Unit #16 - Differential Equations

Some problems and solutions selected or adapted from Hughes-Hallett Calculus.

Growth and Decay

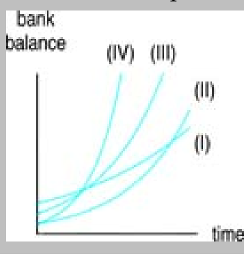
1. Each curve in in the figure below represents the balance in a bank account into which a single deposit was made at time zero. Assuming continuously compounded interest, find:

   (a) The curve representing the largest initial deposit.

   (b) The curve representing the largest interest rate.

   (c) Two curves representing the same initial deposit.

   (d) Two curves representing the same interest rate.



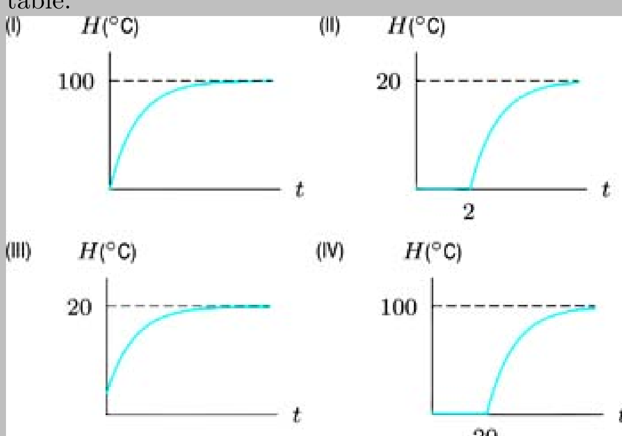
(a) (I) - Initial deposit is the bank balance/y value at time $t = 0$.
(b) (IV) - grows the most quickly/has steepest slopes for same y values
(c) (II) and (IV)
(d) (II) and (III)

2. The graphs in the figure below represent the temperature, H (°C), of four eggs as a function of time, t, in minutes. Match three of the graphs with the descriptions (a)-(c). Write a similar description for the fourth graph, including an interpretation of any intercepts and asymptotes.

   (a) An egg is taken out of the refrigerator (just above 0°C) and put into boiling water.

   (b) Twenty minutes after the egg in part (a) is taken out of the fridge and put into boiling water, the same thing is done with another egg.

   (c) An egg is taken out of the refrigerator at the same time as the egg in part (a) and left to sit on the kitchen table.



(a) (I) - starts at H = 0 at $t = 0$, starts rising towards 100 immediately.
(b) (IV) - The egg stays at 0 degrees for 20 minutes, then rises towards 100 degrees.
(c) (III) - The temperature starts at $H = 0$ at $t = 0$, but immediately begins to rise towards the room temperature, 20°.

The remaining graph, (II), is for an egg that was left in the fridge for 2 minutes, and then put out on the counter (long-term temperatures is only 20 degrees).

3. One model used in medicine is that the rate of growth of a tumor is proportional to the size of the tumor.
   (a) Write a differential equation satisfied by S, the size of the tumor, in mm, as a function of time, t.
   (b) Find the general solution to the differential equation.
   (c) If the tumor is 5 mm across at time $t = 0$, what does that tell you about the solution?
   (d) If, in addition, the tumor is 8 mm across at time $t = 3$, what does that tell you about the solution?

(a) $\dfrac{dS}{dt} = kS$.

(b) You can solve this by separation of variables, or by recognizing that the solution is exponential.
$S = Ae^{kt}$

(c) If $S(0) = 5$, that allows us to find the value of A:

$$\begin{aligned} 5 &= Ae^{k\cdot 0} \\ A &= 5 \end{aligned}$$

(d) If $S(3) = 8$, that allows us to find k:

$$\begin{aligned} 8 &= 5e^{3k} \\ k &= \frac{1}{3}\ln\left(\frac{8}{5}\right) \approx 0.1567 \end{aligned}$$

4. Hydrocodone bitartrate is used as a cough suppressant. After the drug is fully absorbed, the quantity of drug in the body decreases at a rate proportional to the amount left in the body. The half-life of hydrocodone bitartrate in the body is 3.8 hours, and the usual oral dose is 10 mg.
   (a) Write a differential equation for the quantity, Q, of hydrocodone bitartrate in the body at time t, in hours, since the drug was fully absorbed.
   (b) Solve the differential equation given in part (a)
   (c) Use the half-life to find the constant of proportionality, k.
   (d) How much of the 10 mg dose is still in the body after 12 hours?

(a) $\dfrac{dQ}{dt} = -kQ$

(b) The solution will be exponential: $Q = Q_0 e^{-kt}$

(c) If the half-life is 3.8 hours, then

$$\begin{aligned} \frac{1}{2}Q_0 &= Q_0 e^{-k\cdot 3.8} \\ k &= \frac{-1}{3.8}\ln\left(\frac{1}{2}\right) \approx 0.182 \end{aligned}$$

(d) If $Q_0 = 10$, $Q(12) = 10e^{-0.182\cdot 12} \approx 1.126$ mg. So there will be approximately 1.126 mg left in the body after 12 hours.

5. The amount of land in use for growing crops increases as the world's population increases. Suppose $A(t)$ represents the total number of hectares of land in use in year t. (A hectare is about $2\frac{1}{2}$ acres.)
   (a) Explain why it is plausible that $A(t)$ satisfies the equation $A'(t) = kA(t)$. What assumptions are you making about the world's population and its relation to the amount of land used?
   (b) In 1950 about $1 \cdot 10^9$ hectares of land were in use; in 1980 the figure was $2 \cdot 10^9$. If the total amount of land available for growing crops is thought to be $3.2 \cdot 10^9$ hectares, when does this model predict it is exhausted? (Let $t = 0$ in 1950.)

(a) If the world population is growing roughly exponentially, and it requires a fixed amount of arable land to support each person, then it makes sense that the rate of arable land must also be increasing roughly exponentially. An exponential rate of increase can be specified by the DE $A'(t) = kA(t)$.

Obviously, this is a crude approximation. First, you would need to demonstrate that the world population is in fact growing exponentially. Secondly, you would have to discount the effect of technology in both making more land available for agriculture, and its effect on increasing (or decreasing) crop yields per hectare.

(b) The solution to the DE is $A(t) = A_0 e^{kt}$.

$$\begin{aligned}
A(0) = A_0 &= 10^9 && \text{if } t = 0 \text{ represents 1950} \\
A(30) = 2 \times 10^9 &= 10^9 e^{30k} && \text{as } t = 30 \text{ represents 1980} \\
2 &= e^{30k} \\
k &= \frac{1}{30} \ln 2 \approx 0.0231 \\
\text{Solve for } t \text{ when } A(t) = 3.2 \times 10^9 \quad 3.2 \times 10^9 &= 10^9 e^{0.0231t} \\
t &= \frac{\ln(3.2)}{0.0231} \approx 50 \text{ years}
\end{aligned}$$

According to this model, we will have every arable hectare in use at $t = 50$, or in the year 2000. Hmmmm.

6. The radioactive isotope carbon-14 is present in small quantities in all life forms, and it is constantly replenished until the organism dies, after which it decays to stable carbon-12 at a rate proportional to the amount of carbon-14 present, with a half-life of 5730 years. Suppose $C(t)$ is the amount of carbon-14 present at time t.

   (a) Find the value of the constant k in the differential equation $C' = -kC$.

   (b) In 1988 three teams of scientists found that the Shroud of Turin, which was reputed to be the burial cloth of Jesus, contained 91% of the amount of carbon-14 contained in freshly made cloth of the same material. How old is the Shroud of Turin, according to these data?

(a) You can use a half-life formula. We'll solve the DE, though, for practice.

$$\begin{aligned}
C' &= -kC \\
\text{Has solution} \quad C &= C_0 e^{-kt} \\
\text{Half-life is 5730 years:} \quad \frac{C_0}{2} &= C_0 e^{-k\,5730} \\
k &= \frac{-\ln(\frac{1}{2})}{5730} \approx 1.2097 \times 10^{-4}
\end{aligned}$$

(b) In 1988, the amount of carbon is $0.91C_0$:

$$\begin{aligned}
0.91C_0 &= C_0 e^{-kt} \\
t &= \frac{\ln(0.91)}{-k} \approx 780
\end{aligned}$$

From this calculation, it seems that the cloth was made from living plant material roughly 780 years ago. Since the burial of a historic Jesus Christ would have occurred roughly 2,000 years ago, the claims of authenticity by the discoverers are very suspect.

7. Before Galileo discovered that the speed of a falling body with no air resistance is proportional to the time since it was dropped, he mistakenly conjectured that the speed was proportional to the distance it had fallen.

   (a) Assume the mistaken conjecture to be true and write an equation relating the distance fallen, $D(t)$, at time t, and its derivative.

   (b) Using your answer to part (a) and the correct initial conditions, show that D would have to be equal to 0 for all t, and therefore the conjecture must be wrong.

(a)

$$\begin{aligned} \text{Velocity} &\propto \text{Distance} \\ \frac{dD}{dt} &\propto D(t) \\ \frac{dD}{dt} &= kD \end{aligned}$$

(b) If we started at position $D(0) = 0$, this would give a derivative of $\frac{dD}{dt} = k(0) = 0$. A zero derivative indicates that the function itself is constant, so D stays at zero. Since $D = 0$ will lead to $\frac{dD}{dt} = 0$ for any value of t, we get $D(t) = 0$.
In other words, when we let go of an object, and it has not yet moved, it will not accelerate, and so never move. Hmmm....

Other Applications of Differential Equations

8. A yam is put in a 200°C oven and heats up according to the differential equation

$$\frac{dH}{dt} = -k(H - 200)$$

   (a) If the yam is at 20°C when it is put in the oven, solve the differential equation.
   (b) Find k using the fact that after 30 minutes the temperature of the yam is 120°C.

(a)

$$\begin{aligned} & & \frac{dH}{dt} &= -k(H-200) \\ \text{Separate variables:} & & \frac{dH}{H-200} &= -kdt \\ \text{Integrating both sides:} & & \ln|H-200| &= -kt + C \\ \text{Exponentiate both sides:} & & e^{\ln|H-200|} = |H-200| &= e^{-kt+C} \\ & & H - 200 &= Ae^{-kt} \quad \text{if we let } A = \pm e^C \\ & & H &= 200 + Ae^{-kt} \\ \text{If } H(0) = 20, & & 20 &= 200 + Ae^0 \\ & & A &= -180 \\ \text{so} & & H &= 200 - 180e^{-kt} \end{aligned}$$

The temperature of the yam over time is given by the formula $H = 200 - 180e^{-kt}$.

(b)

$$\begin{aligned} \text{If} \quad & H = 200 - 180e^{-kt} \\ \text{and} \quad & H(30) = 120 \\ \text{then} \quad & 120 = 200 - 180e^{-30k} \\ \text{Solve for } k: \quad & e^{-30k} = \frac{-80}{-180} = \frac{4}{9} \\ & k = \frac{1}{30}\ln\left(\frac{4}{9}\right) \approx 0.027. \end{aligned}$$

Note that this constant is correct if we measured t in minutes. The calculation would be identical except for a factor of 60 if you measured t in hours.

9. A detective finds a murder victim at 9 am. The temperature of the body is measured at 90.3°F. One hour later, the temperature of the body is 89.0°F. The temperature of the room has been maintained at a constant 68°F. (For reference, normal body temperature is 98.6°F.)

   (a) Assuming the temperature, T, of the body obeys Newton's Law of Cooling, write a differential equation for T.

   (b) Solve the differential equation to estimate the time the murder occurred.

(a) Let T represent the temperature of the body. Newton's Law of Heating and Cooling states that

$$\frac{dT}{dt} = -k(T - T_{room}) = -k(T - 68)$$

(b) Solve by separation of variables:

$$\begin{aligned}
&& \frac{dT}{T-68} &= -k\ dt \\
&\text{Integrating:} & \ln|T-68| &= -kt + C \\
&\text{Exponentiate:} & T - 68 &= Ae^{-kt} \quad \text{if } A = \pm e^C \\
&& T &= Ae^{-kt} + 68 \\
&\text{Let 9 AM be } t=0\text{:} & 90.3 &= A + 68 \\
&& A &= 22.3 \\
&\text{Let 10 AM be } t=1\text{:} & 89.0 &= 22.3e^{-k} + 68 \\
&& k &= 0.06006 \\
&\text{Solve for time when } T \text{ was } 98.6 = \text{normal body temp:} && \\
&& 98.6 &= 22.3e^{-0.06006t} + 68 \\
&& t &= -5.27
\end{aligned}$$

It looks like the time of death was roughly 5 hours before 9 AM, or just before 4 AM.

10. At 1:00 pm one winter afternoon, there is a power failure at your house in Wisconsin, and your heat does not work without electricity. When the power goes out, it is 68°F in your house. At 10:00 pm, it is 57°F in the house, and you notice that it is 10°F outside.

    (a) Assuming that the temperature, T, in your home obeys Newton's Law of Cooling, write the differential equation satisfied by T.

    (b) Solve the differential equation to estimate the temperature in the house when you get up at 7:00 am the next morning. Should you worry about your water pipes freezing?

    (c) What assumption did you make in part (a) about the temperature outside? Given this (probably incorrect) assumption, would you revise your estimate up or down? Why?

(a) $\dfrac{dT}{dt} = -k(T - T_{ext})$, where T is the temperature in the house, and T_{ext} is the temperature outside the house.

(b) If we assume that the exterior temperature is 10 degrees during the whole power outage, we can solve the DE to get the temperature over time.

We can start with the differential equation $\frac{dT}{dt} = -k(T-10)$, which leads after separation of variables to:

$$\begin{aligned}
& T = Ae^{-kt} + 10 \\
\text{Let 1 PM be } t=0: \quad & 68 = A + 10 \\
& A = 58 \\
\text{At 10 PM, } t=9: \quad & 57 = 58e^{-9k} + 10 \\
& k = \frac{1}{9}\ln\left(\frac{58}{47}\right) \approx 0.023366 \\
\text{At 7 AM, } t=18: \quad & T = 58e^{-18k} + 10 \\
& = 48.06
\end{aligned}$$

At 7 AM, the temperature will have dropped only down to 48 degrees, or well above freezing (32 degrees Fahrenheit). There is no risk of the pipes freezing by morning.

(c) We assumed that the temperature outside the house would always be 10 degrees, which is a gross over-simplification. Since we would expect the outside temperature to drop between 10 PM and 7 AM, that would mean the house cools more during that period than our model predicts, so the temperature at 7 AM would likely be lower than 48 degrees.

11. At time $t = 0$, a bottle of juice at 90°F is stood in a mountain stream whose temperature is 50°F. After 5 minutes, its temperature is 80°F. Let $H(t)$ denote the temperature of the juice at time t, in minutes.

(a) Write a differential equation for $H(t)$ using Newton's Law of Cooling.

(b) Solve the differential equation.

(c) When will the temperature of the juice have dropped to 60°F?

(a) $\frac{dH}{dt} = -k(H-50)$

(b)

$$\begin{aligned}
\int \frac{dH}{H-50} &= \int -k\,dt \\
\ln|H-50| &= -kt + C \\
H &= Ae^{-kt} + 50 \qquad \text{if we define } A = \pm e^C
\end{aligned}$$

(c)

$$\begin{aligned}
H(0) = 90 \quad &\rightarrow \quad A = 40 \\
H(5) = 80 \quad &\rightarrow \quad 30 = 40e^{-k5} \\
& \qquad k = \frac{-1}{5}\ln(3/4) \approx 0.05754 \\
\text{Find } t \text{ when } H = 60 \quad & \quad 60 = 40e^{-kt} + 50 \\
& \qquad t = \frac{\ln(10/40)}{-k} \approx 24\text{minutes}
\end{aligned}$$

It will take around 24 minutes for the drink to cool down to 60 degrees F.

12. Water leaks out of a barrel at a rate proportional to the square root of the depth of the water at that time. If the water level starts at 36 inches and drops to 35 inches in 1 hour, how long will it take for all of the water to leak out of the barrel?

Let $y(t)$ be the depth of water in the barrel at time t. If the barrel is roughly cylindrical, the rate of water flow out will be proportional to the rate of change of the water depth:

$$\frac{dy}{dt} = -k\sqrt{y}$$

$$\text{Separating and integrating:} \qquad \int y^{-1/2}\ dy = \int -k\ dt$$

$$2\sqrt{y} = -kt + C$$

$$\text{At } t = 0,\ y = 36: \qquad 2\sqrt{36} = 0 + C$$

$$C = 12$$

$$y = \frac{1}{2^2}(-kt+12)^2 = \frac{(12-kt)^2}{4}$$

$$\text{At } t = 1,\ y = 35: \qquad 35 = \frac{(12-k(1))^2}{4}$$

$$140 = (12-k)^2$$

$$\sqrt{140} = 12 - k$$

$$k = 12 - \sqrt{140} \approx 0.168$$

$$\text{So our formula for } y \text{ is} \qquad y \approx \frac{(12-(0.168)t)^2}{4}$$

$$\text{This will give } y = 0 \text{ at } t \approx \frac{12}{0.168} \approx 71.4 \text{ hours}$$

It will take around 71 hours, or three days, for the barrel to empty out due to the leak.

13. According to a simple physiological model, an athletic adult needs 20 calories per day per pound of body weight to maintain his weight. If he consumes more or fewer calories than those required to maintain his weight, his weight changes at a rate proportional to the difference between the number of calories consumed and the number needed to maintain his current weight; the constant of proportionality is 1/3500 pounds per calorie. Suppose that a particular person has a constant caloric intake of I calories per day. Let $W(t)$ be the person's weight in pounds at time t (measured in days).

    (a) What differential equation has solution $W(t)$?
    (b) Solve this differential equation.
    (c) Graph $W(t)$ if the person starts out weighing 160 pounds and consumes 3000 calories a day.

(a) Since the rate of change of weight is equal to

$$\text{Rate of weight change} = \frac{1}{3500}(\text{Intake - Amount to maintain weight})$$

$$\text{so the DE is} \qquad \frac{dW}{dt} = \frac{1}{3500}(I - 20W)$$

(b)

$$\int \frac{1}{I-20W} dW = \int \frac{1}{3500}\ dt$$

$$\frac{-1}{20}\ln|I-20W| = \frac{1}{3500}t + C$$

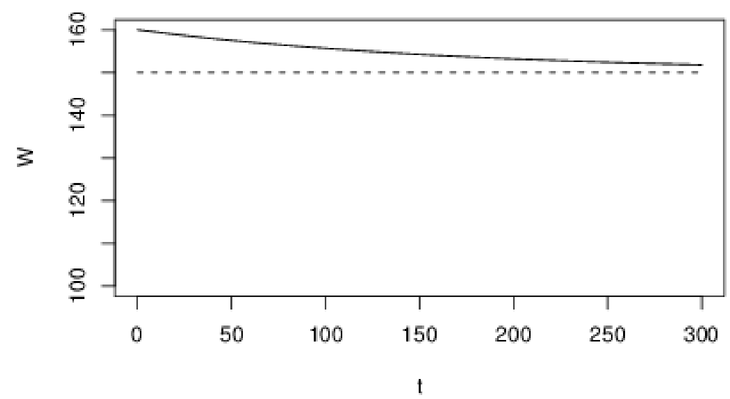
$$\ln|I-20W| = \frac{-2}{350}t + C_2 \quad \text{if } C_2 = -20C$$

$$|I-20W| = e^{\frac{-2}{350}t + C_2} = e^{\frac{-2}{350}t}e^{C_2}$$

$$I - 20W = Ae^{\frac{-2}{350}t} \quad \text{if } A = \pm e^{C_2}$$

$$W = \frac{1}{20}\left(I - Ae^{\frac{-2}{350}t}\right)$$

(c) Using part (b), we have $W = 150 + 10e^{-\frac{2}{350}t}$. This means that $W \to 150$ as $t \to \infty$.



14. Water leaks from a vertical cylindrical tank through a small hole in its base at a rate proportional to the square root of the volume of water remaining. If the tank initially contains 200 liters and 20 liters leak out during the first day, when will the tank be half empty? How much water will there be after 4 days?

$$\frac{dV}{dt} = -k\sqrt{V}$$

Separating and solving leads to
$$\int V^{-1/2}\, dV = \int -k\, dt$$
$$2\sqrt{V} = -kt + C$$
$$\sqrt{V} = -\frac{k}{2}t + C_2 \qquad \left(\text{let } C_2 = \frac{C}{2}\right)$$
$$V = \left(C_2 - \frac{kt}{2}\right)^2$$

Use $V(0) = 200$: $\quad 200 = (C_2 - 0)^2$

So $\quad C_2 = \sqrt{200}$

first day leak of 20 liters implies $\quad V(1) = 200 - 20 = 180$

So $$180 = \left(\sqrt{200} - \frac{k(1)}{2}\right)^2$$
$$\frac{k}{2} = \sqrt{200} - \sqrt{180} \approx 0.726$$
$$\text{so } k \approx 1.452$$

Tank half empty is V = 100: $$100 = \left(\sqrt{200} - \underbrace{(0.726)}_{k/2} t\right)^2$$
$$10 = \sqrt{200} - 0.726t$$
$$t = \frac{\sqrt{200} - 10}{0.726} \approx 5.7 \text{ days to half-full}$$

After 4 days: $$V = \left(\sqrt{200} - (0.726)(4)\right)^2$$
$$\approx 126 \text{ liters}$$

15. As you know, when a course ends, students start to forget the material they have learned. One model (called the Ebbinghaus model) assumes that the rate at which a student forgets material is proportional to the difference between the material currently remembered and some positive constant, a.

    (a) Let $y = f(t)$ be the fraction of the original material remembered t weeks after the course has ended. Set up a differential equation for y. Your equation will contain two constants; the constant a is less than y for all t.

    (b) Solve the differential equation.

    (c) Describe the practical meaning (in terms of the amount remembered) of the constants in the solution $y = f(t)$.

(a) $\dfrac{dy}{dt} = -k(y-a)$

(b)

$$\begin{aligned} \int \frac{1}{y-a}\,dy &= \int -k\,dt \\ \ln|y-a| &= -kt + C \\ |y-a| &= e^{-kt+C} = e^{-kt}e^{C} \\ y-a &= Ae^{-kt} \quad \text{if } A = \pm e^{C} \\ y &= Ae^{-kt} + a \end{aligned}$$

(c) At $t \to \infty$, $y \to a$. Therefore, a represents the final fraction or proportion of the course that sticks with you in the long term. The constant k is related to how quickly you forget all the material beyond a (larger k, faster forgetting).

16. When people smoke, carbon monoxide is released into the air. In a room of volume 60 m^3, air containing 5% carbon monoxide is introduced at a rate of 0.002 m^3/min. (This means that 5% of the volume of the incoming air is carbon monoxide.) The carbon monoxide mixes immediately with the rest of the air and the mixture leaves the room at the same rate as it enters.

    (a) Write a differential equation for $c(t)$, the concentration of carbon monoxide at time t, in minutes.

    (b) Solve the differential equation, assuming there is no carbon monoxide in the room initially.

    (c) What happens to the value of $c(t)$ in the long run?

Questions related to concentration are most easily started by looking at how the underlying *amount* of contaminant moves around. In this case, this means looking at the actual **volume** of CO in the air, relative to the total amount of air. Once we get a DE for the rate of change of CO volume, we can divide by the (constant) volume of the room, to get a DE in terms of the rate of change of concentration. See the Compartmental Analysis section in Section 11.6 for a similar example.

(a) Let $c(t)$ be the fraction (or percentage) of carbon dioxide in the air. If $Q(t)$ represents the quantity (volume, in m^3) of carbon monoxide in the room at time t,

$$c(t) = \frac{Q(t)}{\text{Room volume}} = \frac{Q(t)}{60}$$

$$\begin{aligned} \text{Rate of change of Q} &= (\text{rate in of CO}) \;-\; (\text{rate out of CO}) \\ \text{Rate in} &= 5\%(0.002\text{m}^3/\text{min}) \\ \text{Using unit analysis,}\quad \text{Rate out} &= \text{air flow rate } (\text{m}^3/\text{min}) \times (\text{percentage CO}) \\ &= (0.0002)\frac{Q(t)}{60} \\ \text{So}\qquad \frac{dQ}{dt} &= (0.05)(0.002) - (0.002)\frac{Q(t)}{60} \end{aligned}$$

Since $c(t) = Q(t)/60$, or $Q = 60c$, we can change our variable to $c(t)$:

$$\begin{aligned} \frac{d(60c)}{dt} &= (0.05)(0.002) - (0.002)c \\ \frac{dc}{dt} &= \frac{0.002}{60}(0.05 - c) \end{aligned}$$

(b) Solving the DE, and using the initial condition of no CO at $t = 0$ ($c(0) = 0$):

$$\begin{aligned}
\int \frac{1}{0.05 - c} dc &= \frac{0.002}{60} dt \\
-\ln|0.05 - c| &= \frac{0.002}{60} t + D \\
\ln|0.05 - c| &= \frac{-0.002}{60} t - D \\
|0.05 - c| &= e^{\frac{-0.002}{60} t - D} = e^{\frac{-0.002}{60} t} e^{-D} \\
\text{Solving for } c\text{: } (0.05 - c) &= Ae^{-\frac{0.002}{60} t} \quad \text{if } A = \pm e^{-D} \\
c(0) = 0: \quad 0.05 &= A \\
\text{so} \quad 0.05 - c &= 0.05e^{-\frac{0.002}{60} t} \\
\text{and finally} \quad c &= 0.05 - 0.05e^{-\frac{0.002}{60} t}
\end{aligned}$$

(c) As $t \to \infty$, $e^{-\frac{0.002}{60} t} \to 0$, so $c \to 0.05 + 0 = 0.05$. This means that the CO concentration in the room eventually reaches the concentration of CO in the incoming air, despite starting off clean at $t = 0$.

17. (Continuation of previous problem.) Medical texts warn that exposure to air containing 0.02% carbon monoxide for some time can lead to a coma. How long does it take for the concentration of carbon monoxide in the room in Problem 16 to reach this level?

We found in the earlier question that

$$c = 0.05 - 0.05e^{-\frac{0.002}{60} t}$$

Remembering that 0.02% means a fraction of 0.0002, we set $c = 0.0002$ and solve for t:

$$\begin{aligned}
0.0002 &= 0.05 - 0.05e^{-\frac{0.002}{60} t} \\
\frac{0.0002 - 0.05}{-0.05} &= e^{-\frac{0.002}{60} t} \\
t &= \frac{-60}{0.002} \ln\left(\frac{0.0498}{0.05}\right) \approx 120 \text{ minutes}
\end{aligned}$$

It will take roughly 120 minutes, or 2 hours, for the concentration in the room to reach dangerous levels.

18. An aquarium pool has volume 2×10^6 liters. The pool initially contains pure fresh water. At $t = 0$ minutes, water containing 10 grams/liter of salt is poured into the pool at a rate of 60 liters/minute. The salt water instantly mixes with the fresh water, and the excess mixture is drained out of the pool at the same rate (60 liters/minute).

    (a) Write a differential equation for $S(t)$, the mass of salt in the pool at time t.
    (b) Solve the differential equation to find $S(t)$.
    (c) What happens to $S(t)$ as $t \to \infty$?

(a)

$$\begin{aligned}
\text{Rate of change of salt amount (g/min)} &= \text{ Rate in } - \text{ Rate out} \\
\text{Rate in (g/min)} &= \text{Flow rate} \times \text{Concentration} \\
&= (60 \text{ liters/min}) \times (10 \text{ g/liter}) = 600 \text{ g/min} \\
\text{Rate out (g/min)} &= \text{Flow rate} \times \text{Concentration} \\
&= \text{Flow rate} \times \text{amount (g) / Pool volume (liters)} \\
&= (60 \text{ liters/min})(S(t) \text{ grams})/(2 \times 10^6 \text{ liters}) \\
&= (3 \times 10^{-5})S(t) \\
\text{Finally, we get our DE:} \quad \frac{dS}{dt} &= 600 - (3 \times 10^{-5})S
\end{aligned}$$

(b) The DE will be easier to solve if we factor out the constant in front of S:

$$\frac{dS}{dt} = (3 \times 10^{-5})(2 \times 10^7 - S)$$

Separating and integrating: $$\int \frac{1}{2 \times 10^7 - S} dS = (3 \times 10^{-5}) dt$$

$$-\ln|2 \times 10^7 - S| = (3 \times 10^{-5})t + C$$

$$\ln|2 \times 10^7 - S| = (-3 \times 10^{-5})t - C$$

$$|2 \times 10^7 - S| = e^{(-3\times 10^{-5})t - C}$$

Solve for S: $$2 \times 10^7 - S = Ae^{(-3\times 10^{-5})t} \qquad \text{if } A = \pm e^{-C}$$

$$S = 2 \times 10^7 - Ae^{(-3\times 10^{-5})t}$$

Initially, the pool starts off with pure fresh water, so $S(0) = 0$:

$$0 = 2 \times 10^7 - Ae^0$$

$$A = 2 \times 10^7$$

So $$S = 2 \times 10^7 (1 - e^{(-3\times 10^{-5})t})$$

(c) As $t \to \infty$, $e^{(-3\times 10^{-5})t} \to 0$, so $S \to 2 \times 10^7$ grams. In other words, the salt concentration tends towards 2×10^7 grams/2×10^6 liters $= 10$ grams/liter, the same as the incoming water.

The Logistic Model

19. Table 1 below gives the percentage, P, of households with a DVD player, as a function of year.

Table 1: Percentage of households with a DVD player

Year	1998	1999	2000	2001	2002
P (%)	1	5	13	21	35
Year	2003	2004	2005	2006	
P (%)	50	70	75	81	

(a) Explain why a logistic model is a reasonable one to use for this data.

(b) Use the data to estimate the point of inflection of P. What limiting value L does this point of inflection predict?

(c) The best logistic function for this data turns out to be the one shown below.

$$P = \frac{86.395}{1 + 316.75e^{-0.699t}}$$

What limiting value does this model predict?

(a) We see that the growth follows the pattern we have seen many times before: slow growth when P is low, followed by a rapid growth (around 2002-2004), followed by slower growth (2005-2006).

(b) The inflection point occurs when the rate of change of usage level changes from increasing to a decreasing rate.

i. 2001-2002: 14% increase
ii. 2002-2003: 15% increase
iii. 2003-2004: 20% increase
iv. 2004-2005: 5% increase

The fastest growth occurs between 2003-2004, and then the growth rate declines, so I would use that interval as roughly the maximum slope point. Let's say 2003 was when that occurred (though 2004 would also be a reasonable answer).

If the maximum growth rate occurred in 2003, and the percentage of households with DVD players was 50% then, using our logistic model the maximum number of households with DVD players in the long run will be $2\times 50\%$ or 100% (or everyone: complete market penetration).

(c) As $t \to \infty$, $e^{-0.786t} = 0$ so $P \to \dfrac{86.395}{1+0} = 86.395$. From this more quantitative solution (rather than our *estimates* from part (b)), we expect over time the level of DVD-owning households to level off around 86%.

Our estimate in (b) was crude because of the very rapid DVD player adoption and the fact that we only had data measured every year. If we had had data on a monthly basis, we likely could have come much closer to the real inflection point and so a much more accurate estimate.

20. The growth of an animal population is governed by the equation

$$\frac{1000}{P}\frac{dP}{dt} = 100 - P$$

where $P(t)$ is the number of individuals in the colony at time t. The initial population is known to be 200 individuals. Sketch a graph of $P(t)$. Will there ever be more than 200 individuals in the colony? Will there ever be fewer than 100 individuals? Explain.

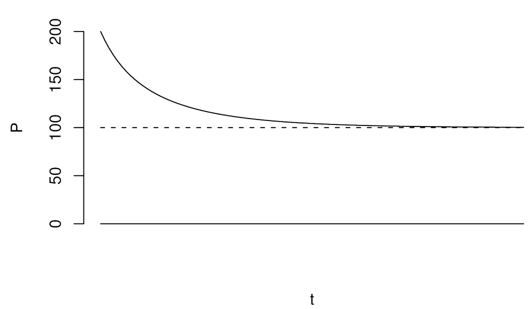
Let's frame the DE in our more standard form for logistic models:

$$\begin{aligned}\frac{dP}{dt} &= \frac{1}{1000}P(100-P)\\ &= kP(L-P)\end{aligned}$$

where k is related to the rate of growth, and $L = 100$ is the limiting population.

If the initial population is 200, then $\frac{dP}{dt}$ will initially be $\frac{dP}{dt} = \frac{1}{1000}(200)(100-200) < 0$, so the population is initially declining. This negative rate will be maintained until the population reaches the limiting population of 100.

There will never be fewer than 100 individuals, because once the population reaches 100, the derivative becomes 0, so the population becomes constant.



21. A model for the population, P, of carp in a landlocked lake at time t is given by the differential equation

$$\frac{dP}{dt} = 0.25P(1 - 0.0004P).$$

   (a) What is the long-term equilibrium population of carp in the lake?

   (b) Under a plan to join the lake to a nearby river, the fish will be able to leave the lake. A net loss of 10% of the carp each year is predicted, but the patterns of birth and death are not expected to change. Revise the differential equation to take this into account. Use the revised differential equation to predict the future development of the carp population.

(a) The equilibrium population occurs when $\frac{dP}{dt} = 0$, which means

$$0 = 0.25P(1 - 0.0004P)$$
$$(P = 0) \text{ or } 1 - 0.0004P = 0 \Rightarrow (P = 2500)$$

Since this model is logistic in form, the $P = 0$ equilibrium will only be reached if we start at zero population; in the long run the population of carp will tend towards the equilibrium at $P = 2500$.

(b) The effect of losing 10% of the fish each year means there is a negative rate of change of $0.1 \times P$ added to the earlier growth rate information. The DE for this new model is therefore

$$\frac{dP}{dt} = \underbrace{0.25P(1 - 0.0004P)}_{\text{original model}} \quad \underbrace{-0.1P}_{\text{10\% loss term}}$$

We search for the equilibrium of this new model, setting $\frac{dP}{dt} = 0$ again:

$$0 = 0.25P(1 - 0.0004P) - 0.1P$$
$$0 = 0.25P - 0.0001P^2 - 0.1P$$
$$0 = P(0.15 - 0.0001P)$$

So the equilibria are at $P = 0$ again (no fish) or at $P = 0.15/0.0001 = 1500$ fish. The regular loss of 10% of the fish will result in a lower equilibrium than with the original model.

22. The population of a species of elk on Reading Island in Canada has been monitored for some years. When the population was 600, the relative birth rate was 35% and the relative death rate was 15%. As the population grew to 800, the corresponding figures were 30% and 20%. The island is isolated so there is no hunting or migration.

   (a) Write a differential equation to model the population as a function of time. Assume that relative growth rate is a linear function of population.

   (b) Find the equilibrium size of the population. Today there are 900 elk on Reading Island. How do you expect the population to change in the future?

   (c) Oil has been discovered on a neighboring island and the oil companies want to move 450 elk of the same species to Reading Island. What effect would this move have on the elk population on Reading Island in the future?

   (d) Assuming the elk are moved to Reading Island, sketch the population on Reading Island as a function of time. Start before the elk are transferred and continue for some time into the future. Comment on the significance of your results.

(a) Let P be the population of elk over time. Initially, when $P = 600$ the net population rate of change was (birth rate) - (death rate) = 35 - 15 = 20%.
When P reaches 800, the net population rate of change was (birth rate) - (death rate) = 30 - 20 = 10%.
If the rate of change of population is linear in P, then we have two points: $R(600) = 0.2$ and $R(800) = 0.1$. If $R(P) = mP + b$ then we find the slope and intercepts and finally $R(P) = \frac{-1}{2000}P + 0.5$

The differential equation that describes the population is then

$$\frac{dP}{dt} = R(t)P = (0.5 - \frac{1}{2000}P)P = \frac{1}{2000}P(1000 - P)$$

This is a logistic growth model.

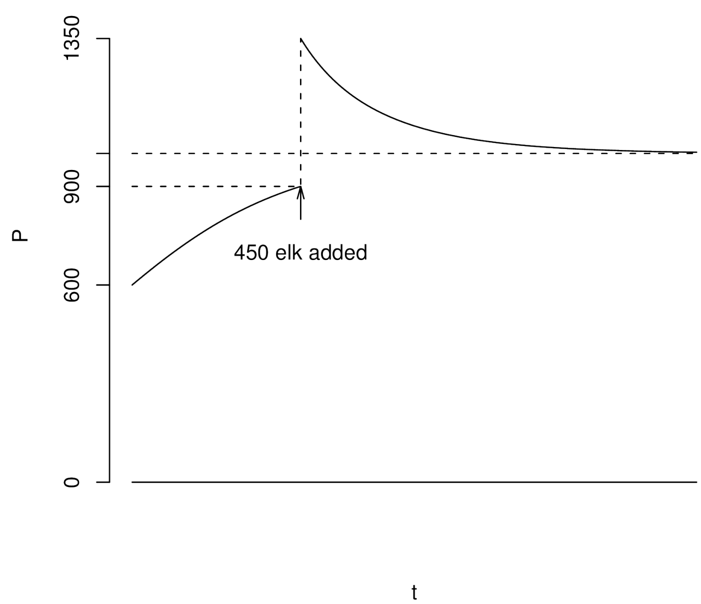
(b) The equilibrium population level will be one for which $\frac{dP}{dt} = 0$: subbing that into the DE gives $P = 0$ or $(1000 - P) = 0 \Rightarrow P = 1000$.

If the current population is 900 elk, we will have a positive rate of growth ($P' > 0$), so the population will keep growing until it approaches 1000.

(c) If 450 elk were added to the 900 elk to make 1350, the population would be greater than the limiting population of 1000, so we would expect the herd size to decline over time due to lack of resources.

More mathematically, if $P = 1350$, $P' = \frac{1}{2000}(1350)(1000 - 1350)$ would be negative, meaning that P is decreasing.

(d) The population grew from 600 (one of our data points) up to 900 elk, at which point the 450 new elk were added.



23. Consider the equation

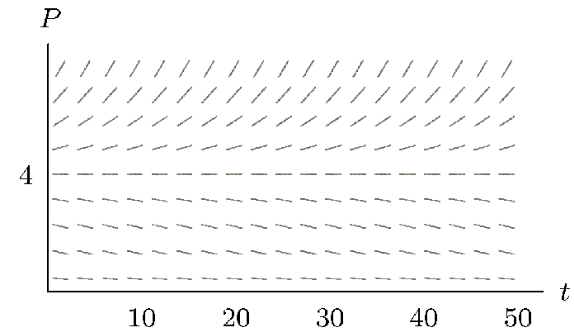
$$\frac{dP}{dt} = 0.02P^2 - 0.08P.$$

(a) Sketch the slope field for this differential equation for $0 \le t \le 50$, $0 \le P \le 8$.

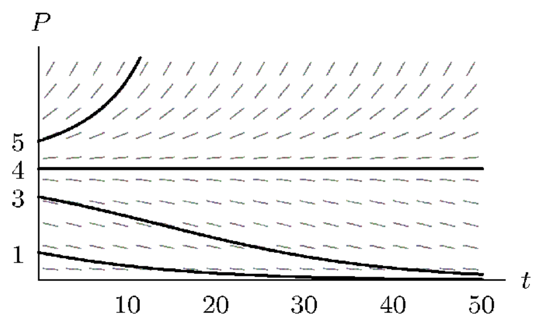
(b) Use your slope field to sketch the general shape of the solutions to the differential equation satisfying the following initial conditions:
(i) $P(0) = 1$, (ii) $P(0) = 3$ (iii) $P(0) = 4$ (iv) $P(0) = 5$

(c) Are there any equilibrium values of the population? If so, are they stable?

(a) The best way to sketch this slope field is to first find the equilibrium values , and then determine the sign of the derivative for other P values. By setting $\frac{dP}{dt} = 0$, solving for P gives $P = 0$ and $P = 4$ as equilibria.



(b)



(c) There are two equilibria: $P = 0$ and $P = 4$. Only the equilibrium at $P = 0$ is stable, because populations that start near $P = 0$ converge towards it.

The equilibrium at $P = 4$ is *not* stable, because populations that start nearby (e.g $P = 3$, or $P = 5$) are pushed away from $P = 4$ over time.

24. Consider the equation

$$\frac{dP}{dt} = P^2 - 6P$$

(a) Sketch a graph of $\frac{dP}{dt}$ against P for positive P.

(b) Use the graph you drew in part (a) to sketch the approximate shape of the solution curve with $P(0) = 5$. To do this, consider the following question. For $0 < P < 6$, is $\frac{dP}{dt}$ positive or negative? What does this tell you about the graph of P against t? As you move along the solution curve with $P(0) = 5$, how does the value of $\frac{dP}{dt}$ change? What does this tell you about the concavity of the graph of P against t?

(c) Use the graph you drew in part (a) to sketch the solution curve with $P(0) = 8$.

(d) Describe the qualitative differences in the behavior of populations with initial value less than 6 and initial value more than 6. Why do you think $P = 6$ is called the threshold population?

(a) See Figure 11.45.

(b) Figure 11.45 shows that for $0 < P < 6$, the sign of dP/dt is negative. This means that P is decreasing over the interval $0 < P < 6$. As P decreases from $P(0) = 5$, the value of dP/dt gets more and more negative until $P = 3$. Thus the graph of P against t is concave down while P is decreasing from 5 to 3. As P decreases below 3, the slope of dP/dt increases toward 0, so the graph of P against t is concave up and asymptotic to the t-axis. At $P = 3$, there is an inflection point. See Figure 11.46.

(c) Figure 11.45 shows that for $P > 6$, the slope of dP/dt is positive and increases with P. Thus the graph of P against t is increasing and concave up. See Figure 11.46.

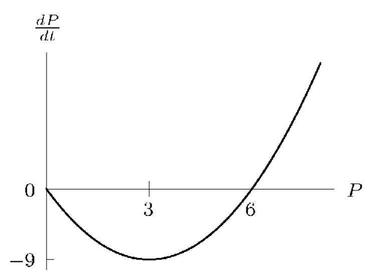


Figure 11.45

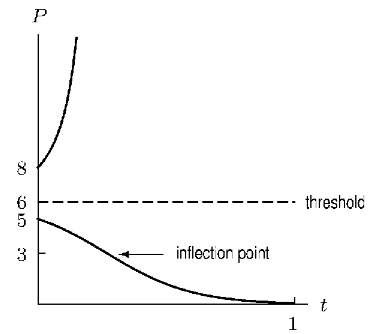


Figure 11.46

(d) For initial populations greater than the threshold value $P = 6$, the population increases without bound. Populations with initial value less than $P = 6$ decrease asymptotically toward 0, i.e. become extinct. Thus the initial population $P = 6$ is the dividing line, or threshold, between populations which grow without bound and those which die out.

25. Consider a population satisfying

$$\frac{dP}{dt} = aP^2 - bP$$

(a) Sketch a graph of $\dfrac{dP}{dt}$ against P.

(b) Use this graph to sketch the shape of solution curves with various initial values. Use your graph from part (a) to decide where $\dfrac{dP}{dt}$ is positive or negative, and where it is increasing or decreasing. What does this tell you about the graph of P against t?

(c) Why is $P = b/a$ called the threshold population? What happens if $P(0) = b/a$? What happens in the long-run if $P(0) > b/a$? What if $P(0) < b/a$?

(a) Don't be thrown off by the rate they ask you to plot: they just want a graph of y versus P, where $y = aP^2 - bP = P(aP - b)$. This function is quadratic, and has roots at $P = 0$ and $P = b/a$. We will only sketch this for $P \geq 0$, since P represents a population.

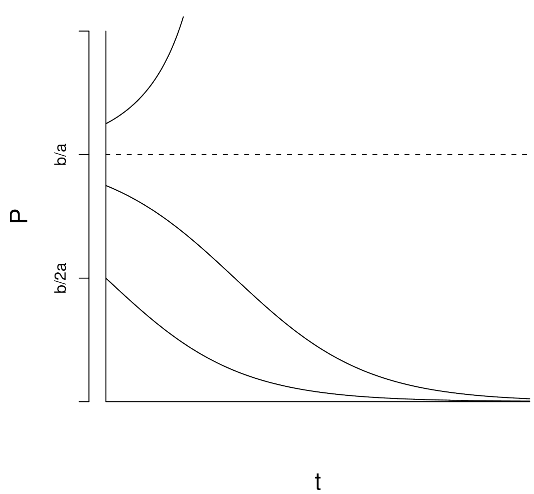
(b) The information in (a) tells us when $\dfrac{dP}{dt}$ is positive and negative, which tells us our population is increasing or decreasing.

i. $P > b/a$ means $dP/dt > 0$ or P is increasing.

ii. $0 < P < b/a$ means $dP/dt < 0$ or P is decreasing.

iii. When $P < \frac{1}{2}b/a$, dP/dt is most negative, or the population is decreasing most quickly.

This sounds a lot like the logistic model, where we get maximum slope at half of some special population, but the rate here is negative. We can try to sketch out these ideas in a graph of P version t:



(c) As we saw with our analysis of $\dfrac{dP}{dt}$, if the population begins below the threshold of b/a, the population will die off towards zero. On the other hand, if the population begins above this threshold, it will continue to grow unboundedly.