Unit #15 - Differential Equations

Some problems and solutions selected or adapted from Hughes-Hallett Calculus.

Basic Differential Equations

1. Show that $y = x + \sin(x) - \pi$ satisfies the initial value problem
$\dfrac{dy}{dx} = 1 + \cos x$

To verify anything is a solution to an equation, we sub it in and verify that the left and right hand sides are equal after the substitution.

$$\begin{aligned}\text{Left side} &= \frac{dy}{dx} = 1 + \cos x - 0 = 1 + \cos x \\ \text{Right side} &= 1 + \cos x\end{aligned}$$

Both sides are equal, so $y = x + \sin(x) - \pi$ is a solution to the differential equation.

2. Find the general solution of the differential equation $\dfrac{dy}{dx} = x^3 + 5$

We can simply integrate both sides:
$y = \dfrac{x^4}{4} + 5x + C$ is the general solution to the equation.

3. Find the solution of the differential equation
$\dfrac{dq}{dz} = 2 + \sin z$, that also satisfies $q = 5$ when $z = 0$.

Integrating both sides with respect to z,

$$\begin{aligned}q &= 2z - \cos z + C \\ \text{If } q(0) = 5\text{, then} \qquad 5 &= 2(0) - \cos(0) + C \\ \text{so } C &= 6 \\ \text{meaning} q &= 2z - \cos(z) + 6 \text{ satisfies the DE and initial condition.}\end{aligned}$$

4. A tomato is thrown upward from a bridge 25 m above the ground at 40 m/sec.
   (a) Give formulas for the acceleration, velocity, and height of the tomato at time t. (Assume that the acceleration due to gravity is $g = 9.8$ m/s^2.)
   (b) How high does the tomato go, and when does it reach its highest point?
   (c) How long is it in the air, assuming it is landing on the ground at the base of the bridge?

(a) Let $y(t)$ be the height of the tomato at any time t. The initial conditions are $y(0) = 25$ (bridge height), and $y'(0) = 40$ (initial velocity upwards).
The differential equation we use is $F = ma = my''$. Since the only force acting on the tomato is gravity, with magnitude $-mg$, the equation of motion is

$$
\begin{aligned}
my'' &= -mg \text{ or } y'' = -g \quad \text{(acceleration)}\\
\text{Integrating both sides with respect to } t: \quad y' &= -gt + C_1\\
\text{Solving for } C_1 \text{ using } y'(0) = 40, \quad 40 &= -g(0) + C_1\\
C_1 &= 40\\
\text{so} \quad y' &= -gt + 40 \quad \text{(velocity)}\\
\text{Integrating again:} \quad y &= -\frac{gt^2}{2} + 40t + C_2\\
\text{Solving for } C_2 \text{ using } y(0) = 25, \quad 25 &= -\frac{g(0)}{2} + 40(0) + C_2\\
C_2 &= 25\\
\text{so} \quad y &= -\frac{gt^2}{2} + 40t + 25 \text{(position)}
\end{aligned}
$$

(b) The maximum height of the tomato occurs when $y'(t) = 0$, at $t = 40/9.8 \approx 4.08$ seconds. The height at this time is $y(4.08) \approx 106.6$ meters.

(c) The tomato is in the air until it hits the ground, at height $y = 0$. Using the quadratic formula, landing $t = \dfrac{-40 \pm \sqrt{40^2 - 4(-4.9)(25)}}{-9.8}$.
This gives $t \approx -0.583$ and ≈ 8.75. We want the positive time, so the tomato lands on the ground approximately 8.75 seconds after it was thrown.

5. Ice is forming on a pond at a rate given by $\dfrac{dy}{dt} = k\sqrt{t}$ where y is the thickness of the ice in inches at time t measured in hours since the ice started forming, and k is a positive constant. Find y as a function of t.

$$
\begin{aligned}
\text{Expressing using powers: } y' &= kt^{1/2}\\
\text{Integrating both sides: } y &= \frac{2}{3}kt^{3/2} + C
\end{aligned}
$$

The thickness of the ice as a function of time is $y = \frac{2}{3}kt^{3/2} + C$.

In this case, we can solve for C, since we were told t *is measured in hours since the ice started forming*, which means that the thickness $y = 0$ when $t = 0$. Using this data point in the general solution,

$$
\begin{aligned}
0 &= \frac{2}{3}k(0) + C\\
\text{so } C &= 0
\end{aligned}
$$

Thus the solution to the differential equation in the scenario given is

$$y = \frac{2}{3}kt^{3/2}$$

6. If a car goes from 0 to 80 km/h in six seconds with constant acceleration, what is that acceleration?

You could just figure the acceleration out by a unit analysis: to get to 80 mph in 6 seconds, the car must be accelerating at
$\dfrac{80 \text{ km/h}}{6 \text{ sec}} = \dfrac{22.2 \text{ m/s}}{6 \text{ s}} = 3.7 \text{ m/s}^2$

The more refined way to do this would be set up the differential equation for constant acceleration, a:

$v' = a$, which, after integrating both sides, gives $v = at + C$. If the initial velocity is $v(0) = 0$, then $C = 0$.

This means $v(t) = at$, and at $t = 6, v(6) = 80$ km/h ≈ 22.2 m/s, so $22.2 \approx 6a$, or $a \approx 22.2/6 \approx 3.7$ m/s^2.

7. Pick out which functions are solutions to which differential equations. (Note: Functions may be solutions to more than one equation or to none; an equation may have more than one solution.)

(a)	$\dfrac{dy}{dx} = -2y$	(I)	$y = 2\sin x$
(b)	$\dfrac{dy}{dx} = 2y$	(II)	$y = \sin 2x$
(c)	$\dfrac{d^2y}{dx^2} = 4y$	(III)	$y = e^{2x}$
(d)	$\dfrac{d^2y}{dx^2} = -4y$	(IV)	$y = e^{-2x}$

The most straightforward approach is to differentiate each solution to see if could satisfy any of the DEs.

	y	y'	y''
(I)	$y = 2\sin x$	$y' = 2\cos x$	$y'' = -2\sin x$
(II)	$y = \sin 2x$	$y' = 2\cos 2x$	$y'' = -4\sin 2x$
(III)	$y = e^{2x}$	$y' = 2e^{2x}$	$y'' = 4e^{2x}$
(IV)	$y = e^{-2x}$	$y' = -2e^{-2x}$	$y'' = 4e^{-2x}$

(a) $\dfrac{dy}{dx} = -2y$ is satisfied by (IV)

(b) $\dfrac{dy}{dx} = 2y$ is satisfied by (III)

(c) $\dfrac{d^2y}{dx^2} = 4y$ is satisfied by (III) and (IV)

(d) $\dfrac{d^2y}{dx^2} = -4y$ is satisfied by (II)

(I) is a solution to none of the DEs.

Modelling With Differential Equations

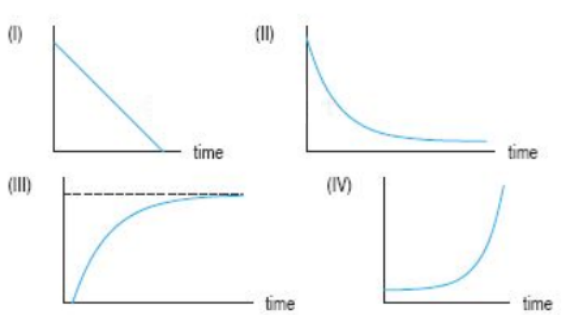
8. Match the graphs in the figure below with the following descriptions.

   (a) The temperature of a glass of ice water left on the kitchen table.

   (b) The amount of money in an interest- bearing bank account into which \$50 is deposited.

   (c) The speed of a constantly decelerating car.

   (d) The temperature of a piece of steel heated in a furnace and left outside to cool.



(a) (III)

(b) (IV)

(c) (I)

(d) (II)

9. Match the graphs in the figure below with the following descriptions.

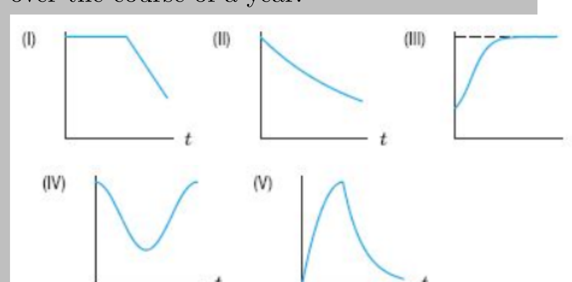
   (a) The population of a new species introduced onto a tropical island

   (b) The temperature of a metal ingot placed in a furnace and then removed

   (c) The speed of a car traveling at uniform speed and then braking uniformly

   (d) The mass of carbon-14 in a historical specimen

   (e) The concentration of tree pollen in the air over the course of a year.



(a) (III), although many graphs would be possible. After eliminating the rest, we find that (III) is a reasonable choice. The population will increase until it reaches an equilibrium. Other possibilities could have included catastrophic extinction, though, for example if there was insufficient food or too much competition.

(b) (V) - Temperature should go up (while in the furnace), and then down (when removed). The only graph that has this shape is (V)

(c) (I) - Uniform speed implies the speed graphs is flat. Followed by constant deceleration means that the speed is a straight line with negative slope.

(d) (II) - The mass of Carbon-14 in a sample will decay exponentially.

(e) (IV) - Concentration will change over time, going both up and down.

10. Show that $y = A + Ce^{kt}$ is a solution to the equation $\frac{dy}{dt} = k(y - A)$.

To show a function is a solution to an equation, we must show that the LHS and RHS of the equation are always equal when we use this formula for y.

$$\begin{aligned} LHS = \frac{dy}{dt} &= \frac{d}{dt}\left(A + Ce^{kt}\right) \\ &= C(ke^{kt}) \\ RHS = k(y - A) &= k(A + Ce^{kt} - A) \\ &= kCe^{kt} \end{aligned}$$

Since the LHS and RHS are equal for all values of t, k, A and C, the function $y = A + Ce^{kt}$ is a solution to the given differential equation.

11. Show that $y = \sin(2t)$ satisfies the differential equation$\frac{d^2y}{dt^2} + 4y = 0$.

Check that when we select $y = \sin 2t$, the left hand side and right hand side of the equation are equal:

$$\begin{aligned} LH &= \frac{d^2y}{dt^2} + 4y \\ &= \left(\frac{d^2}{dt^2}\sin 2t\right) + 4\sin 2t \\ &= \left(\frac{d}{dt}2\cos 2t\right) + 4\sin 2t \\ &= (-4\sin 2t) + 4\sin 2t \\ &= 0 \end{aligned}$$

This is equal to the right hand side of the equation, so $y = \sin 2t$ is a solution to the equation.

12. Find the value(s) of ω for which $y = \cos\omega t$ satisfies $\frac{d^2y}{dt^2} + 9y = 0$.

We try to sub in $y = \cos\omega t$ into both sides of the equation, and see if there are any restrictions on ω.

$$\begin{aligned} LH &= \frac{d^2y}{dt^2} + 9y \\ &= \left(\frac{d^2}{dt^2}\cos\omega t\right) + 9\cos\omega t \\ &= \left(\frac{d}{dt} - \omega\sin\omega t\right) + 9\cos\omega t \\ &= \left(-\omega^2\cos\omega t\right) + 9\cos\omega t \end{aligned}$$

For this to equal the RHS, we must have

$$\begin{aligned} \left(-\omega^2\cos\omega t\right) + 9\cos\omega t &= 0 \\ (9 - \omega^2)\cos(\omega t) &= 0 \end{aligned}$$

Since these two sides must be equal *regardless of* t or *for all values of* t, the cosine term doesn't help us. The only way to make the LHS = 0 is to have $9 - \omega^2$, or $\omega = \pm 3$.

The only solutions of the form $y = \cos\omega t$ are $y = \cos(3t)$ and $y = \cos(-3t)$.

13. Estimate the missing values in the table below if you know that $\frac{dy}{dt} = 0.5y$. Assume the rate of growth given by $\frac{dy}{dt}$ is approximately constant over each unit time interval and that the initial value of y is 8.

t	y
0	8
1	
2	
3	
4	

We are using intervals of $\Delta t = 1$.

To estimate the y values as we move to the right, we use the relationship

$$\begin{aligned} y(b) = y(a) + \Delta y &\approx y(a) + \frac{dy}{dt}\Delta t \\ &= y(a) + 0.5y(a)\Delta t \end{aligned}$$

$$
\begin{aligned}
y(1) &\approx y(0) + \frac{dy}{dt}(0)\cdot 1 = 8 + (0.5(8)) = 12 \\
y(2) &\approx y(1) + \frac{dy}{dt}(1)\cdot 1 = 12 + (0.5(12)) = 18 \\
y(3) &\approx y(2) + \frac{dy}{dt}(2)\cdot 1 = 18 + (0.5(18)) = 27 \\
y(4) &\approx y(3) + \frac{dy}{dt}(3)\cdot 1 = 27 + (0.5(27)) = 40.5
\end{aligned}
$$

Filling in the table, we get

t	y
0	8
1	**12**
2	**18**
3	**27**
4	**40.5**

14. (a) For what values of C and n (if any) is $y = Cx^n$ a solution to the differential equation:
$$x\frac{dy}{dx} - 3y = 0?$$
(b) If the solution satisfies $y = 40$ when $x = 2$, what more (if anything) can you say about C and n?

(a) If $y = Cx^n$ is a solution to the given differential equation, then using that for y and $\frac{dy}{dx} = C(nx^{n-1})$,

$$
\begin{aligned}
LHS &= x\frac{dy}{dx} - 3y \\
&= x(Cnx^{n-1}) - 3(Cx^n) \\
&= Cnx^n - 3Cx^n \\
&= C(n-3)x^n \\
\text{which must } &= \text{ RHS } = 0 \\
\text{so } C(n-3)x^n &= 0
\end{aligned}
$$

From this factored form, one of the three factors must equal zero. Since x changes, $x^n \neq 0$ for most values of x, so we must have either

- $C = 0$, or
- $n - 3 = 0$, implying $n = 3$.

These two options lead to the solutions

- $C = 0$: $y = 0 \cdot x^n = 0$, or
- $n = 3$: $y = Cx^3$

as the set of solutions to the differential equation $x\frac{dy}{dx} - 3y = 0$.

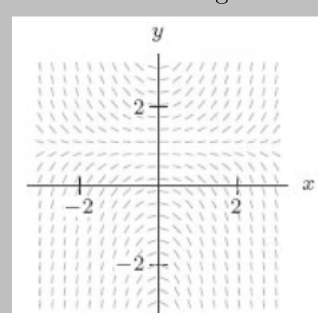
(b) If we add the information that $y(2) = 40$, that won't be satistfied by the $C = 0$, $y = 0$ solution, so we use the solution family $y = Cx^3$:

$$
\begin{aligned}
40 &= C(2)^3 \\
5 &= C
\end{aligned}
$$

so the more specific solution is now $y = 5x^3$. Now both C and n are fixed: $C = 5$ and $n = 3$.

Slope Fields

15. The slope field for the equation $y' = x(y-1)$ is shown in in the figure below.

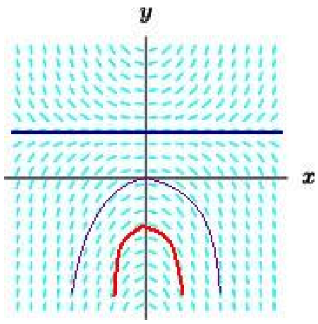


(a) Sketch the solutions passing through the points
(i) (0, 1) (ii) (0, -1) (iii) (0, 0)

(b) From your sketch, write down the equation of the solution with $y(0) = 1$.

(c) Check your solution to part (b) by substituting it into the differential equation.

(a)



(b) The slope lines all look flat around $y = 1$, so the solution would be the flat line $y(x) = 1$.

(c) To check, we sub in $y = 1$ into both sides of the DE and check that the LHS = RHS:

$$\begin{aligned} \text{Left side} &= y' = \frac{d}{dx}1 = 0 \\ \text{Right side} &= x(y-1) = x(1-1) = 0 \\ \text{Left side} &= \text{Right side} \end{aligned}$$

Therefore the constant solution $y = 1$ **is** a solution to the DE.

16. The slope field for the equation $y' = x + y$ is shown in Figure 11.17.

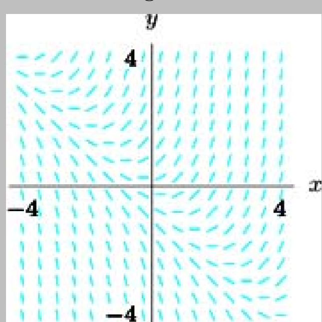


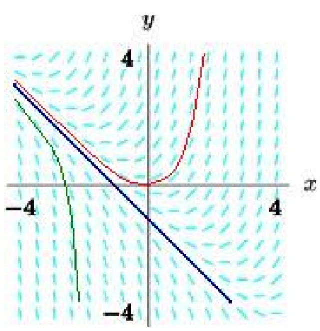
Figure 11.17: $y' = x + y$

(a) Sketch the solutions that pass through the points
(i) (0, 0) (ii) (-3, 1) (iii) (-1, 0)

(b) From your sketch, write the equation of the solution passing through (-1, 0).

(c) Check your solution to part (b) by substituting it into the differential equation.

(a)



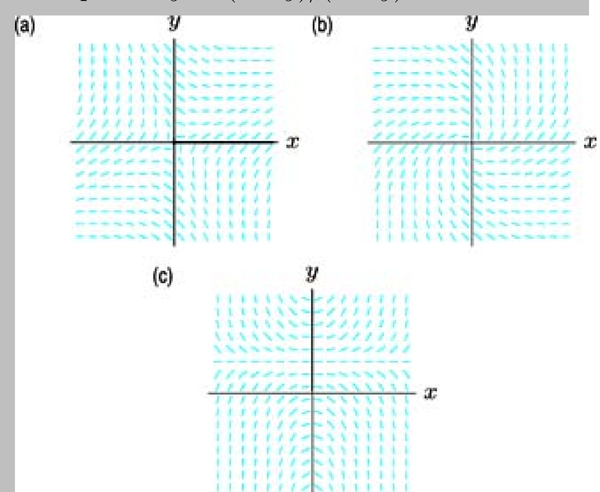
(b) From the slope lines, it looks as if the solution through $(-1, 0)$ is following a line of constant slope down at a 45^o angle. This line would have slope -1, or the equation $y = -1 - x$.

(c) To check, we sub in $y = -1 - x$ into both sides of the DE and check that the LHS = RHS:

$$\begin{aligned} \text{Left side} &= y' = \frac{d}{dx}(-1-x) = -1 \\ \text{Right side} &= x + y = x + (-1-x) = -1 \\ \text{Left side} &= \text{Right side} \end{aligned}$$

Therefore the straight line solution $y = -1 - x$ is a solution to the DE.

17. One of the slope fields on the diagram below has the equation $y' = (x + y)/(x - y)$. Which one?



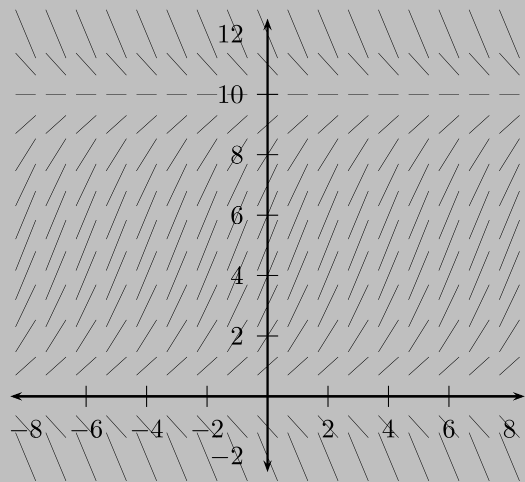
From an earlier question, we already know (c) is the slope field for $y' = x(y-1)$, so (c) can't be the answer. We can check a variety of other features to slopes to determine which of (a) or (b) is correct. Here are the first checks I would try.

- Along the x axis, or the line $y = 0$, we should have slopes $y' = x/x = 1$, except when $x = 0$. This describes only (b).
- Along the y axis, or the line $x = 0$, we should have slopes $y' = y/(-y) = -1$, except at $y = 0$.
- Along the line $y = -x$, the numerator of y' is zero, so we should get horizontal slopes.
- Along the line $y = x$, the denominator of y' is zero, so we should get infinite slopes/vertical slopes. This is only true for (b).

It seems as if (b) is the slope field for the DE $y' = (x + y)/(x - y)$.

Depending on the question, the strategies of looking along the axes, as well as looking for where $y' = 0$ or y' is undefined, can each be useful.

18. The slope field for the equation $dP/dt = 0.1P(10-P)$, for $P \geq 0$, is in the figure below.

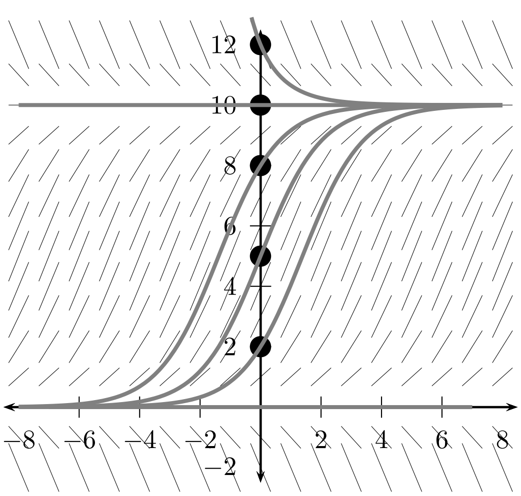


(a) Plot the solutions through the following points:

(i) (0, 0)
(ii) (0, 2)
(iii) (0, 5)
(iv) (0, 8)
(v) (0, 12)

(b) For which positive values of P are the solutions increasing? Decreasing? What is the limiting value of P as t gets large?

(a)



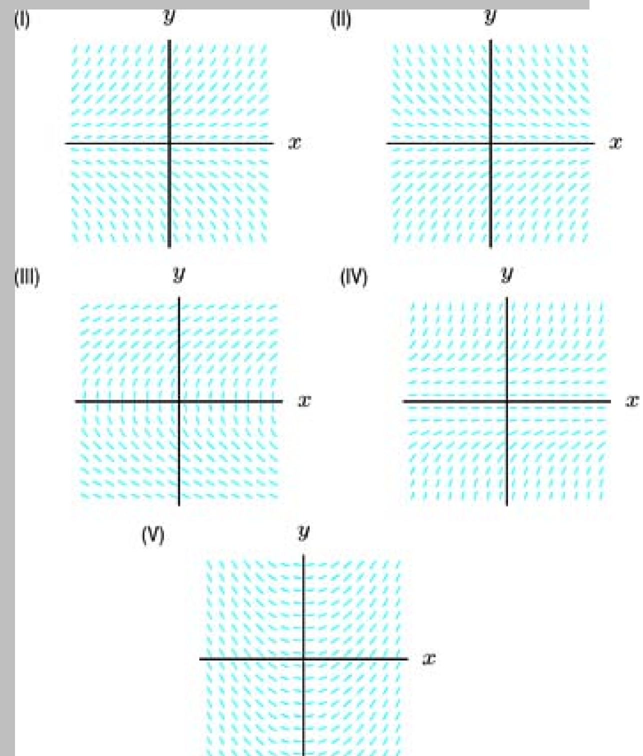
(b) P will be increasing (P' will be positive) when $0 < P < 10$.

P will be decreasing (P' will be negative) when $P > 10$ or $P < 0$.

As $t \to \infty$, $P \to 10$, if the starting value of P was any value greater than zero.

19. Match the slope fields shown below with their differential equations:

(a) $y' = -y$
(b) $y' = y$
(c) $y' = x$
(d) $y' = 1/y$
(e) $y' = y^2$



You will have to infer the vertical & horizontal scaling on the graphs.

- (a) - (II). The slope should be the same for along horizontal lines (constant y values), so (V) is out. Along the x axis, $y = 0$ so $y' = 0$, and (III) is out. Slopes should be negative when y is positive, and positive when y is negative. Only (II) satisfies this.
- (b) - (I). Same as (a), except y' is positive when y is positive and y' is negative when y is negative.
- (c) - (V). Slopes are the same along constant x values, or vertical lines.
- (d) - (III). Slopes are vertical when $y = 0$, and become flatter as y goes away from zero.
- (e) - (IV). Slopes are positive everywhere, and zero along $y = 0$.

20. Match the slope fields shown below with their differential equations:

(a) $y' = 1 + y^2$

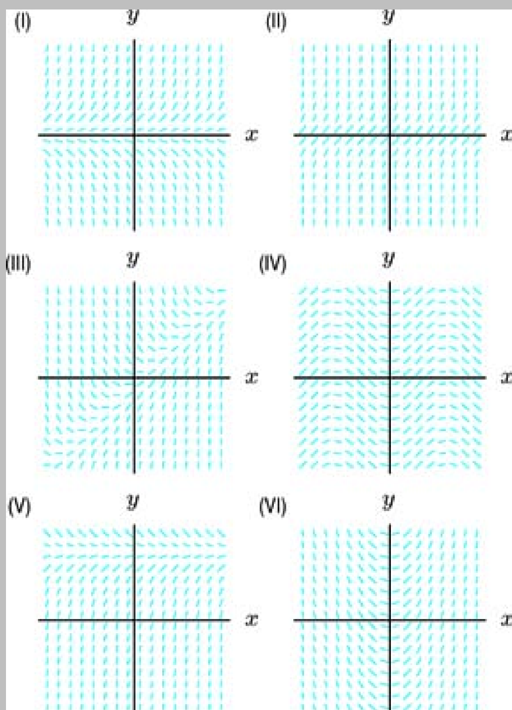
(b) $y' = x$

(c) $y' = \sin x$

(d) $y' = y$

(e) $y' = x - y$

(f) $y' = 4 - y$



Each slope field is graphed for $-5 \leq x \leq 5$, $-5 \leq y \leq 5$.

- (a) - (II). Slopes are positive everywhere. Slopes are 1 along $y = 0$, and steeper as you move away from $y = 0$.
- (b) - (VI). Slopes are constant when x is constant. Slopes have same sign as x and get bigger as x gets bigger.
- (c) - (IV). Slopes change with value of x. Slopes are all between -1 and +1, and change sinusoidally. Since $y' = \sin x$, direct integration tells us that the solution curves should look like $y = -\cos x + C$.
- (d) - (I). Slopes are steeper for large y, and have slope zero along $y = 0$.
- (e) - (III) . Slopes are zero along $y = x$. Slopes are constant along $y = x + C$
- (f) - (V). Along $y = 4$, the slopes should be zero.

Euler's Method

21. Consider the differential equation $y' = x + y$. Use Euler's method with $\Delta x = 0.1$ to estimate y when $x = 0.4$ for the solution curves satisfying

(a) $y(0) = 1$

(b) $y(-1) = 0$

(a)

x	y	$\frac{dy}{dx} = x + y$	$\Delta y = \frac{dy}{dx}\Delta x$
0	1	1	0.1
0.1	1.1	1.2	0.12
0.2	1.22	1.42	0.142
0.3	1.362	1.662	0.1662
0.4	1.5282		

So $y(0.4) \approx 1.5282$.

(b) Keep in mind that we don't usually ask questions that require this many steps. If we do, it's usually because there is a simple repeating pattern, as in this example.

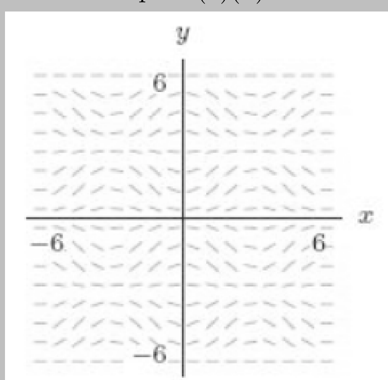
x	y	$\frac{dy}{dx} = x + y$	$\Delta y = \frac{dy}{dx}\Delta x$
-1	0	-1	-0.1
-0.9	-0.1	-1	-0.1
-0.8	-0.2	-1	-0.1
-0.7	-0.3	-1	-0.1
-0.6	-0.4	-1	-0.1
$\vdots$			
0	-1	-1	-0.1
$\vdots$			
0.3	-1.3	-1	-0.1
0.4	-1.4		

So $y(0.4) \approx -1.4$.

22. Consider the differential equation $y' = (\sin x)(\sin y)$.

    (a) Calculate approximate y-values using Euler's method with three steps and $\Delta x = 0.1$, starting at each of the following points:
    (i) (0, 2) (ii)(0, π).

    (b) Use the slope field below to explain your solution to part (a)(ii).



(a) Remember to use radians in your calculator.

	x	y	$\frac{dy}{dx} = \sin(x)\sin(y)$	$\Delta y = \frac{dy}{dx}\Delta x$
	0	2	0	0
(i)	0.1	2	0.091	0.0091
	0.2	2.009	0.18	0.018
	0.3	2.027		

	x	y	$\frac{dy}{dx} = \sin(x)\sin(y)$	$\Delta y = \frac{dy}{dx}\Delta x$
	0	π	0	0
(i)	0.1	π	0	0
	0.2	π	0	0
	0.3	π		

(b) If we consider the slope field at height $y = \pi$, the slopes will always be horizontal there because $\sin(\pi) = 0$, so $\dfrac{dy}{dx} = \sin(x)\sin(y) = 0$ for all $y = \pi$. We see this constant solution coming out of Euler's method in (ii).

23. Consider the differential equation $\dfrac{dy}{dx} = f(x)$ with initial value $y(0) = 0$. Explain why using Euler's method to approximate the solution curve gives the same results as using left Riemann sums to approximate $\displaystyle\int_0^x f(t)dt$.

If we use Euler's method, with $y' = f(x)$, we will start at some $x = x_0$, and count up by intervals of Δx. This will produce estimates of y which will have the following form:

$$\begin{aligned}
y_1 &= y_0 + f(x_0)\Delta x \\
y_2 &= y_1 + f(x_1)\Delta x \\
&\ldots \\
y_n &= y_{n-1} + f(x_{n-1})\Delta x
\end{aligned}$$

where x_n is where we want to stop, y_n is our estimate of the function there, and each $x_{i+1} = x_i + \Delta x$.

Note that if we combine all of these terms together, we get

$$\begin{aligned}
y_n &= y_{n-1} + f(x_{n-1})\Delta x \\
&= \underbrace{(y_{n-2} + f(x_{n-2}\Delta x))}_{y_{n-1}} + f(x_{n-1})\Delta x \\
&\ldots \\
&= y_0 + f(x_0)\Delta x + f(x_1)\Delta x + \ldots + f(x_{n-1})\Delta x \\
&= f(x_0)\Delta x + f(x_1)\Delta x + \ldots + f(x_{n-1})\Delta x \quad \text{since } y(0) = 0
\end{aligned}$$

If instead we try to estimate the integral $\displaystyle\int_0^{x_n} f(x)dx$ using rectangles, we will use

(a) height is $f(x_0)$ or $f(x_1)$, or $\ldots$, or $f(x_{n-1})$

(b) width of Δx

Adding up the area of these rectangles gives us the Riemann sum

$$\text{Area } = f(x_0)\Delta x + f(x_1)\Delta x + \ldots + f(x_{n-1})\Delta x$$

which is exactly the same as the value calculated by Euler's method.

24. Consider the solution of the differential equation $y' = y$ passing through $y(0) = 1$.

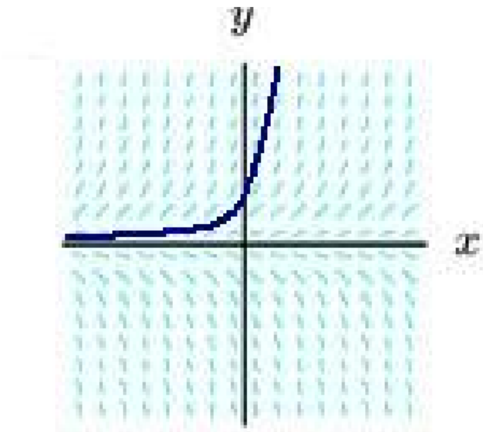
(a) Sketch the slope field for this differential equation, and sketch the solution passing through the point (0, 1).

(b) Use Euler's method with step size $\Delta x = 0.1$ to estimate the solution at $x = 0.1, 0.2, \ldots, 1$.

(c) Plot the estimated solution on the slope field; compare the solution and the slope field.

(d) Check that $y = e^x$ is the solution of $y' = y$ with $y(0) = 1$.

(a) The slopes are steeper further away from the x axis, and zero along that axis. Slopes are positive above and negative below the x axis.



(b)

x	y	$\frac{dy}{dx} = y$	$\Delta y = \frac{dy}{dx}\Delta x$
0	1	1	0.1
0.1	1.1	1.1	0.11
0.2	1.21	1.21	0.121
0.3	1.331	1.331	0.1331
0.4	1.4641	1.4641	0.14641
0.5	1.61051	1.61051	0.161051
0.6	1.77156	1.77156	0.177156
0.7	1.94872	1.94872	0.194872
0.8	2.14359	2.14359	0.214359
0.9	2.35795	2.35795	0.235795
1	2.59374		

(c) The points seem to go up in the same way that the graph does: small changes in y at first, followed by gradually larger and larger steps.

(d) We can check that $y = e^x$ is the solution to $\frac{dy}{dx} = y, y(0) = 1$ by seeing that

$$y(0) = e^0 = 1 \qquad \text{(initial condition)}$$
$$\text{Left side} = \frac{dy}{dx} = \frac{d}{dx}e^x = e^x = y = \text{Right side}$$

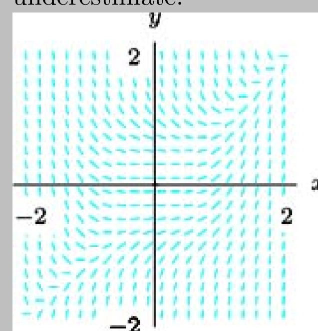
25. (a) Use Euler's method to approximate the value of y at $x = 1$ on the solution curve to the differential equation

$$\frac{dy}{dx} = x^3 - y^3$$

that passes through (0, 0). Use $\Delta x = 1/5$ (i.e., 5 steps).

(b) Using the slope field shown below, sketch the solution that passes through (0, 0). Show the approximation you made in part (a).

(c) Using the slope field, say whether your answer to part (a) is an overestimate or an underestimate.



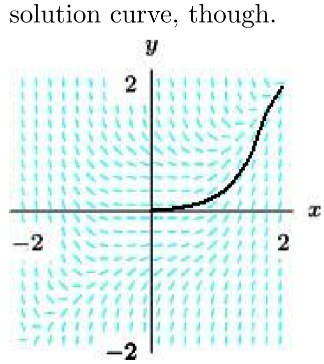
Slope field for $\frac{dy}{dx} = x^3 - y^3$.

(a)

x	y	$\frac{dy}{dx} = x^3 - y^3$	$\Delta y = \frac{dy}{dx}\Delta x$
0	0	0	0
0.2	0	0.008	0.0016
0.4	0.0016	0.064	0.0128
0.6	0.0144	0.216	0.0432
0.8	0.0576	0.51181	0.10236
1	0.15996		

So $y(1) \approx 0.15996$.

(b) It is too difficult to show the solution from part (a) in such a small diagram. Here is a sketch of the solution curve, though.



(c) Because the solution curve is convex up between $x = 0$ and $x = 1$, the solution from part (a) will be an underestimate of the real solution.

26. Consider the differential equation $\frac{dy}{dx} = 2x$, with initial condition $y(0) = 1$.

(a) Use Euler's method with two steps to estimate y when $x = 1$. Then use four steps.

(b) What is the formula for the exact value of y?

(a) With two intervals between $x = 0$ and $x = 1$, we have $\Delta x = 0.5$.

x	y	$\frac{dy}{dx} = 2x$	$\Delta y = \frac{dy}{dx}\Delta x$
0	1	0	0
0.5	1	1	0.5
1	1.5		

With four intervals between $x = 0$ and $x = 1$, we have $\Delta x = 0.25$.

x	y	$\frac{dy}{dx} = 2x$	$\Delta y = \frac{dy}{dx}\Delta x$
0	1	0	0
0.25	1	0.5	0.125
0.50	1.125	1	0.25
0.75	1.375	1.5	0.375
1	1.75		

(b) If $\frac{dy}{dx} = 2x$, we can integrate directly, so $y = x^2 + C$. Since we want a solution going through $y = 1$, $C = 1$ so our analytic (exact) solution becomes $y = x^2 + 1$.

Separable Differential Equations

27. Determine which of the following differential equations is separable. Do not solve the equations.

(a) $y' = y$

(b) $y' = x + y$

(c) $y' = xy$

(d) $y' = \sin(x + y)$

(e) $y' - xy = 0$

(f) $y' = y/x$

(g) $y' = \ln(xy)$

(h) $y' = (\sin x)(\cos y)$

(i) $y' = (\sin x)(\cos xy)$

(j) $y' = x/y$

(k) $y' = 2x$

(l) $y' = (x + y)/(x + 2y)$

(a) Separable. $\frac{dy}{dx} = y$ can be separated as $\frac{1}{y}dy = dx$.

(b) Not separable.

(c) Separable. $\frac{dy}{dx} = xy$ can be separated as $\frac{1}{y}dy = x\ dx$.

(d) Not separable.

(e) Separable. (Rearrange to make $y' = xy$, which is separable.)

(f) Separable. $\frac{dy}{dx} = \frac{y}{x}$ can be separated as $\frac{1}{y}dy = \frac{1}{x}\ dx$.

(g) Not separable.

(h) Separable. $\frac{dy}{dx} = (\sin x)(\cos y)$ can be separated as $\frac{1}{\cos(y)}dy = \sin(x)\ dx$.

(i) Not separable.

(j) Separable. $\frac{dy}{dx} = \frac{x}{y}$ can be separated as $y\ dy = x\ dx$.

(k) Separable. $\frac{dy}{dx} = 2x$ can be separated as $dy = 2x\ dx$.

(l) Not separable.

For Questions 28-36, find the particular solution to the differential equation.

28. $\frac{dP}{dt} = -2P$, $P(0) = 1$

$$\frac{dP}{P} = -2dt$$

Int'te both sides: $$\int \frac{dP}{P} = \int -2dt$$

$$\ln|P| = -2t + C$$

Exp'te both sides: $$e^{\ln|P|} = e^{-2t+C}$$

$$|P| = e^{-2t}e^{C}$$

$$P = C_1 e^{-2t} \text{ where } C_1 = \pm e^{C}$$

We can remove the absolute value signs here, as the change can be factored into the sign of C_1.

$$\begin{aligned} \text{Use } P(0) = 1: \quad 1 &= C_1 e^0 \\ \text{so } \quad C_1 &= 1 \\ \text{and } \quad P(t) &= 1e^{-2t} = e^{-2t} \end{aligned}$$

29. $\dfrac{dL}{dp} = \dfrac{L}{2}, \; L(0) = 100$

$$\begin{aligned} \frac{dL}{L} &= \frac{1}{2}dp \\ \text{Int'te both sides:} \quad \int \frac{dL}{L} &= \int \frac{1}{2}dp \\ \ln|L| &= \frac{1}{2}p + C \\ \text{Exp'te both sides:} \quad e^{\ln|L|} &= e^{\frac{p}{2}+C} \\ L &= C_1 e^{\frac{p}{2}} \quad \text{where } C_1 = \pm e^C \\ \text{Use } L(0) = 100: \quad 100 &= C_1 e^0 \\ \text{so} \quad C_1 &= 100 \\ \text{and} \quad L &= 100e^{\frac{p}{2}} \end{aligned}$$

30. $\dfrac{dy}{dx} + \dfrac{y}{3} = 0, \; y(0) = 10$

$$\begin{aligned} \frac{dy}{dx} &= \frac{-y}{3} \\ \frac{dy}{y} &= \frac{-1}{3}\,dx \\ \text{Int'te both sides:} \quad \int \frac{dy}{y} &= \int \frac{-1}{3}\,dx \\ \ln|y| &= \frac{-1}{3}x + C \\ \text{Exp'te both sides:} \quad |y| &= e^{\frac{-x}{3}+C} \\ y &= C_1 e^{\frac{-x}{3}} \quad \text{where } C_1 = \pm e^C \\ \text{Use } y(0) = 10: \quad 10 &= C_1 e^0 \\ \text{so} \quad C_1 &= 10 \\ \text{and} \quad y &= 10e^{\frac{-x}{3}} \end{aligned}$$

31. $\dfrac{dm}{dt} = 3m, \; m = 5$ when $t = 1$.

$$\begin{aligned} \frac{dm}{m} &= 3dt \\ \text{Int'te both sides:} \quad \int \frac{dm}{m} &= \int 3dt \\ \ln|m| &= 3t + C \\ \text{Exp'te both sides:} \quad |m| &= e^{3t+C} \\ m &= C_1 e^{3t} \text{ where } C_1 = \pm e^C \\ \text{Use } m(1) = 5: \quad 5 &= C_1 e^3 \\ \text{so} \quad C_1 &= 5e^{-3} \\ \text{and} \quad m &= 5e^{-3}e^{3t} = 5e^{3t-3} \end{aligned}$$

32. $\dfrac{1}{z}\dfrac{dz}{dt} = 5, \; z(1) = 5.$

$$\begin{aligned} \frac{dz}{z} &= 5dt \\ \text{Int'te both sides:} \quad \int \frac{dz}{z} &= \int 5dt \\ \ln|z| &= 5t + C \\ \text{Exp'te both sides:} \quad |z| &= e^{5t+C} \\ z &= C_1 e^{5t} \text{ where } C_1 = \pm e^C \\ \text{Use } z(1) = 5: \quad 5 &= C_1 e^5 \\ \text{so} \quad C_1 &= 5e^{-5} \\ \text{and} \quad z &= 5e^{-5}e^{5t} = 5e^{5t-5} \end{aligned}$$

33. $\dfrac{dy}{dt} = 0.5(y - 200), \; y = 50$ when $t = 0$.

$$\begin{aligned} \frac{dy}{(y-200)} &= 0.5dt \\ \text{Int'te both sides:} \quad \int \frac{dy}{(y-200)} &= \int 0.5dt \\ \ln|y - 200| &= 0.5t + C \\ \text{Exp'te both sides:} \quad |y - 200| &= e^{0.5t+C} \\ y - 200 &= C_1 e^{0.5t} \text{ where } C_1 = \pm e^C \\ y &= C_1 e^{0.5t} + 200 \\ \text{Use } y(0) = 50: \quad 50 &= C_1 e^0 + 200 \\ \text{so} \quad C_1 &= -150 \\ \text{and} \quad y &= -150e^{0.5t} + 200 \end{aligned}$$

34. $\dfrac{dm}{dt} = 0.1m + 200, \; m(0) = 1000.$

Depending on how you group the constants (e.g. if you factor out the 0.1 term), you may see different factors along the way in this solution. The final answer should still be the same as the one given here.

$$\frac{dm}{(0.1m+200)} = dt$$

Int'te both sides:

$$\int \frac{dm}{(0.1m+200)} = \int dt$$

$$\frac{\ln|0.1m+200|}{0.1} = t + C$$

$$\ln|0.1m+200| = 0.1t + C_1 \text{ Let } C_1 = 0.1C$$

Exp'te both sides:

$$|0.1m+200| = e^{0.1t+C_1}$$

$$0.1m+200 = C_2e^{0.1t} \text{ where } C_2 = \pm e^{C_1}$$

$$0.1m = C_2e^{0.1t} - 200$$

$$m = 10C_2e^{0.1t} - 2000$$

Use $m(0) = 1000$: $\quad 1000 = 10C_2e^0 - 2000$

so $\quad 10C_2 = 3000$

and $\quad m = 3000e^{0.1t} - 2000$

35. $\dfrac{dz}{dt} = te^z$, through the origin.

$$\frac{dz}{e^z} = t\ dt$$

Int'te both sides: $\quad \displaystyle\int e^{-z}\ dz = \int t\ dt$

$$-e^{-z} = \frac{t^2}{2} + C$$

$$e^{-z} = -\frac{t^2}{2} - C$$

Use $z(0) = 0$: $\quad e^0 = -\dfrac{0^2}{2} - C$

$$C = -1$$

Take ln of both sides: $\quad \ln(e^{-z}) = \ln\left(-\dfrac{t^2}{2} - (-1)\right)$

$$-z = \ln\left(1 - \frac{t^2}{2}\right)$$

$$z = -\ln\left(1 - \frac{t^2}{2}\right)$$

36. $\dfrac{dw}{d\theta} = \theta w^2\sin(\theta^2)$, $w(0) = 1$.

$$\frac{dw}{w^2} = \theta\sin(\theta^2)d\theta$$

Int'te both sides: $\quad \displaystyle\int w^{-2}dw = \int \theta\sin(\theta^2)d\theta$

integrate by substitution, $u = \theta^2$, $\quad -w^{-1} = -\dfrac{1}{2}\cos(\theta^2) + C$

Use $w(0) = 1$: $\quad -1 = -\dfrac{1}{2}\cos(0^2) + C$

$$C = -\frac{1}{2}$$

so $\quad -w^{-1} = -\dfrac{1}{2}\cos(\theta^2) - \dfrac{1}{2}$

$$w^{-1} = \frac{1}{2}\cos(\theta^2) + \frac{1}{2} = \frac{\cos(\theta^2)+1}{2}$$

and finally $\quad w = \dfrac{2}{\cos(\theta^2)+1}$

For Questions 37-40, find the general solution to the differential equations. Assume a, b, and k are nonzero constants.

37. $\dfrac{dR}{dt} = kR$

$$\frac{dR}{R} = kdt$$

Int'te both sides: $\quad \displaystyle\int \frac{dR}{R} = \int kdt$

$$\ln|R| = kt + C$$

Exp'te both sides: $\quad e^{\ln|R|} = e^{kt+C}$

$$R = C_1e^{kt} \text{ where } C_1 = \pm e^C$$

38. $\dfrac{dP}{dt} - aP = b$

$$\frac{dP}{dt} = aP + b$$
$$\frac{dP}{(aP+b)} = dt$$

Int'te both sides:

$$\int \frac{1}{(aP+b)}\ dP = \int dt$$
$$\frac{1}{a}\ln|aP+b| = t + C$$
$$\ln|aP+b| = at + aC$$

Exp'te both sides:

$$|aP+b| = e^{at+aC}$$

Letting $A = +$ or - e^{aC} as needed for the absolute value,

$$P = \frac{1}{a}\left(Ae^{at} - b\right)$$

39. $\dfrac{dy}{dt} = ky^2(1+t^2)$

$$\frac{dy}{y^2} = k(1+t^2)\ dt$$

Int'te both sides: $\displaystyle\int \frac{1}{y^2}\ dy = \int k(1+t^2)dt$

$$-y^{-1} = kt + k\frac{t^3}{3} + C$$
$$y = -\frac{1}{kt + k\frac{t^3}{3} + C}$$

40. $\dfrac{dx}{dt} = \dfrac{x\ln x}{t}$

Note that $x > 0$ because $\ln(x)$ is only defined for those x values. You may further assume $t > 0$.

$$\frac{dx}{x\ln x} = \frac{1}{t}\ dt$$

Int'te both sides: $\displaystyle\int \frac{1}{x\ln x}\ dx = \int \frac{1}{t}dt$

by substitution (let $u = \ln x$) $\quad \ln(\ln x) = \ln(t) + C$

Exp'te both sides $\quad \ln x = e^{\ln t + C} = te^C$

Exponentiate again $\quad x = e^{tC_1}$

if we let $C_1 = e^C$