint of inflection.
2. Find the inverse functin of $f(x) = -2(x+1)^3 + 1$.

Complete Exercise 6E Questions page 235

6F – Graphs of factorised cubic functions

The general cubic function written in **polynomial form** is

$$y = ax^3 + bx^2 + cx + d$$

The graph of a cubic function can have one, two or three x-axis intercepts.

- If a cubic can be written as the product of three linear factors,

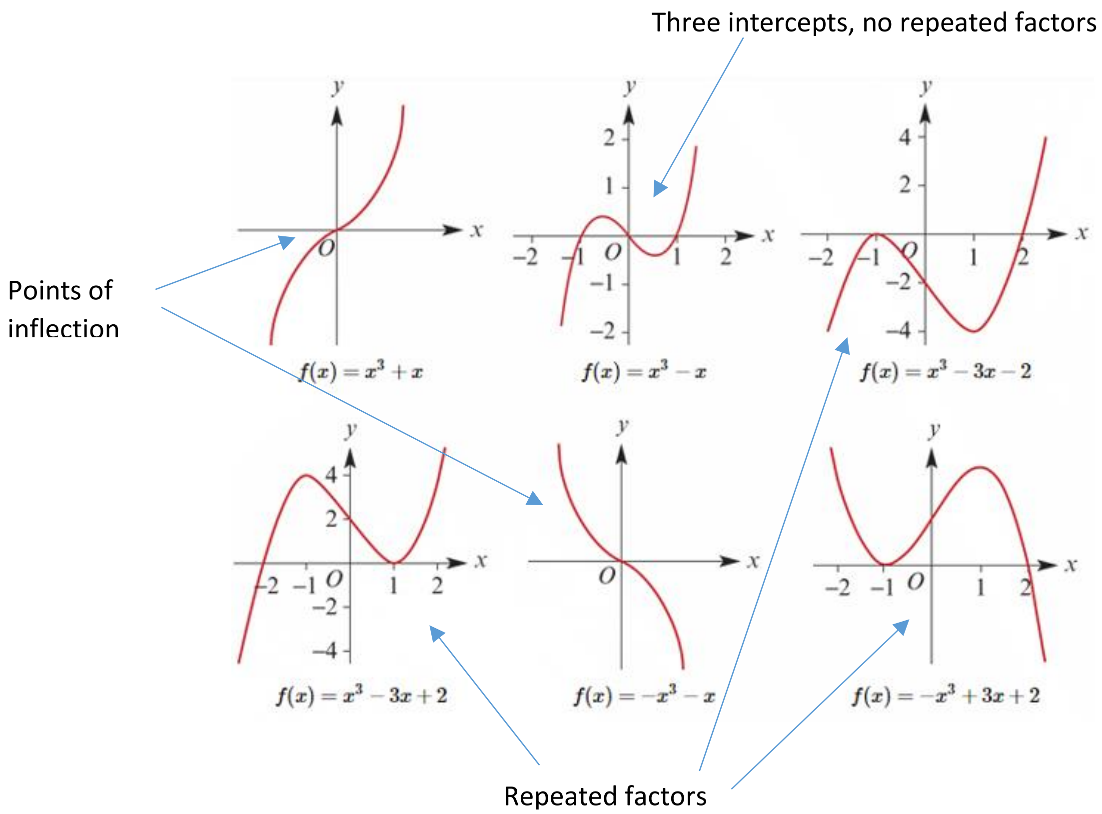
$$y = a(x - \alpha)(x - \beta)(x - \gamma)$$

then its graph can be sketched by following these steps:

- Find the y-axis intercept.
- Find the x-axis intercepts.
- Prepare a sign diagram.
- Consider the y-values as x increases to the right of all x-axis intercepts.
- Consider the y-values as x decreases to the left of all x-axis intercepts.

- If there is a **repeated factor** to the power 2, the y-values have the same sign immediately to the left and right of the corresponding x-axis intercept. Linear factors will be written as $y = a(x - \alpha)(x - \beta)^2$

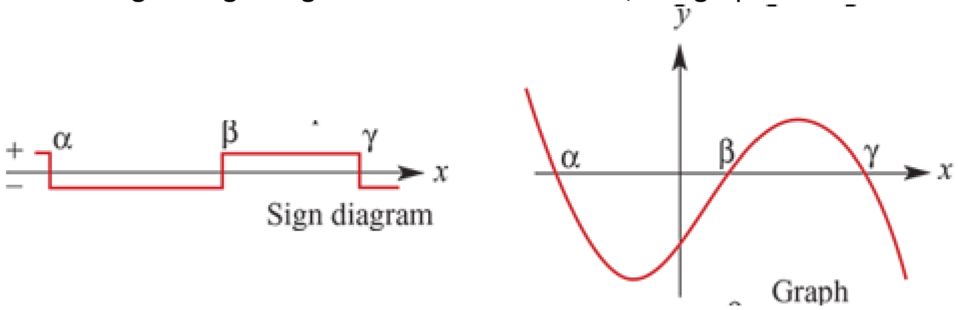
Graphs of some cubic functions



■ Sign diagrams

A sign diagram is a number-line diagram which shows when an expression is positive or negative.

The following is a sign diagram for a cubic function, the graph of which is also shown.



Using a sign diagram requires that the factors, and the x-axis intercepts, be found.

The y −axis intercept and sign diagram can then be used to complete the graph.

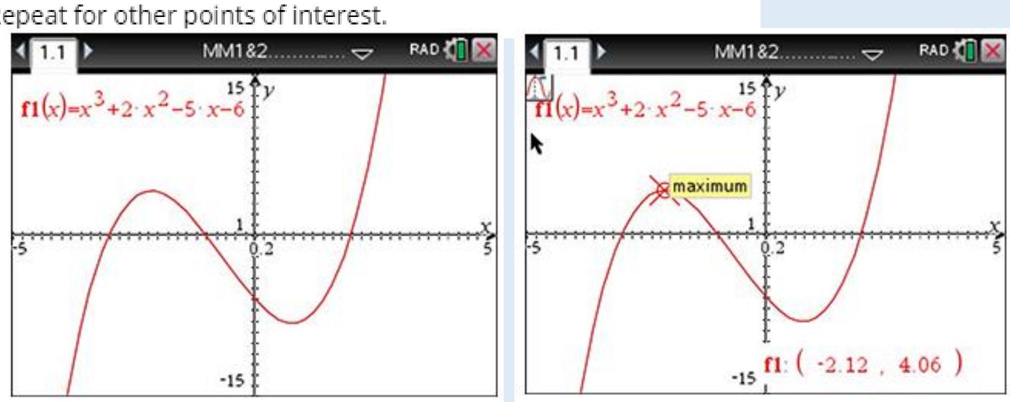
Example:

1. Sketch the graph of $y = (x + 1)(x - 2)(x + 3)$. **Hint: draw a sign diagram first.**
2. Sketch the graph of $y = 3x^3 - 4x^2 - 13x - 6$

– Using the TI-Nspire

In order to provide more detail, the coordinates of the turning points can be found with a CAS calculator.

- Enter $f_1(x) = x^3 + 2x^2 - 5x - 6$ in a **Graphs** application.
- Choose a suitable window (menu > **Window/Zoom** > **Window Settings**).
- Use menu > **Analyze Graph** > **Maximum**.
- Move the cursor to the left of point (lower bound), click, move to the right of point (upper bound) and click to display the coordinates.
- Repeat for other points of interest.



Note:
Alternatively, use menu > **Trace** > **Graph Trace** to find the coordinates of the two turning points. A label will appear near a turning point to indicate that the calculator has found a local maximum or a local minimum.

Complete Exercise 6F Questions page 241

6G – Solving cubic inequalities

As was done for quadratic inequalities, we can solve cubic inequalities by considering the graph of the corresponding polynomial.

Examples:

Solve the following cubic inequalities.

a) $(x+1)(x-2)(x+3) \geq 0$

b) $(x+1)(x-2)(x+3) < 0$

Complete Exercise 6G Questions page 242

6H – Families of cubic polynomial functions

- **Finding rules for cubic polynomialfunctions.**

Apply the same rules that were applied to finding the rule for a quadratic polynomial function.

Use any of the general rules:

$f(x) = a(x-h)^3 + k$

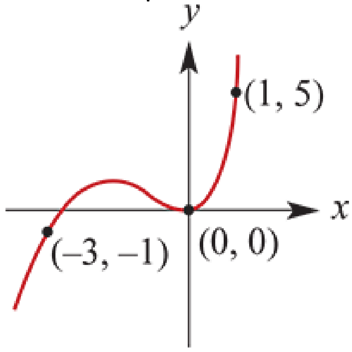
Or

$f(x) = ax^3 + bx^2 + cx + d$

Example:

1. Find a cubic function whose graph touches the x-axis at $x = -4$, cuts it at the origin, and has a value 6 when $x = -3$.

2. Find the equation of the following cubic in the form $y = ax^3 + bx^2$.



Complete Exercise 6H Questions page 246

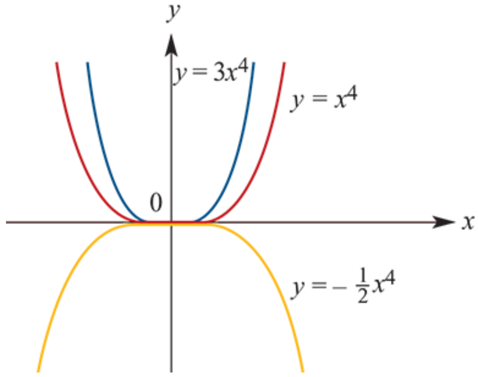
6I – Quartic and other polynomial functions

Quartic functions of the form $f(x) = a(x-h)^4 + k$

- As with other graphs it has been seen that changing the value of a simply narrows or broadens the graph without changing its fundamental shape.
- If $a < 0$, the graph is inverted.
- The significant feature of the graph of a quartic of this form is the **turning point**.
- The turning point of $y = x^4$ is at the origin $(0,0)$.
- For the graph of a quartic function of the form

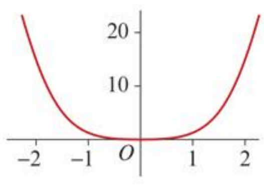
$$\boldsymbol{f(x) = a(x-h)^4 + k},$$

the turning point is at (h, k).

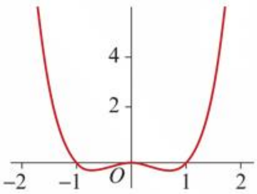


When sketching quartic graphs of the form $y = a(x-h)^4 + k$, first identify the turning point. To add further detail to the graph, find the x-axis and y-axis intercepts.

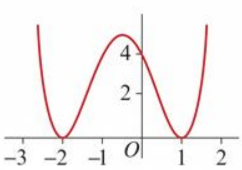
Examples of possible cubic functions and the shape of their graphs.



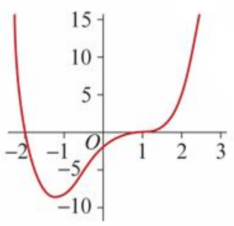
$f(x) = x^4$



$f(x) = x^4 - x^2$



$f(x) = (x-1)^2(x+2)^2$



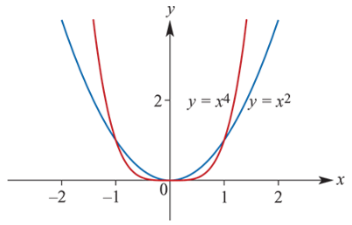
$f(x) = (x-1)^3(x+2)$

■ Odd and even polynomials

Knowing that a function is **even** or that it is **odd** is very helpful when sketching its graph.

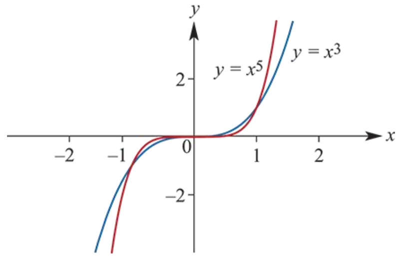
■ A function f is **even** if $f(-x) = f(x)$.

This means that the graph is symmetric about the y-axis. That is, the graph appears the same after reflection in the y-axis.



■ A function f is **odd** if $f(-x) = -f(x)$.

The graph of an **odd** function has rotational symmetry with respect to the origin: the graph remains unchanged after rotation of $180°$ about the origin.



A **power function** is a function f with rule $f(x) = x^r$ where r is a non-zero real number. We will only consider the cases where r is a positive integer or $r \in \{-2, -1, 12, 13\}$.

Even-degree power functions

The functions with rules $f(x) = x^2$ and $f(x) = x^4$ are examples of **even-degree** power functions.

The following are **properties** of all even-degree power functions:

$f(-x) = (x)$ for all x

$f(0) = 0$

As $x \to \pm\infty$, $y \to \infty$.

Odd-degree power functions

The functions with rules $f(x) = x^3$ and $f(x) = x^5$are examples of **odd-degree** power functions.

The following are **properties** of all odd-degree power functions:

$f(-x) = -f(x)$ for all xx

$f(0) = 0$

As $x \to \infty$, $y \to \infty$ and

as $x \to -\infty$, $y \to -\infty$.

Note: if a function has both even and odd powers, it is **neither** even or odd.

Example:

3. State whether the following polynomials is even or odd.
   a) $f(x) = 3x^4 - x^2$

b) $f(x) = 8x^3 + 5x - 9$

4. a) On the same set of axis sketch the graphs of $f(x) = x^4$ and $g(x) = 9x^2$.

b) Solve the equation $f(x) = g(x)$

c) Solve the inequality $f(x) \leq g(x)$

Complete Exercise 6I Questions page 252

6J – Applications of polynomial functions

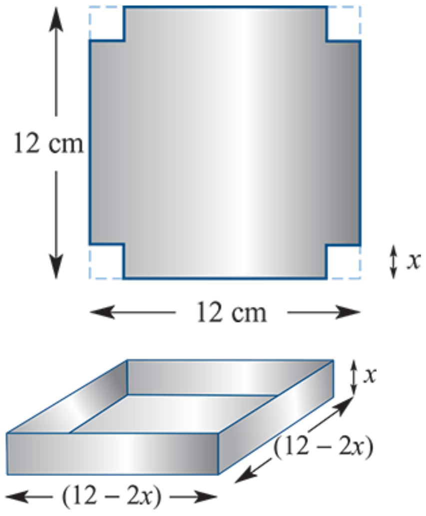
Textbook examples:

1. A square sheet of tin measures $12\ cm \times 12\ cm$.

Four equal squares of edge x cm are cut out of the corners and the sides are turned up to form an open rectangular box. Find:

a) the values of x for which the volume is $100\ cm^3$

b) the maximum volume.



Solution

The figure shows how it is possible to form many open rectangular boxes with dimensions $12 - 2x,\ 12 - 2x$ and x.

The volume of the box is

$$V = x(12 - 2x)^2, \qquad 0 \le x \le 6$$

which is a cubic model.

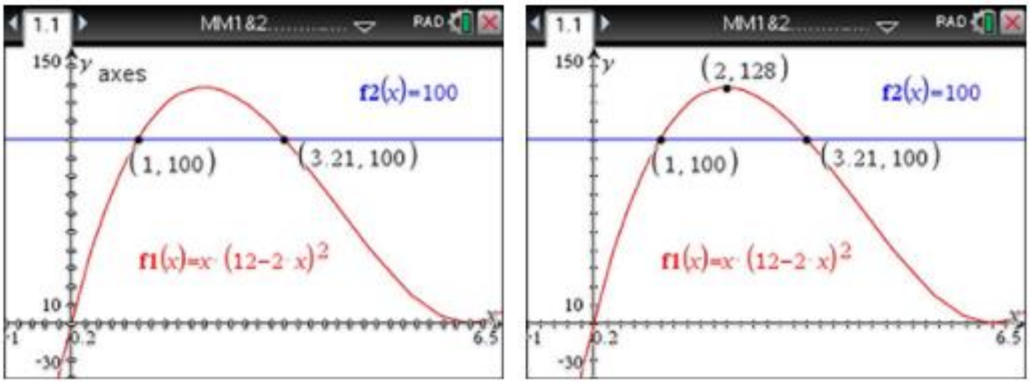
We complete the solution using a CAS calculator as follows.

– Using the TI-Nspire

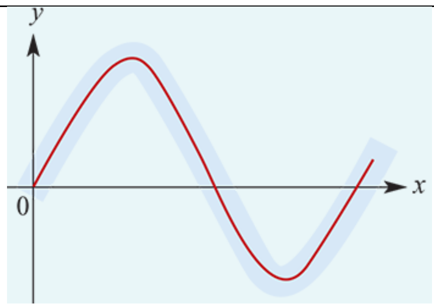
Plot the graph of $V = x(12 - 2x)^2$.

a To find the values of x for which $V = 100$, plot the graph of $V = 100$ on the same screen and find the intersection points using (menu) > **Geometry** > **Points & Lines** > **Intersection Point(s)**.

b To find the maximum volume, use (menu) > **Trace** > **Graph Trace** or (menu) > **Analyze Graph** > **Maximum**.



2. It is found that 250 metres of the path of a stream can be modelled by a cubic function. The cubic passes through the points (0,0), (100,22), (150,−10), (200,−20).

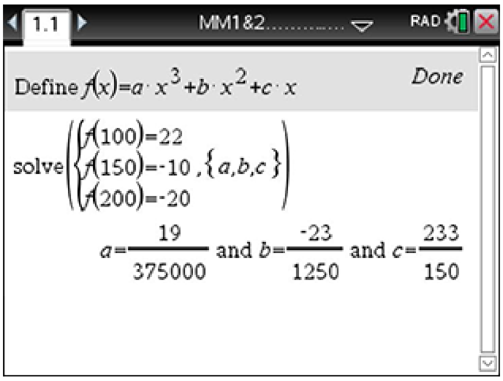


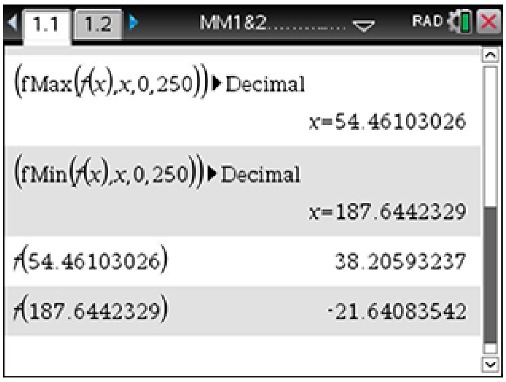
a) Find the equation of the cubic function.
b) Find the maximum deviation of the graph from the x-axis for $x \in [0,250]$.

– Using the TI-Nspire

- **Define** $f(x) = ax^3 + bx^2 + cx$.
- Solve using the **Solve System of Equations** command. Enter using the following function notation:
  $$f(100) = 22, \quad f(150) = -10,$$
  $$f(200) = -20$$
  Proceed as shown in the first screen.
- **Store** these values as a, b and c respectively.
- Use **fMax()** from (menu) > **Calculus** > **Function Maximum** to find where f obtains its maximum value.
- Use **fMin()** from (menu) > **Calculus** > **Function Minimum** to find where f obtains its minimum value.

The maximum deviation is 38.21 metres.





Complete Exercise 6J Questions page 255

Chapter 6 Checklist

This is the minimum requirement, do more where you feel it is necessary.

Exercise 6A: The language of polynomials	Q 2, 3, 4, 5a,c,d,h, 6, 7, 8, 9, 10b
Exercise 6B: Division of polynomials	Q 1a,c, 2b,d, 3b,c, 4a,d, 5, 6, 7a,c,e,f
Exercise 6C: Factorisation of polynomials	Q 1-LHS, 2,b,c, 3, 4, 5-RHS, 7a,d, 8b,c,e,g,h, 9,a,c, 10
Exercise 6D: Solving cubic equations	Q 1b,d, 2a,c, 3c,d, 4-RHS, 5-b,c,f, 6a,c,e, 7b,d
Exercise 6E: Cubic functions $f(x)=a(x-h)^3+k$	Q 2, 3, 4
Exercise 6F: Graphs of factorised cubic functions	Q1a,c, 2b,e,f, 3a,c,d, 4, 5
Exercise 6G: Solving cubic inequalities	Q 1, 2
Exercise 6H: Families of cubic polynomials	Q 1, 3, 5, 6, 7a,c, 8 9a,c,g
Exercise 6I: Quartic and other polynomials	Q 1a,d, 2a,d,e,f, 3-RHS, 4-LHS, 5, 6, 8
Exercise 6J: Applications of polynomials	Q- All
Chapter 6 Review Questions	All