Unit 2 Planning the Unit

In this unit, students write the equations of quadratic functions to model situations and then graph these functions. They study methods of finding solutions to quadratic equations and interpreting these solutions. In the process, students learn about complex numbers.

Vocabulary Development

The key terms for this unit can be found on the Unit Opener page. These terms are divided into Academic Vocabulary and Math Terms. Academic Vocabulary includes terms that have additional meaning outside of math. These terms are listed separately to help students transition from their current understanding of a term to its meaning as a mathematics term. To help students learn new vocabulary:

- Have students discuss meaning and use graphic organizers to record their understanding of new words.
- Remind students to place their graphic organizers in their math notebooks and revisit their notes as their understanding of vocabulary grows.
- As needed, pronounce new words and place pronunciation guides and definitions on the class Word Wall.

Embedded Assessments

Embedded Assessments allow students to do the following:

- Demonstrate their understanding of new concepts.
- Integrate previous and new knowledge by solving real-world problems presented in new settings.

They also provide formative information to help you adjust instruction to meet your students' learning needs.

Prior to beginning instruction, have students unpack the first Embedded Assessment in the unit to identify the skills and knowledge necessary for successful completion of that assessment. Help students create a visual display of the unpacked assessment and post it in your class. As students learn new knowledge and skills, remind them that they will be expected to apply that knowledge to the assessment. After students complete each Embedded Assessment, turn to the next one in the unit and repeat the process of unpacking that assessment with students.

CollegeBoard

AP/College Readiness

Unit 2 continues to prepare students for advanced studies in mathematics by:

- Modeling real-world situations using a quadratic function and interpreting the key features of their graphs in context.
- Learning methods for finding the solutions of quadratic equations.
- Extending their knowledge of number systems to the complex numbers.

Unpacking the Embedded Assessments

The following are the key skills and knowledge students will need to know for each assessment.

Embedded Assessment 1

Applications of Quadratic Functions and Equations, *No Horsing Around*

- Quadratic functions
- Quadratic equations
- Discriminants
- Complex numbers

Embedded Assessment 2

Writing and Transforming Quadratic Functions, *The Safari Experience*

- Standard form of a parabola
- Vertex form of a parabola
- Transformation
- Directrix, focus, and axis of symmetry

Embedded Assessment 3

Graphing Quadratic Functions and Solving Systems, *The Green Monster*

- Graph of a parabola
- Maximum of a parabola
- Domain and range of quadratic functions
- System of equations with a linear equation and a quadratic equation

Suggested Pacing

The following table provides suggestions for pacing using a 45-minute class period. Space is left for you to write your own pacing guidelines based on your experiences in using the materials.

	45-Minute Period	Your Comments on Pacing
Unit Overview/Getting Ready	1	
Activity 7	4	
Activity 8	3	
Activity 9	3	
Embedded Assessment 1	1	
Activity 10	3	
Activity 11	3	
Embedded Assessment 2	1	
Activity 12	5	
Activity 13	2	
Embedded Assessment 3	1	
Total 45-Minute Periods	**27**	

Additional Resources

Additional resources that you may find helpful for your instruction include the following, which may be found in the Teacher Resources at SpringBoard Digital.

- Unit Practice (additional problems for each activity)
- Getting Ready Practice (additional lessons and practice problems for the prerequisite skills)
- Mini-Lessons (instructional support for concepts related to lesson content)

Quadratic Functions

2

Unit Overview

This unit focuses on quadratic functions and equations. You will write the equations of quadratic functions to model situations. You will also graph quadratic functions and other parabolas and interpret key features of the graphs. In addition, you will study methods of finding solutions of quadratic equations and interpreting the meaning of the solutions. You will also extend your knowledge of number systems to the complex numbers.

Key Terms

As you study this unit, add these and other terms to your math notebook. Include in your notes your prior knowledge of each word, as well as your experiences in using the word in different mathematical examples. If needed, ask for help in pronouncing new words and add information on pronunciation to your math notebook. It is important that you learn new terms and use them correctly in your class discussions and in your problem solutions.

Academic Vocabulary

- justify
- derive
- verify
- advantage
- disadvantage
- counterexample

Math Terms

- quadratic equation
- standard form of a quadratic equation
- imaginary number
- complex number
- complex conjugate
- completing the square
- discriminant
- root
- zero
- parabola
- focus
- directrix
- axis of symmetry
- vertex
- quadratic regression
- vertex form

ESSENTIAL QUESTIONS

How can you determine key attributes of a quadratic function from an equation or graph?

How do graphic, symbolic, and numeric methods of solving quadratic equations compare to one another?

EMBEDDED ASSESSMENTS

This unit has three embedded assessments, following Activities 9, 11, and 13. By completing these embedded assessments, you will demonstrate your understanding of key features of quadratic functions and parabolas, solutions to quadratic equations, and systems that include nonlinear equations.

Unit Overview

Ask students to read the unit overview and mark the text to identify key phrases that indicate what they will learn in this unit.

Key Terms

As students encounter new terms in this unit, help them to choose an appropriate graphic organizer for their word study. As they complete a graphic organizer, have them place it in their math notebooks and revisit as needed as they gain additional knowledge about each word or concept.

Essential Questions

Read the essential questions with students and ask them to share possible answers. As students complete the unit, revisit the essential questions to help them adjust their initial answers as needed.

Unpacking Embedded Assessments

Prior to beginning the first activity in this unit, turn to Embedded Assessment 1 and have students unpack the assessment by identifying the skills and knowledge they will need to complete the assessment successfully. Guide students through a close reading of the assessment, and use a graphic organizer or other means to capture their identification of the skills and knowledge. Repeat the process for each Embedded Assessment in the unit.

Developing Math Language

As this unit progresses, help students make the transition from general words they may already know (the Academic Vocabulary) to the meanings of those words in mathematics. You may want students to work in pairs or small groups to facilitate discussion and to build confidence and fluency as they internalize new language. Ask students to discuss new academic and mathematics terms as they are introduced, identifying meaning as well as pronunciation and common usage. Remind students to use their math notebooks to record their understanding of new terms and concepts.

As needed, pronounce new terms clearly and monitor students' use of words in their discussions to ensure that they are using terms correctly. Encourage students to practice fluency with new words as they gain greater understanding of mathematical and other terms.

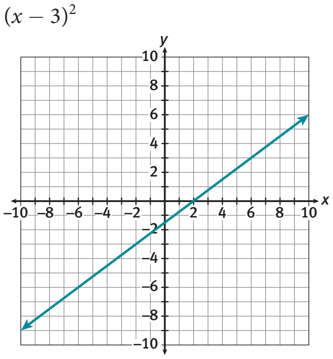
Use some or all of these exercises for formative evaluation of students' readiness for Unit 2 topics.

Prerequisite Skills

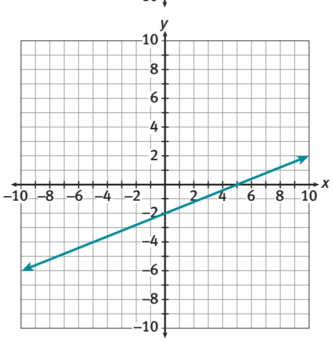
- Factoring polynomials (Items 1–4) HSA-SSE.B.3
- Graphing functions (Items 5–7) HSF-BF.B.3
- Solving quadratic equations (Item 8) HSA-REI.B.4

Answer Key

1. $6x^2y(x + 2y)$
2. $(x - 5)(x + 8)$
3. $(x - 7)(x + 7)$
4. $(x - 3)^2$
5.



6.



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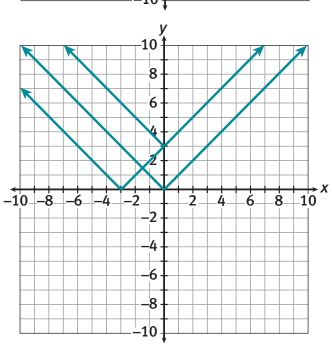


8. $x = \frac{3 \pm \sqrt{29}}{2}$

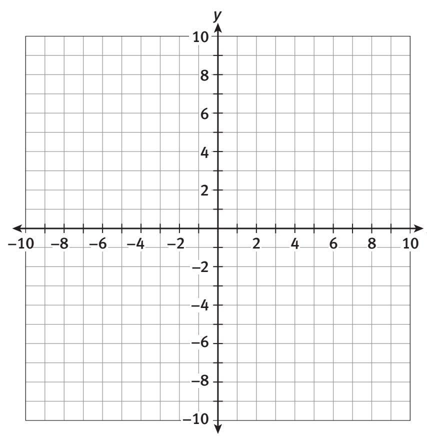
UNIT 2

Getting Ready

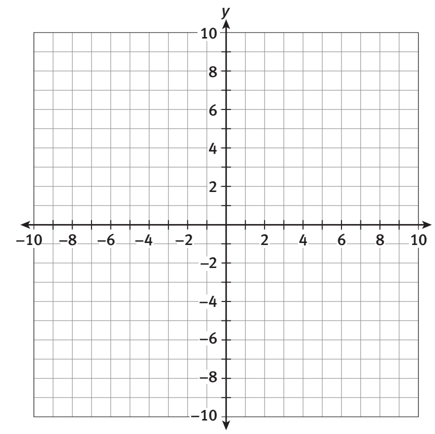
Write your answers on notebook paper. Show your work.

Factor the expressions in Items 1–4 completely.

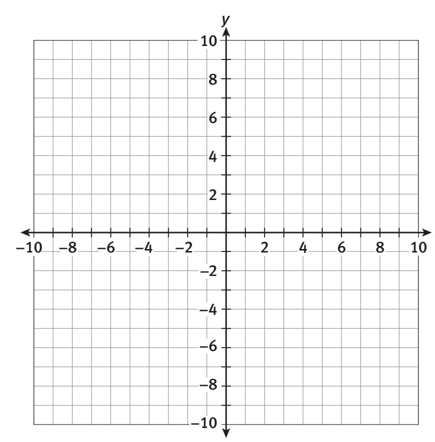
1. $6x^3y + 12x^2y^2$
2. $x^2 + 3x - 40$
3. $x^2 - 49$
4. $x^2 - 6x + 9$
5. Graph $f(x) = \frac{3}{4}x - \frac{3}{2}$.



6. Graph a line that has an x-intercept of 5 and a y-intercept of -2.



7. Graph $y = |x|$, $y = |x + 3|$, and $y = |x| + 3$ on the same grid.



8. Solve $x^2 - 3x - 5 = 0$.

Getting Ready Practice

For students who may need additional instruction on one or more of the prerequisite skills for this unit, Getting Ready practice pages are available in the Teacher Resources at SpringBoard Digital. These practice pages include worked-out examples as well as multiple opportunities for students to apply concepts learned.

Applications of Quadratic Functions

ACTIVITY 7

Fences

Lesson 7-1 Analyzing a Quadratic Function

My Notes

Learning Targets:

- Formulate quadratic functions in a problem-solving situation.
- Graph and interpret quadratic functions.

SUGGESTED LEARNING STRATEGIES: Marking the Text, Guess and Check, Create Representations, Quickwrite, Self Revision/Peer Revision

Fence Me In is a business that specializes in building fenced enclosures. One client has purchased 100 ft of fencing to enclose the largest possible rectangular area in her yard.

Work with your group on Items 1–7. As you share ideas, be sure to explain your thoughts using precise language and specific details to help group members understand your ideas and your reasoning.

1. If the width of the rectangular enclosure is 20 ft, what must be the length? Find the area of this rectangular enclosure.
   The length would be 30 ft and the area of the enclosure would be 600 ft^2.

2. Choose several values for the width of a rectangle with a perimeter of 100 ft. Determine the corresponding length and area of each rectangle. Share your values with members of your class. Then record each set of values in the table below.
   Answers will vary. Common values are included below.

Width (ft)	Length (ft)	Area (ft^2)
1	49	49
5	45	225
10	40	400
20	30	600
25	25	625
30	20	600
40	10	400
45	5	225

3. **Make sense of problems.** What is the relationship between the length and width of a rectangle with perimeter of 100 ft?
   The length plus width equals 50 feet ($\ell + w = 50$).

4. Based on your observations, predict if it is possible for a rectangle with perimeter of 100 ft to have each area. Explain your reasoning.
   Explanations will vary; answers and sample responses follow.

   a. 400 ft^2
   Yes. Dimensions of 10 ft and 40 ft have an area of 400 ft^2.

   b. 500 ft^2 **Yes. Dimensions of 10 ft and 40 ft have an area of 400 ft^2; dimensions of 20 ft and 30 ft have an area of 600 ft^2. There will be**

DISCUSSION GROUP TIP

Reread the problem scenario as needed. Make notes on the information provided in the problem. Respond to questions about the meaning of key information. Summarize or organize the information needed to create reasonable solutions, and describe the mathematical concepts your group will use to create its solutions.

Common Core State Standards for Activity 7

HSA-CED.A.1	Create equations and inequalities in one variable and use them to solve problems.
HSA-CED.A.2	Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.
HSA-SSE.A.1	Interpret expressions that represent a quantity in terms of its context.
HSA-SSE.A.1a	Interpret parts of an expression, such as terms, factors, and coefficients.
HSF-IF.B.4	For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship.
HSF-IF.B.5	Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes.

ACTIVITY 7

Guided

Activity Standards Focus

In Activity 7, students write a quadratic function for a given problem situation. They graph and interpret features of these functions. They factor quadratic expressions, solve quadratic equations, and interpret the meaning of the solutions. Finally they solve quadratic inequalities and graph the solutions to these inequalities. Throughout this activity, emphasize whether the solutions to the equations and inequalities make sense for the given situation.

Lesson 7-1

PLAN

Pacing: 1 class period

Chunking the Lesson

#1–2	#3–4	#5–6
#7–8	#9	#10–11
#12	#13–14	

Check Your Understanding

Lesson Practice

TEACH

Bell-Ringer Activity

Present the following situation to students. Then have them write a function, $c(h)$, that describes this situation.

A canoe livery rents canoes for a flat fee of \$30, plus an additional \$10 per hour.

[$c(h) = 30 + 10h$]

Have students find the amount it costs to rent the canoe for 4 hours.

[$c(4) = 40 + (10)(4) = \$80$]

1–2 Activating Prior Knowledge, Group Presentation Item 1 allows students to review length, width, and area of rectangles. For Item 2, it is essential that students gather adequate amounts of data to establish patterns.

3–4 Look for a Pattern, Guess and Check The fact that the length and width have a sum of 50 becomes apparent by inspecting the first two columns of the table. Some students may express this in other ways. For example, the length equals 50 feet minus the width. Students will likely answer Item 4 from an inductive viewpoint based on the values in their tables. Most groups will have an area of 400 ft^2, no groups will have 500 ft^2 (but will guess it is possible because they have values larger than 500 ft^2 in their table); and no groups will have 700 ft^2. Value all predictions at this time as reasonable guesses.

5–6 Create Representations The patterns students recognize in Item 3 are used to create algebraic representations for the context. It is essential that students understand that the relationship between length and width is necessary to create a function with one independent variable.

7–8 Create Representations Students should graph this function by choosing x-values that result in y-values that are easy to plot. Students can also use their data from Item 2 to plot points. If students remember properties of quadratic functions from Algebra 1, they may use them to graph the function.

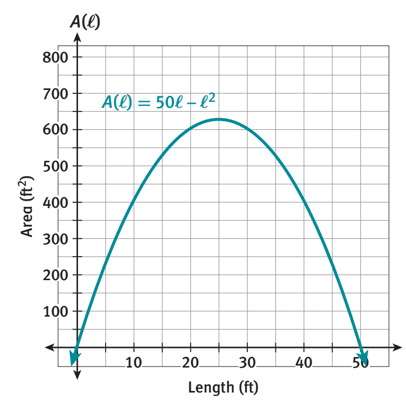
9 Activating Prior Knowledge, Debriefing Remind students when going from the x- and y-values of their graphing calculators to use $A(\ell)$ to represent the function (y) and ℓ to represent the length (x). For additional technology resources, visit SpringBoard Digital. Think about what this graph actually represents. It represents the area of a rectangle, based upon its length. If necessary, review inequalities, interval notation, and set notation. Emphasize that the inequality sign is $<$ and not $\leq$. Students should be prepared to explain why. Also note how this affects the interval notation brackets (rounded rather than square).

c. 700 ft^2
No. No dimensions can be found which have this area.

5. Let l represent the length of a rectangle with a perimeter of 100 ft. Write an expression for the width of the rectangle in terms of l.
$w = 50 - \ell$

6. Express the area $A(l)$ for a rectangle with a perimeter of 100 ft as a function of its length, l.
$A(\ell) = \ell(50 - \ell) = 50\ell - \ell^2 = -\ell^2 + 50\ell$

7. Graph the quadratic function $A(l)$ on the coordinate grid.



8. Use appropriate tools strategically. Now use a graphing calculator to graph the quadratic function $A(l)$. Set your window to correspond to the values on the axes on the graph in Item 7.
Check students' work.

TECHNOLOGY TIP

To graph the function on a graphing calculator, you will first need to substitute y for $A(\ell)$ and x for ℓ before you can enter the equation.

9. Use the function $A(l)$ and your graphs from Items 7 and 8 to complete the following.

a. What is the reasonable domain of the function in this situation? Express the domain as an inequality, in interval notation, and in set notation.
$0 < \ell < 50$; $(0, 50)$; $\{\ell \mid \ell \in \mathbb{R}, 0 < \ell < 50\}$

Lesson 7-1
Analyzing a Quadratic Function

b. Over what interval of the domain is the value of the function increasing? Over what interval of the domain is the value of the function decreasing?
The function increases for $0 < \ell \le 25$ and decreases for $25 \le \ell < 50$.

10. What is the maximum rectangular area that can be enclosed by 100 ft of fencing? ***Justify*** your answer.
625 ft^2. Sample justification: The graph shows that the maximum value of the function occurs when $\ell = 25$, and $A(25) = -25^2 + 50(25) = 625$.

11. a. What is the reasonable range of $A(\ell)$ in this situation? Express the range as an inequality, in interval notation, and in set notation.
$0 < A \le 625$; $(0, 625)$; $\{A \mid A \in \mathbb{R}, 0 < A \le 625\}$

b. Explain how your answer to Item 10 helped you determine the reasonable range.
The maximum value of the function is the same as the maximum value of the range.

12. Reason quantitatively. Revise or confirm your predictions from Item 4. If a rectangle is possible, estimate its dimensions and explain your reasoning. Review the draft of your revised or confirmed predictions. Be sure to check that you have included specific details, the correct mathematical terms to support your explanations, and that your sentences are complete and grammatically correct. You may want to pair-share with another student to critique each other's drafts and make improvements.

a. 400 ft^2
The graph of $A(\ell)$ shows that there are two rectangles that will have this area, a 10 ft $\times$ 40 ft rectangle and a 40 ft $\times$ 10 ft rectangle.

b. 500 ft^2
The graph of $A(\ell)$ shows two possible lengths that will yield an area of 500 ft^2. The lengths are not easily determined from the graph; however, the points of intersection appear to be around 14 ft and 36 ft. There are two rectangles that will have this area, an approximately 14 ft $\times$ 36 ft rectangle and an approximately 36 ft $\times$ 14 ft rectangle.

c. 700 ft^2
Since the graph of $A(\ell)$ never reaches $A(\ell) = 700$, an area of 700 ft^2 is not possible.

My Notes

ACADEMIC VOCABULARY

When you ***justify*** an answer, you show that your answer is correct or reasonable.

CONNECT TO AP

The process of finding the maximum (or minimum) value of a function is called *optimization*, a topic addressed in calculus.

ACTIVITY 7 Continued

10–11 Quickwrite, Self Revision/Peer Revision Students often talk about a maximum point without understanding which part of the ordered pair is indeed a maximum. These items lead to a discussion of the y-value being the maximum that occurs at a particular x-value of the function. Students can verify that the function reaches its maximum value at $x = 25$ because the graph of the function is symmetrical and it crosses the x-axis when $x = 0$ and when $x = 50$.

12 Marking the Text, Summarizing This item provides an opportunity for formative assessment regarding solving quadratic equations algebraically. Students' abilities to solve quadratic equations using factoring and the Quadratic Formula will become apparent when they complete this item. Group presentation on solutions will initiate and enable class discussion.

MINI-LESSON: Quadratic Formula

If students need additional help with using the quadratic formula, a mini-lesson is available to provide practice.

See the Teacher Resources at SpringBoard Digital for a student page for this mini-lesson.

13–14 Quickwrite, Self Revision/ Peer Revision, Debriefing Many students will expect the result to be a square. Some students may disagree, however, mistakenly believing that a square is not a rectangle. Appropriate instruction regarding quadrilaterals and how to reason the answer logically from the definitions of square and rectangle may be necessary.

Check Your Understanding

Debrief students' answers to these items to ensure that they understand concepts related to quadratic functions and to solving quadratic equations by graphing.

Answers

15. Sample answers: The function $A(l) = -l^2 + 50l$ is quadratic, because it can be written in the form $f(x) = ax^2 + bx + c$, with $a = -1$, $b = 50$, and $c = 0$. The function is quadratic, because its graph is a parabola.
16. Sample answer: The graph of a linear function is a line, and the graph of a quadratic function is a parabola. The graph of a linear function has at most one x-intercept, but the graph of a quadratic function can have two x-intercepts. The graph of a quadratic function has a maximum or minimum value, but the graph of a linear function does not.
17. No. A quadratic function has a maximum or a minimum value. If it has a maximum, its range does not include values greater than the maximum. If it has a minimum, its range does not include values less than the minimum.
18. Graph the function $f(x) = x^2 + 2x$. Then find the points on the graph where $f(x) = 3$. The x-coordinates of these points are the solutions of the quadratic equation $x^2 + 2x = 3$.

ASSESS

Students' answers to Lesson Practice problems will provide you with a formative assessment of their understanding of the lesson concepts and their ability to apply their learning.

See the Activity Practice for additional problems for this lesson. You may assign the problems here or use them as a culmination for the activity.

ADAPT

Check students' answers to the Lesson Practice to ensure that they understand how to write an equation that represents the area of a rectangle. For those students requiring additional practice, have them create problems similar to Lesson Practice Items 19–21 to swap with other classmates.

My Notes

Lesson 7-1
Analyzing a Quadratic Function

13. What are the length and width of the largest rectangular area that can be enclosed by 100 ft of fencing?
25 ft by 25 ft

14. The length you gave in Item 13 is the solution of a quadratic equation in terms of l. Write this equation. Explain how you arrived at this equation.
$625 = -\ell^2 + 50\ell$; I found the value of ℓ for which $A(\ell) = 625$, so I solved the equation $625 = -\ell^2 + 50\ell$.

Check Your Understanding

15. Explain why the function $A(l)$ that you used in this lesson is a quadratic function.
16. How does the graph of a quadratic function differ from the graph of a linear function?
17. Can the range of a quadratic function be all real numbers? Explain.
18. Explain how you could solve the quadratic equation $x^2 + 2x = 3$ by graphing the function $f(x) = x^2 + 2x$.

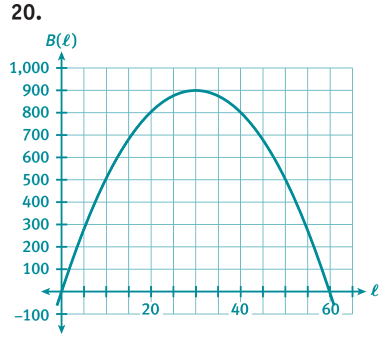
LESSON 7-1 PRACTICE

For Items 19–21, consider a rectangle that has a perimeter of 120 ft.

19. Write a function $B(l)$ that represents the area of the rectangle with length l.
20. Graph the function $B(l)$, using a graphing calculator. Then copy it on your paper, labeling axes and using an appropriate scale.
21. Use the graph of $B(l)$ to find the dimensions of the rectangle with a perimeter of 120 feet that has each area. Explain your answer.
a. 500 ft^2 **b.** 700 ft^2
22. **Critique the reasoning of others.** An area of 1000 ft^2 is not possible. Explain why this is true.
23. How is the maximum value of a function shown on the graph of the function? How would a minimum value be shown?

LESSON 7-1 PRACTICE

19. $B(\ell) = (60 - \ell)\ell = 60\ell - \ell^2$
20.



21. **a.** 10 ft × 50 ft or 50 ft × 10 ft
b. The intersection points of $B(l)$ and $B(l) = 700$ appear to be around (16, 700) and (44, 700). The dimensions are approximately 16 ft × 44 ft and 44 ft × 16 ft.
22. Sample answer: The graph shows that the maximum rectangular area is 900 ft^2.
23. The maximum value is represented by the y-coordinate of the highest point on the graph. The minimum value would be represented by the y-coordinate of the lowest point.

Lesson 7-2
Factoring Quadratic Expressions

Learning Targets:
- Factor quadratic expressions of the form $x^2 + bx + c$.
- Factor quadratic expressions of the form $ax^2 + bx + c$.

SUGGESTED LEARNING STRATEGIES: Interactive Word Wall, Vocabulary Organizer, Marking the Text, Guess and Check, Work Backward, RAFT

In the previous lesson, you used the function $A(l) = -l^2 + 50l$ to model the area in square feet of a rectangle that can be enclosed with 100 ft of fencing.

1. **Reason quantitatively.** What are the dimensions of the rectangle if its area is 525 ft^2? Explain how you determined your answer.
   15 ft × 35 ft or 35 ft × 15 ft. Sample explanation: I found the points on the graph of $A(\ell)$ where $A(\ell) = 525$. The graph shows that $A(\ell) = 525$ when $\ell = 15$ or $\ell = 35$.

2. One way to find the dimensions of the rectangle is to solve a quadratic equation algebraically. What ***quadratic equation*** could you have solved to answer Item 1?
   $525 = -\ell^2 + 50\ell$

3. Write the quadratic equation from Item 2 in the form $al^2 + bl + c = 0$, where $a > 0$. Give the values of a, b, and c.
   $\ell^2 - 50\ell + 525 = 0$; $a = 1$, $b = -50$, $c = 525$

As you have seen, graphing is one way to solve a quadratic equation. However, you can also solve quadratic equations algebraically by factoring.

You can use the graphic organizer shown in Example A on the next page to recall factoring trinomials of the form $x^2 + bx + c = 0$. Later in this activity, you will solve the quadratic equation from Item 3 by factoring.

My Notes

MATH TERMS

A **quadratic equation** can be written in the form $ax^2 + bx + c = 0$, where $a \neq 0$. An expression in the form $ax^2 + bx + c$, $a \neq 0$, is a *quadratic expression*.

Lesson 7-2

PLAN

Pacing: 1 class period

Chunking the Lesson

#1–3 Example A
#4 Example B
Check Your Understanding
Lesson Practice

TEACH

Bell-Ringer Activity

Have students factor the following polynomials completely.

1. $x^2 - 9$ $[(x + 3)(x - 3)]$
2. $f^2 + 9f + 8$ $[(f + 1)(f + 8)]$
3. $y^3 + y^2 - 6y$ $[y(y + 3)(y - 2)]$

Discuss that in this Bell-Ringer Activity, Item 1 is a difference of two squares, Item 2 is a trinomial, and Item 3 has a common factor of y and a trinomial that factors.

1–3 Chunking the Activity, Discussion Groups, Debriefing For these items, have students work with a partner or in small groups. In Item 1, encourage students to trace the graph of $A(\ell) = -\ell^2 + 50\ell$ or to use the table function on their graphing calculators to find the two corresponding values of ℓ when $A(\ell) = 525$. In Items 2 and 3, students will find that this problem could also be solved without the benefit of a graph by writing and solving (by factoring) the quadratic equation represented by this function.

Developing Math Language

Review with students the difference between factoring an expression and solving an equation. In earlier courses, students spend a great deal of time factoring quadratic expressions. Students then move on to solving quadratic equations by factoring, by taking a quadratic expression that is set equal to zero and finding solutions that make a true sentence.

Example A Activating Prior Knowledge, Guess and Check Using the graphic organizer and the example as a guide, students will review factoring skills learned in previous courses. The Try These items provide a formative assessment opportunity for teachers to determine the extent to which students understand the concepts of factor, difference of squares, perfect square trinomials, and factoring trinomials of the form $x^2 + bx + c$. The problem shown in Item g in Try These A should be used to illustrate that factoring out the GCF before finding factors is critical to complete factorization.

Teacher to Teacher

The technique shown in Example A is just one method of factoring. Other methods that students use successfully should be valued and shared.

My Notes

Example A

Factor $x^2 + 12x + 32$.

Step 1: Place x^2 in the upper left box and the constant term 32 in the lower right.

x^2	
	32

Step 2: List factor pairs of 32, the constant term. Choose the pair that has a sum equal to 12, the coefficient b of the x–term.

Factors		Sum
32	1	$32 + 1 = 33$
16	2	$16 + 2 = 18$
8	4	$8 + 4 = 12$

Step 3: Write each factor as coefficients of x and place them in the two empty boxes. Write common factors from each row to the left and common factors for each column above.

	x	8
x	x^2	$8x$
4	$4x$	32

Step 4: Write the sum of the common factors as binomials. Then write the factors as a product.

$(x + 4)(x + 8)$

Solution: $x^2 + 12x + 32 = (x + 4)(x + 8)$

Math Tip

To check that your factoring is correct, multiply the two binomials by distributing.

$(x + 4)(x + 8)$

$= x^2 + 4x + 8x + 32$

$= x^2 + 12x + 32$

Try These A

a. Factor $x^2 - 7x + 12$, using the graphic organizer. Then check by multiplying.
$(x - 3)(x - 4)$

	x	-3
x	x^2	$-3x$
-4	$-4x$	12

Factor, and then check by multiplying. Show your work.

b. $x^2 + 9x + 14$
$(x + 7)(x + 2)$

c. $x^2 - 7x - 30$
$(x - 10)(x + 3)$

d. $x^2 - 12x + 36$
$(x - 6)^2$ or $(x - 6)(x - 6)$

e. $x^2 - 144$
$(x + 12)(x - 12)$

f. $5x^2 + 40x + 75$
$5(x + 3)(x + 5)$

g. $-12x^2 + 108$
$-12(x + 3)(x - 3)$

Math Tip

A difference of squares $a^2 - b^2$ is equal to $(a - b)(a + b)$. A perfect square trinomial $a^2 + 2ab + b^2$ is equal to $(a + b)^2$.

Lesson 7-2
Factoring Quadratic Expressions

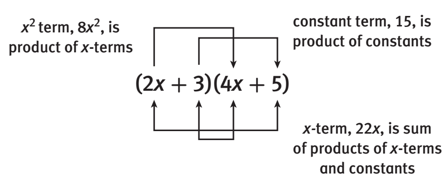
My Notes

Before factoring quadratic expressions $ax^2 + bx + c$, where the leading coefficient $a \neq 1$, consider how multiplying binomial factors results in that form of a quadratic expression.

4. **Make sense of problems.** Use a graphic organizer to multiply $(2x + 3)(4x + 5)$.
   a. Complete the graphic organizer by filling in the two empty boxes.
   b. $(2x + 3)(4x + 5)$
   $= 8x^2 +$ <u>$10x$</u> $+$ <u>$12x$</u> $+ 15$
   $= 8x^2 +$ <u>$22x$</u> $+ 15$

	$2x$	3
$4x$	$8x^2$	$12x$
5	$10x$	15

Using the Distributive Property, you can see the relationship between the numbers in the binomial factors and the terms of the trinomial.



To factor a quadratic expression $ax^2 + bx + c$, work backward from the coefficients of the terms.

Example B
Factor $6x^2 + 13x - 5$. Use a table to organize your work.

Step 1: Identify the factors of 6, which is a, the coefficient of the x^2-term.

Step 2: Identify the factors of -5, which is c, the constant term.

Step 3: Find the numbers whose products add together to equal 13, which is b, the coefficient of the x-term.

Step 4: Then write the binomial factors.

Factors of 6	Factors of -5	Sum $= 13$?
1 and 6	-1 and 5	$1(5) + 6(-1) = -1$
1 and 6	5 and -1	$1(-1) + 6(5) = 29$
2 and 3	-1 and 5	$2(5) + 3(-1) = 7$
2 and 3	5 and -1	$2(-1) + 3(5) = 13$ ✔

Solution: $6x^2 + 13x - 5 = (2x + 5)(3x - 1)$

MATH TIP

Check your answer by multiplying the two binomials.

$(2x + 5)(3x - 1)$

$= 6x^2 - 2x + 15x - 5$

$= 6x^2 + 13x - 5$

ACTIVITY 7 Continued

4 Activating Prior Knowledge This item extends the concept to trinomials of the form $ax^2 + bx + c$, where $a \neq 1$. Students first investigate multiplying binomials using a graphic organizer and then use the organizer to see how the terms of the trinomial relate to the terms in the binomial factors.

Example B Marking the Text, Summarizing, Work Backward In this Example, students move from the visual (graphic organizer) to the abstract method of factoring trinomials with a leading coefficient not equal to 1.

Note that the table in Step 4 does not list all possible combinations of factors of 6 and factors of -5. Other combinations include $-1, -6$ and $-1, 5$; $-1, -6$ and $1, -5$; $-2, -3$ and $-1, 5$; and $-2, -3$ and $1, -5$. When solving these types of problems, students may need to check many combinations before finding the correct binomial factors.

TEACHER to TEACHER

Note that in a trinomial such as $6x^2 + 13x - 5$, the pairs of factors (-1 and -6), as well as (-2 and -3), could be used to represent the factors of 6. While it is much easier to use positive coefficients for the first terms, it should be noted that other factorizations are possible. For instance: $6x^2 + 13x - 5$ can also be factored as $(-2x - 5)(-3x + 1)$. Notice that *every* sign in the binomial factors is the opposite of the answer stated in Example B.

Check Your Understanding

Debrief students' answers to these items to ensure that they understand concepts related to factoring quadratic expressions.

Answers

5. The length of the large square is $x + 5$, and its width is $x + 3$, so its area is $(x + 5)(x + 3)$. The area of the large square is equal to the sum of the areas of the smaller squares: $x^2 + 5x + 3x + 15 = x^2 + 8x + 15$. The expressions $(x + 5)(x + 3)$ and $x^2 + 8x + 15$ are equivalent, because both are equal to the area of the large square.
6. Both constant terms are negative. Sample explanation: The product of the constant terms is equal to c. If c is positive, the constant terms must have the same sign. The sum of the constant terms is equal to b, which is negative. Two positive terms cannot have a negative sum, so the constant terms must both be negative.
7. Sample answer: List factor pairs of -12, the constant term in the quadratic expression. Then find the sum of the factor pairs. Keep trying different factor pairs until you get a sum of -4, the coefficient of the x-term in the quadratic expression. The correct factor pair is 2 and -6. Use those numbers as the constant terms in the factored expression: $(x - 2)(x + 6)$. Check your work by multiplying the binomials to see whether you get the original quadratic expression: $(x - 2)(x + 6) = x^2 + 6x - 2x - 12 = x^2 + 4x - 12$.

ASSESS

Students' answers to Lesson Practice problems will provide you with a formative assessment of their understanding of the lesson concepts and their ability to apply their learning.

See the Activity Practice for additional problems for this lesson. You may assign the problems here or use them as a culmination for the activity.

ADAPT

Check students' answers to the Lesson Practice to ensure that they understand how to factor quadratic expressions whether or not the lead coefficient is equal to one. Make sure students always attempt to find the greatest common factor of a quadratic expression before proceeding. Some students may benefit from making a flow chart or other graphic organizer that describes the process of factoring trinomials.

My Notes

Try These B

Factor, and then check by multiplying. Show your work.

a. $10x^2 + 11x + 3$
(5x + 3)(2x + 1)

b. $4x^2 + 17x - 15$
(4x − 3)(x + 5)

c. $2x^2 - 13x + 21$
(2x − 7)(x − 3)

d. $6x^2 - 19x - 36$
(2x − 9)(3x + 4)

Check Your Understanding

5. Explain how the graphic organizer shows that $x^2 + 8x + 15$ is equal to $(x + 5)(x + 3)$.

	x	5
x	x^2	$5x$
3	$3x$	15

6. **Reason abstractly.** Given that b is negative and c is positive in the quadratic expression $x^2 + bx + c$, what can you conclude about the signs of the constant terms in the factored form of the expression? Explain your reasoning.
7. Write a set of instructions for a student who is absent, explaining how to factor the quadratic expression $x^2 + 4x - 12$.

LESSON 7-2 PRACTICE

Factor each quadratic expression.

8. $2x^2 + 15x + 28$
9. $3x^2 + 25x - 18$
10. $x^2 + x - 30$
11. $x^2 + 15x + 56$
12. $6x^2 - 7x - 5$
13. $12x^2 - 43x + 10$
14. $2x^2 + 5x$
15. $9x^2 - 3x - 2$
16. A customer of Fence Me In wants to increase both the length and width of a rectangular fenced area in her backyard by x feet. The new area in square feet enclosed by the fence is given by the expression $x^2 + 30x + 200$.
 a. Factor the quadratic expression.
 b. **Reason quantitatively.** What were the original length and width of the fenced area? Explain your answer.

LESSON 7-2 PRACTICE

8. $(2x + 7)(x + 4)$
9. $(3x - 2)(x + 9)$
10. $(x + 6)(x - 5)$
11. $(x + 7)(x + 8)$
12. $(2x + 1)(3x - 5)$
13. $(4x - 1)(3x - 10)$
14. $x(2x + 5)$
15. $(3x + 1)(3x - 2)$
16. **a.** $(x + 20)(x + 10)$
 b. 20 ft × 10 ft; The factored expression for the new area shows that the new length is $(x + 20)$ ft, and the new width is $(x + 10)$ ft. The new length is equal to x ft plus the old length, so the old length is 20 ft. The new width is equal to x ft plus the old width, so the old width is 10 ft.

Lesson 7-3
Solving Quadratic Equations by Factoring

Learning Targets:
- Solve quadratic equations by factoring.
- Interpret solutions of a quadratic equation.
- Create quadratic equations from solutions.

SUGGESTED LEARNING STRATEGIES: Marking the Text, Paraphrasing, Think-Pair-Share, Create Representations, Quickwrite

To solve a quadratic equation $ax^2 + bx + c = 0$ by factoring, the equation must be in factored form to use the Zero Product Property.

My Notes

MATH TIP

The *Zero Product Property* states that if $a \cdot b = 0$, then either $a = 0$ or $b = 0$.

Example A
Solve $x^2 + 5x - 14 = 0$ by factoring.

	Original equation	$x^2 + 5x - 14 = 0$
Step 1:	Factor the left side.	$(x + 7)(x - 2) = 0$
Step 2:	Apply the Zero Product Property.	$x + 7 = 0$ or $x - 2 = 0$
Step 3:	Solve each equation for x.	

Solution: $x = -7$ or $x = 2$

Try These A
a. Solve $3x^2 - 17x + 10 = 0$ and check by substitution.

$3x^2 - 17x + 10 = 0$	Original equation
$(3x - 2)(x - 5) = 0$	Factor the left side.
$3x - 2 = 0$ or $x - 5 = 0$	Apply the Zero Product Property.
$x = \frac{2}{3}$ or $x = 5$	Solve each equation for x.

Solve each equation by factoring. Show your work.

b. $12x^2 - 7x - 10 = 0$
$x = -\frac{2}{3}$ or $x = \frac{5}{4}$

c. $x^2 + 8x - 9 = 0$
$x = -9$ or $x = 1$

d. $4x^2 + 12x + 9 = 0$
$x = -\frac{3}{2}$

e. $18x^2 - 98 = 0$
$x = \frac{7}{3}$ or $x = -\frac{7}{3}$

f. $x^2 + 6x = -8$
$x = -2$ or $x = -4$

g. $5x^2 + 2x = 3$
$x = -1$ or $x = \frac{3}{5}$

MATH TIP

You can check your solutions by substituting the values into the original equation.

Lesson 7-3

PLAN

Pacing: 1 class period

Chunking the Lesson

Example A #1 #2–3
Check Your Understanding
Example B
Check Your Understanding
Lesson Practice

TEACH

Bell-Ringer Activity

Review the following as a general summary of steps to factoring:

1. Always look for a common factor.
2. Count the number of terms.
   If two terms: Is it a difference of two squares?
   If three terms: Is it a trinomial square?
   If not, test the pairs of factors of the coefficient of the squared term and the constant.
   If four terms: Try factoring by grouping.
3. Always factor completely!

Example A Marking the Text, Identify a Subtask After students have completed the guided problem in Item a, use the remaining Try These items to assess their progress. Students' understanding of the Zero Product Property is essential.

1 Activating Prior Knowledge, Visualization Remind students that because the sign of 525 in the quadratic equation is *positive*, the binomial factors will have the *same* sign, and since the sign that precedes 50ℓ is *negative*, these signs will both be *negative*. Students need to find a pair of negative factors of 525 whose sum is -50.

For struggling students, encourage them to draw a picture of this rectangle. Remind them that once they find ℓ (length), it must be doubled, then subtracted from 100, and then the difference divided by 2 in order to find the corresponding width. A sketch of this will reduce mistakes and aid the students in visualizing the problem.

2–3 Quickwrite, Debriefing, Marking the Text, Visualization In Item 2, emphasize the wording "side by side." (See ELL Support below.) Explain how to make the negative ℓ^2 term positive by moving the $-\ell^2$ and the 80ℓ terms to the right side of the equation, making the left side equal to zero. Also explain that for the solution to Item 2c, the length of the courts is 40 yards. The width of the combined courts is also 40 yards, but the width of each single court is 20 yards.

ELL Support

To help struggling students who may have a misconception of the phrase "side by side," explain to them that the courts are touching, or they are right up against each other. Then demonstrate this with a drawing or by taking two objects (textbooks) and laying them down on a desktop touching against each other, sharing a common edge. If students have the misconception that these courts are not touching, it alters the perimeter, making the problem unsolvable.

Lesson 7-3
Solving Quadratic Equations by Factoring

My Notes

In the previous lesson, you were asked to determine the dimensions of a rectangle with an area of 525 ft^2 that can be enclosed by 100 ft of fencing. You wrote the quadratic equation $l^2 - 50l + 525 = 0$ to model this situation, where l is the length of the rectangle in feet.

1. a. Solve the quadratic equation by factoring.
$(\ell - 35)(\ell - 15) = 0$; $\ell = 35$ or $\ell = 15$

b. What do the solutions of the equation represent in this situation?
The rectangle could have a length of 35 ft or a length of 15 ft.

c. What are the dimensions of a rectangle with an area of 525 ft^2 that can be enclosed by 100 ft of fencing?
35 ft $\times$ 15 ft or 15 ft $\times$ 35 ft

d. Reason quantitatively. Explain why your answer to part c is reasonable. **Sample answer: The area of a rectangle with a length of 35 ft and a width of 15 ft is $35(15) = 525$ ft^2, and the perimeter is $2(35 + 15) = 100$ ft. The dimensions give the correct area and perimeter, so the answer is reasonable.**

2. A park has two rectangular tennis courts side by side. Combined, the courts have a perimeter of 160 yd and an area of 1600 yd^2.
a. Write a quadratic equation that can be used to find l, the length of the court in yards.
$-\ell^2 + 80\ell = 1600$ or equivalent

b. Construct viable arguments. Explain why you need to write the equation in the form $al^2 + bl + c = 0$ before you can solve it by factoring. **To solve the equation by factoring, you need to apply the Zero Product Property. You can only apply this property when one side of the equation is equal to 0.**

c. Solve the quadratic equation by factoring, and interpret the solution.
$(\ell - 40)^2 = 0$; $\ell = 40$; The length of the court is 40 yd.

d. Explain why the quadratic equation has only one distinct solution.
When the equation is factored, both factors are the same, so there is only one value of ℓ that makes the equation true.

MATH TIP

It is often easier to factor a quadratic equation if the coefficient of the x^2-term is positive. If necessary, you can multiply both sides of the equation by -1 to make the coefficient positive.

My Notes

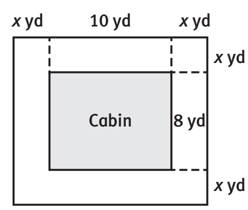
3. The equation $2x^2 + 9x - 3 = 0$ cannot be solved by factoring. Explain why this is true.
The product of 2 and -3 is -6, and no factor pair of -6 has a sum of 9.

Check Your Understanding

4. Explain how to use factoring to solve the equation $2x^2 + 5x = 3$.

5. Critique the reasoning of others. A student incorrectly states that the solution of the equation $x^2 + 2x - 35 = 0$ is $x = -5$ or $x = 7$. Describe the student's error, and solve the equation correctly.

6. Fence Me In has been asked to install a fence around a cabin. The cabin has a length of 10 yd and a width of 8 yd. There will be a space x yd wide between the cabin and the fence on all sides, as shown in the diagram. The area to be enclosed by the fence is 224 yd^2.



a. Write a quadratic equation that can be used to determine the value of x.
b. Solve the equation by factoring.
c. Interpret the solutions.

If you know the solutions to a quadratic equation, then you can write the equation.

Example B

Write a quadratic equation in ***standard form*** with the solutions $x = 4$ and $x = -5$.

Step 1: Write linear equations that correspond to the solutions. $x - 4 = 0$ or $x + 5 = 0$

Step 2: Write the linear expressions as factors. $(x - 4)$ and $(x + 5)$

Step 3: Multiply the factors to write the equation in factored form. $(x - 4)(x + 5) = 0$

Step 4: Multiply the binomials and write the equation in standard form. $x^2 + x - 20 = 0$

Solution: $x^2 + x - 20 = 0$ is a quadratic equation with solutions $x = 4$ and $x = -5$.

MATH TERMS

The **standard form of a quadratic equation** is $ax^2 + bx + c = 0$, where $a \neq 0$.

2–3 (continued) In Item 3, students may attempt to factor the quadratic expression, but they will soon find that it is not possible. It is important for students to realize that not all quadratic trinomials are factorable over the integers.

Check Your Understanding

Debrief students' answers to these items to ensure that they understand concepts related to solving quadratic equations by factoring. For Item 6, have students explain why one of the solutions needs to be excluded in this situation.

Answers

4. First, set the right side of the equation equal to 0 by subtracting 3 from both sides: $2x^2 + 5x - 3 = 0$. Then factor the quadratic expression on the left side of the equation: $(2x - 1)(x + 3) = 0$. Next, use the Zero Product Property to write two equations: $2x - 1 = 0$ or $x + 3 = 0$. Finally, solve each equation for x: $x = \frac{1}{2}$ or $x = -3$.

5. Sample answer: The student factored the equation correctly to get $(x + 7)(x - 5) = 0$, but then used the constant terms of the binomials as the solutions of the equation. Instead, the student should have applied the Zero Product Property to get $x + 7 = 0$ or $x - 5 = 0$. Solving these equations yields the correct solution of $x = -7$ or $x = 5$.

6. a. $4x^2 + 36x + 80 = 224$, or equivalent

b. $4(x - 3)(x + 12) = 0$; $x = 3$ or $x = -12$

c. The solution $x = 3$ shows that the space between the cabin and the fence is 3 yd wide. The solution $x = -12$ should be excluded in this situation, because a negative value for the width does not make sense.

Example B Marking the Text, Paraphrasing, Work Backward, Debriefing Being able to work from solutions to equations is essential for a complete understanding of solving quadratic equations. Students may need practice writing $x = a$ as a linear equation equal to zero: $x - a = 0$.

Example B (continued) The fact that there are infinitely many equations for a given set of solutions may not be apparent to students. Students have to realize that multiplying both sides of any equation by a real number does not change the solutions. The solution with fraction coefficients is correct, but some students may find it difficult to perform operations with the fractions. By multiplying each equation by the denominator to eliminate the fractions, students will be able to find a quadratic equation with integer coefficients.

Check Your Understanding

Debrief students' answers to these items to ensure that they understand concepts related to writing equivalent equations for quadratic equations and writing quadratic equations given their solutions.

Answers

7. $3x^2 - 16x - 12 = 0$
8. The LCD of the fractions in the equation is 6, so multiply both sides of the equation by 6.
   $6\left(x^2 - \frac{7}{6}x + \frac{1}{3}\right) = 6(0)$
   $6x^2 - 7x + 2 = 0$
9. Yes. The solutions of the quadratic equation $x^2 + 4x + 3 = 0$ are $x = -3$ and $x = -1$. Multiplying both sides of this equation by any real number other than 0 or 1 results in another equation whose solutions are $x = -3$ and $x = -1$.
10. Sample answer: Write a linear equation that corresponds to the solution: $x - 4 = 0$. Write the linear expression as a factor: $(x - 4)$. Multiply the factor by itself to write a quadratic equation in factored form: $(x - 4)(x - 4) = 0$. Then multiply the binomials and write the equation in standard form: $x^2 - 8x + 16 = 0$.

Try These B

a. Write a quadratic equation in standard form with the solutions $x = -1$ and $x = -7$.

$(x + 1) = 0, (x + 7) = 0$	Write linear equations that correspond to the solutions.
$(x + 1), (x + 7)$	Write the linear expressions as factors.
$(x + 1)(x + 7) = 0$	Multiply the factors to write the equation in factored form.
$x^2 + 8x + 7 = 0$	Multiply the binomials and write the equation in standard form.

b. Write a quadratic equation in standard form whose solutions are $x = \frac{2}{5}$ and $x = -\frac{1}{2}$. How is your result different from those in Example B?
Sample answers: $x^2 + \frac{1}{10}x - \frac{1}{5} = 0$ and $10x^2 + x - 2 = 0$. The coefficients may be fractions, or the value of *a* may not be 1.

MATH TIP

To avoid fractions as coefficients, multiply the coefficients by the LCD.

Write a quadratic equation in standard form with integer coefficients for each pair of solutions. Show your work.

c. $x = \frac{2}{3}, x = 2$
$3x^2 - 8x + 4 = 0$

d. $x = -\frac{3}{2}, x = \frac{5}{2}$
$4x^2 - 4x - 15 = 0$

Check Your Understanding

7. Write the equation $3x^2 - 6x = 10x + 12$ in standard form.
8. Explain how you could write the equation $x^2 - \frac{7}{6}x + \frac{1}{3} = 0$ with integer values of the coefficients and constants.
9. **Reason quantitatively.** Is there more than one quadratic equation whose solutions are $x = -3$ and $x = -1$? Explain.
10. How could you write a quadratic equation in standard form whose only solution is $x = 4$?

Lesson 7-3
Solving Quadratic Equations by Factoring

My Notes

LESSON 7-3 PRACTICE

Solve each quadratic equation by factoring.

11. $2x^2 - 11x + 5 = 0$

12. $x^2 + 2x = 15$

13. $3x^2 + x - 4 = 0$

14. $6x^2 - 13x - 5 = 0$

Write a quadratic equation in standard form with integer coefficients for which the given numbers are solutions.

15. $x = 2$ and $x = -5$

16. $x = -\frac{2}{3}$ and $x = -5$

17. $x = \frac{3}{5}$ and $x = 3$

18. $x = -\frac{1}{2}$ and $x = \frac{3}{4}$

19. Model with mathematics. The manager of Fence Me In is trying to determine the best selling price for a particular type of gate latch. The function $p(s) = -4s^2 + 400s - 8400$ models the yearly profit the company will make from the latches when the selling price is s dollars.

a. Write a quadratic equation that can be used to determine the selling price that would result in a yearly profit of $1600.

b. Write the quadratic equation in standard form so that the coefficient of s^2 is 1.

c. Solve the quadratic equation by factoring, and interpret the solution(s).

d. Explain how you could check your answer to part c.

CONNECT TO ECONOMICS

The selling price of an item has an effect on how many of the items are sold. The number of items that are sold, in turn, has an effect on the amount of profit a company makes by selling the item.

ASSESS

Students' answers to Lesson Practice problems will provide you with a formative assessment of their understanding of the lesson concepts and their ability to apply their learning.

See the Activity Practice for additional problems for this lesson. You may assign the problems here or use them as a culmination for the activity.

LESSON 7-3 PRACTICE

11. $x = \frac{1}{2}$ or $x = 5$

12. $x = -5$ or $x = 3$

13. $x = -\frac{4}{3}$ or $x = 1$

14. $x = \frac{5}{2}$ or $x = -\frac{1}{3}$

15. $x^2 + 3x - 10 = 0$

16. $3x^2 + 17x + 10 = 0$

17. $5x^2 - 18x + 9 = 0$

18. $8x^2 - 2x - 3 = 0$

19. a. $-4s^2 + 400s - 8400 = 1600$ or equivalent

b. $s^2 - 100s + 2500 = 0$

c. $(s - 50)^2 = 0$; $s = 50$; The selling price that will result in a yearly profit of $1600 is $50.

d. Sample answer: Substitute 50 for s in the function $p(s) = -4s^2 + 400s - 8400$, and check that $p(50) = 1600$.

ADAPT

Check students' answers to the Lesson Practice to ensure that they understand how to use the Zero Product Property to solve a quadratic equation. Students should also know how to write a quadratic equation in standard form given the solutions to the equation. For those students requiring additional practice, have them work in pairs to create quadratic equations for each other to solve. Students will practice the main skills in this lesson by creating and then solving each other's equations.

Lesson 7-4

PLAN

Pacing: 1 class period

Chunking the Lesson

Example A

#1–4 #5–7

Check Your Understanding

Lesson Practice

TEACH

Bell-Ringer Activity

Have students solve the following inequalities and describe their graphs.

1. $x + 5 \le -3$ [*$x \le -8$; a number line with a closed point plotted at −8, with a ray to the left*]
2. $-2x > 4$ [*$x < -2$; a number line with an open point plotted at −2, with a ray to the left. Note: change of inequality sign due to dividing both sides by a negative*]
3. $5x - 2 \ge 13$ [*$x \ge 3$; a number line with a closed point plotted at 3, with a ray to the right*]

Example A Marking the Text, Identify a Subtask, Guess and Check After solving quadratic equations by factoring, it is natural to extend factoring to solutions of inequalities. Use the number line as a guide to determine intervals on which each factor is positive or negative, and then choose the appropriate interval(s) for the solution. As with linear inequalities, it is important that students realize that there are infinitely many solutions to quadratic inequalities, and that solutions are best written as intervals.

My Notes

MATH TIP

For a product of two numbers to be positive, both factors must have the same sign. If the product is negative, then the factors must have opposite signs.

Learning Targets:

- Solve quadratic inequalities.
- Graph the solutions to quadratic inequalities.

SUGGESTED LEARNING STRATEGIES: Identify a Subtask, Guess and Check, Think Aloud, Create Representations, Quickwrite

Factoring is also used to solve quadratic inequalities.

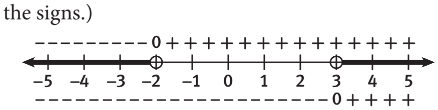
Example A

Solve $x^2 - x - 6 > 0$.

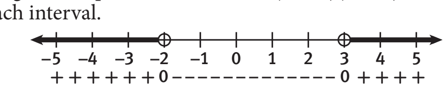
Step 1: Factor the quadratic expression on the left side. $(x + 2)(x - 3) > 0$

Step 2: Determine where each factor equals zero. $(x + 2) = 0$ at $x = -2$; $(x - 3) = 0$ at $x = 3$

Step 3: Use a number line to visualize the intervals for which each factor is positive and negative. (Test a value in each interval to determine the signs.) $(x + 2)$, $(x - 3)$



Step 4: Identify the sign of the product of the two factors on each interval. $(x + 2)(x - 3)$

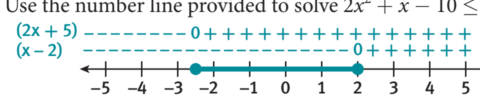


Step 5: Choose the appropriate interval. Since $x^2 - x - 6$ is positive (> 0), the intervals that show $(x + 2)(x - 3)$ as positive represent the solutions.

Solution: $x < -2$ or $x > 3$

Try These A

a. Use the number line provided to solve $2x^2 + x - 10 \le 0$. $-\frac{5}{2} \le x \le 2$



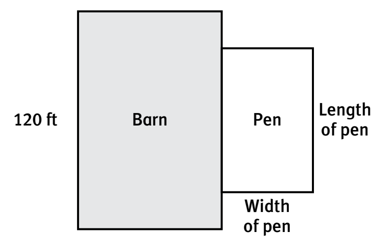
Solve each quadratic inequality.

b. $x^2 + 3x - 4 < 0$
$-4 < x < 1$

c. $3x^2 + x - 10 \ge 0$
$x \ge \frac{5}{3}$ or $x \le -2$

Lesson 7-4
More Uses for Factors

A farmer wants to enclose a rectangular pen next to his barn. A wall of the barn will form one side of the pen, and the other three sides will be fenced. He has purchased 100 ft of fencing and has hired Fence Me In to install it so that it encloses an area of at least 1200 ft^2.



Work with your group on Items 1–5. As you share ideas with your group, be sure to explain your thoughts using precise language and specific details to help group members understand your ideas and your reasoning.

1. **Attend to precision.** If Fence Me In makes the pen 50 ft in length, what will be the width of the pen? What will be its area? Explain your answers.
   Width: 25 ft; The length plus twice the width is equal to 100 ft. So, $50 + 2w = 100$ and $w = 25$.
   Area: 1250 ft^2; The length is 50 ft, and the width is 25 ft, so the area is $50(25) = 1250$ ft^2.

2. Let *l* represent the length in feet of the pen. Write an expression for the width of the pen in terms of *l*.
   $50 - \frac{\ell}{2}$ or equivalent

3. Write an inequality in terms of *l* that represents the possible area of the pen. Explain what each part of your inequality represents.
   $\ell\left(50 - \frac{\ell}{2}\right) \geq 1200$ or equivalent. Sample explanation: The left side shows the area of the pen as the length times the expression for the width. The symbol $\geq$ shows that the area is at least 1200 ft^2.

4. Write the inequality in standard form with integer coefficients.
   $\ell^2 - 100\ell + 2400 \leq 0$ (or $-\ell^2 + 100\ell - 2400 \geq 0$)

5. Use factoring to solve the quadratic inequality.
   $(\ell - 40)(\ell - 60) \leq 0$; $40 \leq \ell \leq 60$

My Notes

DISCUSSION GROUP TIP

Reread the problem scenario as needed. Make notes on the information provided in the problem. Respond to questions about the meaning of key information. Summarize or organize the information needed to create reasonable solutions, and describe the mathematical concepts your group will use to create solutions.

MATH TIP

If you multiply or divide both sides of an inequality by a negative number, you must reverse the inequality symbol.

ACTIVITY 7 Continued

1–4 Debriefing, Marking the Text, Think-Pair-Share In Item 1, remember there is only one length of the pen to consider when setting up the equation to find the width, because the other length is against the barn. In Item 2, some students may need assistance arriving at the answer shown: if $2w + \ell = 100$, then to isolate the w term, $2w = 100 - \ell$. To solve for w, divide by 2 on both sides to arrive at $w = 50 - \frac{1}{2}\ell$ or $w = 50 - \frac{\ell}{2}$. In Item 3, open up a class discussion about the use of the formula $A = \ell w$, as well as the use of the inequality symbol $\geq$. In Item 4, highlight the phrases "standard form" and "integer coefficients." Ask students to explain how to get rid of the fraction.

5–7 Visualization, Debriefing, Identify a Subtask In Item 5, students must factor and solve: $\ell^2 - 100\ell + 2400 = 0$. Once they have the solutions of 40 and 60, have them draw a number line to guide them to the inequality solution, similar to the one shown in Example A.

ELL Support

Students who are speaking English as a second language, or struggling students in general, may be having difficulty translating the phrase "at least" to its corresponding symbol of $\geq$. Discuss that the phase "at least" indicates "the same as or more," or "no less than." You could further explain that the wording "at most" corresponds to the inequality symbol of $\leq$. "At most" indicates "the same as or less," or "no more than."

5–7 (continued) Item 6 is just a matter of explaining (in words) what the inequality $40 \leq \ell \leq 60$ represents. Item 7 involves substituting the extreme values of 40 and 60 into the expression for width in Item 2.

Check Your Understanding

Debrief students' answers to these items to ensure that they understand concepts related to factoring quadratic expressions.

Answers

8. a. First determine the value of x for which $(x + 4)$ is equal to 0: $(x + 4) = 0$ when $x = -4$. Then test a value of x less than -4 to check whether it makes $(x + 4)$ positive or negative: When $x < -4$, the factor $(x + 4)$ is negative. Next, test a value of x greater than -4 to check whether it makes $(x + 4)$ positive or negative: When $x > -4$, the factor $(x + 4)$ is positive. Repeat these steps for the factor $(x - 5)$ to find that it is negative when $x < 5$ and positive when $x > 5$.

b. For intervals on which both factors are positive or both factors are negative, the product $(x + 4)(x - 5)$ is positive. For intervals on which one factor is positive and one factor is negative, the product $(x + 4)(x - 5)$ is negative.

c. The solutions are values of x for which $(x + 4)(x - 5) \geq 0$. So, the solutions are intervals for which the product $(x + 4)(x - 5)$ is positive.

9. The equation $x^2 + 5x - 24 = 0$ has two solutions: $x = -8$ and $x = 3$. The solutions of the inequality $x^2 + 5x - 24 \leq 0$ also include $x = -8$ and $x = 3$ as well as all values of x between -8 and 3. So, the solutions of the inequality are $-8 \leq x \leq 3$.

10. Sample answer: The expression x^2 is never negative, so the sum of x^2 and 4 is never negative. Because the expression on the left side of the inequality can never be less than 0, the inequality has no real solutions.

ASSESS

Students' answers to Lesson Practice problems will provide you with a formative assessment of their understanding of the lesson concepts and their ability to apply their learning.

See the Activity Practice for additional problems for this lesson. You may assign the problems here or use them as a culmination for the activity.

LESSON 7-4 PRACTICE

11. $x \leq -5$ or $x \geq 2$

12. $-3 < x < \frac{3}{2}$

13. $-6 \leq x \leq -3$

14. $x < -\frac{2}{3}$ or $x > 4$

15. $3 < x < 9$

16. $x < -2$ or $x > -\frac{2}{5}$

17. a. $-500s^2 + 15{,}000s - 100{,}000 \geq 8000$

b. $s^2 - 30s + 216 \leq 0$

c. $(s - 12)(s - 18) \leq 0$; $12 \leq s \leq 18$; The company will make a profit of at least \$8000 when the installation price of the wooden fencing is at least \$12 per foot and no more than \$18 per foot.

ADAPT

Check students' answers to the Lesson Practice to ensure that they understand how to use a number line to solve a quadratic inequality. Although some students may be able to determine the solution through visualization of the parabola, it is essential that students master the number line method. This method will be used again when students solve rational and higher order polynomial inequalities.

My Notes

6. Interpret the solutions of the inequality.
The length of the pen must be at least 40 ft and no more than 60 ft.

7. Use the possible lengths of the pen to determine the possible widths.
The width of the pen must be at least 20 ft and no more than 30 ft.

Check Your Understanding

8. Consider the inequality $(x + 4)(x - 5) \geq 0$.
   a. Explain how to determine the intervals on a number line for which each of the factors $(x + 4)$ and $(x - 5)$ is positive or negative.
   b. **Reason abstractly.** How do you determine the sign of the product $(x + 4)(x - 5)$ on each interval?
   c. Once you know the sign of the product $(x + 4)(x - 5)$ on each interval, how do you identify the solutions of the inequality?

9. Explain how the solutions of $x^2 + 5x - 24 = 0$ differ from the solutions of $x^2 + 5x - 24 \leq 0$.

10. Explain why the quadratic inequality $x^2 + 4 < 0$ has no real solutions.

LESSON 7-4 PRACTICE

Solve each inequality.

11. $x^2 + 3x - 10 \geq 0$

12. $2x^2 + 3x - 9 < 0$

13. $x^2 + 9x + 18 \leq 0$

14. $3x^2 - 10x - 8 > 0$

15. $x^2 - 12x + 27 < 0$

16. $5x^2 + 12x + 4 > 0$

17. The function $p(s) = -500s^2 + 15{,}000s - 100{,}000$ models the yearly profit Fence Me In will make from installing wooden fences when the installation price is s dollars per foot.
   a. Write a quadratic inequality that can be used to determine the installation prices that will result in a yearly profit of at least \$8000.
   b. Write the quadratic inequality in standard form so that the coefficient of s^2 is 1.
   c. **Make sense of problems.** Solve the quadratic inequality by factoring, and interpret the solution(s).

ACTIVITY 7 PRACTICE

Write your answers on notebook paper. Show your work.

Lesson 7-1

A rectangle has perimeter 40 cm. Use this information for Items 1–7.

1. Write the dimensions and areas of three rectangles that fit this description.
2. Let the length of one side be x. Then write a function $A(x)$ that represents that area of the rectangle.
3. Graph the function $A(x)$ on a graphing calculator. Then sketch the graph on grid paper, labeling the axes and using an appropriate scale.
4. An area of 96 cm^2 is possible. Use $A(x)$ to demonstrate this fact algebraically and graphically.
5. An area of 120 cm^2 is not possible. Use $A(x)$ to demonstrate this fact algebraically and graphically.
6. What are the reasonable domain and reasonable range of $A(x)$? Express your answers as inequalities, in interval notation, and in set notation.
7. What is the greatest area that the rectangle could have? Explain.

Use the quadratic function $f(x) = x^2 - 6x + 8$ for Items 8–11.

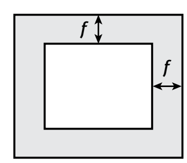
8. Graph the function.
9. Write the domain and range of the function as inequalities, in interval notation, and in set notation.
10. What is the function's y-intercept?
    A. 0 **B.** 2
    C. 4 **D.** 8
11. Explain how you could use the graph of the function to solve the equation $x^2 - 6x + 8 = 3$.

Lesson 7-2

12. Factor $x^2 + 11x + 28$ by copying and completing the graphic organizer. Then check by multiplying.

	?	?
?	x^2	?
?	?	28

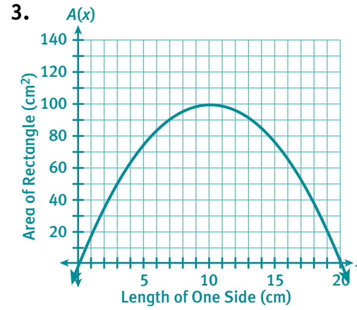
13. Factor each quadratic expression.
    a. $2x^2 - 3x - 27$ **b.** $4x^2 - 121$
    c. $6x^2 + 11x - 10$ **d.** $3x^2 + 7x + 4$
    e. $5x^2 - 42x - 27$ **f.** $4x^2 - 4x - 35$
    g. $36x^2 - 100$ **h.** $12x^2 + 60x + 75$
14. Given that b is positive and c is negative in the quadratic expression $x^2 + bx + c$, what can you conclude about the signs of the constant terms in the factored form of the expression? Explain your reasoning.
15. The area in square inches of a framed photograph is given by the expression $4f^2 + 32f + 63$, where f is the width in inches of the frame.

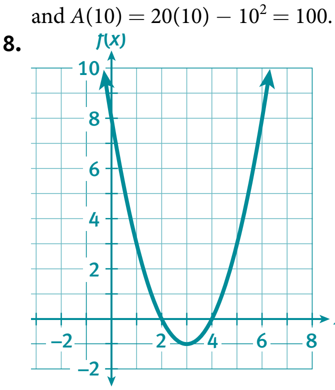


    a. Factor the quadratic expression.
    b. What are the dimensions of the opening in the frame? Explain your answer.
    c. If the frame is 2 inches wide, what are the overall dimensions of the framed photograph? Explain your answer.

ACTIVITY 7 Continued

ACTIVITY PRACTICE

1. Sample answers: 5 cm × 15 cm, area = 75 cm^2; 9 cm × 11 cm, area = 99 cm^2; 2 cm × 18 cm, area = 36 cm^2
2. $A(x) = (20 - x)x = 20x - x^2$
3. 

4. $20x - x^2 = 96$ has solutions of $x = 8$ and $x = 12$ represented graphically by the points of intersection of $A(x) = 96$ and $A(x)$.
5. The equation $20x - x^2 = 120$ has no real-number solutions, which is represented graphically by there being no point of intersection between $A(x)$ and $A(x) = 120$.
6. domain: $0 < x < 20$, $(0, 20)$, $\{x \mid x \in \mathbb{R}, 0 < x < 20\}$; range: $0 < A \le 100$, $(0, 100]$, $\{A \mid A \in \mathbb{R}, 0 < A \le 100\}$
7. 100 cm^2; Sample explanation: The graph shows that the maximum value of the function occurs when $x = 10$, and $A(10) = 20(10) - 10^2 = 100$.
8. 

9. domain: $-\infty < x < \infty$, $(-\infty, \infty)$, $\{x \mid x \in \mathbb{R}\}$; range: $y \ge -1$ $[-1, \infty)$, $\{y \mid y \in \mathbb{R}, y \ge -1\}$
10. D
11. Find the points on the graph of $f(x)$ where $f(x) = 3$. The x-coordinates of these points are the solutions of $x^2 - 6x + 8 = 3$. Because $f(x) = 3$ when $x = 1$ and when $x = 5$, the solutions of $x^2 - 6x + 8 = 3$ are $x = 1$ and $x = 5$.
12. $x^2 + 11x + 28 = (x + 4)(x + 7)$

	x	4
x	x^2	$4x$
7	$7x$	28

13. **a.** $(2x - 9)(x + 3)$
    b. $(2x + 11)(2x - 11)$
    c. $(2x + 5)(3x - 2)$
    d. $(3x + 4)(x + 1)$
    e. $(5x + 3)(x - 9)$
    f. $(2x + 5)(2x - 7)$
    g. $4(3x + 5)(3x - 5)$
    h. $3(2x + 5)^2$
14. One constant term is positive, and the other is negative. Sample explanation: The product of the constant terms is equal to c. If c is negative, the constant terms must have opposite signs.
15. **a.** $(2f + 9)(2f + 7)$
    b. The length is 9 in., and the width is 7 in. The factored expression for the area shows that the overall length is $(2f + 9)$ in. and the overall width is $(2f + 7)$ in. The overall length is equal to 2 times f plus the length of the opening, so the length of the opening is 9 in. The overall width is equal to 2 times f plus the width of the opening, so the width of the opening is 7 in.
    c. 13 in. × 11 in.; The expression for the overall length in inches is $2f + 9$. If $f = 2$, the overall length is $2(2) + 9 = 13$ in. The expression for the overall width in inches is $2f + 7$. If $f = 2$, the overall length is $2(2) + 7 = 11$ in.

16. a. $x = -\frac{3}{2}; x = 4$

b. $x = -\frac{1}{3}; x = -2$

c. $x = \frac{5}{2}$

d. $x = \frac{2}{3}; x = -\frac{2}{3}$

e. $x = \frac{4}{3}; x = -\frac{1}{2}$

17. More than one correct equation is possible; other correct equations would be real-number multiples of the equations given.
a. $x^2 + 3x - 40 = 0$
b. $3x^2 - 14x + 8 = 0$
c. $10x^2 + 9x - 7 = 0$
d. $x^2 - 12x + 36 = 0$

18. No. Sample explanation: The student is assuming that if a product is equal to 2, then one of the factors must be equal to 2. This assumption is incorrect. For example, the product $4\left(\frac{1}{2}\right)$ is equal to 2, but neither of the factors is equal to 2.

19. B

20. $b^2 + 30b - 5400 = 0$

21. $(b + 90)(b - 60) = 0$; $b = -90$ or $b = 60$; The solution $b = -90$ must be excluded, because b represents the base of a triangle, and it does not make sense for the base to be negative. The solution $b = 60$ shows that the base of the triangle measures 60 ft.

22. $x < -4$ or $x > 6$; Sample explanation: The factor $(x + 4)$ is negative for $x < -4$ and positive for $x > -4$. The factor $(x - 6)$ is negative for $x < 6$ and positive for $x > 6$. Both factors are negative, which means their product is positive when $x < -4$; and both factors are positive, which also means their product is positive when $x > 6$.

23. a. $-1 \le x \le 4$
b. $x < -\frac{2}{3}$ or $x > 3$
c. no real solutions
d. $x \le -3$ or $x \ge -1$
e. $-3 \le x \le 7$
f. $-\frac{2}{5} < x < 3$

24. $-16t^2 + 20t + 6 \ge 10$

25. $16t^2 - 20t + 4 \le 0$

26. $\frac{1}{4} \le t \le 1$; The ball is at least 10 ft above the ground between $\frac{1}{4}$ second and 1 second after it is thrown.

ADDITIONAL PRACTICE

If students need more practice on the concepts in this activity, see the Teacher Resources at SpringBoard Digital for additional practice problems.

Lesson 7-3

16. Solve each quadratic equation by factoring.
a. $2x^2 - 5x - 12 = 0$
b. $3x^2 + 7x = -2$
c. $4x^2 - 20x + 25 = 0$
d. $27x^2 - 12 = 0$
e. $6x^2 - 4 = 5x$

17. For each set of solutions, write a quadratic equation in standard form.
a. $x = 5, x = -8$ **b.** $x = \frac{2}{3}, x = 4$
c. $x = -\frac{7}{5}, x = \frac{1}{2}$ **d.** $x = 6$

18. A student claims that you can find the solutions of $(x - 2)(x - 3) = 2$ by solving the equations $x - 2 = 2$ and $x - 3 = 2$. Is the student's reasoning correct? Explain why or why not.

One face of a building is shaped like a right triangle with an area of 2700 ft^2. The height of the triangle is 30 ft greater than its base. Use this information for Items 19–21.

19. Which equation can be used to determine the base b of the triangle in feet?
A. $b(b + 30) = 2700$
B $\frac{1}{2}b(b + 30) = 2700$
C. $b(b - 30) = 2700$
D. $\frac{1}{2}b(b - 30) = 2700$

20. Write the quadratic equation in standard form so that the coefficient of b^2 is 1.

21. Solve the quadratic equation by factoring, and interpret the solutions. If any solutions need to be excluded, explain why.

Lesson 7-4

22. For what values of x is the product $(x + 4)(x - 6)$ positive? Explain.

23. Solve each quadratic inequality.
a. $x^2 - 3x - 4 \le 0$ **b.** $3x^2 - 7x - 6 > 0$
c. $x^2 - 16x + 64 < 0$ **d.** $2x^2 + 8x + 6 \ge 0$
e. $x^2 - 4x - 21 \le 0$ **f.** $5x^2 - 13x - 6 < 0$

The function $h(t) = -16t^2 + 20t + 6$ models the height in feet of a football t seconds after it is thrown. Use this information for Items 24–26.

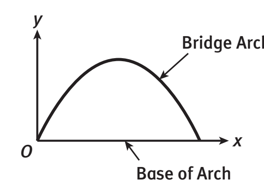
24. Write a quadratic inequality that can be used to determine when the football will be at least 10 ft above the ground.

25. Write the quadratic inequality in standard form.

26. Solve the quadratic inequality by factoring, and interpret the solution(s).

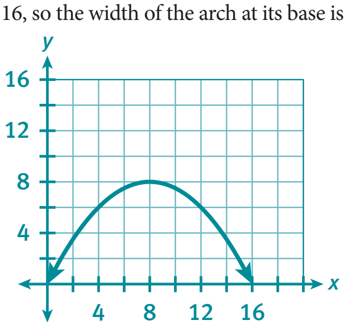
MATHEMATICAL PRACTICES
Make Sense of Problems and Persevere in Solving Them

27. The graph of the function $y = -\frac{1}{8}x^2 + 2x$ models the shape of an arch that forms part of a bridge, where x and y are the horizontal and vertical distances in feet from the left end of the arch.



a. The greatest width of the arch occurs at its base. Use a graph to determine the greatest width of the arch. Explain how you used the graph to find the answer.
b. Now write a quadratic equation that can help you find the greatest width of the arch. Solve the equation by factoring, and explain how you used the solutions to find the greatest width.
c. Compare and contrast the methods of using a graph and factoring an equation to solve this problem.

27. a. 16 ft; Sample explanation: The graph shows that $y = 0$ when $x = 0$ and when $x = 16$. The distance between the points (0, 0) and (16, 0) is 16, so the width of the arch at its base is 16 ft.



b. $-\frac{1}{8}x^2 + 2x = 0$; $x = 0$ or $x = 16$; Sample explanation: The solutions show that $y = 0$ when $x = 0$ and when $x = 16$. The right end of the base of the arch is 16 ft from the left end of the base of the arch.

c. Sample answer: Both methods involve finding the values of x for which $y = 0$. When using a graph, the values of x are found by observing where the graph of the function intersects the x-axis. When using an equation, these values of x are found by substituting 0 for y in the equation of the function and then solving.

Introduction to Complex Numbers

ACTIVITY 8

Cardano's Imaginary Numbers
Lesson 8-1 The Imaginary Unit, *i*

My Notes

Learning Targets:

- Know the definition of the complex number *i*.
- Know that complex numbers can be written as $a + bi$, where *a* and *b* are real numbers.
- Graph complex numbers on the complex plane.

SUGGESTED LEARNING STRATEGIES: Create Representations, Interactive Word Wall, Marking the Text, Think-Pair-Share, Quickwrite

The equation $x^2 + 1 = 0$ has special historical and mathematical significance. At the beginning of the sixteenth century, mathematicians believed that the equation had no solutions.

1. Why would mathematicians of the early sixteenth century think that $x^2 + 1 = 0$ had no solutions?
If $x^2 + 1 = 0$, then $x^2 = -1$, and the square root of a negative number is not a real number.

A breakthrough occurred in 1545 when the talented Italian mathematician Girolamo Cardano (1501–1576) published his book, *Ars Magna* (*The Great Art*). In the process of solving one cubic (third-degree) equation, he encountered—and was required to make use of—the square roots of negative numbers. While skeptical of their existence, he demonstrated the situation with this famous problem: Find two numbers with the sum 10 and the product 40.

2. Make sense of problems. To better understand this problem, first find two numbers with the sum 10 and the product 21.
$3 + 7 = 10$ and $3 \cdot 7 = 21$.
(This solution will likely be discovered by guess and check.)

3. Letting *x* represent one number, write an expression for the other number in terms of *x*. Use the expressions to write an equation that models the problem in Item 2: "find two numbers with the product 21."
$x(10 - x) = 21$

Common Core State Standards for Activity 8

HSN-CN.A.1	Know there is a complex number *i* such that $i^2 = -1$, and every complex number has the form $a + bi$ with *a* and *b* real.
HSN-CN.A.2	Use the relation $i^2 = -1$ and the commutative, associative, and distributive properties to add, subtract, and multiply complex numbers.
HSN-CN.A.3(+)	Find the conjugate of a complex number; use conjugates to find moduli and quotients of complex numbers.
HSN-CN.B.4(+)	Represent complex numbers on the complex plane in rectangular and polar form (including real and imaginary numbers), and explain why the rectangular and polar forms of a given complex number represent the same number.

[Note: HSN-CN.A.3 and HSN-CN.B.4 are introduced in this activity but are also addressed in higher level mathematics courses.]

ACTIVITY 8
Directed

Activity Standards Focus

In Activity 8, students explore complex numbers. They begin by defining the complex number *i* and writing complex numbers in the form $a + bi$, where *a* and *b* are real numbers. Students also graph complex numbers in the complex plane. Students then add, subtract and multiply complex numbers. They factor quadratic expressions with complex conjugates. Finally, students solve equations with complex solutions. Throughout this activity, make connections between properties of real numbers and properties of complex numbers.

Lesson 8-1

PLAN

Pacing: 1 class period

Chunking the Lesson

#1 #2–4
#5–6 Example A
Check Your Understanding
#11
Check Your Understanding
Example B
Check Your Understanding
Lesson Practice

TEACH

Bell-Ringer Activity

Have students solve the following equations containing square roots.

1. $\sqrt{x} = 4$ $[x = 16]$
2. $\sqrt{x + 2} = 7$ $[x = 47]$
3. $x - 1 = \sqrt{4x - 8}$ $[x = 3]$

Discuss the methods used to solve each equation.

1 Activating Prior Knowledge This item provides a good opportunity for formative assessment regarding solving quadratic equations of the form $x^2 + a = 0$.

2–4 Guess and Check, Create Representations, Activating Prior Knowledge, Debriefing Most students will likely be able to answer Item 2 using a guess-and-check strategy. For Items 3 and 4, opportunities exist for formative assessment of students' abilities to create a quadratic equation from a verbal description and for them to solve the equation using factoring and the Quadratic Formula.

5–6 Create Representations, Activating Prior Knowledge After debriefing Items 3 and 4, students should be able to write the equation in Item 5 more easily. In Item 6, students should recognize the equation may be solved using the Quadratic Formula, but not by factoring.

Developing Math Language

In mathematics, imaginary numbers are not "make-believe"; they are a set of numbers that do exist. Imaginary numbers exist so that negative numbers can have square roots and certain equations can have solutions. Additionally, imaginary numbers have significant technological applications, particularly in the fields of electronics and engineering.

Universal Access

A misconception that some students have is to not realize that $\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$ is true for nonnegative numbers only. For example, students may incorrectly solve this problem as follows:

$$-\sqrt{-3} \cdot \sqrt{-11}$$
$$= -\sqrt{-3 \cdot -11}$$
$$= -\sqrt{33}$$

However, this must be solved by rewriting the radicals using i:

$$-\sqrt{-3} \cdot \sqrt{-11}$$
$$= -i\sqrt{3} \cdot i\sqrt{11}$$
$$= -i^2\sqrt{3 \cdot 11}$$
$$= -(-1)\sqrt{33}$$
$$= \sqrt{33}$$

CONNECT TO HISTORY

The *Ars Magna* was first published under the title *Artis Magnæ, Sive de Regulis Algebraicis Liber Unus (Book Number One about The Great Art, or The Rules of Algebra)*. The main focus of this work involved methods of solving third- and fourth-degree equations. The "great art" of algebra described by Cardano was in comparison to the "lesser art" of the day, arithmetic.

Rafael Bombelli (1526–1572), an Italian architect and engineer, was intrigued by Cardano's methods and formalized the rules for operations with complex numbers.

My Notes

MATH TIP

You can solve a quadratic equation by graphing, by factoring, or by using the Quadratic Formula, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. You can use it to solve quadratic equations in the form

$ax^2 + bx + c = 0$, where $a \neq 0$.

CONNECT TO HISTORY

When considering his solutions, Cardano dismissed "mental tortures" and ignored the fact that $\sqrt{x} \cdot \sqrt{x} = x$ only when $x \geq 0$.

MATH TERMS

An **imaginary number** is any number of the form bi, where b is a real number and $i = \sqrt{-1}$.

4. Solve your equation in Item 3 in two different ways. Explain each method.

$x(10 - x) = 21$, so $x^2 - 10x + 21 = 0$.
By factoring, $(x - 3)(x - 7) = 0$, so $x = 3$ or $x = 7$.
Using the Quadratic Formula on $x^2 - 10x + 21 = 0$ yields
$x = x = \frac{10 \pm \sqrt{100 - 4(1)21}}{2} = \frac{10 \pm 4}{2} = 5 \pm 2$, so $x = 3$ or $x = 7$.

5. Write an equation that represents the problem that Cardano posed.

$x(10 - x) = 40$

6. Cardano claimed that the solutions to the problem are $x = 5 + \sqrt{-15}$ and $x = 5 - \sqrt{-15}$. Verify his solutions by using the Quadratic Formula with the equation in Item 5.

$x = \frac{10 \pm \sqrt{100 - 4(1)40}}{2} = \frac{10 \pm \sqrt{-60}}{2} = \frac{10 \pm 2\sqrt{-15}}{2} = 5 \pm \sqrt{-15}$

Cardano avoided any more problems in *Ars Magna* involving the square root of a negative number. However, he did demonstrate an understanding about the properties of such numbers. Solving the equation $x^2 + 1 = 0$ yields the solutions $x = \sqrt{-1}$ and $x = -\sqrt{-1}$. The number $\sqrt{-1}$ is represented by the symbol i, the imaginary unit. You can say $i = \sqrt{-1}$. The imaginary unit i is considered the solution to the equation $x^2 + 1 = 0$, or $x^2 = -1$.

To simplify an ***imaginary number*** $\sqrt{-s}$, where s is a positive number, you can write $\sqrt{-s} = i\sqrt{s}$.

Common Core State Standards for Activity 8 *(continued)*

HSN-CN.C.7	Solve quadratic equations with real coefficients that have complex solutions.
HSN-CN.C.8	Extend polynomial identities to the complex numbers.

Lesson 8-1
The Imaginary Unit, *i*

My Notes

Example A
Write the numbers $\sqrt{-17}$ and $\sqrt{-9}$ in terms of i.

		$\sqrt{-17}$	$\sqrt{-9}$
Step 1:	Definition of $\sqrt{-s}$	$= i \cdot \sqrt{17}$	$= i \cdot \sqrt{9}$
Step 2:	Take the square root of 9.	$= i\sqrt{17}$	$= i \cdot 3$
			$= 3i$

Solution: $\sqrt{-17} = i\sqrt{17}$ and $\sqrt{-9} = 3i$

Try These A
Write each number in terms of i.

a. $\sqrt{-25}$ **$5i$**

b. $\sqrt{-7}$ **$i\sqrt{7}$**

c. $\sqrt{-12}$ **$2i\sqrt{3}$**

d. $\sqrt{-150}$ **$5i\sqrt{6}$**

WRITING MATH
Write $i\sqrt{17}$ instead of $\sqrt{17}i$, which may be confused with $\sqrt{17i}$.

CONNECT TO HISTORY
René Descartes (1596–1650) was the first to call these numbers *imaginary*. Although his reference was meant to be derogatory, the term *imaginary number* persists. Leonhard Euler (1707–1783) introduced the use of i for the imaginary unit.

Check Your Understanding

7. Make use of structure. Rewrite the imaginary number $4i$ as the square root of a negative number. Explain how you determined your answer.

8. Simplify each of these expressions: $-\sqrt{20}$ and $\sqrt{-20}$. Are the expressions equivalent? Explain.

9. Write each number in terms of i.

a. $\sqrt{-98}$

b. $-\sqrt{-27}$

c. $\sqrt{(-8)(3)}$

d. $\sqrt{25 - 4(2)(6)}$

10. Why do you think imaginary numbers are useful for mathematicians?

11. Write the solutions to Cardano's problem, $x = 5 + \sqrt{-15}$ and $x = 5 - \sqrt{-15}$, using the imaginary unit i.

$5 + i\sqrt{15}$ and $5 - i\sqrt{15}$

Example A Discussion Groups, Activating Prior Knowledge, Debriefing The concept of the imaginary unit will be new, and somewhat confusing, to many students. Tell students that complex numbers, although presented in a theoretical way, play an important part in advanced studies of applied sciences like physics and electrical engineering.

Check Your Understanding
Debrief students' answers to these items to ensure that they understand concepts related to basic concepts of imaginary numbers.

Answers
7. $\sqrt{-16}$. Sample explanation: First write $4i$ as a square root: $4i = \sqrt{(4i)^2}$. Apply the Power of a Product Property: $\sqrt{(4i)^2} = \sqrt{4^2 \cdot i^2}$. i^2 is equal to -1, so $\sqrt{4^2 \cdot i^2} = \sqrt{16 \cdot (-1)} = \sqrt{-16}$.

8. $-\sqrt{20} = -2\sqrt{5}$ and $\sqrt{-20} = 2i\sqrt{5}$; The expressions are not equivalent. Sample explanation: $-\sqrt{20}$ represents the opposite of the square root of a positive number. The square root of a positive number is a real number, so its opposite is also a real number. $\sqrt{-20}$ represents the square root of a negative number. The square root of any negative number is an imaginary number.

9. a. $7i\sqrt{2}$

b. $-3i\sqrt{3}$

c. $2i\sqrt{6}$

d. $i\sqrt{23}$

10. Sample answer: Imaginary numbers can be used when solving quadratic equations that do not have real solutions.

11 Activating Prior Knowledge, Think-Pair-Share Have students work in pairs to apply what they have learned so far, by going back to Item 6. Cardano's solutions to a problem where two numbers have a sum of 10 and a product of 40 are:
$x = 5 + \sqrt{-15}$ and $x = 5 - \sqrt{-15}$.

These can be rewritten as follows:
$x = 5 + \sqrt{-1} \cdot \sqrt{15}$ and
$x = 5 - \sqrt{-1} \cdot \sqrt{15}$
$x = 5 + i\sqrt{15}$ and $x = 5 - i\sqrt{15}$

Ask a student volunteer to present these to the class.

Check Your Understanding

Debrief students' answers to these items to ensure that they understand concepts related to identifying the parts of complex numbers.

Answers

12. **a.** real part: 5; imaginary part: $8i$
b. real part: 8; imaginary part: $0i$
c. real part: 0; imaginary part: $i\sqrt{10}$
d. real part: $\frac{5}{2}$; imaginary part: $\frac{3}{2}i$

13. A complex number has the form $a + bi$, where a and b are real numbers. If $b = 0$, a complex number has the form $a + 0i = a$. So, the real numbers are the complex numbers $a + bi$ for which $b = 0$.

14. A complex number has the form $a + bi$, where a and b are real numbers. If $a = 0$, a complex number has the form $0 + bi = bi$, and bi is an imaginary number. So, the imaginary numbers are the complex numbers $a + bi$ for which $a = 0$.

Example B Summarizing, Debriefing, Construct an Argument As students should already be aware, every real number corresponds to a point on a number line.

Similarly, every complex number corresponds to a number in the complex plane. In the complex plane, a complex number without a real part will be graphed on the imaginary (vertical) axis (like point A), and a complex number without an imaginary part will be graphed on the real (horizontal) axis (like point D). In other words, points graphed on the vertical axis represent pure imaginary numbers, while points graphed on the horizontal axis represent real numbers. Any point graphed within any one of the quadrants is a complex number.

Developing Math Language

A complex number $a + bi$ is made up of two parts. The real part of the complex number is a, and the imaginary part of the complex number is bi. A complex number is *pure imaginary* if it has a real part of 0, like $0 + 2i$, or just $2i$. If a complex number is *real*, it has an imaginary part of 0, like $-5 + 0i$, or just -5.

ACTIVITY 8 continued

My Notes

Lesson 8-1
The Imaginary Unit, *i*

The set of complex numbers consists of the real numbers and the imaginary numbers. A ***complex number*** has two parts: the real part a and the imaginary part bi. For example, in $2 + 3i$, the real part is 2 and the imaginary part is $3i$.

MATH TERMS

A **complex number** is a number in the form $a + bi$, where a and b are real numbers and $i = \sqrt{-1}$.

Check Your Understanding

12. Identify the real part and the imaginary part of each complex number.
a. $5 + 8i$ **b.** 8
c. $i\sqrt{10}$ **d.** $\frac{5 + 3i}{2}$

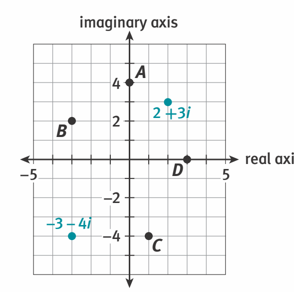
13. Using the definition of complex numbers, show that the set of real numbers is a subset of the complex numbers.

14. Using the definition of complex numbers, show that the set of imaginary numbers is a subset of the complex numbers.

Complex numbers in the form $a + bi$ can be represented geometrically as points in the *complex plane*. The complex plane is a rectangular grid, similar to the Cartesian plane, with the horizontal axis representing the real part a of a complex number and the vertical axis representing the imaginary part bi of a complex number. The point (a, b) on the complex plane represents the complex number $a + bi$.

Example B

Point A represents $0 + 4i$.
Point B represents $-3 + 2i$.
Point C represents $1 - 4i$.
Point D represents $3 + 0i$.



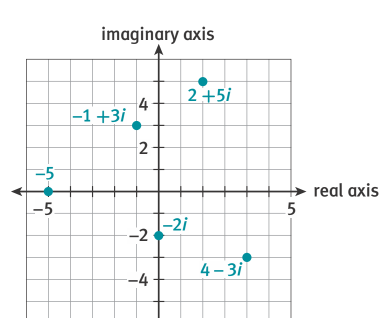
Try These B

a. Graph $2 + 3i$ and $-3 - 4i$ on the complex plane above.

Graph each complex number on a complex plane grid.
b. $2 + 5i$ **c.** $4 - 3i$ **d.** $-1 + 3i$ **e.** $-2i$ **f.** -5

Try These B

Answers



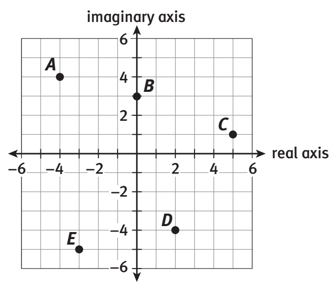
My Notes

Check Your Understanding

15. **Reason abstractly.** Compare and contrast the Cartesian plane with the complex plane.

16. What set of numbers do the points on the real axis of the complex plane represent? Explain.

17. Name the complex number represented by each labeled point on the complex plane below.



LESSON 8-1 PRACTICE

18. Write each expression in terms of i.

a. $\sqrt{-49}$ **b.** $\sqrt{-13}$

c. $3+\sqrt{-8}$ **d.** $5-\sqrt{-36}$

19. Identify the real part and the imaginary part of the complex number $16-i\sqrt{6}$.

20. **Reason quantitatively.** Is π a complex number? Explain.

21. Draw the complex plane. Then graph each complex number on the plane.

a. $6i$ **b.** $3+4i$ **c.** $-2-5i$

d. $4-i$ **e.** $-3+2i$

22. The sum of two numbers is 8, and their product is 80.

a. Let x represent one of the numbers, and write an expression for the other number in terms of x. Use the expressions to write an equation that models the situation given above.

b. Use the Quadratic Formula to solve the equation. Write the solutions in terms of i.

MATH TIP

π is the ratio of a circle's circumference to its diameter. π is an irrational number, and its decimal form neither terminates nor repeats.

Check Your Understanding

Debrief students' answers to these items to ensure that they understand concepts related to graphing complex numbers on the complex plane.

Answers

15. Sample answer: Both are formed by the intersection of a horizontal axis and a vertical axis. In the Cartesian plane, both axes (the x-axis and the y-axis) represent real numbers. In the complex plane, the horizontal, or real, axis represents the real numbers, and the vertical, or imaginary, axis represents the imaginary numbers. On the Cartesian plane, an ordered pair (x, y) gives the location of a point that is a horizontal distance of x units from the origin and a vertical distance of y units from the origin. On the complex plane, an ordered pair (a, b) represents the location of the complex number $a + bi$.

16. The set of real numbers; Points on the real axis represent complex numbers with an imaginary part that is equal to 0. In other words, the numbers have the form $a + 0i = a$, where a is a real number.

17. A: $-4 + 4i$
B: $3i$
C: $5 + i$
D: $2 - 4i$
E: $-3 - 5i$

ASSESS

Students' answers to Lesson Practice problems will provide you with a formative assessment of their understanding of the lesson concepts and their ability to apply their learning.

See the Activity Practice for additional problems for this lesson. You may assign the problems here or use them as a culmination for the activity.

ADAPT

Check students' answers to the Lesson Practice to ensure that they understand both the symbolic and the graphical representations of complex numbers. Students should also be comfortable solving a quadratic equation that has complex solutions. Some students may benefit from a Venn diagram depicting the set of complex numbers and all of its subsets. Students can write examples of each type of number within the Venn diagram as a reference.

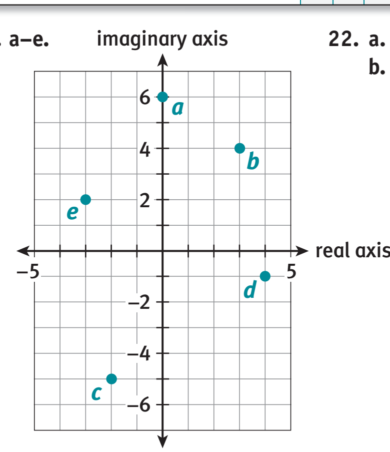
LESSON 8-1 PRACTICE

18. a. $7i$
b. $i\sqrt{13}$
c. $3+2i\sqrt{2}$
d. $5-6i$

19. real part: 16; imaginary part: $-\sqrt{6}$

20. Yes. π is an irrational number, and all irrational numbers are real numbers. Because π is a real number, it is also a complex number that can be written as $\pi + 0i$.

21. a–e.



22. a. $8 - x$; $x(8 - x) = 80$
b. $x = 4 + 8i$ or $x = 4 - 8i$

Lesson 8-2

PLAN

Pacing: 1 class period

Chunking the Lesson

Example A #1–2
Check Your Understanding
Example B #6–7
Check Your Understanding
#12–13 Example C #14
Check Your Understanding
Lesson Practice

TEACH

Bell-Ringer Activity

Have students solve the following four problems by adding or subtracting.

1. $(4 + x) + (-12 - 4x)$ $[-8 - 3x]$
2. $(-9 + x) + (3 + x)$ $[-6 + 2x]$
3. $(2y + 1) - (-y + 5)$ $[3y - 4]$
4. $(y + 4) - (-5y - 3)$ $[6y + 7]$

A review of *combining like terms* will help with the addition and subtraction of complex numbers. Discuss these examples with students prior to moving on with the lesson.

Example A Debriefing, Activating Prior Knowledge To add complex numbers, add the real parts and the imaginary parts separately. It is assumed that *i* acts like a real-number variable, following the Commutative, Associative and Distributive Properties. Therefore, tell students to treat *i* as they would a variable, and combine like terms. Emphasize that the elements of the table in this Example demonstrate that the addition of complex numbers is just like binomial addition.

1–2 Activating Prior Knowledge, Group Presentation, Debriefing In creating a rule for adding or subtracting two complex numbers, students will use properties of real numbers—Commutative, Associative, and Distributive—although they may not state them explicitly. In debriefing on Item 1, stress the reliance on such properties in the generalization of the rule.

My Notes

Learning Targets:

- Add and subtract complex numbers.
- Multiply and divide complex numbers.

SUGGESTED LEARNING STRATEGIES: Group Presentation, Self Revision/Peer Revision, Look for a Pattern, Quickwrite

Perform addition of complex numbers as you would for addition of binomials of the form $a + bx$. To add such binomials, you collect like terms.

Example A

		Addition of Binomials	Addition of Complex Numbers
		$(5 + 4x) + (-2 + 3x)$	$(5 + 4i) + (-2 + 3i)$
Step 1	Collect like terms.	$= (5 - 2) + (4x + 3x)$	$= (5 - 2) + (4i + 3i)$
Step 2	Simplify.	$= 3 + 7x$	$= 3 + 7i$

Try These A

Add the complex numbers.

a. $(6 + 5i) + (4 - 7i)$ $10 - 2i$

b. $(-5 + 3i) + (-3 - i)$ $-8 + 2i$

c. $(2 + 3i) + (-2 - 3i)$ 0

1. **Express regularity in repeated reasoning.** Use Example A above and your knowledge of operations of real numbers to write general formulas for the sum and difference of two complex numbers.

$(a + bi) + (c + di) =$ $(a + c) + (b + d)i$

$(a + bi) - (c + di) =$ $(a - c) + (b - d)i$

My Notes

2. Find each sum or difference of the complex numbers.
   a. $(12 - 13i) - (-5 + 4i)$ **17 − 17*i***
   b. $\left(\frac{1}{2} - i\right) + \left(\frac{5}{2} + 9i\right)$ **3 + 8*i***
   c. $(\sqrt{2} - 7i) + (2 + i\sqrt{3})$ $\mathbf{(2+\sqrt{2})+(\sqrt{3}-7)i}$
   d. $(8 - 5i) - (3 + 5i) + (-5 + 10i)$ **0 + 0*i* = 0**

Check Your Understanding

3. Recall that the sum of a number and its additive inverse is equal to 0. What is the additive inverse of the complex number $3 - 5i$? Explain how you determined your answer.
4. **Reason abstractly.** Is addition of complex numbers commutative? In other words, is $(a + bi) + (c + di)$ equal to $(c + di) + (a + bi)$? Explain your reasoning.
5. Give an example of a complex number you could subtract from $8 + 3i$ that would result in a real number. Show that the difference of the complex numbers is equal to a real number.

Perform multiplication of complex numbers as you would for multiplication of binomials of the form $a + bx$. The only change in procedure is to substitute i^2 with -1.

Example B

Multiply Binomials	Multiply Complex Numbers
$(2 + 3x)(4 - 5x)$	$(2 + 3i)(4 - 5i)$
$2(4) + 2(-5x) + 3x(4) + 3x(-5x)$	$2(4) + 2(-5i) + 3i(4) + 3i(-5i)$
$8 - 10x + 12x - 15x^2$	$8 - 10i + 12i - 15i^2$
$8 + 2x - 15x^2$	$8 + 2i - 15i^2$
	Now substitute -1 for i^2.
	$8 + 2i - 15i^2 = 8 + 2i - 15(-1)$
	$= 23 + 2i$

Try These B

Multiply the complex numbers.
a. $(6 + 5i)(4 - 7i)$ **59 − 22*i***
b. $(2 - 3i)(3 - 2i)$ **0 − 13*i* = −13*i***
c. $(5 + i)(5 + i)$ **24 + 10*i***

ACTIVITY 8 Continued

1–2 (continued) Item 2c allows for an opportunity for formative assessment as students have to add integer and radical values.

Check Your Understanding

Debrief students' answers to these items to ensure that they understand concepts related to adding and subtracting complex numbers.

Answers

3. $-3 + 5i$. Sample explanation: The real part of $3 - 5i$ is 3, so the additive inverse of the real part is -3. The imaginary part of $3 - 5i$ is -5, so the additive inverse of the imaginary part is 5. Therefore, the additive inverse of $3 - 5i$ is $-3 + 5i$.
4. Yes. The sum $(a + bi) + (c + di)$ is $(a + c) + (b + d)i$. The sum $(c + di) + (a + bi)$ is $(c + a) + (d + b)i$, which is equivalent to $(a + c) + (b + d)i$. So, $(a + bi) + (c + di)$ results in the same sum as $(c + di) + (a + bi)$, which means that addition of complex numbers is commutative.
5. Accept any complex number with an imaginary part of 3. Sample answer: $6 + 3i$; $(8 + 3i) - (6 + 3i) = 2 + 0i = 2$, and 2 is a real number.

Example B Activating Prior Knowledge, Quickwrite Have students write about a method from Algebra 1 that the multiplication of complex numbers resembles. [*Sample answer: The process of multiplying complex numbers mimics the FOIL (First, Outer, Inner, Last) method of multiplying binomials.*]

Also have students explain in their own words why $i^2 = -1$. [*Sample answer: because* $i = \sqrt{-1}$, $i^2 = (\sqrt{-1})^2 = -1$.]

6–7 Activating Prior Knowledge, Look for a Pattern, Group Presentation, Debriefing Students will again use properties of real numbers to generalize the rules for multiplication of complex numbers. Make certain that students acknowledge the use of these properties in their discussions and explanations. Additionally, the fact that $i^2 = -1$ and the pattern that evolves with powers of i in the accompanying Math Tip should be highlighted and explored with at least a few additional examples.

Check Your Understanding

Debrief students' answers to these items to ensure that they understand concepts related to multiplying complex numbers.

Answers

8. **a.** 34
   b. 52
   c. 65
9. Sample answer: Each problem has the form $(a + bi)(a - bi)$. All of the products are real numbers.
10. Sample answer: For $(a + bi)(c + di)$, the imaginary terms of the product are $(ad)i$ and $(bc)i$. If these imaginary terms are opposites, their sum will be $0i$ or 0, leaving only the real terms of the product.
11. Yes. Sample explanation: Let bi and ci be any 2 imaginary numbers, where b and c are real. Find their product: $(bi) \cdot (ci) = (bc)i^2 = (bc)(-1) = -bc$. Because b and c are real numbers, the product $-bc$ is a real number.

My Notes

6. **Express regularity in repeated reasoning.** Use Example B and your knowledge of operations of real numbers to write a general formula for the multiplication of two complex numbers.

   $(a + bi) \cdot (c + di) =$ **$(ac - bd) + (ad + bc)i$**

7. Use operations of complex numbers to verify that the two solutions that Cardano found, $x = 5 + \sqrt{-15}$ and $x = 5 - \sqrt{-15}$, have a sum of 10 and a product of 40.

   $\left(5 + i\sqrt{15}\right) + \left(5 - i\sqrt{15}\right) = 10$ **and**

   $\left(5 + i\sqrt{15}\right)\left(5 - i\sqrt{15}\right) = 25 - 5i\sqrt{15} + 5i\sqrt{15} - 15i^2 = 25 + 15 = 40$

MATH TIP

Since $i = \sqrt{-1}$, the powers of i can be evaluated as follows:

$i^1 = i$
$i^2 = -1$
$i^3 = i^2 \cdot i = -1i = -i$
$i^4 = i^2 \cdot i^2 = (-1)^2 = 1$

Since $i^4 = 1$, further powers repeat the pattern shown above.

$i^5 = i^4 \cdot i = i$
$i^6 = i^4 \cdot i^2 = i^2 = -1$
$i^7 = i^4 \cdot i^3 = i^3 = -i$
$i^8 = i^4 \cdot i^4 = i^4 = 1$

Check Your Understanding

8. Find each product.
   a. $(5 + 3i)(5 - 3i)$ **b.** $(-6 - 4i)(-6 + 4i)$
   c. $(8 + i)(8 - i)$
9. What patterns do you observe in the products in Item 8?
10. Explain how the product of two complex numbers can be a real number, even though both factors are not real numbers.
11. **Critique the reasoning of others.** A student claims that the product of any two imaginary numbers is a real number. Is the student correct? Explain your reasoning.

MINI-LESSON: Powers of *i*

If students need additional help with writing expressions as a power of *i* with an exponent between 1 and 4, a mini-lesson is available to provide practice.

See the Teacher Resources at SpringBoard Digital for a student page for this mini-lesson.

Lesson 8-2
Operations with Complex Numbers

My Notes

The ***complex conjugate*** of $a + bi$ is defined as $a - bi$. For example, the complex conjugate of $2 + 3i$ is $2 - 3i$.

MATH TERMS

The **complex conjugate** of a complex number $a + bi$ is $a - bi$.

12. A special property of multiplication of complex numbers occurs when a number is multiplied by its conjugate. Multiply each number by its conjugate and then describe the product when a number is multiplied by its conjugate.

a. $2 - 9i$ **$(2 - 9i)(2 + 9i) = 4 - 81i^2 = 4 + 81 = 85$**

b. $-5 + 2i$ **$(-5 + 2i)(-5 - 2i) = 25 - 4i^2 = 29$**

Sample answer: A complex number multiplied by its conjugate results in a real number. The product is the sum of the squares of *a* and *b*.

13. Write an expression to complete the general formula for the product of a complex number and its complex conjugate.

$(a + bi)(a - bi) =$ **$a^2 - (bi)^2 = a^2 - b^2i^2 = a^2 + b^2$**

To divide two complex numbers, start by multiplying both the dividend and the divisor by the conjugate of the divisor. This step results in a divisor that is a real number.

Example C

Divide $\frac{4 - 5i}{2 + 3i}$.

Step 1: Multiply the numerator and denominator by the complex conjugate of the divisor.
$$\frac{4-5i}{2+3i} = \frac{4-5i}{2+3i} \cdot \frac{(2-3i)}{(2-3i)}$$

Step 2: Simplify and substitute -1 for i^2.
$$= \frac{8 - 22i + 15i^2}{4 - 6i + 6i - 9i^2}$$
$$= \frac{8 - 22i - 15}{4 + 9}$$

Step 3: Simplify and write in the form $a + bi$.
$$= \frac{-7 - 22i}{13} = -\frac{7}{13} - \frac{22}{13}i$$

Solution: $\frac{4-5i}{2+3i} = -\frac{7}{13} - \frac{22}{13}i$

Developing Math Language

The *complex conjugate* of a complex number $a + bi$ is $a - bi$. What is special about the relationship between a complex number and its conjugate is that their product is always a real number. In mathematics, a *conjugate* is a binomial formed from another binomial by changing the sign of the second term. The other place students have seen a conjugate is when rationalizing the denominator containing a radical. For example, $\frac{1}{1+\sqrt{2}}$ is rationalized by multiplying both the numerator and the denominator by the conjugate of the denominator:

$$\frac{1}{1+\sqrt{2}} \cdot \frac{1-\sqrt{2}}{1-\sqrt{2}} = \frac{1-\sqrt{2}}{1-2} = -1+\sqrt{2}.$$

So, just as multiplying a binomial with a radical by its conjugate "clears" the radical sign, multiplying a binomial with an imaginary number "clears" the imaginary number.

12–13 Activating Prior Knowledge While the process of finding the conjugate of a complex number is relatively simple, many students confuse this with finding the opposite of a complex number by changing the signs of both the real and imaginary parts. Make sure students realize that the product of a complex number and its conjugate will always be a real number, generalizing that result in Item 13.

Example C Quickwrite, Activating Prior Knowledge Have students summarize the process of dividing complex numbers and explain why you can perform this process using the complex conjugate. Ask them to write down anything they have already learned in algebra that this process reminds them of.

Some may articulate that you can multiply by the complex conjugate as a way to get the quotient in standard form of a complex number, $a + bi$. Make sure students understand that multiplying both the numerator and denominator by the complex conjugate of the denominator does not change the "value" because you are multiplying by an expression that is equal to 1. This process has some similarities to rationalizing the denominator.

Example C (continued) Two different yet appropriate answers exist for Try These C Item a, and both should be explored regardless of student response. The identity of multiplication (because the number is essentially multiplied by 1) and inverse operations (multiply the quotient by the denominator for a product equal to the numerator) are concepts that warrant exploration.

Technology Tip

Students should learn to use their calculators to perform operations using *i*. This will give them another tool to check their work. Help them find *i* on their calculator — on TI graphing calculators, press [2ND] [·]. Remind students to be careful to close parentheses when entering operations.

14 Group Presentation, Debriefing Students generalize the operation of division of complex numbers. While this may appear to be a more daunting task than previous generalizations of operations, students will be successful, especially if they recognize that the denominator can be found easily using Item 13.

Check Your Understanding

Debrief students' answers to these items to ensure that they understand concepts related to dividing complex numbers.

Answers

15. The quotient of 2 imaginary numbers is a real number. Sample example:

$$\frac{3i}{4i} = \frac{3i}{4i} \bullet \frac{(-4i)}{(-4i)} = \frac{-12i^2}{-16i^2} = \frac{12}{16} = \frac{3}{4}$$

16. The quotient of a real number and an imaginary number is an imaginary number. Sample example:

$$\frac{3}{4i} = \frac{3}{4i} \bullet \frac{(-4i)}{(-4i)} = \frac{-12i}{-16i^2} = \frac{-12i}{16} = -\frac{3}{4}i$$

17. D

18. Sample explanation: I know that i^{-1} is equal to $\frac{1}{i}$. To divide 1 by *i*, multiply the dividend and the divisor by the conjugate of *i*:

$$\frac{1}{i} = \frac{1}{i} \bullet \frac{(-i)}{(-i)} = \frac{-i}{-i^2} = \frac{-i}{1} = -i.$$

My Notes

TECHNOLOGY TIP

Many graphing calculators have the capability to perform operations on complex numbers.

Try These C

a. In Example C, why is the quotient $-\frac{7}{13} - \frac{22}{13}i$ equivalent to the original expression $\frac{4-5i}{2+3i}$? **There are two acceptable responses.**

First, multiplication by $\frac{2-3i}{2-3i}$ is the equivalent of multiplication by 1, the multiplicative identity. Therefore the result is equivalent to the original expression.

Also, multiplying $-\frac{7}{13} - \frac{22}{13}i$ by the divisor, $2 + 3i$, yields $4 - 5i$, as division is the inverse operation of multiplication.

$$\left(-\frac{7}{13} - \frac{22}{13}i\right)(2+3i) = -\frac{14}{13} - \frac{21}{13}i - \frac{44}{13}i - \frac{66}{13}i^2 = \left(\frac{66}{13} - \frac{14}{13}\right) + \left(-\frac{21}{13} - \frac{44}{13}\right)i$$
$$= \frac{52}{13} - \frac{65}{13}i = 4 - 5i$$

Divide the complex numbers. Write your answers on notebook paper. Show your work.

b. $\frac{5i}{2+3i}$ $\frac{15}{13} + \frac{10}{13}i$ **c.** $\frac{5+2i}{3-4i}$ $\frac{7}{25} + \frac{26}{25}i$ **d.** $\frac{1-i}{\sqrt{3}+4i}$ $\frac{\sqrt{3}-4}{19} + \frac{-4-\sqrt{3}}{19}i$

14. Express regularity in repeated reasoning. Use Example C and your knowledge of operations of real numbers to write a general formula for the division of two complex numbers.

$$\frac{(a+bi)}{(c+di)} = \frac{(ac+bd)}{c^2+d^2} + \frac{(bc-ad)}{c^2+d^2}i$$

MATH TIP

For Item 17, $n^{-1} = \frac{1}{n}$ for $n \neq 0$.
So, $i^{-1} = \frac{1}{i}$.

Check Your Understanding

15. Make a conjecture about the quotient of two imaginary numbers where the divisor is not equal to 0*i*. Is the quotient real, imaginary, or neither? Give an example to support your conjecture.

16. Make a conjecture about the quotient of a real number divided by an imaginary number not equal to 0*i*. Is the quotient real, imaginary, or neither? Give an example to support your conjecture.

17. Which of the following is equal to i^{-1}?

A. 1 **B.** -1 **C.** i **D.** $-i$

18. Explain your reasoning for choosing your answer to Item 17.

Lesson 8-2
Operations with Complex Numbers

My Notes

LESSON 8-2 PRACTICE

19. Find each sum or difference.
a. $(6 - 5i) + (-2 + 6i)$ **b.** $(4 + i) + (-4 + i)$
c. $(5 - 3i) - (3 - 5i)$ **d.** $(-3 + 8i) - \left(\frac{3}{2} + \frac{1}{2}i\right)$

20. Multiply. Write each product in the form $a + bi$.
a. $(2 + 9i)(3 - i)$ **b.** $(-5 + 8i)(2 - i)$
c. $(8 + 15i)(8 - 15i)$ **d.** $(8 - 4i)(5i)$

21. Divide. Write each quotient in the form $a + bi$.
a. $\frac{1+4i}{4-i}$ **b.** $\frac{-2+5i}{3-4i}$ **c.** $\frac{7-3i}{i}$

22. Use substitution to show that the solutions of the equation $x^2 - 4x + 20 = 0$ are $x = 2 + 4i$ and $x = 2 - 4i$.

23. **Make use of structure.** What is the sum of any complex number $a + bi$ and its complex conjugate?

24. Explain how to use the Commutative, Associative, and Distributive Properties to add the complex numbers $5 + 8i$ and $6 + 2i$.

ASSESS

Students' answers to Lesson Practice problems will provide you with a formative assessment of their understanding of the lesson concepts and their ability to apply their learning.

See the Activity Practice for additional problems for this lesson. You may assign the problems here or use them as a culmination for the activity.

LESSON 8-2 PRACTICE

19. **a.** $4 + i$
b. $2i$
c. $2 + 2i$
d. $-\frac{9}{2} + \frac{15}{2}i$

20. **a.** $15 + 25i$
b. $-2 + 21i$
c. $289 + 0i$
d. $20 + 40i$

21. **a.** $0 + i$
b. $-\frac{26}{25} + \frac{7}{25}i$
c. $-3 - 7i$

22.
$$(2 + 4i)^2 - 4(2 + 4i) + 20 = 0$$
$$4 + 8i + 8i + 16i^2 - 8 - 16i + 20 = 0$$
$$(4 - 16 - 8 + 20) + (8i + 8i - 16i) = 0$$
$$0 + 0i = 0$$
$$0 = 0 \checkmark$$
$$(2 - 4i)^2 - 4(2 - 4i) + 20 = 0$$
$$4 - 8i - 8i + 16i^2 - 8 + 16i + 20 = 0$$
$$(4 - 16 - 8 + 20) + (-8i - 8i + 16i) = 0$$
$$0 + 0i = 0$$
$$0 = 0 \checkmark$$

23. $2a$

24. Sample answer: Use the Commutative Property to write the sum $5 + 8i + 6 + 2i$ as $5 + 6 + 8i + 2i$. Then use the Associative Property to group the real addends and the imaginary addends: $(5 + 6) + (8i + 2i)$. Add the real addends, and then use the Distributive Property to add the imaginary addends: $11 + (8 + 2)i = 11 + 10i$.

ADAPT

Check students' answers to the Lesson Practice to ensure that they have mastered arithmetic operations involving complex numbers. In addition, make sure that students have a thorough understanding of complex conjugates. Provide additional practice with operations on complex numbers by having students solve quadratic equations with complex solutions and checking the solutions through substitution.

Lesson 8-3

PLAN

Pacing: 1 class period

Chunking the Lesson

#1 #2
Check Your Understanding
Example A
Check Your Understanding
Lesson Practice

TEACH

Bell-Ringer Activity

Remind students about the factoring process they learned in algebra, and conduct a review of factoring in general. Remind students that to factor a *difference of two squares,* $a^2 - b^2$, use the rule $(a + b)(a - b)$. However, one cannot factor the sum of two squares over the set of real numbers. Present sums of squares and differences of squares and ask students which can be factored and which cannot.

1 Summarizing, Debriefing The sum of two squares cannot be factored in the real number system; however, in the complex number system, it can. The factorization of the sum of two squares is $a^2 + b^2 = (a + bi)(a - bi)$. By definition, these factors are *complex conjugates* of each other.

2 Think-Pair-Share, Activating Prior Knowledge, Debriefing By applying the generalization demonstrated in Item 1 and drawing from their prior experience with factoring, students should be able to successfully factor these sums of two squares. Remind students to go back to the first thing they were told to look for when factoring, a common factor amongst the terms. If there is a common factor, this might help transform the expression to look more like a sum of two squares. In Item 2d, students will likely need assistance, as the coefficients of 3 and 20 are not square numbers and have no common factors.

My Notes

Learning Targets:

- Factor quadratic expressions using complex conjugates.
- Solve quadratic equations with complex roots by factoring.

SUGGESTED LEARNING STRATEGIES: Discussion Groups, Look for a Pattern, Quickwrite, Self Revision/Peer Revision, Paraphrasing

1. Look back at your answer to Item 13 in the previous lesson.
   a. Given your answer, what are the factors of the expression $a^2 + b^2$? Justify your answer.
   $a + bi$ and $a - bi$. Sample explanation: Multiply the factors: $a^2 + b^2$: $(a + bi)(a - bi) = a^2 - abi + abi - b^2i^2 = a^2 - b^2(-1) = a^2 + b^2$

   b. What is the relationship between the factors of $a^2 + b^2$?
   They are complex conjugates.

MATH TIP

You can check your answers to Item 2 by multiplying the factors. Check that the product is equal to the original expression.

You can use complex conjugates to factor quadratic expressions that can be written in the form $a^2 + b^2$. In other words, you can use complex conjugates to factor the sum of two squares.

2. **Express regularity in repeated reasoning.** Use complex conjugates to factor each expression.
   a. $16x^2 + 25$
   $(4x)^2 + 5^2 = (4x + 5i)(4x - 5i)$

   b. $36x^2 + 100y^2$
   $4(9x^2 + 25y^2) = 4[(3x)^2 + (5y)^2] = 4(3x + 5yi)(3x - 5yi)$

   c. $2x^2 + 8y^2$
   $2(x^2 + 4y^2) = 2[x^2 + (2y)^2] = 2(x + 2yi)(x - 2yi)$

   d. $3x^2 + 20y^2$
   $\left(x\sqrt{3}\right)^2 + \left(2y\sqrt{5}\right)^2 = \left(x\sqrt{3} + 2yi\sqrt{5}\right)\left(x\sqrt{3} - 2yi\sqrt{5}\right)$

Lesson 8-3
Factoring with Complex Numbers

My Notes

Check Your Understanding

3. Explain how to factor the expression $81x^2 + 64$.
4. Compare and contrast factoring an expression of the form $a^2 - b^2$ and an expression of the form $a^2 + b^2$.
5. **Critique the reasoning of others.** A student incorrectly claims that the factored form of the expression $4x^2 + 5$ is $(4x + 5i)(4x - 5i)$.
   a. Describe the error that the student made.
   b. How could the student have determined that his or her answer is incorrect?
   c. What is the correct factored form of the expression?

You can solve some quadratic equations with complex solutions by factoring.

Example A

Solve $9x^2 + 16 = 0$ by factoring.

	Original equation	$9x^2 + 16 = 0$
Step 1:	Factor the left side.	$(3x + 4i)(3x - 4i) = 0$
Step 2:	Apply the Zero Product Property.	$3x + 4i = 0$ or $3x - 4i = 0$
Step 3:	Solve each equation for x.	

Solution: $x = -\frac{4}{3}i$ or $x = \frac{4}{3}i$

Try These A

a. Solve $x^2 + 81 = 0$ and check by substitution.

$x^2 + 81 = 0$	Original equation
$(x + 9i)(x - 9i) = 0$	Factor the left side.
$x + 9i = 0$ or $x - 9i = 0$	Apply the Zero Product Property.
$x = -9i$ or $x = 9i$	Solve each equation for x.

Solve each equation by factoring. Show your work.

b. $100x^2 + 49 = 0$
$x = -\frac{7}{10}i, x = \frac{7}{10}i$

c. $25x^2 = -4$
$x = -\frac{2}{5}i, x = \frac{2}{5}i$

d. $2x^2 + 36 = 0$
$x = -3i\sqrt{2}, x = 3i\sqrt{2}$

e. $4x^2 = -45$
$x = -\frac{3\sqrt{5}}{2}i, x = \frac{3\sqrt{5}}{2}i$

Check Your Understanding

Debrief students' answers to these items to ensure that they understand concepts related to factoring quadratic expressions.

Answers

3. Sample answer: The terms have no common factors, so start by writing the expression in the form $a^2 + b^2$: $(9x)^2 + 8^2$. Then factor the sum of squares by writing it in the form $(a + bi)(a - bi)$: $(9x)^2 + 8^2 = (9x + 8i)(9x - 8i)$.
4. Sample answer: The expression $a^2 - b^2$ is a difference of squares, and its factored form is $(a + b)(a - b)$. The expression $a^2 + b^2$ is a sum of squares, and its factored form is $(a + bi)(a - bi)$. The factored form of $a^2 - b^2$ is the product of 2 binomials. The first terms of each binomial are the same, and the second terms are opposites. The factored form of $a^2 + b^2$ is the product of 2 complex numbers. The real parts of each complex number are the same, and the imaginary parts are opposites.
5. a. The student factored an expression having the form $a^2x^2 + b^2$ as $(a^2x + b^2i)(a^2x - b^2i)$, instead of correctly factoring it as $(ax + bi)(ax - bi)$.
   b. The student could have used multiplication to find that $(4x + 5i)(4x - 5i)$ is equal to $16x^2 + 25$ instead of $4x^2 + 5$.
   c. $\left(2x + i\sqrt{5}\right)\left(2x - i\sqrt{5}\right)$

Example A Debriefing In the Try These A, Item b is very straightforward, written as a sum of two squares, a being $10x$ and b being 7. Item c requires the student to add 4 to both sides to make it a sum of two squares equal to zero, where a is $5x$ and b is 2. Item d is a little more complex because students may or may not factor out the GCF of 2. The solution will be the same either way, but the steps will look very different, so be prepared to address this. If students do not factor out a common factor of 2, they should have $(x\sqrt{2} + 6i)(x\sqrt{2} + 6i) = 0$, for which they will have to rationalize a denominator in the process of isolating x. If they do factor out a common factor of 2, they should have $x^2 + 18 = 0$ to solve, where $a = x$ and $b = \sqrt{18}$, or $3\sqrt{2}$. Item e requires students to add 45 to both sides to make it a sum of two squares equal to zero, where $a = 2x$ and $b = 3\sqrt{5}$.

Check Your Understanding

Debrief students' answers to these items to ensure that they understand concepts related to the types of solutions of quadratic equations.

Answers

6. a. Real solutions; Sample explanation: The left side of the equation can be factored as a difference of squares: $(x+12)(x-12)=0$. The solutions of the equation are $x=-12$ and $x=12$, which are both real.
   b. Imaginary solutions; Sample explanation: The left side of the equation can be factored as a sum of squares: $(x+12i)(x-12i)=0$. The solutions of the equation are $x=-12i$ and $x=12i$, which are both imaginary.
7. a. $x=-\frac{b}{a}i$ and $x=\frac{b}{a}i$; The left side of the equation can be factored as a sum of squares: $(ax+bi)(ax-bi)=0$. Apply the Zero Product Property: $ax+bi=0$ or $ax-bi=0$. Then solve each equation for x: $x=-\frac{b}{a}i$ and $x=\frac{b}{a}i$.
   b. Answers may vary but should be one of the following: The solutions are opposites. The solutions are additive inverses. The solutions are complex conjugates.
8. Sample answer: Substitute 0 for $f(x)$: $0=x^2+225$. Factor the right side as a sum of squares: $0=(x+15i)(x-15i)$. Then solve for x: $x=-15i$ or $x=15i$. The solutions of the function are $-15i$ and $15i$.

ASSESS

Students' answers to Lesson Practice problems will provide you with a formative assessment of their understanding of the lesson concepts and their ability to apply their learning.

See the Activity Practice for additional problems for this lesson. You may assign the problems here or use them as a culmination for the activity.

ADAPT

Check students' answers to the Lesson Practice to ensure that they understand how to factor the sum of two squares using complex numbers. Further, students should be able to use such factors to solve quadratic equations using the Zero Product Property. Many students will benefit from repeated comparing and contrasting of the sum of two squares and the difference of two squares.

My Notes

Lesson 8-3
Factoring with Complex Numbers

Check Your Understanding

6. Tell whether each equation has real solutions or imaginary solutions and explain your answer.
   a. $x^2-144=0$ b. $x^2+144=0$
7. a. What are the solutions of a quadratic equation that can be written in the form $a^2x^2+b^2=0$, where a and b are real numbers and $a\neq 0$? Show how you determined the solutions.
   b. What is the relationship between the solutions of a quadratic equation that can be written in the form $a^2x^2+b^2=0$?
8. Explain how you could find the solutions of the quadratic function $f(x)=x^2+225$ when $f(x)=0$.

LESSON 8-3 PRACTICE

Use complex conjugates to factor each expression.

9. $3x^2+12$
10. $5x^2+80y^2$
11. $9x^2+11$
12. $2x^2+63y^2$

Solve each equation by factoring.

13. $2x^2+50=0$
14. $3x^2=-54$
15. $4x^2+75=0$
16. $32x^2=-98$
17. **Reason quantitatively.** Solve the equations $9x^2-64=0$ and $9x^2+64=0$ by factoring. Then describe the relationship between the solutions of $9x^2-64=0$ and the solutions of $9x^2+64=0$.

LESSON 8-3 PRACTICE

9. $3(x+2i)(x-2i)$
10. $5(x+4yi)(x-4yi)$
11. $\left(3x+i\sqrt{11}\right)\left(3x-i\sqrt{11}\right)$
12. $\left(x\sqrt{2}+3yi\sqrt{7}\right)\left(x\sqrt{2}-3yi\sqrt{7}\right)$
13. $x=-5i, x=5i$
14. $x=-3i\sqrt{2}, x=3i\sqrt{2}$
15. $x=-\frac{5\sqrt{3}}{2}i, x=\frac{5\sqrt{3}}{2}i$
16. $x=-\frac{7}{4}i, x=\frac{7}{4}i$
17. The solutions of $9x^2-64=0$ are $x=-\frac{8}{3}$ and $x=\frac{8}{3}$. The solutions of $9x^2+64=0$ are $x=-\frac{8}{3}i$ and $x=\frac{8}{3}i$. The solutions of $9x^2-64=0$ are real, and the solutions of $9x^2+64=0$ are imaginary. The real parts of the solution of $9x^2-64=0$ are the same as the imaginary parts of the solutions of $9x^2+64=0$.

ACTIVITY 8 PRACTICE

Write your answers on notebook paper. Show your work.

Lesson 8-1

1. Write each expression in terms of i.

a. $\sqrt{-64}$

b. $\sqrt{-31}$

c. $-7+\sqrt{-12}$

d. $5-\sqrt{-50}$

2. Which expression is equivalent to $5i$?

A. $\sqrt{-5}$ **B.** $-\sqrt{5}$

C. $\sqrt{-25}$ **D.** $-\sqrt{25}$

3. Use the Quadratic Formula to solve each equation.

a. $x^2+5x+9=0$

b. $2x^2-4x+5=0$

4. The sum of two numbers is 12, and their product is 100.

a. Let x represent one of the numbers. Write an expression for the other number in terms of x. Use the expressions to write an equation that models the situation given above.

b. Use the Quadratic Formula to solve the equation. Write the solutions in terms of i.

5. Explain why each of the following is a complex number, and identify its real part and its imaginary part.

a. $5+3i$ **b.** $\sqrt{2}-i$

c. $-14i$ **d.** $\frac{3}{4}$

6. Draw the complex plane on grid paper. Then graph each complex number on the plane.

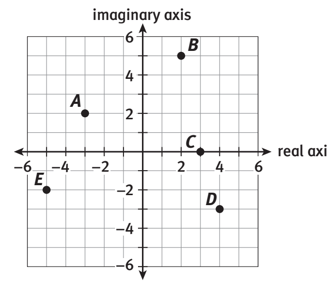
a. $-4i$ **b.** $6+2i$

c. $-3-4i$ **d.** $3-5i$

e. $-2+5i$

7. What complex number does the ordered pair $(5, -3)$ represent on the complex plane? Explain.

8. Name the complex number represented by each labeled point on the complex plane below.



Lesson 8-2

9. Find each sum or difference.

a. $(5-6i)+(-3+9i)$

b. $(2+5i)+(-5+3i)$

c. $(9-2i)-(1+6i)$

d. $(-5+4i)-\left(\frac{7}{3}+\frac{1}{6}i\right)$

10. Find each product, and write it in the form $a+bi$.

a. $(1+4i)(5-2i)$

b. $(-2+3i)(3-2i)$

c. $(7+24i)(7-24i)$

d. $(8-3i)(4-2i)$

11. Find each quotient, and write it in the form $a+bi$.

a. $\frac{3+2i}{5-2i}$ **b.** $\frac{-1+i}{5-2i}$

c. $\frac{10-2i}{5i}$ **d.** $\frac{3+i}{3-i}$

12. Explain how to use the Commutative, Associative, and Distributive Properties to perform each operation.

a. Subtract $(3+4i)$ from $(8+5i)$.

b. Multiply $(-2+3i)$ and $(4-6i)$.

ACTIVITY 8 Continued

ACTIVITY PRACTICE

1. a. $8i$ **b.** $i\sqrt{31}$

c. $-7+2i\sqrt{3}$ **d.** $5-5i\sqrt{2}$

2. C

3. a. $x=-\frac{5}{2}\pm\frac{\sqrt{11}}{2}i$

b. $x=1\pm\frac{\sqrt{6}}{2}i$

4. a. $12-x$; $x(12-x)=100$

b. $x=6-8i$, $x=6+8i$

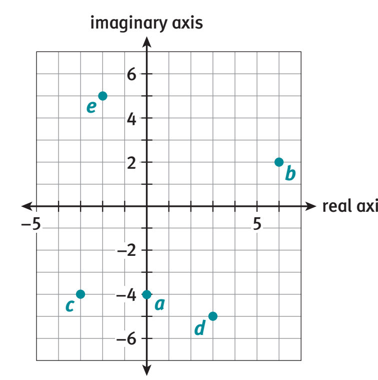
5. a. $5+3i$ is a complex number because it has the form $a+bi$. The real part is 5, and the imaginary part is $3i$.

b. $\sqrt{2}-i$ is a complex number because it can be written in the form $a+bi$: $\sqrt{2}+(-1)i$. The real part is $\sqrt{2}$, and the imaginary part is $-1i$.

c. $-14i$ is a complex number because it can be written in the form $a+bi$: $0+(-14)i$. The real part is 0, and the imaginary part is $-14i$.

d. $\frac{3}{4}$ is a complex number because it can be written in the form $a+bi$: $\frac{3}{4}+0i$. The real part is $\frac{3}{4}$, and the imaginary part is $0i$.

6. a–e.



7. $5-3i$; The first number in the ordered pair is the real part of the complex number, and the second number in the ordered pair is the imaginary part of the complex number.

8. A: $-3+2i$
B: $2+5i$
C: 3
D: $4-3i$
E: $-5-2i$

9. a. $2+3i$

b. $-3+8i$

c. $8-8i$

d. $-\frac{22}{3}+\frac{23}{6}i$

10. a. $13+18i$

b. $0+13i$

c. $625+0i$

d. $26-28i$

11. a. $\frac{11}{29}+\frac{16}{29}i$

b. $-\frac{7}{29}+\frac{3}{29}i$

c. $-\frac{2}{5}-2i$

d. $\frac{4}{5}+\frac{3}{5}i$

12. a. Sample answer: Use the Distributive Property to rewrite subtraction as addition of the opposite: $(8+5i)-(3+4i)=(8+5i)+[-3+(-4)i]$. Then use the Commutative and Associative Properties to group the real addends and the imaginary addends: $[8+(-3)]+[5i+(-4)i]$. Add the real addends, and then use the Distributive Property to add the imaginary addends: $5+[5+(-4)]i=5+i$.

b. Sample answer: First, apply the Distributive Property to multiply: $(-2+3i)(4-6i)=-8+12i+12i+18$. Then use the Commutative and Associative Properties to group the real addends and the imaginary addends: $(-8+18)+(12i+12i)$. Add the real addends, and then use the Distributive Property to add the imaginary addends: $10+(12+12)i=10+24i$.

13. Accept any complex number with a real part of -4. Sample answer: $-4 + 2i$; $(4 - 8i) + (-4 + 2i) = 0 - 6i = -6i$, and $-6i$ is an imaginary number.

14. A

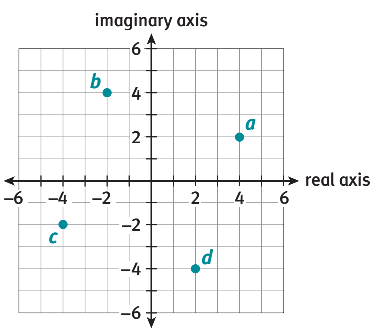
15. a. 1 **b.** -6
c. $-8i$ **d.** $-\frac{9}{4}$

16. $2bi$

17.
$$(3 + 5i)^2 - 6(3 + 5i) + 34 = 0$$
$$9 + 15i + 15i + 25i^2 - 18 - 30i + 34 = 0$$
$$(9 - 25 - 18 + 34) + (15i + 15i - 30i) = 0$$
$$0 + 0i = 0$$
$$0 = 0$$

$$(3 - 5i)^2 - 6(3 - 5i) + 34 = 0$$
$$9 - 15i - 15i + 25i^2 - 18 + 30i + 34 = 0$$
$$(9 - 25 - 18 + 34) + (-15i - 15i + 30i) = 0$$
$$0 + 0i = 0$$
$$0 = 0$$

18. a–d.



b. $-2 + 4i$
c. $-4 - 2i$
d. $2 - 4i$
e. Sample answers: Each of the complex numbers is the same distance from the origin on the complex plane. Each number represents the previous number rotated by 90° on the complex plane. The real part of each complex number is the opposite of the imaginary part of the previous complex number. The imaginary part of each complex number is the same as the real part of the previous complex number.
f. The product is equal to $-b + ai$.

19. a. $(x + 11i)(x - 11i)$
b. $2(x + 8yi)(x - 8yi)$
c. $4\left(x + yi\sqrt{15}\right)\left(x - yi\sqrt{15}\right)$
d. $\left(3x + 2yi\sqrt{35}\right)\left(3x - 2yi\sqrt{35}\right)$

ADDITIONAL PRACTICE

If students need more practice on the concepts in this activity, see the Teacher Resources at SpringBoard Digital for additional practice problems.

ACTIVITY 8 *continued*

Introduction to Complex Numbers
Cardano's Imaginary Numbers

13. Give an example of a complex number you could add to $4 - 8i$ that would result in an imaginary number. Show that the sum of the complex numbers is equal to an imaginary number.

14. What is the complex conjugate of $-3 + 7i$?
A. $-3 - 7i$ **B.** $3 - 7i$
C. $3 + 7i$ **D.** $7 - 3i$

15. Simplify each expression.
a. $-i^2$ **b.** $-6i^4$
c. $(2i)^3$ **d.** $\left(\frac{3}{2i^3}\right)^2$

16. What is the difference of any complex number $a + bi$ and its complex conjugate?

17. Use substitution to show that the solutions of the equation $x^2 - 6x + 34 = 0$ are $x = 3 + 5i$ and $x = 3 - 5i$.

18. a. Graph the complex number $4 + 2i$ on a complex plane.
b. Multiply $4 + 2i$ by i, and graph the result.
c. Multiply the result from part b by i, and graph the result.
d. Multiply the result from part c by i, and graph the result.
e. Describe any patterns you see in the complex numbers you graphed.
f. What happens when you multiply a complex number $a + bi$ by i?

Lesson 8-3

19. Use complex conjugates to factor each expression.
a. $x^2 + 121$ **b.** $2x^2 + 128y^2$
c. $4x^2 + 60y^2$ **d.** $9x^2 + 140y^2$

20. Explain how to solve the equation $2x^2 + 100 = 0$ by factoring.

21. Solve each equation by factoring.
a. $x^2 + 64 = 0$ **b.** $x^2 = -120$
c. $4x^2 + 169 = 0$ **d.** $25x^2 = -48$

22. Which equation has solutions of $x = -\frac{2}{3}i$ and $x = \frac{2}{3}i$?
A. $3x^2 - 2 = 0$ **B.** $3x^2 + 2 = 0$
C. $9x^2 - 4 = 0$ **D.** $9x^2 + 4 = 0$

23. What are the solutions of each quadratic function?
a. $f(x) = x^2 + 1$
b. $f(x) = 25x^2 + 36$

24. Without solving the equation, explain how you know that $x^2 + 48 = 0$ has imaginary solutions.

MATHEMATICAL PRACTICES
Look for and Express Regularity in Repeated Reasoning

25. Find the square of each complex number.
a. $(4 + 5i)$
b. $(2 + 3i)$
c. $(4 - 2i)$
d. Use parts a–c and your knowledge of operations of real numbers to write a general formula for the square of a complex number $(a + bi)$.

---

20. Factor 2 from both terms of the left side: $2(x^2 + 50) = 0$. Write $x^2 + 50$ as a sum of squares: $x^2 + \left(5\sqrt{2}\right)^2 = 0$. Factor the left side: $2\left(x + 5i\sqrt{2}\right)\left(x - 5i\sqrt{2}\right) = 0$. Solve for x: $x = -5i\sqrt{2}$ or $x = 5i\sqrt{2}$.

21. a. $x = -8i, x = 8i$
b. $x = -2i\sqrt{30}, x = 2i\sqrt{30}$
c. $x = -\frac{13}{2}i, x = \frac{13}{2}i$
d. $x = -\frac{4\sqrt{3}}{5}i, x = \frac{4\sqrt{3}}{5}i$

22. D

23. a. $-i$ and i
b. $-\frac{6}{5}i$ and $\frac{6}{5}i$

24. Sample answer: If you subtract 48 from both sides of the equation, you get $x^2 = -48$. To solve for x, you must find the positive and negative square roots of a negative number which are imaginary, so the solutions are imaginary.

25. a. $(4 + 5i)^2 = -9 + 40i$
b. $(2 + 3i)^2 = -5 + 12i$
c. $(4 - 2i)^2 = 12 - 16i$
d. $(a + bi)^2 = (a^2 - b^2) + 2abi$

Solving $ax^2 + bx + c = 0$

ACTIVITY 9

Deriving the Quadratic Formula

Lesson 9-1 Completing the Square and Taking Square Roots

Learning Targets:
- Solve quadratic equations by taking square roots.
- Solve quadratic equations $ax^2 + bx + c = 0$ by completing the square.

SUGGESTED LEARNING STRATEGIES: Marking the Text, Group Presentation, Quickwrite, Create Representations

To solve equations of the form $ax^2 + c = 0$, isolate x^2 and take the square root of both sides of the equation.

Example A

Solve $5x^2 - 23 = 0$ for x.

$5x^2 - 23 = 0$

Step 1: Add 23 to both sides. $5x^2 = 23$

Step 2: Divide both sides by 5. $\frac{5x^2}{5} = \frac{23}{5}$

Step 3: Simplify to isolate x^2. $x^2 = \frac{23}{5}$

Step 4: Take the square root of both sides. $x = \pm\frac{\sqrt{23}}{\sqrt{5}}$

Step 5: Rationalize the denominator. $x = \pm\frac{\sqrt{23}}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}}$

Step 6: Simplify. $x = \pm\frac{\sqrt{115}}{5}$

Solution: $x = \pm\frac{\sqrt{115}}{5}$

Try These A

Make use of structure. Solve for x. Show your work.

a. $9x^2 - 49 = 0$

$x = \frac{7}{3}, x = -\frac{7}{3}$

b. $25x^2 - 7 = 0$

$x = \frac{\sqrt{7}}{5}, x = -\frac{\sqrt{7}}{5}$

c. $5x^2 - 16 = 0$

$x = \frac{4}{\sqrt{5}} = \frac{4\sqrt{5}}{5}$,

$x = -\frac{4}{\sqrt{5}} = -\frac{4\sqrt{5}}{5}$

d. $4x^2 + 15 = 0$

$x = \frac{i\sqrt{15}}{2}, x = -\frac{i\sqrt{15}}{2}$

My Notes

MATH TIP

When taking the square root of both sides of an equation, include both positive and negative roots. For example,

$x^2 = 4$

$x = \pm\sqrt{4}$

$x = \pm 2$

MATH TIP

To rewrite an expression so that there are no radicals in the denominator, you must *rationalize the denominator* by multiplying both the numerator and denominator by the radical.

Example:

$\frac{7}{\sqrt{3}} = \frac{7}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{7\sqrt{3}}{3}$

Common Core State Standards for Activity 9

HSN-CN.C.7 Solve quadratic equations with real coefficients that have complex solutions.

ACTIVITY 9

Guided

Activity Standards Focus

In Activity 9, students solve quadratic equations using a variety of techniques. They solve quadratic equations by taking square roots and by completing the square. Students derive the quadratic formula and then use it to solve equations. They use the discriminant to determine the nature of the solutions. Throughout this activity, emphasize when to use each solution method and compare and contrast these solution methods and the process of solving by factoring.

Lesson 9-1

PLAN

Pacing: 1 class period

Chunking the Lesson

Example A #1

Example B #2

Check Your Understanding

Example C

Check Your Understanding

Lesson Practice

TEACH

Bell-Ringer Activity

Have students rationalize the denominators in the following expressions.

1. $\frac{5}{\sqrt{2}}$ $\left[\frac{5\sqrt{2}}{2}\right]$
2. $\frac{24}{\sqrt{8}}$ $[6\sqrt{2}]$
3. $\frac{40\sqrt{10}}{\sqrt{2}}$ $[40\sqrt{5}]$

Example A Marking the Text

Students may ask why the final answer is not $x = \pm\frac{\sqrt{23}}{\sqrt{5}}$. Irrational solutions with radicals in the denominator will be rationalized in this activity.

Note that it is customary to write radical expressions in simplest radical form. A radical expression is in simplest radical form if the radicand has no perfect square factor and no radicals appear in the denominator. In this activity, denominators will be rationalized so that the solutions obtained by completing the square will match the form of those found using the quadratic formula. You may wish to compare numerical approximations of unrationalized solutions to those of rationalized solutions so that students can understand that the two representations are equal in value.

1 Activating Prior Knowledge, Quickwrite Students will solve these equations using the same method. However, the solutions differ in type.

Teacher to Teacher

Another way to solve the equations that are differences is to factor the two binomials. In Example A, you could state: $5x^2 - 23 = 0$

$(\sqrt{5}x - \sqrt{23})(\sqrt{5}x + \sqrt{23}) = 0$

$x = \pm\frac{\sqrt{23}}{\sqrt{5}} = x = \pm\frac{\sqrt{115}}{5}$

Example B Marking the Text As in Example A, students will take the square root of both sides of the equation to solve the quadratic equation. Have students share their methods of solving the items in Try These B.

2 Activating Prior Knowledge, Quickwrite, Debriefing Again, students will recognize that some solutions are rational, some are irrational, and some are complex. For the rational solutions, students may emphasize the distinction between those solutions that can be rationalized and those that need not be rationalized. How well students identify and operate with these solutions will provide you with formative assessment opportunities.

My Notes

CONNECT TO AP

In calculus, *rationalizing a numerator* is a skill used to evaluate certain types of limit expressions.

1. Compare and contrast the solutions to the equations in Try These A.
Sample answers:
Parts a, b, and c are real-number solutions; part d is complex.
Part a is rational; parts b and c are irrational.
Parts b and d did not need to be rationalized; part c was rationalized.

To solve the equation $2(x - 3)^2 - 5 = 0$, you can use a similar process.

Example B

Solve $2(x - 3)^2 - 5 = 0$ for x.

$2(x - 3)^2 - 5 = 0$

Step 1: Add 5 to both sides. $2(x - 3)^2 = 5$

Step 2: Divide both sides by 2. $(x - 3)^2 = \frac{5}{2}$

Step 3: Take the square root of both sides. $x - 3 = \pm\frac{\sqrt{5}}{\sqrt{2}}$

Step 4: Rationalize the denominator and solve for x. $x - 3 = \pm\frac{\sqrt{10}}{2}$

Solution: $x = 3 \pm \frac{\sqrt{10}}{2}$

Try These B

Solve for x. Show your work.

a. $4(x + 5)^2 - 49 = 0$
$x = -\frac{3}{2}, x = -\frac{17}{2}$

b. $3(x - 2)^2 - 16 = 0$
$x = 2 \pm \frac{4\sqrt{3}}{3}$

c. $5(x + 1)^2 - 8 = 0$
$x = -1 \pm \frac{2\sqrt{10}}{5}$

d. $4(x + 7)^2 + 25 = 0$
$x = -7 \pm \frac{5}{2}i$

2. Reason quantitatively. Describe the differences among the solutions to the equations in Try These B.
Sample answers:
Parts a, b, and c are real-number solutions; part d is complex.
Part a is a rational solution; parts b and c are irrational.
Parts b and c were rationalized.

Lesson 9-1
Completing the Square and Taking Square Roots

My Notes

Check Your Understanding

3. Use Example A to help you write a general formula for the solutions of the equation $ax^2 - c = 0$, where a and c are both positive.
4. Is the equation solved in Example B a quadratic equation? Explain.
5. Solve the equation $-2(x + 4)^2 + 3 = 0$, and explain each of your steps.
6. a. Solve the equation $3(x - 5)^2 = 0$.
   b. **Make use of structure.** Explain why the equation has only one solution and not two solutions.

The standard form of a quadratic equation is $ax^2 + bx + c = 0$. You can solve equations written in standard form by ***completing the square***.

MATH TERMS

Completing the square is the process of adding a constant to a quadratic expression to transform it into a perfect square trinomial.

Example C

Solve $2x^2 + 12x + 5 = 0$ by completing the square.

$$2x^2 + 12x + 5 = 0$$

Step 1: Divide both sides by the leading coefficient and simplify.

$$\frac{2x^2}{2} + \frac{12x}{2} + \frac{5}{2} = \frac{0}{2}$$

$$x^2 + 6x + \frac{5}{2} = 0$$

Step 2: Isolate the variable terms on the left side.

$$x^2 + 6x = -\frac{5}{2}$$

Step 3: Divide the coefficient of the linear term by 2 [$6 \div 2 = 3$], square the result [$3^2 = 9$], and add it [9] to both sides. This completes the square.

$$x^2 + 6x + \square = -\frac{5}{2} + \square$$

$$x^2 + 6x + \boxed{9} = -\frac{5}{2} + \boxed{9}$$

MATH TIP

You can factor a perfect square trinomial $x^2 + 2xy + y^2$ as $(x + y)^2$.

Step 4: Factor the perfect square trinomial on the left side into two binomials.

$$(x + 3)^2 = \frac{13}{2}$$

Step 5: Take the square root of both sides of the equation.

$$x + 3 = \pm\sqrt{\frac{13}{2}}$$

Step 6: Rationalize the denominator and solve for x.

$$x + 3 = \pm\frac{\sqrt{13}}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \pm\frac{\sqrt{26}}{2}$$

Solution: $x = -3 \pm \frac{\sqrt{26}}{2}$

Answers

3. $x = \pm\sqrt{\frac{c}{a}} = \pm\frac{\sqrt{ac}}{a}$
4. Yes. Sample explanation: The equation can be rewritten in the standard form for quadratic equations as $2x^2 - 12x + 13 = 0$.
5. $x = -4 \pm \frac{\sqrt{6}}{2}$. Sample explanation:
   - Step 1: Subtract 3 from both sides. $-2(x + 4)^2 = -3$
   - Step 2: Divide both sides by -2. $(x + 4)^2 = \frac{3}{2}$
   - Step 3: Take the square root of both sides. $x + 4 = \pm\frac{\sqrt{3}}{\sqrt{2}}$
   - Step 4: Rationalize the denominator. $x + 4 = \pm\frac{\sqrt{6}}{2}$
   - Step 5: Subtract 4 from both sides. $x = -4 \pm \frac{\sqrt{6}}{2}$
6. a. $x = 5$
   b. Sample answer: When solving, you must take the square root of both sides of the equation $(x - 5)^2 = 0$. This step results in $x - 5 = \pm\sqrt{0}$. Because $\sqrt{0} = 0$, and 0 is neither positive nor negative, you can eliminate the symbol $\pm$. You are left with the equation $x - 5 = 0$, which has a single solution: $x = 5$.

ACTIVITY 9 Continued

Check Your Understanding

Debrief students' answers to these items to ensure that they understand concepts related to quadratic functions and solving quadratic equations by taking square roots.

Developing Math Language

The process of completing the square is a series of steps that can be used to solve *any* quadratic equation. The process transforms a quadratic expression into a perfect square trinomial that factors into the square of a binomial. Here is a summary of the steps shown in Example C for the standard form of a quadratic expression $ax^2 + bx + c = 0$:

1. Divide both sides of the equation $ax^2 + bx + c = 0$ by a (when $a \neq 1$) and then subtract $\frac{c}{a}$ from both sides.
2. Now take the value of the coefficient of x ($\frac{b}{a}$, or b if $a = 1$), divide it by 2, and square it, to get $\left(\frac{b}{2a}\right)^2$. Add this value to both sides of the equation. This is when you are actually *completing the square*. The result is a perfect square trinomial on the left side of the equation: $x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a}$.
3. Factor the perfect square trinomial you have just created (refer to the Math Tip alongside Example C). The factored form of the left side of the equation is $\left(x + \left(\frac{b}{2a}\right)\right)^2$.
4. Take the square root of both sides of the equation, and simplify and rationalize as necessary. These final steps will be covered in more detail in the next lesson when students derive the quadratic formula.

Example C Marking the Text, Activating Prior Knowledge, Create Representations, Debriefing The method utilized in the example is used to derive the quadratic formula, allowing students an opportunity for success in Item 1 of the next lesson. Students must exercise care to ensure that equivalent quantities exist on both sides of the equation.

Again, rationalizing the denominator is essential to obtain solutions that are the same in format as those obtained by using the Quadratic Formula.

It may be worthwhile to emphasize the meaning of the words *completing the square*. Essentially, the goal of the process is to obtain an equation that can be solved by taking the square root of both sides of the equation. That can only occur if there is a perfect square on one side—hence the need to complete the perfect square trinomial by the addition of the appropriate constant term.

Check Your Understanding

Debrief students' answers to these items to ensure that they understand concepts related to quadratic functions and solving quadratic equations by completing the square.

Answers

7. Divide the coefficient of the x-term by 2: $8 \div 2 = 4$. Next, square the result: $4^2 = 16$. Then, add the final result to the quadratic expression: $x^2 + 8x + 16$.

8. Completing the square lets you write one side of the quadratic equation as a perfect square trinomial. After you factor the perfect square trinomial, you can solve the equation by taking the square root of both sides.

9. Sample answer: I would solve the equation by factoring. I can tell by using mental math that the factors of -12 that have a sum of 1 are 4 and -3. The equation can be factored as $(x + 4)(x - 3) = 0$, which means that its solutions are $x = -4$ and $x = 3$. If I solved the equation by completing the square, I would need to perform many more steps: isolating the variable terms, completing the square, factoring the perfect square trinomial, taking the square root of both sides, and then solving for x.

ASSESS

Students' answers to Lesson Practice problems will provide you with a formative assessment of their understanding of the lesson concepts and their ability to apply their learning.

See the Activity Practice for additional problems for this lesson. You may assign the problems here or use them as a culmination for the activity.

ADAPT

Check students' answers to the Lesson Practice to ensure that they understand how to solve a quadratic equation in standard form by completing the square. If students are proficient at solving by completing the square, they will also be proficient at solving an equation by taking the square root of both sides. Completing the square can be a difficult process for students; however, most will master the method through practice. Mastery of completing the square makes the derivation of the Quadratic Formula much more meaningful.

Lesson 9-1
Completing the Square and Taking Square Roots

My Notes

Try These C

Solve for x by completing the square.

a. $4x^2 + 16x - 5 = 0$

$x = -2 \pm \frac{\sqrt{21}}{2}$

b. $5x^2 - 30x - 3 = 0$

$x = 3 \pm \frac{4\sqrt{15}}{5}$

c. $2x^2 - 6x - 1 = 0$

$x = \frac{3}{2} \pm \frac{\sqrt{11}}{2}$

d. $2x^2 - 4x + 7 = 0$

$x = 1 \pm \frac{i\sqrt{10}}{2}$

Check Your Understanding

7. Explain how to complete the square for the quadratic expression $x^2 + 8x$.

8. How does completing the square help you solve a quadratic equation?

9. **Construct viable arguments.** Which method would you use to solve the quadratic equation $x^2 + x - 12 = 0$: factoring or completing the square? Justify your choice.

LESSON 9-1 PRACTICE

10. Use the method for completing the square to make a perfect square trinomial. Then factor the perfect square trinomial.

a. $x^2 + 10x$ **b.** $x^2 - 7x$

11. Solve each quadratic equation by taking the square root of both sides of the equation. Identify the solutions as rational, irrational, or complex conjugates.

a. $9x^2 - 64 = 0$ **b.** $5x^2 - 12 = 0$
c. $16(x - 2)^2 - 25 = 0$ **d.** $2(x - 3)^2 - 15 = 0$
e. $4x^2 + 49 = 0$ **f.** $3(x - 1)^2 + 10 = 0$

12. Solve by completing the square.

a. $x^2 - 4x - 12 = 0$ **b.** $2x^2 - 5x - 3 = 0$
c. $x^2 + 6x - 2 = 0$ **d.** $3x^2 + 9x + 2 = 0$
e. $x^2 - x + 5 = 0$ **f.** $5x^2 + 2x + 3 = 0$

13. The diagonal of a rectangular television screen measures 42 in. The ratio of the length to the width of the screen is $\frac{16}{9}$.

a. **Model with mathematics.** Write an equation that can be used to determine the length l in inches of the television screen.

b. Solve the equation, and interpret the solutions.

c. What are the length and width of the television screen, to the nearest half-inch?

CONNECT TO GEOMETRY

The length, width, and diagonal of the television screen form a right triangle.

LESSON 9-1 PRACTICE

10. a. $x^2 + 10x + 25$; $(x + 5)^2$

b. $x^2 - 7x + \frac{49}{4}$; $\left(x - \frac{7}{2}\right)^2$

11. a. $x = \pm\frac{8}{3}$; rational

b. $x = \pm\frac{2\sqrt{15}}{5}$; irrational

c. $x = \frac{3}{4}$, $x = \frac{13}{4}$; rational

d. $x = 3 \pm \frac{\sqrt{30}}{2}$; irrational

e. $x = \pm\frac{7}{2}i$; complex conjugates

f. $x = 1 \pm \frac{i\sqrt{30}}{3}$; complex conjugates

12. a. $x = 6$, $x = -2$

b. $x = -\frac{1}{2}$, $x = 3$

c. $x = -3 \pm \sqrt{11}$

d. $x = -\frac{3}{2} \pm \frac{\sqrt{57}}{6}$

e. $x = \frac{1}{2} \pm \frac{\sqrt{19}}{2}i$

f. $x = -\frac{1}{5} \pm \frac{\sqrt{14}}{5}i$

13. a. $l^2 + \left(\frac{9}{16}l\right)^2 = 42^2$

b. $l = \pm\frac{672\sqrt{337}}{337}$ or $l \approx \pm 36.61$; The length is about 36.61 in. The negative solution can be excluded because it does not make sense for the length to be negative.

c. Length: $36\frac{1}{2}$ in.; width: $20\frac{1}{2}$ in.

Learning Targets:

- Derive the Quadratic Formula.
- Solve quadratic equations using the Quadratic Formula.

SUGGESTED LEARNING STRATEGIES: Create Representations, Discussion Groups, Self Revision/Peer Revision, Think-Pair-Share, Quickwrite

Previously you learned that solutions to the general quadratic equation $ax^2 + bx + c = 0$ can be found using the Quadratic Formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \text{ where } a \neq 0$$

You can ***derive*** the quadratic formula by completing the square on the general quadratic equation.

1. **Reason abstractly and quantitatively.** Derive the quadratic formula by completing the square for the equation $ax^2 + bx + c = 0$. (Use Example C from Lesson 9-1 as a model.)

If $ax^2 + bx + c = 0$

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

My Notes

ACADEMIC VOCABULARY

When you ***derive*** a formula, you use logical reasoning to show that the formula is correct. In this case, you will derive the Quadratic Formula by solving the standard form of a quadratic equation, $ax^2 + bx + c = 0$, for x.

Lesson 9-2

PLAN

Pacing: 1 class period

Chunking the Lesson

#1–2

Check Your Understanding

Lesson Practice

TEACH

Bell-Ringer Activity

Have students solve the following items using the method given.

1. By factoring:
   $x^2 - 2x - 15 = 0$ $[x = 5, -3]$
2. By completing the square:
   $x^2 - 2x - 5 = 0$ $\left[x = 1 \pm \sqrt{6}\right]$
3. By using the quadratic formula:
   $3x^2 + 7x - 20 = 0$ $\left[x = \frac{5}{3}, -4\right]$

This will provide students with a brief overview of the various methods that can be used to solve a quadratic equation.

ELL Support

In this activity, the word *derive* is used. By applying the method of completing the square to the general equation $ax^2 + bx + c = 0$, we are able to trace the steps that it takes to arrive at the *quadratic formula*. Students should follow these steps so that they have a better understanding of how the *quadratic formula* originated and why it can be used to solve any quadratic equation.

1–2 Create Representations, Group Presentations, Debriefing The derivation of the quadratic formula is a nontrivial task for most students. A student who masters this easily is likely one who has mastered the abstraction of algebraic methods. Many students may struggle with the derivation, so monitor student progress closely. A whole-class discussion and debriefing following the allotted time period is a critical strategy.

Universal Access

Make sure that students who are struggling with Item 2 understand the purpose of the item. It asks them to use both completing the square and the quadratic formula as a means to practice both methods as well as to check their solutions to see if they match. Let students know another way they can check their solutions would be to substitute them back into the original quadratic equations to see if they create true statements.

Check Your Understanding

Debrief students' answers to these items to ensure that they understand concepts related to the derivation of the quadratic formula.

Answers

3. To complete the square, add the square of one-half of the coefficient of the x-term.

4. $x = 0$ or $x = -\frac{b}{a}$. Sample derivation:

$ax^2 + bx = 0$	Original equation
$x(ax + b) = 0$	Factor the left side.
$x = 0$ or $ax + b = 0$	Apply the Zero Product Property.
$x = 0$ or $x = -\frac{b}{a}$	Solve for x.

5. Sample answer: I preferred using the Quadratic Formula. The Quadratic Formula required fewer steps than completing the square. It also involved fewer operations with fractions.

6. **a.** $x = 3 \pm \sqrt{2}$
b. No. There is no factor pair of 7 that has a sum of -6.

My Notes

ACADEMIC VOCABULARY

When you ***verify*** a solution, you check that it is correct.

2. Solve $2x^2 - 5x + 3 = 0$ by completing the square. Then ***verify*** that the solution is correct by solving the same equation using the Quadratic Formula.

Completing the square:

$2x^2 - 5x + 3 = 0$

$x^2 - \frac{5}{2}x + \frac{3}{2} = 0$

$x^2 - \frac{5}{2}x + \boxed{\frac{25}{16}} = -\frac{3}{2} + \boxed{\frac{25}{16}}$

$\left(x - \frac{5}{4}\right)^2 = \frac{1}{16}$

$x - \frac{5}{4} = \pm\frac{1}{4}$

therefore, $x = \frac{5}{4} + \frac{1}{4} = \frac{3}{2}$ or

$x = \frac{5}{4} - \frac{1}{4} = 1$

Using the Quadratic Formula:

$a = 2, b = -5, c = 3$, therefore

$x = \frac{5 \pm \sqrt{5^2 - 4(2)(3)}}{2(2)}$

$x = \frac{5 \pm \sqrt{25 - 24}}{4} = \frac{5 \pm 1}{4}$

therefore, $x = \frac{5}{4} + \frac{1}{4} = \frac{3}{2}$ or

$x = \frac{5}{4} - \frac{1}{4} = 1$

Check Your Understanding

3. In Item 1, why do you need to add $\left(\frac{b}{2a}\right)^2$ to both sides?

4. Derive a formula for solving a quadratic equation of the form $ax^2 + bx = 0$, where $a \neq 0$.

5. **Construct viable arguments.** Which method did you prefer for solving the quadratic equation in Item 2: completing the square or using the Quadratic Formula? Justify your choice.

6. Consider the equation $x^2 - 6x + 7 = 0$.
a. Solve the equation by using the Quadratic Formula.
b. Could you have solved the equation by factoring? Explain.

ASSESS

Students' answers to Lesson Practice problems will provide you with a formative assessment of their understanding of the lesson concepts and their ability to apply their learning.

See the Activity Practice for additional problems for this lesson. You may assign the problems here or use them as a culmination for the activity.

LESSON 9-2 PRACTICE

7. a. $x = -1 \pm \frac{\sqrt{14}}{2}$
   b. $x = -\frac{7}{6} \pm \frac{\sqrt{71}}{6}i$
   c. $x = \frac{9 \pm \sqrt{85}}{2}$
   d. $x = \frac{5 \pm 3\sqrt{17}}{8}$
   e. $x = \frac{7 \pm \sqrt{73}}{4}$
   f. $x = \frac{-3 \pm i\sqrt{87}}{8}$
8. Sample answers are given.
   a. $x = -3$; factoring; The left side of the equation is a perfect square trinomial, which is easy to factor.
   b. $x = \frac{-5 \pm \sqrt{217}}{16}$; Quadratic Formula; The coefficient of the x^2-term is not 1, which makes the other methods of solving more difficult.
   c. $x = -10$ or $x = 2$; taking the square root of both sides; When you add 36 to both sides of the equation, each side is a perfect square.
   d. $x = -1 \pm 2\sqrt{2}$; completing the square; The variable terms are already isolated on one side, the coefficient of the x^2-term is 1, and the coefficient of the x-term is even, all of which make completing the square easier.

ADAPT

Check students' answers to the Lesson Practice to ensure that they understand how to solve a quadratic equation using the Quadratic Formula. Additionally, check to see that students are choosing and correctly using all possible methods for solving quadratic equations. Students may wish to create a graphic organizer of solution methods for quadratic equations to aid in their mastery of the methods.

Lesson 9-2
The Quadratic Formula

ACTIVITY 9 *continued*

LESSON 9-2 PRACTICE

7. Solve each equation by using the Quadratic Formula.
   a. $2x^2 + 4x - 5 = 0$
   b. $3x^2 + 7x + 10 = 0$
   c. $x^2 - 9x - 1 = 0$
   d. $-4x^2 + 5x + 8 = 0$
   e. $2x^2 - 3 = 7x$
   f. $4x^2 + 3x = -6$
8. Solve each quadratic equation by using any of the methods you have learned. For each equation, tell which method you used and why you chose that method.
   a. $x^2 + 6x + 9 = 0$
   b. $8x^2 + 5x - 6 = 0$
   c. $(x + 4)^2 - 36 = 0$
   d. $x^2 + 2x = 7$
9. a. **Reason abstractly.** Under what circumstances will the radicand in the Quadratic Formula, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, be negative?
   b. If the radicand is negative, what does this tell you about the solutions of the quadratic equation? Explain.
10. A player shoots a basketball from a height of 7 ft with an initial vertical velocity of 18 ft/s. The equation $-16t^2 + 18t + 7 = 10$ can be used to determine the time t in seconds at which the ball will have a height of 10 ft—the same height as the basket.
    a. Solve the equation by using the Quadratic Formula.
    b. **Attend to precision.** To the nearest tenth of a second, when will the ball have a height of 10 ft?
    c. Explain how you can check that your answers to part b are reasonable.

My Notes

MATH TIP

A *radicand* is an expression under a radical symbol. For $\sqrt{b^2 - 4ac}$, the radicand is $b^2 - 4ac$.

CONNECT TO PHYSICS

The function $h(t) = -16t^2 + v_0t + h_0$ can be used to model the height h in feet of a thrown object t seconds after it is thrown, where v_0 is the initial vertical velocity of the object in ft/s and h_0 is the initial height of the object in feet.

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9. a. when $4ac$ is greater than b^2
   b. Sample answer: If the radicand is negative, the solutions involve the square root of a negative number. The square root of a negative number is imaginary, so the solutions of the equation will be complex numbers.
10. a. $t = \frac{9 \pm \sqrt{33}}{16}$
    b. about 0.2 s and 0.9 s after the ball is thrown
    c. Sample answer: Substitute the times from part b into the original equation to check that they make the left side of the equation approximately equal to 10.

$$-16(0.2)^2 + 18(0.2) + 7 \approx 10$$
$$-0.64 + 3.6 + 7 \approx 10$$
$$9.96 \approx 10 \checkmark$$

$$-16(0.9)^2 + 18(0.9) + 7 \approx 10$$
$$-12.96 + 16.2 + 7 \approx 10$$
$$10.24 \approx 10 \checkmark$$

Lesson 9-3

PLAN

Pacing: 1 class period

Chunking the Lesson

#1–2

Check Your Understanding

#7–8

Check Your Understanding

Lesson Practice

TEACH

Bell-Ringer Activity

Emphasize to students that in order to correctly identify a, b, and c for use in the quadratic formula, the quadratic equation must first be written in the form $ax^2 + bx + c = 0$.

For example, if you must solve the quadratic equation $x^2 - 4x = -13$, 13 would have to be added to both sides of the equation to get the general form: $x^2 - 4x + 13 = 0$.

Thus $a = 1$, $b = -4$, and $c = 13$. Present students with equations that they need to rewrite in the form $ax^2 + bx + c = 0$.

1–2 Activating Prior Knowledge, Look for a Pattern, Group Presentation, Debriefing Students are asked not only to solve equations using the quadratic formula, but also to discuss the number and type of solutions. This item may allow students to recognize characteristics of the equations, or similarly, characteristics of the quadratic formula expression that results from the equation. Class discussion and group presentation will allow the free exchange of ideas that may segue naturally to a discussion of the discriminant.

My Notes

MATH TIP

The complex numbers include the real numbers, so real solutions are also complex solutions. However, when asked to classify solutions as real or complex, you can assume that "complex" does not include the reals.

Learning Targets:

- Solve quadratic equations using the Quadratic Formula.
- Use the discriminant to determine the nature of the solutions of a quadratic equation.

SUGGESTED LEARNING STRATEGIES: Look for a Pattern, Group Presentation, Self Revision/Peer Revision, Think-Pair-Share, Quickwrite

1. Solve each equation by using the Quadratic Formula. For each equation, write the number of solutions. Tell whether the solutions are real or complex, and, if real, whether the solutions are rational or irrational.

a. $4x^2 + 5x - 6 = 0$

solutions: $x = -2, x = \frac{3}{4}$

number of solutions: 2

real or complex: real

rational or irrational: irrational

b. $4x^2 + 5x - 2 = 0$

solutions: $x = \frac{-5 \pm \sqrt{57}}{8}$

number of solutions: 2

real or complex: real

rational or irrational: irrational

c. $4x^2 + 4x + 1 = 0$

solutions: $x = -\frac{1}{2}$

number of solutions: 1

real or complex: real

rational or irrational: irrational

d. $4x^2 + 4x + 5 = 0$

solutions: $x = -\frac{1}{2} \pm i$

number of solutions: 2

real or complex: complex

rational or irrational: not applicable (since this applies to real numbers only)

Lesson 9-3
Solutions of Quadratic Equations

My Notes

2. **Express regularity in repeated reasoning.** What patterns can you identify from your responses to Item 1?
   Patterns that students may recognize are that if $b^2 - 4ac$ is positive, the solutions are real, and if $b^2 - 4ac$ is negative, the solutions are complex.

Check Your Understanding

3. **a.** In Item 1, was the expression under the square root symbol of the Quadratic Formula positive, negative, or zero when there were two real solutions?
   b. What about when there was one real solution?
   c. What about when there were two complex solutions?
4. In Item 1, how did you determine whether the real solutions of a quadratic equation were rational or irrational?
5. **Reason quantitatively.** The quadratic function related to the equation in Item 1a is $f(x) = 4x^2 + 5x - 6$. Without graphing the function, determine how many x-intercepts it has and what their values are. Explain how you determined your answer.
6. Make a conjecture about the relationship between the solutions of a quadratic equation that has complex roots.

1–2 (continued) For Item 2, have students informally discuss the patterns they see. The next items will have them formalize the information the discriminant reveals about the solutions.

Differentiating Instruction

In a problem such as Item 1d, a common error made by students occurs during the last step or steps of the solution. Some students have the misconception that with an expression such as $\frac{-4 \pm 8i}{8}$, they can divide out the 8's and simplify the fraction to $-4 \pm i$. Students must realize that if you cancel one term of the numerator by a common factor, they must be able to divide all the terms of the numerator by that factor. In this case, the greatest factor they all have in common is 4.

Check Your Understanding

Debrief students' answers to these items to ensure that they understand concepts related to the nature of solutions of quadratic equations.

Answers

3. **a.** positive
   b. zero
   c. negative
4. Sample answer: If the solutions can be written as integers or as fractions with integer numerators and integer denominators, then solutions are rational. If the solutions involve square roots of non-perfect squares, then the solutions are irrational.
5. The function has two x-intercepts: -2 and $\frac{3}{4}$. Sample explanation: The x-intercepts are the real values of x for which $f(x) = 0$, so the x-intercepts are the real solutions of the equation $0 = 4x^2 + 5x - 6$. Item 1a shows that this equation has two real solutions of $x = -2$ and $x = \frac{3}{4}$, so the x-intercepts of the related function are -2 and $\frac{3}{4}$.
6. Sample answers: The solutions are complex conjugates. The solutions have the same real parts and opposite imaginary parts.

Developing Math Language

The expression under the radical sign of the quadratic formula, $b^2 - 4ac$, is called the *discriminant*. The value of this expression enables one to determine, or *discriminate*, among the possible types of solutions of its corresponding quadratic equation.

Another word often used to refer to the solution(s) of an equation is *root(s)*. In the case of a quadratic equation of general form $ax^2 + bx + c = 0$, the root(s) refers to the zeros of the equation, or the location(s) where the graph of its function crosses the x-axis. The terms *root*, *zero*, and *solution* are all essentially synonymous.

7–8 Identify a Subtask, Debriefing Students use the definition of the discriminant to evaluate and interpret earlier results. As students review the table, be sure they understand that a, b, and c must be rational numbers for the nature of the solutions described in the table to be true. For example, $x^2 + 2\pi x + \pi^2 = 0$ has an irrational solution of π. Evaluating the discriminant of the equations in Item 7 gives students the opportunity to formalize the conjectures that they may have put forth in their discussions.

Note: Be sure that students recognize the synonymous use of the words *solution* and *root*.

My Notes

MATH TERMS

The **discriminant** is the expression $b^2 - 4ac$ under the radical sign in the Quadratic Formula.

MATH TERMS

A solution to an equation is also called a **root** of the equation.

The roots of a quadratic equation $ax^2 + bx + c = 0$ represent the **zeros** (or x-intercepts) of the quadratic function $y = ax^2 + bx + c$.

MATH TIP

If the values of a, b, and c are integers and the discriminant $b^2 - 4ac$ is a perfect square, then the quadratic expression $ax^2 + bx + c$ is factorable over the integers.

The ***discriminant*** of a quadratic equation $ax^2 + bx + c = 0$ is defined as the expression $b^2 - 4ac$. The value of the discriminant determines the *nature of the solutions* of a quadratic equation in the following manner.

Discriminant	Nature of Solutions
$b^2 - 4ac > 0$ and $b^2 - 4ac$ is a perfect square	Two real, rational solutions
$b^2 - 4ac > 0$ and $b^2 - 4ac$ is *not* a perfect square	Two real, irrational solutions
$b^2 - 4ac = 0$	One real, rational solution (a double ***root***)
$b^2 - 4ac < 0$	Two complex conjugate solutions

7. Compute the value of the discriminant for each equation in Item 1 to determine the number and nature of the solutions.

a. $4x^2 + 5x - 6 = 0$
121; Since $b^2 - 4ac$ is positive and a perfect square, there are two real, rational roots.

b. $4x^2 + 5x - 2 = 0$
57; Since $b^2 - 4ac$ is positive and not a perfect square, there are two real, irrational roots.

c. $4x^2 + 4x + 1 = 0$
0; Since $b^2 - 4ac$ is zero, there is one real, rational root.

d. $4x^2 + 4x + 5 = 0$
−64; Since $b^2 - 4ac$ is negative, there are two complex conjugate roots.

Lesson 9-3
Solutions of Quadratic Equations

My Notes

8. For each equation below, compute the value of the discriminant and describe the solutions without solving.

a. $2x^2 + 5x + 12 = 0$
discriminant = −71; two complex conjugate roots

b. $3x^2 - 11x + 4 = 0$
discriminant = 73; two real, irrational roots

c. $5x^2 + 3x - 2 = 0$
discriminant = 49; two real, rational roots

d. $4x^2 - 12x + 9 = 0$
discriminant = 0; one real, rational root

Check Your Understanding

9. Critique the reasoning of others. A student solves a quadratic equation and gets solutions of $x = -\frac{7}{3}$ and $x = 8$. To check the reasonableness of his answer, the student calculates the discriminant of the equation and finds it to be −188. Explain how the value of the discriminant shows that the student made a mistake when solving the equation.

10. One of the solutions of a quadratic equation is $x = 6 + 4i$. What is the other solution of the quadratic equation? Explain your answer.

11. The discriminant of a quadratic equation is 225. Are the roots of the equation rational or irrational? Explain.

12. Consider the quadratic equation $2x^2 + 5x + c = 0$.
a. For what value(s) of c does the equation have two real solutions?
b. For what value(s) of c does the equation have one real solution?
c. For what value(s) of c does the equation have two complex conjugate solutions?

7–8 (continued) Item 8 provides students with the opportunity to identify the nature of the solutions (roots) without actually solving the equation. Students may notice that they've done most of the work in applying the Quadratic Formula by finding the discriminant.

Check Your Understanding

Debrief students' answers to these items to ensure that they understand concepts related to the discriminant of a quadratic equation.

Answers

9. The discriminant is negative, which means that the quadratic equation has two complex conjugate solutions. The student's answers of $x = -\frac{7}{3}$ and $x = 8$ are not complex conjugates, so the student must have made a mistake when solving the equation.

10. $x = 6 - 4i$; The given solution is a complex number. When a quadratic equation has complex solutions, the solutions are complex conjugates. The other solution must be the complex conjugate of $6 + 4i$.

11. The roots are rational because the discriminant of 225 is a perfect square ($15^2 = 225$).

12. a. $c < \frac{25}{8}$

b. $c = \frac{25}{8}$

c. $c > \frac{25}{8}$

ASSESS

Students' answers to Lesson Practice problems will provide you with a formative assessment of their understanding of the lesson concepts and their ability to apply their learning.

See the Activity Practice for additional problems for this lesson. You may assign the problems here or use them as a culmination for the activity.

LESSON 9-3 PRACTICE

13. a. discriminant = 49, two rational roots, solutions are $x = -6$ and $x = 1$

b. discriminant = 169, two rational roots, solutions are $x = -\frac{3}{2}$ and $x = 5$

c. discriminant = 0, one rational (double) root, solution is $x = 4$

d. discriminant = −24, two complex roots, solutions are $x = \frac{2}{5} \pm \frac{\sqrt{6}}{5}i$

e. discriminant = −79, two complex roots, solutions are $x = -\frac{9}{4} \pm \frac{\sqrt{79}}{4}i$

f. discriminant = 37, two irrational roots, solutions are $x = \frac{5}{6} \pm \frac{\sqrt{37}}{6}$

14. Sample answer: The discriminant is $b^2 - 4ac$ and originates from the radicand in the Quadratic Formula. When it is a perfect square, the radical disappears, yielding rational solutions. When it is positive but not a perfect square, then the radical remains, yielding irrational solutions. When the discriminant is negative, there is a negative under the radical, yielding complex solutions.

15. The discriminant is positive and a perfect square, so the quadratic equation has two rational roots.

16. Answers will vary, but the discriminant of the equation should be positive and not a perfect square. Sample answer: $x^2 - 4x + 2 = 0$. The discriminant is 8, which is positive and not a perfect square, so the quadratic equation has two irrational roots.

ADAPT

Check students' answers to the Lesson Practice to ensure that they understand how to use the discriminant to determine the nature of the solutions of a quadratic equation. To check for full conceptual understanding, pair students up and ask one student to explain the discriminant to the other student as if that student had been absent. The student playing the role of absentee can evaluate the explanation.

My Notes

LESSON 9-3 PRACTICE

13. For each equation, evaluate the discriminant and determine the nature of the solutions. Then solve each equation using the Quadratic Formula to verify the nature of the roots.

a. $x^2 + 5x - 6 = 0$ **b.** $2x^2 - 7x - 15 = 0$

c. $x^2 - 8x + 16 = 0$ **d.** $5x^2 - 4x + 2 = 0$

e. $2x^2 + 9x + 20 = 0$ **f.** $3x^2 - 5x - 1 = 0$

14. Reason abstractly. What is the discriminant? How does the value of the discriminant affect the solutions of a quadratic equation?

15. The discriminant of a quadratic equation is 1. What can you conclude about the solutions of the equation? Explain your reasoning.

16. Give an example of a quadratic equation that has two irrational solutions. Use the discriminant to show that the solutions of the equation are irrational.

17. Make sense of problems. A baseball player throws a ball from a height of 6 ft with an initial vertical velocity of 32 ft/s. The equation $-16t^2 + 32t + 6 = 25$ can be used to determine the time t in seconds at which the ball will reach a height of 25 ft.

a. Evaluate the discriminant of the equation.

b. What does the discriminant tell you about whether the ball will reach a height of 25 ft?

MATH TIP

In Item 17, remember to write the equation in standard form before you evaluate the discriminant.

17. a. −192

b. The discriminant is negative, which means that the solutions of the equation are not real. There are no real values of the time t for which the height of the ball will reach 25 ft.

ACTIVITY 9 PRACTICE

Write your answers on notebook paper. Show your work.

Lesson 9-1

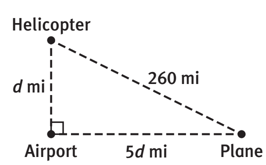
For Items 1–8, solve each equation by taking the square root of both sides.

1. $4x^2 - 49 = 0$
2. $5x^2 = 36$
3. $9x^2 - 32 = 0$
4. $(x + 4)^2 - 25 = 0$
5. $3(x + 2)^2 = 15$
6. $-2(x - 4)^2 = 16$
7. $4(x - 8)^2 - 10 = 14$
8. $6(x + 3)^2 + 20 = 12$

9. Which of the following represents a formula that can be used to solve quadratic equations of the form $a(x - h)^2 + k = 0$, where $a \neq 0$?

A. $x = -h \pm \sqrt{-\frac{k}{a}}$
B. $x = -h \pm \sqrt{\frac{k}{a}}$
C. $x = h \pm \sqrt{-\frac{k}{a}}$
D. $x = h \pm \sqrt{\frac{k}{a}}$

10. A plane begins flying due east from an airport at the same time as a helicopter begins flying due north from the airport. After half an hour, the plane and helicopter are 260 mi apart, and the plane is five times the distance from the airport as the helicopter.



Not to scale

a. Write an equation that can be used to determine d, the helicopter's distance in miles from the airport after half an hour.
b. Solve the equation and interpret the solutions.
c. What are the average speeds of the plane and the helicopter? Explain.

For Items 11–14, complete the square for each quadratic expression. Then factor the perfect square trinomial.

11. $x^2 + 10x$
12. $x^2 - 16x$
13. $x^2 + 9x$
14. $x^2 - x$

For Items 15–20, solve each equation by completing the square.

15. $x^2 + 2x + 5 = 0$
16. $x^2 - 10x = 26$
17. $x^2 + 5x - 9 = 0$
18. $2x^2 + 8x - 7 = 0$
19. $3x^2 - 15x = 20$
20. $6x^2 + 16x + 9 = 0$

Lesson 9-2

For Items 21–28, solve each equation by using the Quadratic Formula.

21. $x^2 + 12x + 6 = 0$
22. $3x^2 - 5x + 3 = 0$
23. $2x^2 + 6x = 25$
24. $42x^2 + 11x - 20 = 0$
25. $x^2 + 6x + 8 = 4x - 3$
26. $10x^2 - 5x = 9x + 8$
27. $4x^2 + x - 12 = 3x^2 - 5x$
28. $x^2 - 20x = 6x^2 - 2x + 20$

29. Write a formula that represents the solutions of a quadratic equation of the form $mx^2 + nx + p = 0$. Explain how you arrived at your formula.

30. Derive a formula for solving a quadratic equation of the form $x^2 + bx + c = 0$.

ACTIVITY PRACTICE

1. $x = \pm\frac{7}{2}$
2. $x = \pm\frac{6\sqrt{5}}{5}$
3. $x = \pm\frac{4\sqrt{2}}{3}$
4. $x = -9, x = 1$
5. $x = -2 \pm \sqrt{5}$
6. $x = 4 \pm 2i\sqrt{2}$
7. $x = 8 \pm \sqrt{6}$
8. $x = -3 \pm \frac{2i\sqrt{3}}{3}$
9. C
10. **a.** $d^2 + (5d)^2 = 260^2$ or equivalent
b. $d = \pm 10\sqrt{26}$; The helicopter is $10\sqrt{26}$ mi, or about 51 mi, from the airport; the negative solution can be excluded, because it does not make sense for the distance to be negative.
c. helicopter: 51 mi ÷ 0.5 h = 102 mi/h; plane: 5(51) mi ÷ 0.5 h = 510 mi/h
11. $x^2 + 10x + 25$; $(x + 5)^2$
12. $x^2 - 16x + 64$; $(x - 8)^2$
13. $x^2 + 9x + \frac{81}{4}$; $\left(x + \frac{9}{2}\right)^2$
14. $x^2 - x + \frac{1}{4}$; $\left(x - \frac{1}{2}\right)^2$
15. $x = -1 \pm 2i$
16. $x = 5 \pm \sqrt{51}$
17. $x = -\frac{5}{2} \pm \frac{\sqrt{61}}{2}$
18. $x = -2 \pm \frac{\sqrt{30}}{2}$
19. $x = \frac{5}{2} \pm \frac{\sqrt{465}}{6}$
20. $x = -\frac{4}{3} \pm \frac{\sqrt{10}}{6}$
21. $x = -6 \pm \sqrt{30}$
22. $x = \frac{5 \pm i\sqrt{11}}{6}$
23. $x = \frac{-3 \pm \sqrt{59}}{2}$
24. $x = -\frac{5}{6}, x = \frac{4}{7}$
25. $x = -1 \pm i\sqrt{10}$
26. $x = \frac{7 \pm \sqrt{129}}{10}$
27. $x = -3 \pm \sqrt{21}$
28. $x = \frac{-9 \pm i\sqrt{19}}{5}$
29. $x = \frac{-n \pm \sqrt{n^2 - 4mp}}{2m}$; Use the Quadratic Formula with $a = m$, $b = n$, and $c = p$.

30. $x = \frac{-b \pm \sqrt{b^2 - 4c}}{2}$. Sample derivation:

$x^2 + bx + c = 0$	Original equation
$x^2 + bx = -c$	Subtract c from both sides.
$x^2 + bx + \left(\frac{b}{2}\right)^2 = -c + \left(\frac{b}{2}\right)^2$	Complete the square.
$\left(x + \frac{b}{2}\right)^2 = \frac{b^2 - 4c}{4}$	Factor the left side. Simplify the right side.
$x + \frac{b}{2} = \pm\frac{\sqrt{b^2 - 4c}}{2}$	Take the square root of both sides.
$x = \frac{-b \pm \sqrt{b^2 - 4c}}{2}$	Solve for x.

31–36. Sample answers are given.

31. $x = 2, x = -8$; taking the square root of both sides; When you add 25 to both sides of the equation, each side is a perfect square.

32. $x = \frac{9 \pm \sqrt{41}}{4}$; Quadratic Formula; The coefficient of the x^2-term is not 1, which makes the other methods of solving more difficult.

33. $x = -4, x = -3$; factoring; Mental math shows that 12 has a factor pair of 4 and 3 with a sum of 7.

34. $x = -\frac{7}{3}, x = 2$; Quadratic Formula; The coefficient of the x^2-term is not 1, which makes the other methods of solving more difficult.

35. $x = -4 \pm \sqrt{23}$; completing the square; The variable terms are already isolated on one side, the coefficient of the x^2-term is 1, and the coefficient of the x-term is even, all of which make completing the square easier.

36. $x = \pm\frac{\sqrt{33}}{2}$; taking the square root of both sides; The equation has the form $ax^2 + c = 0$.

37. a. $-2t^2 + 82t + 5 = 301$ or equivalent

b. $t = 4, t = 37$; Megan bought 4 tickets; the solution $t = 37$ can be excluded because customers may buy no more than 15 tickets.

c. $75.25

38. -23, two complex roots

39. 0, one real (double) root

40. 529, two real, rational roots

41. 284, two real, irrational roots

42. 136, two real, irrational roots

43. D

44. a. $a < 3$ and $a \neq 0$

b. $a = 3$

c. $a > 3$

45. a. $-14s^2 + 440s - 2100 = 1200$ or equivalent

b. 8800

c. Sample answer: The discriminant is positive, so the equation has two real solutions. However, if one or both values of s are negative, they would need to be excluded in this situation.

46. Sample answers:

a. $ax^2 + c = 0$; taking the square root of both sides; You can solve the equation for x^2 and then take the square root of both sides to solve for x.

b. $ax^2 + bx = 0$; factoring; You can factor x from the left side of the equation to get $x(ax + b) = 0$.

c. $x^2 + bx = -c$, where b is even; completing the square; The variable terms are already isolated on one side, the coefficient of the x^2-term is 1, and the coefficient of the x-term is even, all of which make completing the square easier.

d. $x^2 + bx + c = 0$, where c has a factor pair with a sum of b; factoring; The coefficient of the x^2-term is 1, and the information about b and c show that the equation is easy to factor.

e. $ax^2 + bx + c = 0$, where a, b, and c are each greater than 10; Quadratic Formula; The coefficient of the x^2-term is not 1 and the values of b and c are large, which makes the other methods of solving more difficult.

ADDITIONAL PRACTICE

If students need more practice on the concepts in this activity, see the Teacher Resources at SpringBoard Digital for additional practice problems.

For Items 31–36, solve each equation, using any method that you choose. For each equation, tell which method you used and why you chose that method.

31. $(x + 3)^2 - 25 = 0$

32. $2x^2 - 9x + 5 = 0$

33. $x^2 + 7x + 12 = 0$

34. $3x^2 + x - 14 = 0$

35. $x^2 + 8x = 7$

36. $4x^2 - 33 = 0$

37. The more concert tickets a customer buys, the less each individual ticket costs. The function $c(t) = -2t^2 + 82t + 5$ gives the total cost in dollars of buying t tickets to the concert. Customers may buy no more than 15 tickets.

a. Megan spent a total of $301 on concert tickets. Write a quadratic equation that can be used to determine the number of tickets Megan bought.

b. Use the Quadratic Formula to solve the equation. Then interpret the solutions.

c. What was the cost of each ticket Megan bought?

Lesson 9-3

For each equation, find the value of the discriminant and describe the nature of the solutions.

38. $2x^2 + 3x + 4 = 0$

39. $9x^2 + 30x + 25 = 0$

40. $6x^2 - 7x - 20 = 0$

41. $5x^2 + 12x - 7 = 0$

42. $x^2 - 8x = 18$

43. The discriminant of a quadratic equation is -6. What types of solutions does the equation have?

A. 1 real solution
B. 2 rational solutions
C. 2 irrational solutions
D. 2 complex conjugate solutions

44. Consider the quadratic equation $ax^2 - 6x + 3 = 0$, where $a \neq 0$.

a. For what value(s) of a does the equation have two real solutions?

b. For what value(s) of a does the equation have one real solution?

c. For what value(s) of a does the equation have two complex conjugate solutions?

45. The function $p(s) = -14s^2 + 440s - 2100$ models the monthly profit in dollars made by a small T-shirt company when the selling price of its shirts is s dollars.

a. Write an equation that can be used to determine the selling price that will result in a monthly profit of $1200.

b. Evaluate the discriminant of the equation.

c. What does the discriminant tell you about whether the company can have a monthly profit of $1200?

MATHEMATICAL PRACTICES
Look for and Make Use of Structure

46. Tell which method you would use to solve each quadratic equation having the given form. Then explain why you would use that method.

a. $ax^2 + c = 0$

b. $ax^2 + bx = 0$

c. $x^2 + bx = -c$, where b is even

d. $x^2 + bx + c = 0$, where c has a factor pair with a sum of b

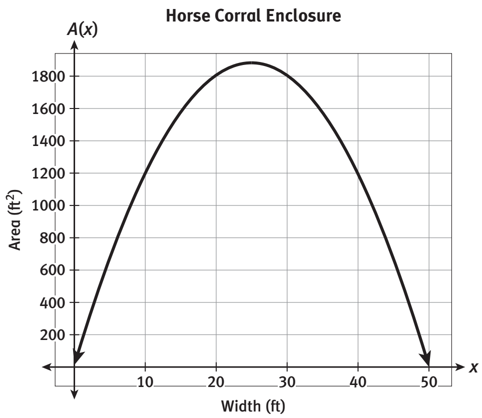
e. $ax^2 + bx + c = 0$, where a, b, and c are each greater than 10

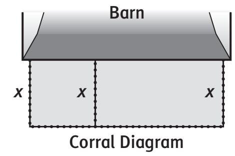
Applications of Quadratic Functions and Equations

Embedded Assessment 1

Use after Activity 9

NO HORSING AROUND





1. Kun-cha has 150 feet of fencing to make a corral for her horses. The barn will be one side of the partitioned rectangular enclosure, as shown in the diagram above. The graph illustrates the function that represents the area that could be enclosed.

a. Write a function, $A(x)$, that represents the area that can be enclosed by the corral.

b. What information does the graph provide about the function?

c. Which ordered pair indicates the maximum area possible for the corral? Explain what each coordinate tells about the problem.

d. What values of x will give a total area of 1000 ft^2? 2000 ft^2?

2. **Critique the reasoning of others.** Tim is the punter for the Bitterroot Springs Mustangs football team. He wrote a function $h(t) = 16t^2 + 8t + 1$ that he thinks will give the height of a football in terms of t, the number of seconds after he kicks the ball. Use two different methods to determine the values of t for which $h(t) = 0$. Show your work. Is Tim's function correct? Why or why not?

3. Tim has been studying complex numbers and quadratic equations. His teacher, Mrs. Pinto, gave the class a quiz. Demonstrate your understanding of the material by responding to each item below.

a. Write a quadratic equation that has two solutions, $x = 2 + 5i$ and $x = 2 - 5i$.

b. Solve $3x^2 + 2x - 8 = 0$, using an algebraic method.

c. Rewrite $\frac{4 + i}{3 - 2i}$ in the form $a + bi$, where a and b are rational numbers.

Common Core State Standards for Embedded Assessment 1

HSA-CED.A.1	Create equations and inequalities in one variable and use them to solve problems. Include equations arising from linear functions.
HSA-CED.A.2	Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.
HSF-IF.B.4	For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship.
HSF-IF.B.5	Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes.

Embedded Assessment 1

Assessment Focus

- Quadratic functions
- Quadratic equations
- Discriminants
- Complex numbers

Answer Key

1. a. $A(x) = x(150 - 3x) = -3x^2 + 150x$

b. Students may note:
- the x-intercepts for the graph appear to be at $x = 0$ and $x = 50$
- the graph is a parabola that is symmetric about the line $x = 25$
- the maximum y-value appears to be between 1800 and 1900

c. The point indicating the maximum area for the enclosure is (25, 1875), where 25 is the length in feet of each part of the enclosure that is perpendicular to the barn, and 1875 is the total area in square feet enclosed by the corral.

d. For 1000 ft^2 of area, $x \approx 7.922$ ft or 42.078 ft. No value of x gives 2000 ft^2 of total area.

2. Tim's equation is written incorrectly since the value of the function is only equal to 0 when t is a negative value. This indicates that the height of the ball is never 0 ft after the ball is kicked, which would mean that the ball would never return to the ground. Students can choose one of several methods.
- A graph of the function shows that there is no positive value of the time for which the height of the ball is 0 ft. It also shows that for positive values of the time, the height of the ball keeps increasing without ever decreasing.
- Letting $h(t) = 0$ and solving for t by using the Quadratic Formula or completing the square gives $t = \frac{-8 \pm \sqrt{0}}{32} = -\frac{1}{4}$, which shows that there is no positive value of the time at which the ball hits the ground.

3. a. $x^2 - 4x + 29 = 0$ or equivalent

b. $x = \frac{4}{3}, x = -2$

c. $\frac{10}{13} + \frac{11}{13}i$

Teacher to Teacher

Students may or may not realize that Tim's equation does not make sense as a model for the height of the football over time. Tim's equation is for a parabola with a minimum at the vertex instead of a parabola with a maximum height at the vertex.

Teacher to Teacher

You may wish to read through the scoring guide with students and discuss the differences in the expectations at each level. Check that students understand the terms used.

Unpacking Embedded Assessment 2

Once students have completed this Embedded Assessment, turn to Embedded Assessment 2 and unpack it with them. Use a graphic organizer to help students understand the concepts they will need to know to be successful on Embedded Assessment 2.

Embedded Assessment 1

Use after Activity 9

Applications of Quadratic Functions and Equations

NO HORSING AROUND

Scoring Guide	Exemplary	Proficient	Emerging	Incomplete
	The solution demonstrates these characteristics:			
Mathematics Knowledge and Thinking (Items 1c, 1d, 2, 3a-c)	• Effective understanding of and accuracy in solving quadratic equations algebraically or graphically • Clear and accurate understanding of the key features of graphs of quadratic functions and the relationship between zeros and solutions to quadratic equations • Clear and accurate understanding of how to perform operations with complex numbers	• Adequate understanding of solving quadratic equations algebraically or graphically, leading to solutions that are usually correct • Largely correct understanding of the key features of graphs of quadratic functions and the relationship between zeros and solutions to quadratic equations • Largely correct understanding of how to perform operations with complex numbers	• Partial understanding of and some difficulty solving quadratic equations algebraically or graphically • Partial understanding of the key features of graphs of quadratic functions and the relationship between zeros and solutions to quadratic equations • Difficulty performing operations with complex numbers	• Inaccurate or incomplete understanding of solving quadratic equations algebraically or graphically • Little or no understanding of the key features of graphs of quadratic functions and the relationship between zeros and solutions to quadratic equations • Little or no understanding of how to perform operations with complex numbers
Problem Solving (Items 1c, 1d, 2)	• An appropriate and efficient strategy that results in a correct answer	• A strategy that may include unnecessary steps but results in a correct answer	• A strategy that results in some incorrect answers	• No clear strategy when solving problems
Mathematical Modeling / Representations (Item 1)	• Effective understanding of how to write a quadratic equation or function from a verbal description, graph or diagram • Clear and accurate understanding of how to interpret features of the graphs of quadratic functions and the solutions to quadratic equations	• Adequate understanding of how to write a quadratic equation or function from a verbal description, graph or diagram • Largely correct understanding of how to interpret features of the graphs of quadratic functions and the solutions to quadratic equations	• Partial understanding of how to write a quadratic equation or function from a verbal description, graph or diagram • Some difficulty with interpreting the features of graphs of quadratic functions and the solutions to quadratic equations	• Little or no understanding of how to write a quadratic equation or function from a verbal description, graph or diagram • Inaccurate or incomplete interpretation of the features of graphs of quadratic functions and the solutions to quadratic equations
Reasoning and Communication (Items 1b, 1c, 2)	• Precise use of appropriate math terms and language to relate equations and graphs of quadratic functions and their key features to a real-world scenario • Clear and accurate use of mathematical work to justify or refute a claim	• Adequate descriptions to relate equations and graphs of quadratic functions and their key features to a real-world scenario • Correct use of mathematical work to justify or refute a claim	• Misleading or confusing descriptions to relate equations and graphs of quadratic functions and their key features to a real-world scenario • Partially correct use of mathematical work to justify or refute a claim	• Incomplete or inaccurate descriptions to relate equations and graphs of quadratic functions and their key features to a real-world scenario • Incorrect or incomplete use of mathematical work to justify or refute a claim

Common Core State Standards for Embedded Assessment 1 *(cont.)*

HSN-CN.A.1 Know there is a complex number i such that $i^2 = -1$, and every complex number has the form $a + bi$ with a and b real.

HSN-CN.A.2 Use the relation $i^2 = -1$ and the commutative, associative, and distributive properties to add, subtract, and multiply complex numbers.

HSN-CN.C.7 Solve quadratic equations with real coefficients that have complex solutions.

HSN-CN.C.8 Extend polynomial identities to the complex numbers.

Writing Quadratic Equations

ACTIVITY 10

What Goes Up Must Come Down
Lesson 10-1 Parabolas and Quadratic Equations

My Notes

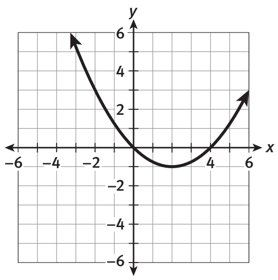
Learning Targets:

- Derive a general equation for a parabola based on the definition of a parabola.
- Write the equation of a parabola given a graph and key features.

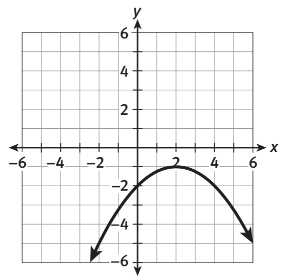
SUGGESTED LEARNING STRATEGIES: Predict and Confirm, Discussion Groups, Interactive Word Wall, Create Representations, Close Reading

Take a look at the graphs shown below.

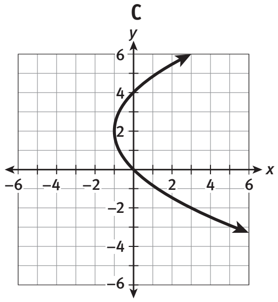
A



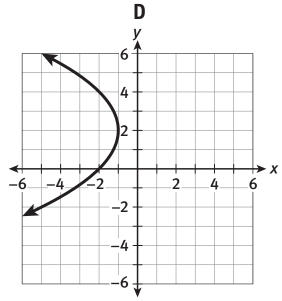
B



C



D



1. **Make use of structure.** Match each equation with one of the graphs above.

$x = \frac{1}{4}(y-2)^2 - 1$

C

$y = \frac{1}{4}(x-2)^2 - 1$

A

$y = -\frac{1}{4}(x-2)^2 - 1$

B

$x = -\frac{1}{4}(y-2)^2 - 1$

D

Common Core State Standards for Activity 10

HSA-CED.A.1 Create equations and inequalities in one variable and use them to solve problems.

HSA-CED.A.2 Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.

ACTIVITY 10

Investigative

Activity Standards Focus

In Activity 10, students write equations of parabolas given a graph or key features of the parabola. They determine a quadratic function given three points on a plane that the function passes through. They also find a quadratic model for a given set of data values and use the model to make predictions about the data. Throughout this activity, emphasize the definition of a parabola and how the equation of a parabola relates to a quadratic function.

Lesson 10-1

PLAN

Pacing: 1 class period

Chunking the Lesson

#1–3 #4–6

Check Your Understanding

#10–11 #12–13 #14–17

Check Your Understanding

#21

Check Your Understanding

Lesson Practice

TEACH

Bell-Ringer Activity

Have students make a table of values for the equations $y = 2x$, $y = 2x^2$, and $y = 2x^3$ using domain values of −3, −2, −1, 0, 1, 2, 3. Then have them graph the equations.

1–3 Think-Pair-Share, Critique Reasoning Students will likely use several different methods to match the graphs to the equations. After students have shared their methods, ask students to determine the most efficient method.

4–6 Activating Prior Knowledge, Graphic Organizer When comparing the graphs of A and B to the graphs of C and D, students may note that A and B are functions while C and D are not. While this is true, it is important that students note that the orientations of the axes of symmetry are different. The notion of functions will be covered in Item 8.

Teacher to Teacher

Parabolas are defined both geometrically and algebraically. Geometrically, a parabola is a conic section and meets the geometric criteria set forth in this activity. Algebraically, a parabola is the graph of any quadratic equation.

Technology Tip

Point out that splitting an equation that includes the $\pm$ symbol results in two separate equations that are reflections of each other. Have students enter each equation separately. For additional technology resources, visit SpringBoard Digital.

My Notes

Technology Tip

If an equation includes the $\pm$ symbol, you will need to enter it in a graphing calculator as two separate equations. For example, enter the equation $y = 2 \pm \sqrt{x}$ as $y = 2 + \sqrt{x}$ and $y = 2 - \sqrt{x}$.

2. Explain how you matched each equation with one of the graphs.
Sample answer: A is the only graph that includes (4, 0), B is the only graph that includes (0, −2), C is the only graph that includes (0, 4), and D is the only graph that includes (−2, 0). Use substitution to determine which of these ordered pairs is a solution of each equation.

3. Use appropriate tools strategically. Use a graphing calculator to confirm your answers to Item 1. Which equations must be rewritten to enter them in the calculator? Rewrite any equations from Item 1 as necessary so that you can use them with your calculator.
Rewrite $x = \frac{1}{4}(y-2)^2 - 1$ as $y = 2 \pm 2\sqrt{x+1}$; rewrite $x = -\frac{1}{4}(y-2)^2 - 1$ as $y = 2 \pm 2\sqrt{-x-1}$; check students' calculator graphs.

4. a. How do graphs A and B differ from graphs C and D?
Sample answer: A and B are parabolas that open up or down. They are symmetric about a vertical line. C and D are parabolas that open right or left. They are symmetric about a horizontal line.

b. How do the equations of graphs A and B differ from the equations of graphs C and D?
Sample answer: The equations for A and B are solved for *y*, and the expression equal to *y* is a quadratic expression in terms of *x*. The equations for C and D are solved for *x*, and the expression equal to *x* is a quadratic expression in terms of *y*.

Lesson 10-1
Parabolas and Quadratic Equations

5. Work with your group. Consider graphs A and B and their equations.

a. Describe the relationship between the graphs.

Sample answer: The graphs are reflections of each other across the line $y = -1$.

b. What part of the equation determines whether the graph opens up or down? How do you know?

The equations are identical except for the sign of $\frac{1}{4}$, so the sign of this number determines whether the graph opens up or down. If the sign is positive, the graph opens up; if it is negative, the graph opens down.

c. Attend to precision. What are the coordinates of the lowest point on graph A? What are the coordinates of the highest point on graph B? How do the coordinates of these points relate to the equations of the graphs?

A: (2, −1); B: (2, −1); The *x*-coordinate is the number subtracted from *x* inside the parentheses. The *y*-coordinate is the number added outside the parentheses.

6. Continue to work with your group. Consider graphs C and D and their equations.

a. Describe the relationship between the graphs.

Sample answer: The graphs are reflections of each other across the line $x = -1$.

b. What part of the equation determines whether the graph opens to the right or left? How do you know?

The equations are identical except for the sign of $\frac{1}{4}$, so the sign of this number determines whether the graph opens to the right or left. If the sign is positive, the graph opens to the right; if it is negative, the graph opens to the left.

c. What are the coordinates of the leftmost point on graph C? What are the coordinates of the rightmost point on graph D? How do the coordinates of these points relate to the equations of the graphs?

C: (−1, 2); D: (−1, 2); The *x*-coordinate is the number added outside the parentheses. The *y*-coordinate is the number subtracted from *y* inside the parentheses.

My Notes

DISCUSSION GROUP TIP

As you share ideas for Items 5 and 6 in your group, ask your group members or your teacher for clarification of any language, terms, or concepts that you do not understand.

MATH TIP

A graph is said to open upward when both ends of the graph point up. A graph is said to open downward when both ends of the graph point down.

The vertex of a graph that opens upward is the *minimum* of the graph, and is its lowest point. The vertex of a graph that opens downward is the *maximum* of the graph, and is its highest point.

ACTIVITY 10 Continued

4–6 (continued) Help students create a graphic organizer to summarize the connections between the equation and the graph of a parabola. There should be three levels of organization: What is the orientation of the parabola? Which specific direction does the parabola open? What is the vertex of the parabola?

Differentiating Instruction

Challenge students to determine why a parabola with a negative x^2 coefficient opens down and one with a positive x^2 coefficient opens up. Have students create a table of values in which the domain values tend toward infinity and negative infinity for the equations $y = -2x^2$ and $y = 2x^2$. As students create these tables, they should note that squaring always results in a positive value and the lead coefficient will determine the sign of the *y*-value. They should also note that as domain values tend toward infinity and negative infinity, the *y*-values will either increase or decrease without bound.

Check Your Understanding

Debrief students' answers to these items to ensure that they understand concepts related to graphing quadratic equations. Ask students whether the graph in Item 9 represents a function. [*yes*]

Answers

7. B

8. The equations $y = \frac{1}{4}(x-2)^2 - 1$ and $y = -\frac{1}{4}(x-2)^2 - 1$ represent functions because the graphs of these equations show that there is only 1 value of y for each value of x. The equations $x = \frac{1}{4}(y-2)^2 - 1$ and $x = -\frac{1}{4}(y-2)^2 - 1$ are not functions because the graphs of these equations show that there are values of x for which there is more than one value of y.

9. To the left; Sample explanation: The equation is solved for x, and the expression equal to x is a quadratic expression in terms of y, which indicates that the graph opens to the right or to the left. The sign of the number multiplied by the squared quantity is negative, so the graph opens to the left.

Developing Math Language

Have students make four drawings in their notes.

1. a focus and directrix that would result in a parabola that opens up
2. a focus and directrix that would result in a parabola that opens down
3. a focus and directrix that would result in a parabola that opens left
4. a focus and directrix that would result in a parabola that opens right.

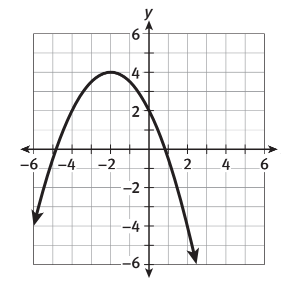
Check Your Understanding

7. Which equation does the graph at right represent? Explain your answer.

A. $y = -\frac{1}{2}(x+2)^2 - 4$

B. $y = -\frac{1}{2}(x+2)^2 + 4$

C. $y = -\frac{1}{2}(x-2)^2 + 4$



8. Construct viable arguments. Which of the equations in Item 1 represent functions? Explain your reasoning.

9. Consider the equation $x = -2(y+4)^2 - 1$. Without graphing the equation, tell which direction its graph opens. Explain your reasoning.

MATH TERMS

A **parabola** is the set of points in a plane that are equidistant from a fixed point and a fixed line.

The fixed point is called the **focus**.

The fixed line is called the **directrix**.

MATH TIP

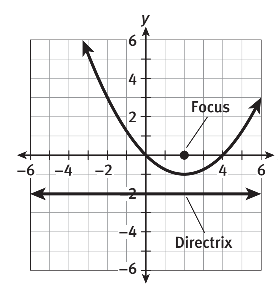
The distance between two points (x_1, y_1) and (x_2, y_2) is given by $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

MATH TIP

The distance between a point and a horizontal line is the length of the vertical segment with one endpoint at the point and one endpoint on the line.

The graphs shown at the beginning of this lesson are all parabolas. A ***parabola*** can be defined as the set of points that are the same distance from a point called the ***focus*** and a line called the ***directrix***.

10. The focus of graph A, shown below, is (2, 0), and the directrix is the horizontal line $y = -2$.



a. The point (−2, 3) is on the parabola. Find the distance between this point and the focus.

distance to focus: $\sqrt{(2-(-2))^2 + (0-3)^2} = 5$

My Notes

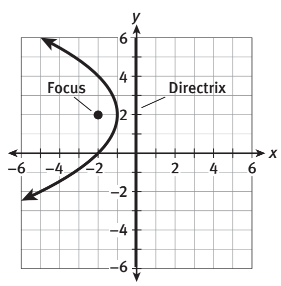
b. Find the distance between the point (–2, 3) and the directrix.

distance to directrix: $\sqrt{(-2-(-2))^2+(-2-3)^2}=5$; $x=2$

c. Reason quantitatively. Compare your answers in parts a and b. What do you notice?

The point (–2, 3) on the parabola is the same distance from the focus as from the directrix.

11. The focus of graph D, shown below, is (–2, 2), and the directrix is the vertical line $x = 0$.



a. The point (–2, 4) is on the parabola. Show that this point is the same distance from the focus as from the directrix.

distance to focus: $\sqrt{(-2-(-2))^2+(2-4)^2}=2$;

distance to directrix: $\sqrt{(0-(-2))^2+(4-4)^2}=2$

b. The point (–5, –2) is also on the parabola. Show that this point is the same distance from the focus as from the directrix.

distance to focus: $\sqrt{(-2-(-5))^2+(2-(-2))^2}=5$;

distance to directrix: $\sqrt{(0-(-5))^2+(-2-(-2))^2}=5$

MATH TIP

The distance between a point and a vertical line is the length of the horizontal segment with one endpoint at the point and one endpoint on the line.

ACTIVITY 10 Continued

10–11 Create Representations, Identify a Subtask For students who struggle with either remembering or using the distance formula correctly, remind them that it is possible to determine the distance between two points that fall on a horizontal or vertical line by counting. Also, you can determine the distance between two points that do not fall on a vertical or horizontal line by constructing a right triangle and using the Pythagorean Theorem to determine the hypotenuse of the triangle.

Technology Tip

Use a dynamic mathematics software program such as GeoGebra to create a parabola by selecting a point and a directrix and using the parabola tool. You can then construct points on the parabola and measure the distances between the point and the focus and the point and the directrix.

For additional technology resources, visit SpringBoard Digital.

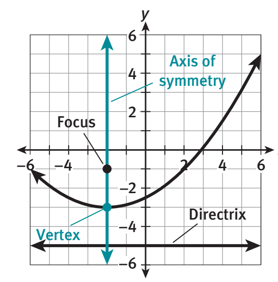
12–13 Create Representations, Summarizing Students commonly reverse the equations of horizontal and vertical lines. Be sure to emphasize the relationship between the equation of the axis of symmetry and the coordinates of the vertex.

Differentiating Instruction

For students who need a challenge, present them with the following task: Sketch a line through the focus of a parabola that is perpendicular to the axis of symmetry. This line will intersect the parabola in two points, P_1 and P_2. Explain why the distance between P_1 and P_2 is double the distance between the focus and the directrix of the parabola.

My Notes

The focus of the parabola shown below is (−2, −1), and the directrix is the line $y = -5$.



MATH TERMS

The **axis of symmetry** is a line that divides the parabola into two congruent halves. The axis of symmetry passes through the focus and is perpendicular to the directrix.

The **vertex** is the point on the parabola that lies on the axis of symmetry. The vertex is the midpoint of the segment connecting the focus and the directrix.

12. **a.** Draw and label the ***axis of symmetry*** on the graph above. What is the equation of the axis of symmetry?
$x = -2$

b. Explain how you identified the axis of symmetry of the parabola.
Sample answer: The directrix is horizontal, so I drew a vertical line through the focus.

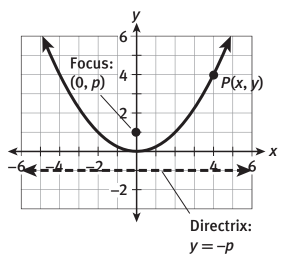
13. **a.** Draw and label the ***vertex*** on the graph above. What are the coordinates of the vertex?
(−2, −3)

b. Explain how you identified the vertex of the parabola.
Sample answer: I drew a point where the axis of symmetry intersects the parabola.

c. What is another way you could have identified the vertex?
Sample answer: I could have drawn a vertical segment from the focus to the directrix. Then I could have drawn a point at the midpoint of this segment.

My Notes

You can use what you have learned about parabolas to derive a general equation for a parabola whose vertex is located at the origin. Start with a parabola that has a vertical axis of symmetry, a focus of $(0, p)$, and a directrix of $y = -p$. Let $P(x, y)$ represent any point on the parabola.



14. Write, but do not simplify, an expression for the distance from point P to the focus.

$\sqrt{(x-0)^2+(y-p)^2}$ **or equivalent**

15. Write, but do not simplify, an expression for the distance from point P to the directrix.

$\sqrt{(x-x)^2+(y-(-p))^2}$ **or equivalent**

16. Make use of structure. Based on the definition of a parabola, the distance from point P to the focus is the same as the distance from point P to the directrix. Set your expressions from Items 14 and 15 equal to each other, and then solve for y.

$$\sqrt{(x-0)^2+(y-p)^2}=\sqrt{(x-x)^2+(y-(-p))^2}$$
$$(x-0)^2+(y-p)^2=(x-x)^2+(y-(-p))^2$$
$$x^2+(y-p)^2=(y+p)^2$$
$$x^2+y^2-2py+p^2=y^2+2py+p^2$$
$$x^2-2py=2py$$
$$x^2=4py$$
$$\frac{1}{4p}x^2=y$$

MATH TIP

In Item 16, start by squaring each side of the equation to eliminate the square root symbols. Next, simplify each side and expand the squared terms.

ACTIVITY 10 Continued

14–17 Think-Pair-Share, Look for a Pattern Have volunteers share their answers to these items. Some students may choose to simplify the expressions under the radicals prior to squaring both sides. Watch for the common student error of improperly expanding a squared binomial.

Differentiating Instruction

Have students determine how the value of p will affect the shape of the parabola $y = \frac{1}{4p}x^2$. In particular, have students answer these questions: What happens to the shape of the parabola as the value of p increases toward infinity? What happens to the shape of the parabola as the value of p gets closer and closer to zero?

Check Your Understanding

Debrief students' answers to these items to ensure that they understand concepts related to the general equation of a parabola.

Answers

18. $x = \frac{1}{4p}y^2$. Sample derivation:
distance from P to focus = distance from P to directrix

$$\sqrt{(x-p)^2 + (y-0)^2} = \sqrt{(x-(-p))^2 + (y-y)^2}$$

$$(x-p)^2 + (y-0)^2 = (x-(-p))^2 + (y-y)^2$$
$$(x-p)^2 + y^2 = (x+p)^2$$
$$x^2 - 2px + p^2 + y^2 = x^2 + 2px + p^2$$
$$-2px + y^2 = 2px$$
$$y^2 = 4px$$
$$\frac{1}{4p}y^2 = x$$

19. $y = -\frac{1}{12}x^2$; The vertex and the focus of the parabola are on the y-axis, so the y-axis is the axis of symmetry. The parabola has its vertex at the origin and a vertical axis of symmetry, so its equation has the form $y = \frac{1}{4p}x^2$, where p is the y-coordinate of the focus. The focus is $(0, -3)$, and the equation of the parabola is $y = \frac{1}{4(-3)}x^2 = -\frac{1}{12}x^2$.

20. a. $y = 4$; The directrix is vertical, so the axis of symmetry is a horizontal line through the focus. The focus has a y-coordinate of 4, so the axis of symmetry is the line $y = 4$.

b. $(1, 4)$; The vertex is the midpoint of the segment that connects the focus and the directrix. The endpoints of this segment have coordinates $(3, 4)$ and $(-1, 4)$, so the vertex has coordinates $(1, 4)$.

c. To the right; The axis of symmetry is horizontal and the focus is to the right of the directrix, so the parabola opens to the right.

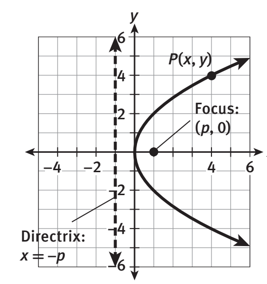
My Notes

17. What is the general equation for a parabola with its vertex at the origin, a focus of $(0, p)$, and a directrix of $y = -p$?

$y = \frac{1}{4p}x^2$ **or equivalent**

Check Your Understanding

18. See the diagram at right. Derive the general equation of a parabola with its vertex at the origin, a horizontal axis of symmetry, a focus of $(p, 0)$, and a directrix of $x = -p$. Solve the equation for x.



19. Model with mathematics. The vertex of a parabola is at the origin and its focus is $(0, -3)$. What is the equation of the parabola? Explain your reasoning.

20. A parabola has a focus of $(3, 4)$ and a directrix of $x = -1$. Answer each question about the parabola, and explain your reasoning.

a. What is the axis of symmetry?

b. What is the vertex?

c. In which direction does the parabola open?

MATH TIP

A parabola always opens toward the focus and away from the directrix.

You can also write general equations for parabolas that do not have their vertex at the origin. You will derive these equations later in this activity.

	Vertical Axis of Symmetry	Horizontal Axis of Symmetry
Vertex	(h, k)	(h, k)
Focus	$(h, k + p)$	$(h + p, k)$
Directrix	horizontal line $y = k - p$	vertical line $x = h - p$
Equation	$y = \frac{1}{4p}(x-h)^2 + k$	$x = \frac{1}{4p}(y-k)^2 + h$

Lesson 10-1
Parabolas and Quadratic Equations

My Notes

MATH TIP

You may find it helpful to make a quick sketch of the information you are given.

21. Reason quantitatively. Use the given information to write the equation of each parabola.

a. axis of symmetry: $y = 0$; vertex: (0, 0); directrix: $x = \frac{1}{2}$

$x = -\frac{1}{2}y^2$

b. vertex: (3, 4); focus: (3, 6)

$y = \frac{1}{8}(x - 3)^2 + 4$

c. vertex: (−2, 1); directrix: $y = 4$

$y = -\frac{1}{12}(x + 2)^2 + 1$

d. focus: (−4, 0); directrix: $x = 4$

$x = -\frac{1}{16}y^2$

e. opens up; focus: (5, 7); directrix: $y = 3$

$y = \frac{1}{8}(x - 5)^2 + 5$

21 Create Representations, Identify a Subtask, Debriefing Prior to writing the equation of each parabola, ask students to supply the missing information for each parabola. For example, students would determine the vertex and the axis of symmetry for Part d.

TEACHER to TEACHER

Parabolas have many applications in the real world. One application is the use of parabolic reflectors in reflecting telescopes. These parabolic reflectors range in diameter from 3 inches in home telescopes to 200 inches in research telescopes. A parabolic reflector is a paraboloid which is formed by rotating a parabola about its axis of symmetry. Help students to visualize this rotation and the formation of the paraboloid.

Check Your Understanding

Debrief students' answers to these items to ensure that they understand concepts related to equations of parabolas.

Answers

23. No. Sample explanation: To write the equation of a parabola, you need to know the value of p. To determine the value of p given the vertex, you would also need to know either the focus or the directrix of the parabola.

24. vertex: (1, 2); axis of symmetry: $y = 2$; focus: (3, 2); directrix: $x = -1$

ASSESS

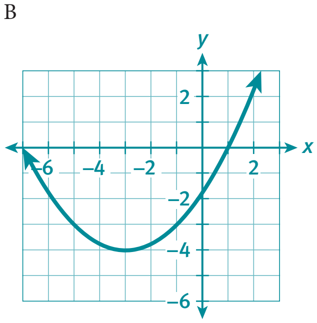
Students' answers to Lesson Practice problems will provide you with a formative assessment of their understanding of the lesson concepts and their ability to apply their learning.

See the Activity Practice for additional problems for this lesson. You may assign the problems here or use them as a culmination for the activity.

LESSON 10-1 PRACTICE

25. B

26.



27. axis of symmetry: $y = 2$;
vertex: $(-3, 2)$; opens to the right

28. $y = -\frac{1}{2}x^2$

29. $x = \frac{1}{16}y^2$

30. $x = -\frac{1}{20}(y-5)^2$

31. $y = \frac{1}{12}(x-3)^2 - 4$

32. $x = \frac{1}{4}(y-4)^2 - 2$

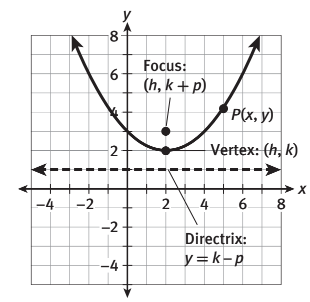
ADAPT

Check students' answers to the Lesson Practice to ensure that they understand the geometric definition of a parabola, its component parts, and the general form of the equation of a parabola. Students should be able to match graphs to their equations and vice versa. Encourage students who require extra practice to create their own problems using Lesson Practice Items 28–32 as a template. Students can check their own work by graphing the equation they write on a graphing calculator.

My Notes

Check Your Understanding

22. See the diagram at right. Derive the general equation of a parabola with its vertex at (h, k), a vertical axis of symmetry, a focus of $(h, k + p)$, and a directrix of $y = k - p$. Solve the equation for y.



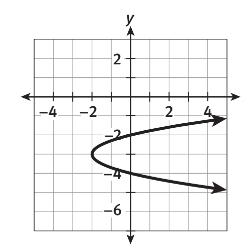
23. **Construct viable arguments.** Can you determine the equation of a parabola if you know only its axis of symmetry and its vertex? Explain.

24. The equation of a parabola is $x = \frac{1}{8}(y-2)^2 + 1$. Identify the vertex, axis of symmetry, focus, and directrix of the parabola.

LESSON 10-1 PRACTICE

25. Which equation does the graph at right represent?

A. $x = -2(y+3)^2 - 2$
B. $x = 2(y+3)^2 - 2$
C. $y = -2(x+3)^2 - 2$
D. $y = 2(x+3)^2 - 2$



26. Graph the parabola given by the equation $y = \frac{1}{4}(x+3)^2 - 4$.

27. **Make sense of problems.** The focus of a parabola is (0, 2), and its directrix is the vertical line $x = -6$. Identify the axis of symmetry, the vertex, and the direction the parabola opens.

Use the given information to write the equation of each parabola.

28. vertex: (0, 0); focus: $\left(0, -\frac{1}{2}\right)$

29. focus: (4, 0); directrix: $x = -4$

30. opens to the left; vertex: (0, 5); focus: $(-5, 5)$

31. axis of symmetry: $x = 3$; focus: $(3, -1)$; directrix: $y = -7$

32. vertex: $(-2, 4)$; directrix: $x = -3$

22. Sample derivation:

distance from P to focus = distance from P to directrix

$$\sqrt{(x-h)^2 + (y-(k+p))^2} = \sqrt{(x-x)^2 + (y-(k-p))^2}$$

$$(x-h)^2 + (y-(k+p))^2 = (x-x)^2 + (y-(k-p))^2$$

$$(x-h)^2 + y^2 - 2(k+p)y + (k+p)^2 = y^2 - 2(k-p)y + (k-p)^2$$

$$(x-h)^2 + y^2 - 2ky - 2py + k^2 + 2pk + p^2 = y^2 - 2ky + 2py + k^2 - 2pk + p^2$$

$$(x-h)^2 - 2py + 2pk = 2py - 2pk$$

$$(x-h)^2 + 4pk = 4py$$

$$\frac{1}{4p}(x-h)^2 + k = y$$

Lesson 10-2
Writing a Quadratic Function Given Three Points

My Notes

Learning Targets:

- Explain why three points are needed to determine a parabola.
- Determine the quadratic function that passes through three given points on a plane.

SUGGESTED LEARNING STRATEGIES: Create Representations, Quickwrite, Questioning the Text, Create Representations, Identify a Subtask

Recall that if you are given any two points on the coordinate plane, you can write the equation of the line that passes through those points. The two points are said to determine the line because there is only one line that can be drawn through them.

Do two points on the coordinate plane determine a parabola? To answer this question, work through the following items.

1. Follow these steps to write the equation of a quadratic function whose graph passes through the points (2, 0) and (5, 0).

a. Write a quadratic equation in standard form with the solutions $x = 2$ and $x = 5$.

$x^2 - 7x + 10 = 0$ or a nonzero multiple of this equation

b. Replace 0 in your equation from part a with y to write the corresponding quadratic function.

Answers may vary depending on the equation in part a. Sample answer: $y = x^2 - 7x + 10$

c. Use substitution to check that the points (2, 0) and (5, 0) lie on the function's graph.

$0 \stackrel{?}{=} (2)^2 - 7(2) + 10$	$0 \stackrel{?}{=} (5)^2 - 7(5) + 10$
$0 \stackrel{?}{=} 4 - 14 + 10$	$0 \stackrel{?}{=} 25 - 35 + 10$
$0 = 0$ ✓	$0 = 0$ ✓

MATH TIP

To review writing a quadratic equation when given its solutions, see Lesson 7-3.

2. a. Use appropriate tools strategically. Graph your quadratic function from Item 1 on a graphing calculator.

Check students' work.

b. On the same screen, graph the quadratic functions $y = 2x^2 - 14x + 20$ and $y = -x^2 + 7x - 10$.

Check students' work.

Lesson 10-2

PLAN

Pacing: 1 class period

Chunking the Lesson

#1–3 #4–6 #7–10

Check Your Understanding

Lesson Practice

TEACH

Bell-Ringer Activity

Have students write the equation in slope-intercept form of a line that passes through the points (3, 2) and (−5, 6), using the following procedures:

1. Substitute the point (3, 2) into the equation $y = mx + b$.
2. Substitute the point (−5, 6) into the equation $y = mx + b$.
3. Solve the system of equations using substitution or Gaussian elimination.

1–3 Interactive Word Wall, Create Representations Encourage the use of proper math vocabulary to describe the similarities and differences of the three parabolas. Students could use the terms *vertex*, *axis of symmetry*, *maximum*, *minimum*, *y-intercept*, and *x-intercepts* in their descriptions.

Universal Access

Students may question where the equations $y = 2x^2 - 14x + 10$ and $y = -x^2 + 7x - 10$ in Item 2b came from. Return to Item 1 and lead students through writing binomial factors other than $(x - 2)$ and $(x - 5)$ that lead to a quadratic equation with solution set {2, 5}.

4–6 Predict and Confirm, Create Representations, Visualization Ask students to plot the three points on a coordinate plane and make predictions about the vertex of the parabola. Have students share answers and note that it is impossible to accurately predict the vertex. This will motivate the algebraic solution process.

Universal Access

Using algebraic methods to solve a system of equations in three variables can be time-consuming and frustrating to students. Consider allowing students to use matrix equations and their graphing calculators to determine the values of a, b, and c. Doing this will keep the lesson focused on finding the equation of the parabola.

My Notes

Lesson 10-2
Writing a Quadratic Function Given Three Points

c. Describe the graphs. Do all three parabolas pass through the points (2, 0) and (5, 0)?
Answers may vary, but students should note that all three parabolas pass through the points (2, 0) and (5, 0).
Sample answer: Two of the parabolas open upward, and one opens downward. One parabola is narrower than the others. However, all of the parabolas have the same *x*-intercepts: 2 and 5.

3. Reason abstractly. Do two points on the coordinate plane determine a parabola? Explain.
No. Sample explanation: My graph of the three parabolas shows that more than one parabola can be drawn through the same pair of points, (2, 0) and (5, 0). So, two points are not enough to determine a parabola.

MATH TIP

Three or more points are *collinear* if they lie on the same straight line.

Three points in the coordinate plane that are not on the same line determine a parabola given by a quadratic function. If you are given three noncollinear points on the coordinate plane, you can write the equation of the quadratic function whose graph passes through them.

Consider the quadratic function whose graph passes through the points (1, 2), (3, 0), and (5, 6).

4. Write an equation by substituting the coordinates of the point (1, 2) into the standard form of a quadratic function, $y = ax^2 + bx + c$.
$2 = a + b + c$ or equivalent

5. Write a second equation by substituting the coordinates of the point (3, 0) into the standard form of a quadratic function.
$0 = 9a + 3b + c$ or equivalent

6. Write a third equation by substituting the coordinates of the point (5, 6) into the standard form of a quadratic function.
$6 = 25a + 5b + c$ or equivalent

My Notes

7. Use your equations from Items 4–6 to write a system of three equations in the three variables *a*, *b*, and *c*.

$$\begin{cases} a+b+c=2 \\ 9a+3b+c=0 \\ 25a+5b+c=6 \end{cases}$$ or equivalent

8. Use substitution or Gaussian elimination to solve your system of equations for *a*, *b*, and *c*.

$a=1, b=-5, c=6$

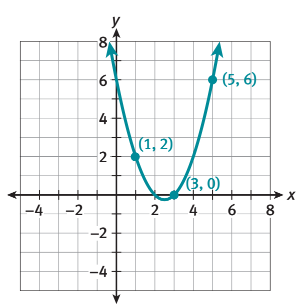
MATH TIP

To review solving a system of three equations in three variables, see Lesson 3-2.

9. Now substitute the values of *a*, *b*, and *c* into the standard form of a quadratic function.

$y=x^2-5x+6$

10. **Model with mathematics.** Graph the quadratic function to confirm that it passes through the points (1, 2), (3, 0), and (5, 6).



ACTIVITY 10 Continued

7–10 Debriefing, Identify a Subtask, Summarizing Ask students to provide a summary of the method used to determine the equation of a parabola in standard form that passes through three given points.

TEACHER to TEACHER

If an additional example is needed, students can create examples for a partner by working backward. Have students graph a parabola in standard form on a calculator and use the table or trace function to identify three integral points that lie on the parabola. Students can exchange points and then check each other's work.

Check Your Understanding

Debrief students' answers to these items to ensure that they understand concepts related to writing equations of quadratic functions.

Answers

11. Substitute the coordinates of each point into the standard form of a quadratic function, $y = ax^2 + bx + c$. Write the 3 resulting equations as a system of equations. Then solve the system for the values of a, b, and c. Finally, use the values of a, b, and c to write the equation of the quadratic function in standard form.

12. a. You find that $a = 0$, $b = -1$, and $c = 4$, which results in the function $f(x) = -x + 4$. This function is linear, not quadratic.

b. The 3 points are on the same line, which means that you cannot write the equation of a quadratic function whose graph passes through the points.

13. a. $(-6, 0)$. Sample explanation: For a quadratic function, the axis of symmetry is a vertical line that passes through the vertex, so the axis of symmetry is $(x = -2)$. The point $(2, 0)$ is 4 units to the right of the axis of symmetry, so there will be another point on the graph of the function 4 units to the left of the axis of symmetry with the same y-coordinate. This point has coordinates $(-6, 0)$.

b. $y = x^2 + 4x - 12$

ASSESS

Students' answers to Lesson Practice problems will provide you with a formative assessment of their understanding of the lesson concepts and their ability to apply their learning.

See the Activity Practice for additional problems for this lesson. You may assign the problems here or use them as a culmination for the activity.

ADAPT

Check students' answers to the Lesson Practice to ensure that they understand how to write the equation of a parabola in standard form given three points that lie on the parabola. For additional practice, students can make up their own problems. Have them select three points and write the equation of the parabola using the method learned in this activity. Show students how they can check their work using quadratic regression on their graphing calculators.

My Notes

Check Your Understanding

11. Describe how to write the equation of a quadratic function whose graph passes through three given points.

12. a. What happens when you try to write the equation of the quadratic function that passes through the points $(0, 4)$, $(2, 2)$, and $(4, 0)$?

b. What does this result indicate about the three points?

13. a. **Reason quantitatively.** The graph of a quadratic function passes through the point $(2, 0)$. The vertex of the graph is $(-2, -16)$. Use symmetry to identify another point on the function's graph. Explain how you determined your answer.

b. Write the equation of the quadratic function.

MATH TIP

A sequence is an ordered list of numbers or other items. Each number or item in a sequence is called a term.

LESSON 10-2 PRACTICE

Write the equation of the quadratic function whose graph passes through each set of points.

14. $(-3, 2)$, $(-1, 0)$, $(1, 6)$

15. $(-2, -5)$, $(0, -3)$, $(1, 4)$

16. $(-1, -5)$, $(1, -9)$, $(4, 0)$

17. $(-3, 7)$, $(0, 4)$, $(1, 15)$

18. $(1, 0)$, $(2, -7)$, $(5, -16)$

19. $(-2, -11)$, $(-1, -12)$, $(1, 16)$

20. The table below shows the first few terms of a sequence. This sequence can be described by a quadratic function, where $f(n)$ represents the nth term of the sequence. Write the quadratic function that describes the sequence.

Term Number, n	1	2	3	4	5
Term of Sequence, $f(n)$	2	6	12	20	30

CONNECT TO GEOMETRY

A regular hexagon is a six-sided polygon with all sides having the same length and all angles having the same measure.

21. A quadratic function $A(s)$ gives the area in square units of a regular hexagon with a side length of s units.

a. Use the data in the table below to write the equation of the quadratic function.

Side Length, s	2	4	6
Area, $A(s)$	$6\sqrt{3}$	$24\sqrt{3}$	$54\sqrt{3}$

b. **Attend to precision.** To the nearest square centimeter, what is the area of a regular hexagon with a side length of 8 cm?

LESSON 10-2 PRACTICE

14. $y = x^2 + 3x + 2$

15. $y = 2x^2 + 5x - 3$

16. $y = x^2 - 2x - 8$

17. $y = 3x^2 + 8x + 4$

18. $y = x^2 - 10x + 9$

19. $y = 5x^2 + 14x - 3$

20. $f(n) = n^2 + n$

21. a. $A(s) = \frac{3\sqrt{3}}{2}s^2$

b. 166 cm^2

Lesson 10-3
Quadratic Regression

Learning Targets:
- Find a quadratic model for a given table of data.
- Use a quadratic model to make predictions.

SUGGESTED LEARNING STRATEGIES: Think Aloud, Discussion Groups, Create Representations, Interactive Word Wall, Quickwrite, Close Reading, Predict and Confirm, Look for a Pattern, Group Presentation

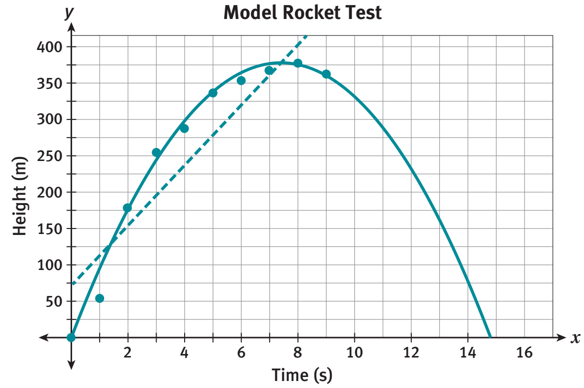
A model rocketry club placed an altimeter on one of its rockets. An altimeter measures the altitude, or height, of an object above the ground. The table shows the data the club members collected from the altimeter before it stopped transmitting a little over 9 seconds after launch.

Model Rocket Test

Time Since Launch (s)	0	1	2	3	4	5	6	7	8	9
Height (m)	0	54	179	255	288	337	354	368	378	363

1. Predict the height of the rocket 12 seconds after launch. Explain how you made your prediction.
Predictions and explanations will vary.

2. Model with mathematics. Make a scatter plot of the data on the coordinate grid below.



My Notes

CONNECT TO PHYSICS

A model rocket is not powerful enough to escape Earth's gravity. The maximum height that a model rocket will reach depends in part on the weight and shape of the rocket, the amount of force generated by the rocket motor, and the amount of fuel the motor contains.

ACTIVITY 10 Continued

Lesson 10-3

PLAN

Pacing: 1 class period

Chunking the Lesson

#1–2 #3–6

#7–9 #10–11

Check Your Understanding

Lesson Practice

TEACH

Bell-Ringer Activity

Have students determine whether the function $y = x^2 - 4x + 3$ is a good model for the data points $(-5, 50)$, $(-1, 8)$, $(0, 2)$, $(4, 2)$, and $(5, 7)$. Make sure that students justify their answers.

1–2 Construct an Argument, Visualization, Think Aloud Some students will struggle to understand the physics behind the problem. Help students to realize that while the rocket is launched with an initial velocity, gravity slows the rocket down as it ascends until it finally reaches a point of zero velocity. Once the rocket begins falling back to earth, gravity serves to increase its velocity as it descends.

MINI-LESSON: Second Differences

If students need additional help with how to use second differences to determine if a set of data is a good candidate for a quadratic model, a mini-lesson is available to provide practice.

See the Teacher Resources at SpringBoard Digital for a student page for this mini-lesson.

3–6 Create Representations, Note Taking, Discussion Groups Have students work in groups to perform the quadratic regression. Some students may need to record the keystrokes in their notes for future reference.

Teacher to Teacher

How well a regression model fits a set of data can be mathematically represented by the correlation coefficient, r^2, which measures the percent of variability in the y-values that has been explained by the regression equation. This statistic ranges from 0 to 1, with a value of 1 indicating that 100% of the variability of y has been explained by the regression equation. In other words, a value of 1 would occur if the graph of the model passed through each point of the data set.

Developing Math Language

Emphasize that regression, no matter what model, is an attempt to find a sufficiently good functional relationship between an independent variable and a dependent variable. In this case, the independent variable is time and the dependent variable is height.

Lesson 10-3
Quadratic Regression

My Notes

3. Enter the rocket data into a graphing calculator. Enter the time data as List 1 (L1) and the height data as List 2 (L2). Then use the calculator to perform a linear regression on the data. Write the equation of the linear model that results from the regression. Round coefficients and constants to the nearest tenth.
$y = 41.4x + 71.4$

4. Use a dashed line to graph the linear model from Item 3 on the coordinate grid showing the rocket data.
See graph below Item 2.

5. a. Attend to precision. To the nearest meter, what height does the linear model predict for the rocket 12 seconds after it is launched?
568 m

b. How does this prediction compare with the prediction you made in Item 1?
Answers will vary.

6. Construct viable arguments. Do you think the linear model is a good model for the rocket data? Justify your answer.
Sample answer: No. The linear model indicates that the rocket was already about 71 m off the ground at the time it was launched, when its actual height at this time was 0 m. Also, the linear model indicates that the rocket's height would continue to increase with time without the rocket ever landing. The actual data show that the rocket's height is starting to decrease after 8 seconds.

MATH TIP

A calculator may be able to generate a linear model for a data set, but that does not necessarily mean that the model is a good fit or makes sense in a particular situation.

A linear regression is the process of finding a linear function that best fits a set of data. A ***quadratic regression*** is the process of finding a quadratic function that best fits a set of data. The steps for performing a quadratic regression on a graphing calculator are similar to those for performing a linear regression.

MATH TERMS

Quadratic regression is the process of determining the equation of a quadratic function that best fits the given data.

7. Use these steps to perform a quadratic regression for the rocket data.

- Check that the data set is still entered as List 1 and List 2.
- Press STAT to select the Statistics menu. Then move the cursor to highlight the Calculate (CALC) submenu.
- Select 5:QuadReg to perform a quadratic regression on the data in Lists 1 and 2. Press ENTER.
- The calculator displays the values of *a*, *b*, and *c* for the standard form of the quadratic function that best fits the data.

Write the equation of the quadratic model that results from the regression. Round coefficients and constants to the nearest tenth.

$y = -6.9x^2 + 103.8x - 11.8$

8. Graph the quadratic model from Item 7 on the coordinate grid showing the rocket data.

See graph below Item 2.

9. Construct viable arguments. Contrast the graph of the linear model with the graph of the quadratic model. Which model is a better fit for the data?

Sample answer: The quadratic model is a better fit for the data because the data points are closer to the parabola overall than to the line. Unlike the linear model, the quadratic model shows that the rocket will eventually return to ground level.

10. a. To the nearest meter, what height does the quadratic model predict for the rocket 12 seconds after it is launched?

Predictions should be close to 240 m.

b. How does this prediction compare with the prediction you made in Item 1?

Answers will vary.

11. Reason quantitatively. Use the quadratic model to predict when the rocket will hit the ground. Explain how you determined your answer.

Answers may vary but should be close to 15 s.
Sample explanation: I set the height *y* of the quadratic model equal to 0, and used the Quadratic Formula to solve for the time *x*. The solutions show that the rocket will hit the ground after about 14.9 s.

My Notes

TECHNOLOGY TIP

You can graph the equation from a quadratic regression by using these steps: After selecting **5:QuadReg** as described at the left, do not press ENTER. Instead, press VARS to select the VARS menu. Then move the cursor to highlight the Y-VARS submenu. Select **1:Function**. Then select **1:Y1**. Press ENTER. The equation from the quadratic regression is now assigned to Y1. You can press GRAPH to view the graph of the equation.

ACTIVITY 10 Continued

7–9 Predict and Confirm Point out to students that while the quadratic model generated by the calculator is mathematically the best fit for the data, the equation is still imperfect. For example, the model suggests that the rocket is at −11.8 feet at 0 seconds, which clearly is not true. However, the quadratic model is a much better fit than the linear model and will result in better estimates for when the model returns to the ground.

Teacher to Teacher

Have students display a scatter plot of the data in Lists 1 and 2 on their calculators. Students will need to access the statplot menu and ensure that Plot 1 is turned on and that the *X*list is L_1 and the *Y*list is L_2. Also have students graph the regression equation along with the scatter plot.

10–11 Discussion Groups, Debriefing Have students use the CALC function on a graphing calculator to determine the vertex of the quadratic regression equation. Ask students to interpret the meaning of the *x*- and *y*-coordinates of the vertex and how it relates to the time that the rocket returns to the ground. For additional technology resources, visit SpringBoard Digital.

Check Your Understanding

Debrief students' answers to these items to ensure that they understand concepts related to quadratic regression.

Answers

12. An underestimate; The parachute slows the rocket down, which means that it will take the rocket longer to reach the ground than the model predicts. The prediction from the quadratic model is an underestimate of the time at which the rocket will reach the ground.

13. a. Yes. Three noncollinear points determine a parabola, so you can perform a quadratic regression if you have at least 3 data points.

b. The model would fit the data set exactly because there is only 1 parabola representing a quadratic function that can be drawn through any set of 3 noncollinear points.

c. If the 3 points lie on the same line, the quadratic regression would show that the coefficient of the x^2-term is 0. In other words, the quadratic regression would result in a linear model. The linear model would fit the data exactly, because the 3 points lie on the same line.

ASSESS

Students' answers to Lesson Practice problems will provide you with a formative assessment of their understanding of the lesson concepts and their ability to apply their learning.

See the Activity Practice for additional problems for this lesson. You may assign the problems here or use them as a culmination for the activity.

ADAPT

Check students' answers to the Lesson Practice to ensure that they understand how to use their graphing calculator to complete quadratic regression. Also make sure that students understand the concept of regression such that they can justify an appropriate regression model given a set of data. Many students will require additional practice. Create additional data sets for students by writing a quadratic function in standard form and using it to provide approximate points that fall on the curve.

My Notes

Check Your Understanding

12. Make sense of problems. Most model rockets have a parachute or a similar device that releases shortly after the rocket reaches its maximum height. The parachute helps to slow the rocket so that it does not hit the ground with as much force. Based on this information, do you think your prediction from Item 11 is an underestimate or an overestimate if the rocket has a parachute? Explain.

13. a. Could you use a graphing calculator to perform a quadratic regression on three data points? Explain.

b. How closely would the quadratic model fit the data set in this situation? Explain.

c. How would your answers to parts a and b change if you knew that the three points lie on the same line?

LESSON 10-3 PRACTICE

Tell whether a linear model or a quadratic model is a better fit for each data set. Justify your answer, and give the equation of the better model.

14.

x	10	12	14	16	18	20	22	24
y	19	15	13	11	9	9	10	11

15.

x	2	4	6	8	10	12	14	16
y	10	22	26	35	45	50	64	66

The tables show time and height data for two other model rockets.

Rocket A

Time (s)	0	1	2	3	4	5	6	7
Height (m)	0	54	179	255	288	337	354	368

Rocket B

Time (s)	0	1	2	3	4	5	6	7
Height (m)	0	37	92	136	186	210	221	229

16. Use appropriate tools strategically. Use a graphing calculator to perform a quadratic regression for each data set. Write the equations of the quadratic models. Round coefficients and constants to the nearest tenth.

17. Use your models to predict which rocket had a greater maximum height. Explain how you made your prediction.

18. Use your models to predict which rocket hit the ground first and how much sooner. Explain how you made your prediction.

LESSON 10-3 PRACTICE

14. Sample justification: A quadratic model is a better fit. A graph of both models shows that the data points are closer to the quadratic model. Also, the values of y first decrease and then begin to increase as x increases, which indicates the shape of a quadratic, not a linear, model. Quadratic model: $y = 0.1x^2 - 4.1x + 49.3$

15. Sample justification: A linear model is a better fit. The values of y increase as x increases without ever decreasing, which indicates the shape of a linear, not a quadratic, model. Linear model: $y = 4.1x + 3.1$

16. Rocket A: $y = -7.6x^2 + 107.9x - 14.9$; Rocket B: $y = -3.9x^2 + 62.1x - 10.4$

17. Rocket A. Sample explanation: Graph both quadratic models on the same coordinate grid. The graphs show that Rocket A reaches a greater height than Rocket B.

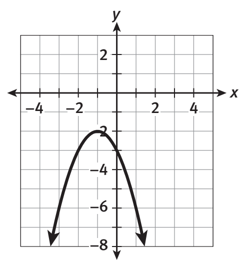
18. Predictions may vary but should indicate that Rocket A will hit the ground about 1.7 seconds sooner than Rocket B. Sample explanation: I set the height y of each quadratic model equal to 0 and used the Quadratic Formula to solve for the time x. The solutions show that Rocket A will hit the ground after about 14.1 seconds and Rocket B will hit the ground after about 15.8 seconds, or about 1.7 seconds later.

Writing Quadratic Equations
What Goes Up Must Come Down

ACTIVITY 10 PRACTICE
Write your answers on notebook paper. Show your work.

Lesson 10-1

Use the parabola shown in the graph for Items 1 and 2.



1. What is the equation of the parabola?
A. $y = -(x-1)^2 - 2$ **B.** $y = -(x+1)^2 - 2$
C. $y = (x-1)^2 - 2$ **D.** $y = (x+1)^2 + 2$

2. The focus of the parabola is $\left(-1, -\frac{9}{4}\right)$, and the directrix is the line $y = -\frac{7}{4}$. Show that the point $(-2, -3)$ on the parabola is the same distance from the focus as from the directrix.

3. Graph the parabola given by the equation $x = \frac{1}{2}(y-3)^2 + 3$.

4. Identify the following features of the parabola given by the equation $y = \frac{1}{8}(x-4)^2 + 3$.
a. vertex **b.** focus
c. directrix **d.** axis of symmetry
e. direction of opening

5. Describe the relationships among the vertex, focus, directrix, and axis of symmetry of a parabola.

6. The focus of a parabola is $(3, -2)$, and its directrix is the line $x = -5$. What are the vertex and the axis of symmetry of the parabola?

For Items 7–11, use the given information to write the equation of each parabola.

7. vertex: (0, 0); focus: (0, 5)

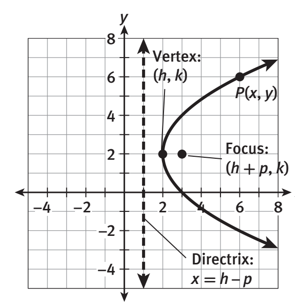
8. vertex: (0, 0); directrix: $x = -3$

9. vertex: (2, 2); axis of symmetry: $y = 2$; focus: (1, 2)

10. opens downward; vertex: $(-1, -2)$; directrix: $y = -1$

11. focus: $(-1, 3)$; directrix: $x = -5$

12. Use the diagram below to help you derive the general equation of a parabola with its vertex at (h, k), a horizontal axis of symmetry, a focus of $(h + p, k)$, and a directrix of $x = h - p$. Solve the equation for x.



Lesson 10-2

Write the equation of the quadratic function whose graph passes through each set of points.

13. $(-3, 0), (-2, -3), (2, 5)$

14. $(-2, -6), (1, 0), (2, 10)$

15. $(-5, -3), (-4, 0), (0, -8)$

16. $(-3, 10), (-2, 0), (0, -2)$

17. $(1, 0), (4, 6), (7, -6)$

18. $(-2, -9), (-1, 0), (1, -12)$

ACTIVITY 10 Continued

ACTIVITY PRACTICE

1. B

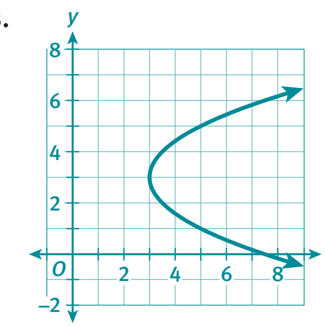
2. distance to focus:

$$\sqrt{(-1-(-2))^2 + \left(-\frac{9}{4}-(-3)\right)^2} = \frac{5}{4}$$

distance to directrix:

$$\sqrt{(-2-(-2))^2 + \left(-\frac{7}{4}-(-3)\right)^2} = \frac{5}{4}$$

3.



4. a. (4, 3)
b. (4, 5)
c. $y = 1$
d. $x = 4$
e. upward

5. Sample answer: The axis of symmetry is perpendicular to the directrix. The focus and the vertex lie on the axis of symmetry. The vertex is the midpoint of the segment that lies on the axis of symmetry and has its endpoints at the focus and on the directrix.

6. Vertex: $(-1, -2)$; axis of symmetry: $y = -2$

7. $y = \frac{1}{20}x^2$

8. $x = \frac{1}{12}y^2$

9. $x = -\frac{1}{4}(y-2)^2 + 2$

10. $y = -\frac{1}{4}(x+1)^2 - 2$

11. $x = \frac{1}{8}(y-3)^2 - 3$

13. $y = x^2 + 2x - 3$

14. $y = 2x^2 + 4x - 6$

15. $y = -x^2 - 6x - 8$

16. $y = 3x^2 + 5x - 2$

17. $y = -x^2 + 7x - 6$

18. $y = -5x^2 - 6x - 1$

12. Sample derivation:

distance from P to focus = distance from P to directrix

$$\sqrt{(x-(h+p))^2 + (y-k)^2} = \sqrt{(x-(h-p))^2 + (y-y)^2}$$
$$(x-(h+p))^2 + (y-k)^2 = (x-(h-p))^2 + (y-y)^2$$
$$x^2 - 2(h+p)x + (h+p)^2 + (y-k)^2 = x^2 - 2(h-p)x + (h-p)^2$$
$$x^2 - 2hx - 2px + h^2 + 2hp + p^2 + (y-k)^2 = x^2 - 2hx + 2px + h^2 - 2hp + p^2$$
$$-2px + 2hp + (y-k)^2 = 2px - 2hp$$
$$(y-k)^2 + 4hp = 4px$$
$$\frac{1}{4p}(y-k)^2 + h = x$$

19. Answers may vary, but equations should be a nonzero multiple of $y = x^2 + 2x - 48$. Sample answer: The parabolas given by the equations $y = x^2 + 2x - 48$ and $y = -x^2 - 2x + 48$ both pass through the points $(-8, 0)$ and $(6, 0)$.

20. a. $(-1, 5)$. Sample explanation: For a quadratic function, the axis of symmetry is a vertical line that passes through the vertex, so the axis of symmetry is $x = 3$. The point $(7, 5)$ is 4 units to the right of the axis of symmetry, so there will be another point on the graph of the function 4 units to the left of the axis of symmetry with the same y-coordinate. This point has coordinates $(-1, 5)$.

b. $f(x) = \frac{1}{4}x^2 - \frac{3}{2}x + \frac{13}{4}$

21. Sample justification: A linear model is a better fit. The values of y increase as x increases without ever decreasing, which indicates the shape of a linear, not a quadratic, model. Linear model: $y = 4.9x + 18.2$

22. Sample justification: A quadratic model is a better fit. A graph of both models shows that the data points are closer to the quadratic model. Also, the values of y first decrease and then begin to increase as x increases, which indicates the shape of a quadratic, not a linear, model. Quadratic model: $y = 0.3x^2 - 5.0x + 23.8$

23. car: $y = 0.047x^2 + 2.207x + 0.214$; truck: $y = 0.064x^2 + 2.210x - 0.500$

24. Predictions should be close to 42 feet.

25. No. Sample explanation: Based on the quadratic model, the stopping distance for the truck at 60 mi/h is about 363 feet. This distance is greater than the distance between the truck and the intersection, so the driver will not be able to stop in time.

26. a. $y = -2.2x^2 + 454.9x - 12{,}637.0$

b. Yes. Sample explanation: A graph of the quadratic model and the data from the table shows that the graph of the model is close to the data points. Also, the monthly revenue increases and then decreases as the selling price increases, which indicates a quadratic model could be a good fit for the data.

c. Answers should be close to \$103.

ADDITIONAL PRACTICE

If students need more practice on the concepts in this activity, see the Teacher Resources at SpringBoard Digital for additional practice problems.

19. Demonstrate that the points $(-8, 0)$ and $(6, 0)$ do not determine a unique parabola by writing the equations of two different parabolas that pass through these two points.

20. a. The graph of a quadratic function passes through the point $(7, 5)$. The vertex of the graph is $(3, 1)$. Use symmetry to identify another point on the function's graph. Explain your answer.

b. Write the equation of the quadratic function.

Lesson 10-3

Tell whether a linear model or a quadratic model is a better fit for each data set. Justify your answer and give the equation of the better model.

21.

x	0	2	4	6	8	10	12	14
y	17	29	40	45	59	63	76	88

22.

x	2	4	6	8	10	12	14	16
y	15	9	5	2	6	7	16	22

The stopping distance of a vehicle is the distance the vehicle travels between the time the driver recognizes the need to stop and the time the vehicle comes to a stop. The table below shows how the speed of two vehicles affects their stopping distances.

Speed (mi/h)	Stopping distance (ft)	
	Car	Truck
10	27	28
15	44	47
20	63	69
25	85	95
30	109	123
35	135	155
40	164	190

23. Use a graphing calculator to perform a quadratic regression on the data for each vehicle. Write the equations of the quadratic models. Round coefficients and constants to the nearest thousandth.

24. Use your models to predict how much farther it would take the truck to stop from a speed of 50 mi/h than it would the car.

25. Suppose the truck is 300 ft from an intersection when the light at the intersection turns yellow. If the truck's speed is 60 mi/h when the driver sees the light change, will the driver be able to stop without entering the intersection? Explain how you know.

MATHEMATICAL PRACTICES
Use Appropriate Tools Strategically

26. A shoe company tests different prices of a new type of athletic shoe at different stores. The table shows the relationship between the selling price and the monthly revenue per store the company made from selling the shoes.

Selling Price (\$)	Monthly Revenue per Store (\$)
80	9680
90	10,520
100	11,010
110	10,660
120	10,400
130	9380

a. Use a graphing calculator to determine the equation of a quadratic model that can be used to predict y, the monthly revenue per store in dollars when the selling price is x dollars. Round values to the nearest tenth.

b. Is a quadratic model a good model for the data set? Explain.

c. Use your model to determine the price at which the company should sell the shoes to generate the greatest revenue.

Transformations of $y = x^2$

ACTIVITY 11

Parent Parabola

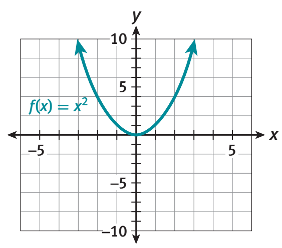
Lesson 11-1 Translations of Parabolas

Learning Targets:

- Describe translations of the parent function $f(x) = x^2$.
- Given a translation of the function $f(x) = x^2$, write the equation of the function.

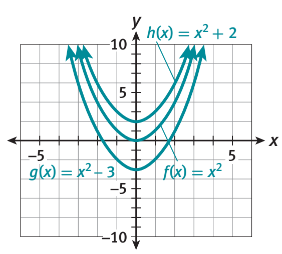
SUGGESTED LEARNING STRATEGIES: Create Representations, Quickwrite, Group Presentation, Look for a Pattern, Discussion Groups

1. Graph the parent quadratic function, $f(x) = x^2$, on the coordinate grid below. Include the points that have x-values $-2, -1, 0, 1,$ and 2.



The points on the parent function graph that have x-values $-2, -1, 0, 1,$ and 2 are *key points* that can be used when graphing any quadratic function as a transformation of the parent quadratic function.

2. Graph $f(x) = x^2$ on the coordinate grid below. Then graph and label $g(x) = x^2 - 3$ and $h(x) = x^2 + 2$.



3. Make use of structure. Identify and describe the transformations of the graph of $f(x) = x^2$ that result in the graphs of $g(x)$ and $h(x)$.
Sample answer: The transformations moved the graph vertically but did not change the shape.

My Notes

MATH TIP

A *parent function* is the simplest function of a particular type. For example, the parent linear function is $f(x) = x$. The parent absolute value function is $f(x) = |x|$.

MATH TIP

A *transformation* of a graph of a parent function is a change in the position, size, or shape of the graph.

Common Core State Standards for Activity 11

HSF-BF.B.3	Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $kf(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology.
HSF-IF.C.8	Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.

ACTIVITY 11

Guided

Activity Standards Focus

In Activity 11, students explore transformations of parabolas. Students also write quadratic functions in vertex form. Throughout this activity, emphasize the effects of coefficients and constants on the graphs of functions.

Lesson 11-1

PLAN

Pacing: 1 class period

Chunking the Lesson

#1 #2–3 #4–5 #6
Check Your Understanding
#11–12
Check Your Understanding
Lesson Practice

TEACH

Bell-Ringer Activity

Spend some time reviewing with students the following characteristics of linear equations:

- The parent function is $y = x$. The slope is 1, and it has a y-intercept of zero.
- Changing the parent function to $y = -x$ changes the slope from 1 to -1.
- When a coefficient other than 1 precedes the x-term, $y = mx$, it changes the slope or steepness of the line.
- The addition of a y-intercept, $y = mx + b$, changes the point at which the line crosses the y-axis (vertical translation).

1 Create Representations Students will likely be familiar with the parent quadratic function, but emphasis in this item should be on the five key points that are listed.

2–3 Create Representations, Quickwrite Students may use graphing calculators to visualize the graphs. When drawing the graphs of these two functions, students must recognize that the y-coordinates of the key points are changing. This will help identify the transformations as vertical translations (or an equivalent verbal form that describes the vertical movement of the graph without changing the shape). Be certain that students are graphing the functions accurately, and not just drawing a rough sketch.

ELL Support

For students for whom English is a second language, explain that the use of the word *parent* is not referring to a mother and/or father. However, just as a mother and father are the building blocks of a family; in mathematics, a "parent function" serves as a building block to more complicated forms of a function family.

4–5 Create Representations Students may need a graphing calculator to assist in graphing these functions. Those who use tables may find only half of the parabola, especially if they use the same *x*-coordinates of the key points for each function. Make sure that all the key points are graphed on the students' graphs.

6 Create Representations, Group Presentation, Debriefing Students have the opportunity to graph the functions by hand, without the use of a graphing calculator. Emphasis should be on the ease of graphing once the transformations are known. Group presentations will allow students to realize that the order of the translations does not matter when graphing the transformed function.

Universal Access

Refer students back to a term they learned in Geometry to explain *translations* as "sliding" a shape without rotating, flipping, or dilating. When a shape is translated in the geometric sense, it looks exactly the same, just in a different place.

Because this lesson is titled *Translations of Parabolas*, the *shapes* of the parabolas presented in the items thus far have not changed. The only changes have been horizontal and vertical shifts, or "slides." Because they have the same shape, you could stack the parabolas on top of one another, and they would be the same.

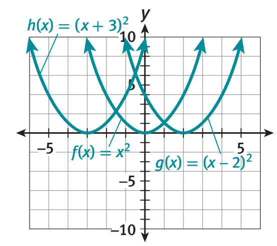
In upcoming lessons, students will discover that transformations of parabolas involve more than translations.

My Notes

MATH TIP

Translations are transformations that change the location of a graph but maintain the original shape of a graph. For this reason, they are known as *rigid transformations*.

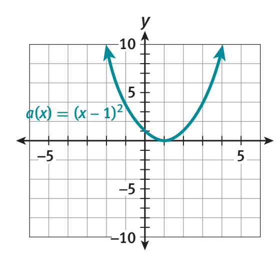
4. Model with mathematics. Graph $f(x) = x^2$ on the coordinate grid below. Then graph and label $g(x) = (x - 2)^2$ and $h(x) = (x + 3)^2$.



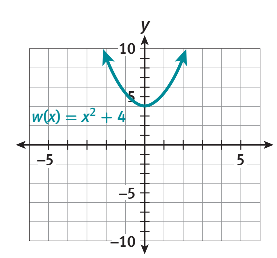
5. Identify and describe the transformations of the graph of $f(x) = x^2$ that result in the graphs of $g(x)$ and $h(x)$.
Sample answer: The transformations moved the graph horizontally but did not change the shape.

6. Describe each function as a transformation of $f(x) = x^2$. Then use that information to graph each function on the coordinate grid.

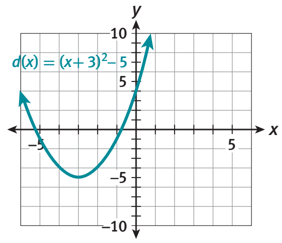
a. $a(x) = (x - 1)^2$ **translated 1 unit right**



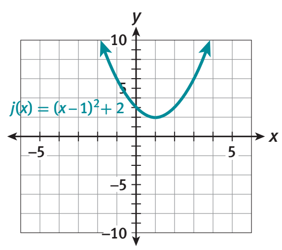
b. $w(x) = x^2 + 4$ **translated 4 units up**



c. $d(x) = (x + 3)^2 - 5$ translated 3 units left and 5 units down



d. $j(x) = (x - 1)^2 + 2$ translated 1 unit right and 2 units up



My Notes

Check Your Understanding

7. **Express regularity in repeated reasoning.** The graph of each function below is a translation of the graph of $f(x) = x^2$ by k units, where $k > 0$. For each function, tell which direction the graph of $f(x)$ is translated.
   a. $g(x) = x^2 + k$ **b.** $h(x) = (x + k)^2$
   c. $j(x) = x^2 - k$ **d.** $m(x) = (x - k)^2$
8. What is the vertex of the function $p(x) = x^2 - 5$? Justify your answer in terms of a translation of $f(x) = x^2$.
9. What is the axis of symmetry of the function $q(x) = (x + 1)^2$? Justify your answer in terms of a translation of $f(x) = x^2$.
10. **Reason abstractly.** The function $r(x)$ is a translation of the function $f(x) = x^2$. What can you conclude about the direction in which the parabola given by $r(x)$ opens? Justify your answer.

MATH TIP

If you need help with Item 7, try substituting a positive number for k and then graphing each function.

ACTIVITY 11 Continued

Check Your Understanding

Debrief students' answers to these items to ensure that they understand concepts related to translations of the parent parabola.

Answers

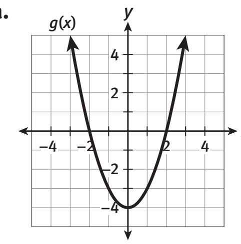
7. **a.** k units up
   b. k units to the left
   c. k units down
   d. k units to the right
8. $(0, -5)$. Sample explanation: The graph of $p(x)$ is the graph of $f(x) = x^2$ translated 5 units down. The vertex of $f(x)$ is $(0, 0)$, so the vertex of $p(x)$ will be 5 units down from $(0, 0)$ at $(0, -5)$.
9. $x = -1$. Sample explanation: The graph of $q(x)$ is the graph of $f(x) = x^2$ translated 1 unit to the left. The axis of symmetry of the graph of $f(x)$ is the line $x = 0$, so the axis of symmetry of the graph of $q(x)$ will be 1 unit to the left of the line $x = 0$ at $x = -1$.
10. The parabola given by $r(x)$ opens upward. A translation shifts a graph to the left, to the right, up, and/or down but does not change the direction that the parabola opens. Because the parabola given by $f(x) = x^2$ opens upward, all translations of $f(x) = x^2$ will open upward (assuming no other transformations are performed).

11–12 Chunking the Activity, Predict and Confirm, Think-Pair-Share Pair students of varying abilities to work through graphing these parabolas by hand as well as on their graphing calculators. Prior to graphing, ask students to predict what they think the graph will look like. Then students can confirm their predictions with sketches and calculator graphs. Point out that as graphs shift horizontally, their input values will need to adjust accordingly.

My Notes

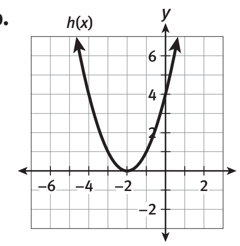
11. Each function graphed below is a translation of $f(x) = x^2$. Describe the transformation. Then write the equation of the transformed function.

a.



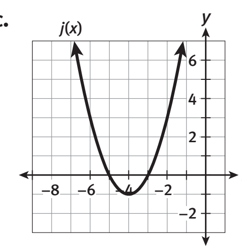
translated 4 units down, $g(x) = x^2 - 4$

b.



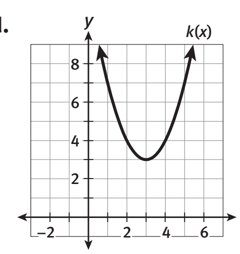
translated 2 units left, $h(x) = (x + 2)^2$

c.



translated 4 units left and 1 unit down, $j(x) = (x + 4)^2 - 1$

d.



translated 3 units right and 3 units up, $k(x) = (x - 3)^2 + 3$

Differentiating Instruction

If students are having trouble predicting how the graphs are going to translate, have them make a chart like the one shown to summarize what has been covered so far and to use as a source of reference.

Translations of $y = x^2$			
Vertical		**Horizontal**	
$y = x^2 + 2$	moves 2 units up	$y = (x + 2)^2$	moves 2 units left
$y = x^2 - 2$	moves 2 units down	$y = (x - 2)^2$	moves 2 units right

Lesson 11-1
Translations of Parabolas

My Notes

12. Use a graphing calculator to graph each of the equations you wrote in Item 11. Check that the graphs on the calculator match those shown in Item 11. Revise your answers to Item 11 as needed.
Check students' work.

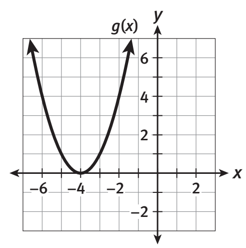
TECHNOLOGY TIP

When you graph a function on a graphing calculator, the distance between tick marks on the *x*-axis is not always the same as the distance between tick marks on the *y*-axis. To make these distances the same, press ZOOM, and select **5 : ZSquare**. This step will make it easier to compare your calculator graphs to the graphs in Item 11.

Check Your Understanding

13. Explain how you determined the equation of $k(x)$ in Item 11d.

14. Critique the reasoning of others. The graph shows a translation of $f(x) = x^2$. A student says that the equation of the transformed function is $g(x) = (x - 4)^2$. Is the student correct? Explain.



15. The graph of $h(x)$ is a translation of the graph of $f(x) = x^2$. If the vertex of the graph of $h(x)$ is $(-1, -2)$, what is the equation of $h(x)$? Explain your answer.

LESSON 11-1 PRACTICE

Make sense of problems. Describe each function as a transformation of $f(x) = x^2$.

16. $g(x) = x^2 - 6$

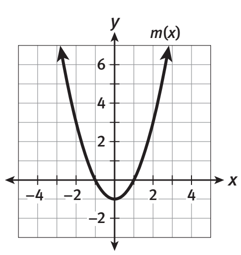
17. $h(x) = (x + 5)^2$

18. $j(x) = (x - 2)^2 + 8$

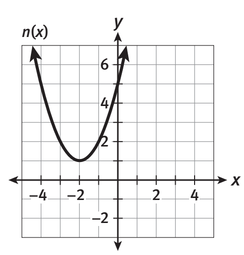
19. $k(x) = (x + 6)^2 - 4$

Each function graphed below is a translation of $f(x) = x^2$. Describe the transformation. Then write the equation of the transformed function.

20.



21.



22. What is the vertex of the function $p(x) = (x - 5)^2 + 4$? Justify your answer in terms of a translation of $f(x) = x^2$.

23. What is the axis of symmetry of the function $q(x) = (x + 8)^2 - 10$? Justify your answer in terms of a translation of $f(x) = x^2$.

LESSON 11-1 PRACTICE

16. translation 6 units down

17. translation 5 units left

18. translation 2 units right and 8 units up

19. translation 6 units left and 4 units down

20. translation 1 unit down; $m(x) = x^2 - 1$

21. translation 2 units left and 1 unit up; $n(x) = (x + 2)^2 + 1$

22. (5, 4). Sample explanation: The graph of $p(x)$ is the graph of $f(x) = x^2$ translated 5 units right and 4 units up. The vertex of $f(x)$ is (0, 0), so the vertex of $p(x)$ will be 5 units to the right and 4 units up from (0, 0) at (5, 4).

23. $x = -8$. Sample explanation: The graph of $q(x)$ is the graph of $f(x) = x^2$ translated 8 units left and 10 units down. The axis of symmetry of $f(x)$ is $x = 0$, so the axis of symmetry of $q(x)$ will be 8 units to the left of $x = 0$ at $x = -8$.

ACTIVITY 11 Continued

Check Your Understanding

Debrief students' answers to these items to ensure that they understand writing equations of quadratic functions given information about a translation.

Answers

13. Sample answer: The vertex of the graph of $k(x)$ is (3, 3), which means that $k(x)$ is a translation of $f(x)$ by 3 units right and 3 units up. To show a translation 3 units right, subtract 3 from x inside the parentheses. To show a translation 3 units up, add 3 to the squared term. So, $k(x) = (x - 3)^2 + 3$.

14. No. Sample explanation: If the number subtracted from x is positive, the translation is to the right. The graph of $g(x)$ is the graph of $f(x)$ translated 4 units to the left, so to write the equation of $g(x)$, you need to subtract -4 inside the parentheses. The correct equation is $g(x) = (x - (-4))^2$, which simplifies to $g(x) = (x + 4)^2$.

15. $h(x) = (x + 1)^2 - 2$. Sample explanation: The vertex of the graph of $h(x)$ indicates that $h(x)$ is a translation of $f(x)$ by 1 unit to the left and 2 units down. To write the equation of $h(x)$, subtract -1 from x inside the parentheses and add -2 to the squared term: $h(x) = (x - (-1))^2 + (-2) = (x + 1)^2 - 2$.

ASSESS

Students' answers to Lesson Practice problems will provide you with a formative assessment of their understanding of the lesson concepts and their ability to apply their learning.

See the Activity Practice for additional problems for this lesson. You may assign the problems here or use them as a culmination for the activity.

ADAPT

Check students' answers to the Lesson Practice to ensure that they understand vertical and horizontal translations of the parent quadratic function $y = x^2$. Make sure that students can match equations to their graphs. Have students create a matching game using note cards. Each game should contain ten cards that have quadratic equations and ten cards that have the graphs of those equations. These sets of cards can be distributed to other students in the classroom as additional practice.

Lesson 11-2

PLAN

Pacing: 1 class period

Chunking the Lesson

#1–2 #3–6 #7

Check Your Understanding

#11–12 #13–16

Check Your Understanding

#20 #21

Check Your Understanding

Lesson Practice

TEACH

Bell-Ringer Activity

Before moving on to the next lesson in this activity, review the following terminology. This will help build the students' foundation prior to performing additional parabolic transformations.

- The vertex of a parabola is its highest or lowest point.
- The axis of symmetry of a parabola is a line that passes through the vertex of the parabola that divides the parabola into mirror images.

1–2 Create Representations When using a graphing calculator to graph these functions, students may understand the concept of a vertical stretch or shrink yet lack sufficient precision in their graphs to see the changes. Again, emphasize what is happening to the key points. Some key points may not fit on the grids, but adjusting the viewing window so that the differences between those points that do fit is essential. Students should recognize the change in the y-coordinate of the key points.

TEACHER to TEACHER

Some students may identify these transformations without recognizing the vertical nature of them. A vertical stretch appears the same as a horizontal shrink and a horizontal stretch as a vertical shrink. Emphasize the change in the y-value when discussing vertical transformations and changes in the x-value when discussing horizontal transformations.

Lesson 11-2
Shrinking, Stretching, and Reflecting Parabolas

My Notes

MATH TIP

Unlike a rigid transformation, a *vertical stretch* or *vertical shrink* will change the shape of the graph.

A vertical stretch stretches a graph away from the x-axis by a factor and a vertical shrink shrinks the graph toward the x-axis by a factor.

Learning Targets:

- Describe transformations of the parent function $f(x) = x^2$.
- Given a transformation of the function $f(x) = x^2$, write the equation of the function.

SUGGESTED LEARNING STRATEGIES: Create Representations, Look for a Pattern, Group Presentation, Quickwrite, Identify a Subtask

1. Graph the function $f(x) = x^2$ as Y1 on a graphing calculator. Then graph each of the following functions as Y2. Describe the graph of each function as a transformation of the graph of $f(x) = x^2$.

a. $g(x) = 2x^2$

a vertical stretch by a factor of 2

b. $h(x) = 4x^2$

a vertical stretch by a factor of 4

c. $j(x) = \frac{1}{2}x^2$

a vertical shrink by a factor of $\frac{1}{2}$

d. $k(x) = \frac{1}{4}x^2$

a vertical shrink by a factor of $\frac{1}{4}$

2. Express regularity in repeated reasoning. Describe any patterns you observed in the graphs from Item 1.

Sample answer: The graphs have the same vertex (0, 0) and the same axis of symmetry ($x = 0$). All the graphs open upward. The graphs are vertical stretches when the number multiplied by x^2 is greater than 1 and vertical shrinks when the number multiplied by x^2 is a fraction less than 1.

MATH TIP

Reflections over axes do not change the shape of the graph, so they are also rigid transformations.

3. Graph the function $f(x) = x^2$ as Y1 on a graphing calculator. Then graph each of the following functions as Y2. Identify and describe the graph of each function as a transformation of the graph of $f(x) = x^2$.

a. $g(x) = -x^2$

a reflection over the x-axis

Lesson 11-2
Shrinking, Stretching, and Reflecting Parabolas

b. $h(x) = -4x^2$
a reflection over the *x*-axis and a vertical stretch by a factor of 4

c. $j(x) = -\frac{1}{4}x^2$
a reflection over the *x*-axis and a vertical stretch by a factor of $\frac{1}{4}$

4. Describe any patterns you observed in the graphs from Item 3.
Sample answer: The graphs have the same vertex (0, 0) and the same axis of symmetry ($x = 0$). All the graphs open downward. The graphs are vertical stretches when the number multiplied by x^2 is less than -1 and vertical shrinks when the number multiplied by x^2 is between -1 and 0.

5. Make a conjecture about how the sign of k affects the graph of $g(x) = kx^2$ compared to the graph of $f(x) = x^2$. Assume that $k \neq 0$.
If *k* is positive, the graph of *g*(*x*) is not reflected over the *x*-axis compared to the graph of *f*(*x*). If *k* is negative, the graph of *g*(*x*) is reflected over the *x*-axis compared to the graph of *f*(*x*).

6. Make a conjecture about whether the absolute value of k affects the graph of $g(x) = kx^2$ when compared to the graph of $f(x) = x^2$. Assume that $k \neq 0$ and write your answer using absolute value notation.
If $|k| > 1$, the graph of *g*(*x*) is a vertical stretch of the graph of *f*(*x*) by a factor of $|k|$. If $|k| < 1$, the graph of *g*(*x*) is a vertical shrink of the graph of *f*(*x*) by a factor of $|k|$.

7. Make use of structure. Without graphing, describe each function as a transformation of $f(x) = x^2$.
a. $h(x) = 6x^2$
a vertical stretch by a factor of 6

b. $j(x) = -\frac{1}{10}x^2$
a vertical shrink by a factor of $\frac{1}{10}$ and a reflection over the *x*-axis

My Notes

MATH TIP

In Item 6, consider the situation in which $|k| > 1$ and the situation in which $|k| < 1$.

ACTIVITY 11 Continued

3–6 Summarizing, Debriefing Another way of summarizing these items is to explain to students that the value of k determines whether a parabola opens upward or downward (in other words, has a vertex that is a lowest point or a highest point). Additionally, the value of k determines the shape of the parabola as it compares to the parent function of $y = x^2$. As $|k|$ increases, the parabola stretches vertically, becoming narrower. As $|k|$ decreases, the parabola shrinks vertically, becoming wider.

7 Predict and Confirm, Look for a Pattern Students should be able to look back at the outcomes of Items 1–6 to predict (without graphing) what will result from changing the value of k in this item.

Check Your Understanding

Debrief students' answers to these items to ensure that they understand concepts related to vertical shrinks and stretches of the parent parabola.

Answers

8. $g(x) = \frac{1}{6}x^2$
9. $h(x) = 7x^2$. Sample explanation: The graph of $f(x)$ passes through the point (1, 1). This point is stretched away from the x-axis by a factor of 7 to the corresponding point (1, 7) on the graph of $h(x)$. The graph of $h(x)$ is a vertical stretch of the graph of $f(x)$ by a factor of 7.
10. The value of k is negative. Sample justification: If the graph of $j(x) = kx^2$ opens downward, it is a reflection of the graph of $f(x) = x^2$ in the x-axis. Therefore, k must be negative.

11–12 Debriefing For these items, have students make comparisons between the coefficients of x and their corresponding horizontal shrink or stretch factors. Do students notice any relationship between each pair? [*Each pair is a reciprocal of each other*.]

c. $p(x) = -9x^2$
a vertical stretch by a factor of 9 and a reflection over the x-axis

d. $q(x) = \frac{1}{5}x^2$
a vertical shrink by a factor of $\frac{1}{5}$

Check Your Understanding

8. The graph of $g(x)$ is a vertical shrink of the graph of $f(x) = x^2$ by a factor of $\frac{1}{6}$. What is the equation of $g(x)$?
9. **Reason quantitatively.** The graph of $h(x)$ is a vertical stretch of the graph of $f(x) = x^2$. If the graph of $h(x)$ passes through the point (1, 7), what is the equation of $h(x)$? Explain your answer.
10. The graph of $j(x) = kx^2$ opens downward. Based on this information, what can you conclude about the value of k? Justify your conclusion.

MATH TIP

A horizontal stretch stretches a graph away from the y-axis by a factor and a vertical shrink shrinks the graph toward the y-axis by a factor.

11. Graph the function $f(x) = x^2$ as Y1 on a graphing calculator. Then graph each of the following functions as Y2. Identify and describe the graph of each function as a horizontal stretch or shrink of the graph of $f(x) = x^2$.
a. $g(x) = (2x)^2$
a horizontal shrink by a factor of $\frac{1}{2}$

b. $h(x) = (4x)^2$
a horizontal shrink by a factor of $\frac{1}{4}$

c. $j(x) = \left(\frac{1}{2}x\right)^2$
a horizontal stretch by a factor of 2

d. $k(x) = \left(\frac{1}{4}x\right)^2$
a horizontal stretch by a factor of 4

Lesson 11-2
Shrinking, Stretching, and Reflecting Parabolas

My Notes

Work with your group on Items 12–16.

12. Describe any patterns you observed in the graphs from Item 11.

Sample answer: The graphs have the same vertex (0, 0) and the same axis of symmetry ($x = 0$). All the graphs open upward. The graphs are horizontal shrinks when the number multiplied by x inside the parentheses is greater than 1 and horizontal stretches when the number multiplied by x inside the parentheses is a fraction less than 1.

13. a. Use appropriate tools strategically. Graph the function $f(x) = x^2$ as Y1 on a graphing calculator. Then graph $h(x) = (-x)^2$ as Y2. Describe the result.

The graph of $h(x)$ is the same as the graph of $f(x)$.

b. Reason abstractly. Explain why this result makes sense.

Sample answer: The expression $(-x)^2$ is equal to $(-1 \cdot x)^2 = (-1)^2 \cdot x^2 = 1 \cdot x^2 = x^2$. So, the rule for $h(x)$ is equivalent to the rule for $f(x)$.

14. Make a conjecture about how the sign of k affects the graph of $g(x) = (kx)^2$ compared to the graph of $f(x) = x^2$. Assume that $k \neq 0$.

The sign of k has no effect on the graph of $g(x)$ compared to the graph of $f(x)$.

15. Make a conjecture about whether the absolute value of k affects the graph of $g(x) = (kx)^2$ when compared to the graph of $f(x) = x^2$. Assume that $k \neq 0$.

If $|k| > 1$, the graph of $g(x)$ is a horizontal shrink of the graph of $f(x)$ by a factor of $\left|\frac{1}{k}\right|$. If $|k| < 1$, the graph of $g(x)$ is a horizontal stretch of the graph of $f(x)$ by a factor of $\left|\frac{1}{k}\right|$.

16. Describe each function as a transformation of $f(x) = x^2$.

a. $p(x) = (6x)^2$

a horizontal shrink by a factor of $\frac{1}{6}$

b. $q(x) = \left(\frac{1}{10}x\right)^2$

a horizontal stretch by a factor of 10

DISCUSSION GROUP TIP

In your discussion groups, read the text carefully to clarify meaning. Reread definitions of terms as needed to help you comprehend the meanings of words, or ask your teacher to clarify vocabulary terms.

ACTIVITY 11 Continued

13–16 Summarizing, Debriefing

Emphasize that a horizontal shrink implies the parabola is narrower, and a horizontal stretch implies the parabola is wider. As addressed in the Debriefing for Items 11 and 12, demonstrate to students that when a graph is in the form $g(x) = (kx)^2$, the factor by which it shrinks or stretches horizontally is the reciprocal of k, or $\frac{1}{k}$. Furthermore, when $k > 1$, the graph is a horizontal shrink, and when $k < 1$, the graph is a horizontal stretch.

ELL Support

Since the terms *shrink* and *stretch* are referenced with frequency in this activity, it is beneficial to give ELL students an analogy of these concepts with something other than parabolas. Demonstrate stretching by pulling on a rubber band or a piece of elastic. Demonstrate shrinking by writing a letter on the board and writing a much smaller version of that same number beside it to show that it shrinks. Point out that some materials, when stretched in one direction, will shrink in the perpendicular direction. Ask students if they can think of anything in the real world that commonly shrinks or stretches. [*Samples with laundry/clothing: If you wash a wool item, it will shrink. If you wash certain clothing items in hot water, they may shrink. If you wear an article of clothing that is too small, it will stretch beyond its original size.*]

Check Your Understanding

Debrief students' answers to these items to ensure that they understand concepts related to horizontal shrinks and stretches of the parent parabola.

Answers

17. Sample answer: The graph of $g(x) = 4x^2$ is a vertical stretch of the graph of $f(x) = x^2$ by a factor of 4. Each point (x, y) on the graph of $f(x)$ maps to the point $(x, 4y)$ on the graph of $g(x)$. The graph of $h(x) = (4x)^2$ is a horizontal shrink of the graph of $f(x) = x^2$ by a factor of $\frac{1}{4}$. Each point (x, y) on the graph of $f(x)$ maps to the point $\left(\frac{1}{4}x, y\right)$ on the graph of $h(x)$.

18. $g(x) = \left(\frac{1}{5}x\right)^2$

19. $h(x) = (5x)^2$. Sample explanation: The graph of $f(x)$ passes through the point (5, 25). This point is shrunk toward the y-axis by a factor of $\frac{1}{5}$ to the corresponding point (1, 25) on the graph of $h(x)$. The graph of $h(x)$ is a horizontal shrink of the graph of $f(x)$ by a factor of $\frac{1}{5}$.

20 Quickwrite, Discussion Groups, Group Presentation Have students work in pairs or small groups of varying abilities to analyze and summarize the transformations shown in this item. Emphasize that the transformations of $f(x) = x^2$ to $g(x) = kx^2$ can only be described as a vertical transformation. However, the transformations of $f(x) = x^2$ to $h(x) = (kx)^2$ can be described as either a horizontal or vertical transformation. Have groups see if they can arrive at all the possibilities. Have student volunteers explain the descriptions of the transformations to their peers.

My Notes

Check Your Understanding

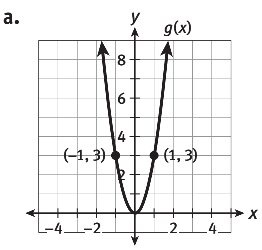
17. Describe how the graph of $g(x) = 4x^2$ differs from the graph of $h(x) = (4x)^2$.

18. The graph of $g(x)$ is a horizontal stretch of the graph of $f(x) = x^2$ by a factor of 5. What is the equation of $g(x)$?

19. Reason quantitatively. The graph of $h(x)$ is a horizontal shrink of the graph of $f(x) = x^2$. If the graph of $h(x)$ passes through the point (1, 25), what is the equation of $h(x)$? Explain your answer.

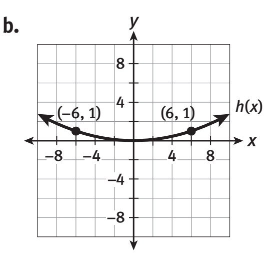
20. Each function graphed below is a transformation of $f(x) = x^2$. Describe the transformation. Then write the equation of the transformed function.

a.



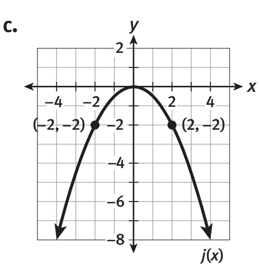
vertically stretched by a factor of 3, $g(x) = 3x^2$

b.



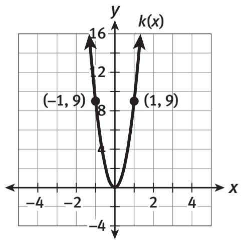
horizontally stretched by a factor of 6, $h(x) = \left(\frac{1}{6}x\right)^2$ (or vertically shrunk by a factor of $\frac{1}{36}$, $h(x) = \frac{1}{36}x^2$)

c.



vertically shrunk by a factor of $\frac{1}{2}$ and reflected over the x-axis, $j(x) = -\frac{1}{2}x^2$

d.

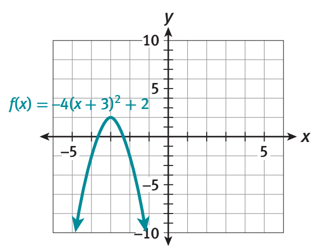


horizontally shrunk by a factor of $\frac{1}{3}$, $k(x) = (3x)^2$ (or vertically stretched by a factor of 9, $k(x) = 9x^2$)

21. Model with mathematics. Multiple transformations can be represented in the same function. Describe the transformations from the parent function. Then graph the function, using your knowledge of transformations only.

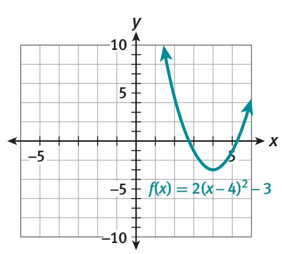
a. $f(x) = -4(x + 3)^2 + 2$

translate 3 left, reflect over *x*-axis, stretch vertically by factor of 4, translate 2 up



b. $f(x) = 2(x - 4)^2 - 3$

translate 4 right, stretch vertically by factor of 2, translate 3 down



My Notes

MATH TIP

When graphing multiple transformations of quadratic functions, follow this order:

1. horizontal translation
2. horizontal shrink or stretch
3. reflection over the x-axis and/or vertical shrink or stretch
4. vertical translation

21 Identify a Subtask, Create Representations, Group Presentation, Debriefing All of the transformations introduced so far in this activity are combined in this culminating item. Students may correctly graph these transformations in different orders. For instance, from an order of operations approach, if students first complete any transformations that involve multiplication (stretch/shrink or reflect), then apply any transformations that involve addition/subtraction (translations), they will have an accurate graph. Conversely, if they view the translations as “moving the origin,” then apply the reflection and stretch/shrink transformations, they will also have a correct graph.

Careful monitoring while students are working and class debriefing after a group presentation are essential after this item.

Check Your Understanding

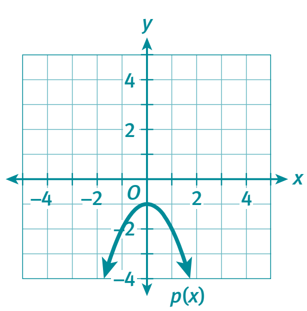
Debrief students' answers to these items to ensure that they understand concepts related to transformations of quadratic functions and writing equations given information about the transformation of a quadratic function.

Answers

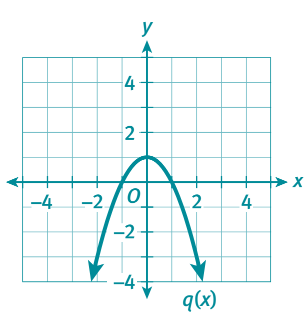
22. Sample answer: The graph appears to be a vertical stretch of $f(x) = x^2$. The graph of $f(x) = x^2$ passes through the point (1, 1). This point is stretched away from the x-axis by a factor of 3 to the corresponding point (1, 3) on the graph of $g(x)$. The graph of $g(x)$ is a vertical stretch of the graph of $f(x)$ by a factor of 3. Therefore, $g(x) = 3x^2$.

23. (3, 4). Sample explanation: The graph of $h(x)$ is a translation 3 units to the right of the graph of $f(x)$. This translation moves the vertex from (0, 0) to (3, 0). The translation is followed by a vertical stretch by a factor of 2. This stretch does not change the position of the vertex. The stretch is then followed by a translation 4 units up. This translation moves the vertex from (3, 0) to (3, 4).

24. a.



b.



c. Yes; the graph of $p(x)$ is different from the graph of $q(x)$, so the order in which the transformations are performed matters.

d. $p(x) = -x^2 - 1$ or equivalent; $q(x) = -x^2 + 1$ or equivalent

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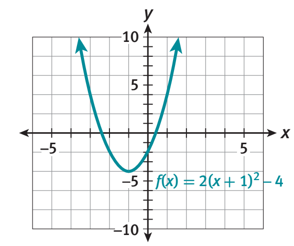
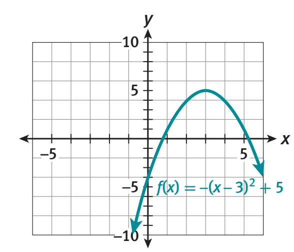
c. $f(x) = 2(x + 1)^2 - 4$

translate 1 left, stretch vertically by a factor of 2, translate 4 down



d. $f(x) = -(x - 3)^2 + 5$

translate 3 right, reflect over x-axis, translate 5 up



Check Your Understanding

22. Explain how you determined the equation of $g(x)$ in Item 20a.

23. Without graphing, determine the vertex of the graph of $h(x) = 2(x - 3)^2 + 4$. Explain how you found your answer.

24. a. Start with the graph of $f(x) = x^2$. Reflect it over the x-axis and then translate it 1 unit down. Graph the result as the function $p(x)$.

b. Start with the graph of $f(x) = x^2$. Translate it 1 unit down and then reflect it over the x-axis. Graph the result as the function $q(x)$.

c. Construct viable arguments. Does the order in which the two transformations are performed matter? Explain.

d. Write the equations of $p(x)$ and $q(x)$.

Lesson 11-2
Shrinking, Stretching, and Reflecting Parabolas

LESSON 11-2 PRACTICE

My Notes

Describe each function as a transformation of $f(x) = x^2$.

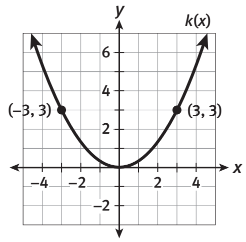
25. $g(x) = -5x^2$

26. $h(x) = (8x)^2$

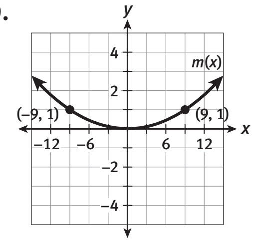
27. Make sense of problems. The graph of $j(x)$ is a horizontal stretch of the graph of $f(x) = x^2$ by a factor of 7. What is the equation of $j(x)$?

Each function graphed below is a transformation of $f(x) = x^2$. Describe the transformation. Then write the equation of the transformed function.

28.



29.

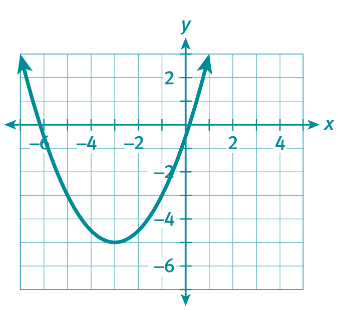


Describe the transformations from the parent function. Then graph the function, using your knowledge of transformations only.

30. $n(x) = -3(x - 4)^2$

31. $p(x) = \frac{1}{2}(x + 3) - 5$

31. translated 3 units left, vertically shrunk by a factor of $\frac{1}{2}$ and translated 5 units down



ACTIVITY 11 Continued

ASSESS

Students' answers to Lesson Practice problems will provide you with a formative assessment of their understanding of the lesson concepts and their ability to apply their learning.

See the Activity Practice for additional problems for this lesson. You may assign the problems here or use them as a culmination for the activity.

LESSON 11-2 PRACTICE

25. reflected over the x-axis and vertically stretched by a factor of 5

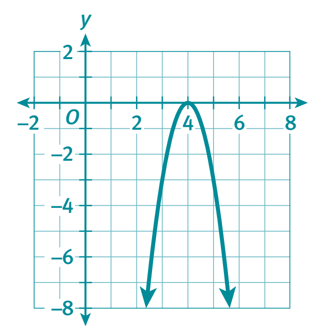
26. horizontal shrink by a factor of $\frac{1}{8}$

27. $j(x) = \left(\frac{1}{7}x\right)^2$

28. vertical shrink by a factor of $\frac{1}{3}$; $k(x) = \frac{1}{3}x^2$

29. horizontal stretch by a factor of 9; $m(x) = \left(\frac{1}{9}x\right)^2$ (or vertical shrink by a factor of 81; $m(x) = \frac{1}{81}x^2$)

30. translated 4 units right, reflected over the x-axis and vertically stretched by a factor of 3



ADAPT

Check students' answers to the Lesson Practice to ensure that they understand reflections over both axes and vertical and horizontal shrinks and stretches of the parent function $y = x^2$. Make sure that students can match equations to their graphs. Once all transformations have been covered, students may benefit from creating a graphic organizer of transformations and their effects on the graph. Students will need to apply these transformations to different parent functions throughout future lessons.

Lesson 11-3

PLAN

Pacing: 1 class period

Chunking the Lesson

Example A #1–2

Check Your Understanding

Lesson Practice

TEACH

Bell-Ringer Activity

Have students complete the square in the following expressions. Reviewing this concept will help prepare them for writing quadratic equations in *vertex form*.

1. $x^2 + 8x +$ ______ $[16]$
2. $y^2 + 5y +$ ______ $\left[\frac{25}{4}\right]$
3. $a^2 - \frac{1}{2}a +$ ______ $\left[\frac{1}{16}\right]$

Developing Math Language

Just as a linear equation has a standard form of $Ax + By = C$, the standard form of a quadratic equation is $y = ax^2 + bx + c$. However, when graphing linear equations, mathematicians usually use the slope-intercept form of the equation because it is easy to graph the y-intercept and plot a second point from it using the slope. Once you have two points, you can draw a line. When graphing quadratic equations, you usually use the vertex form of the equation, $y = a(x - h)^2 + k$, because you can easily discern and plot the vertex (h, k). The horizontal translation of the parabola can be determined from the value of h, and the vertical translation of the parabola can be determined from the value of k. Additionally, the value of a determines both the direction and shape of the parabola.

Example A Activating Prior Knowledge, Debriefing, Create a Plan There are a few subtle differences between the steps of Example A and the steps you followed when completing the square to solve quadratic equations. The reason for these differences is due to the fact that students should only be able to work on one side of the equation. Students should group the $ax^2 + bx$ terms and factor out a, leaving $a\left(x^2 + \frac{b}{a}x\right)$. Now complete the square as before. Note that the value $\frac{b^2}{4a^2}$, which is added inside the parentheses after the term $\frac{b}{a}x$, when multiplied by the a preceding the parentheses, gives the product $\frac{b^2}{4a}$.

Example A (continued) This value is subtracted from c outside the parentheses in order to maintain equality with the original function. This sum of $-\frac{b^2}{4a}$ and c will determine the value of k. Once you have completed the square, you factor the perfect square trinomial just like before. You should now have an equation in vertex form.

ACTIVITY 11 *continued*

Lesson 11-3 Vertex Form

My Notes

MATH TERMS

The **vertex form** of a quadratic function is $f(x) = a(x - h)^2 + k$, where the vertex of the function is (h, k). Notice that the transformations of $f(x) = x^2$ are apparent when the function is in vertex form.

Learning Targets:

- Write a quadratic function in vertex form.
- Use transformations to graph a quadratic function in vertex form.

SUGGESTED LEARNING STRATEGIES: Interactive Word Wall, Marking the Text, Create Representations, Group Presentation, RAFT

A quadratic function in standard form, $f(x) = ax^2 + bx + c$, can be changed into ***vertex form*** by completing the square.

Example A

Write $f(x) = 3x^2 - 12x + 7$ in vertex form.

Step 1: Factor the leading coefficient from the quadratic and linear terms. $f(x) = 3(x^2 - 4x) + 7$

Step 2: Complete the square by taking half the linear coefficient $[0.5(-4) = -2]$, squaring it $[(-2)^2 = 4]$, and then adding it inside the parentheses. $f(x) = 3(x^2 - 4x + \quad) + 7$ ↑ $+4$

Step 3: To maintain the value of the expression, multiply the leading coefficient [3] by the number added inside the parentheses [4]. Then subtract that product [12].
$f(x) = 3(x^2 - 4x + 4) - 3(4) + 7$
$f(x) = 3(x^2 - 4x + 4) - 12 + 7$

Step 4: Write the trinomial inside the parentheses as a perfect square. The function is in vertex form. $f(x) = 3(x - 2)^2 - 5$

Solution: The vertex form of $f(x) = 3x^2 - 12x + 7$ is $f(x) = 3(x - 2)^2 - 5$.

Try These A

Make use of structure. Write each quadratic function in vertex form. Show your work.

a. $f(x) = 5x^2 + 40x - 3$
$f(x) = 5(x + 4)^2 - 83$

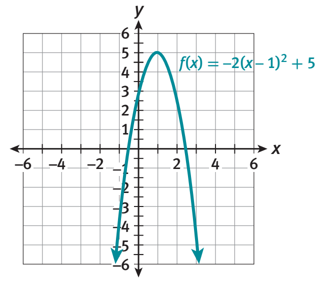
b. $g(x) = -4x^2 - 12x + 1$
$g(x) = -4\left(x + \frac{3}{2}\right)^2 + 10$

Lesson 11-3
Vertex Form

My Notes

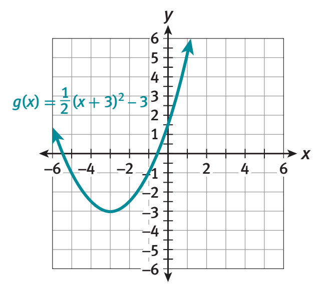
1. **Make sense of problems.** Write each function in vertex form. Then describe the transformation(s) from the parent function and graph without the use of a graphing calculator.
   a. $f(x) = -2x^2 + 4x + 3$

   $f(x) = -2(x - 1)^2 + 5$; translate 1 right, reflect over the x-axis, vertical stretch by factor of 2, and translate 5 up



   b. $g(x) = \frac{1}{2}x^2 + 3x + \frac{3}{2}$

   $g(x) = \frac{1}{2}(x + 3)^2 - 3$; translate 3 left, vertical shrink by factor of $\frac{1}{2}$, and translate 3 down



2. Consider the function $f(x) = 2x^2 - 16x + 34$.
   a. Write the function in vertex form.
   $f(x) = 2(x - 4)^2 + 2$

   b. What is the vertex of the graph of the function? Explain your answer.
   (4, 2). Sample explanation: In the vertex form of the equation, the value of h is 4 and the value of k is 2.

MATH TIP

You can check that you wrote the vertex form correctly by rewriting the vertex form in standard form and checking that the rewritten standard form equation matches the original equation.

ACTIVITY 11 Continued

1–2 Marking the Text, Group Presentation, Debriefing Make sure that students understand that after changing functions into vertex form, they are able to graph the function by using their knowledge of transformations.

Many students may have difficulty changing quadratics with a negative leading coefficient into vertex form. Group presentation and whole-class debriefing will allow students to see how others handle this challenge.

Universal Access

Students should be careful to avoid a common error during the process of completing the square when converting a quadratic function to its vertex form. Because students are accustomed to performing the same mathematical operations to both sides of an equation, they may be inclined to add the value to complete the square, first inside the parentheses and then again outside the parentheses, adding it to c. Emphasize that they must subtract the value from c because they are working only on one side of the equation.

Check Your Understanding (p.188)

Debrief students' answers to these items to ensure that they understand concepts related to writing equations for parabolas and quadratic functions in vertex form.

Answers

3. Sample answer:
   (1) Write the variable terms in parentheses:
   $f(x) = (x^2 + 6x) + 11.$
   (2) Next, decide what number to add inside the parentheses to complete the square. Divide the coefficient of the x-term by 2: $6 \div 2 = 3$. Then square the result: $3^2 = 9$. You need to add 9 to complete the square.
   (3) Add 9 inside the parentheses. To keep the expression on the right side of the equation balanced, subtract 9 outside the parentheses:
   $f(x) = (x^2 + 6x + 9) - 9 + 11.$
   (4) Factor the expression in parentheses:
   $f(x) = (x + 3)^2 - 9 + 11.$
   (5) Combine the constant terms:
   $f(x) = (x + 3)^2 + 2.$
   The equation of the function is now in vertex form, $f(x) = a(x - h)^2 + k$, with $a = 1$, $h = -3$ and $k = 2$.

4. Sample answer: It is easier to determine the vertex of the graph of the function when the equation is written in vertex form. It is also easier to graph the function as a set of transformations of the parent function when the equation is written in vertex form.

5. Write 1 in the first box because adding 1 completes the square for the quadratic expression $x^2 - 2x$ inside the parentheses. Write 4 in the second box because subtracting 4 outside the parentheses keeps the expression on the right side of the equation balanced.

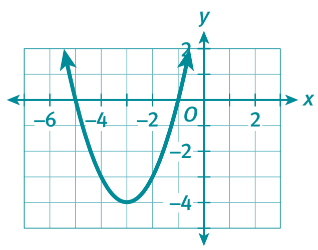
ASSESS

Students' answers to Lesson Practice problems will provide you with a formative assessment of their understanding of the lesson concepts and their ability to apply their learning.

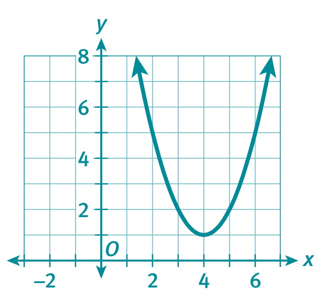
See the Activity Practice for additional problems for this lesson. You may assign the problems here or use them as a culmination for the activity.

LESSON 11-3 PRACTICE

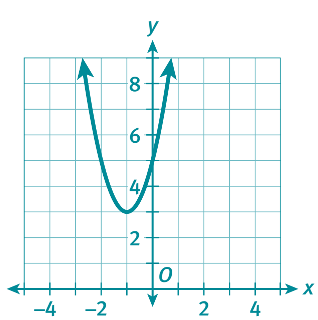
6. $g(x) = (x + 3)^2 - 4$; translated 3 units left and 4 units down



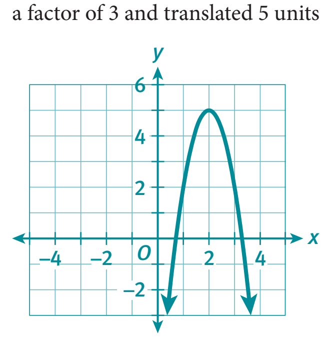
7. $h(x) = (x - 4)^2 + 1$; translated 4 units right and 1 unit up



8. $j(x) = 2(x + 1)^2 + 3$; translated 1 unit left, vertically stretched by a factor of 2 and translated 3 units up



9. $k(x) = -3(x - 2)^2 + 5$; translated 2 units right, reflected over the x-axis, vertically stretched by a factor of 3 and translated 5 units up



10. $f(x) = (x - 10)^2 + 7$; vertex: (10, 7); axis of symmetry: $x = 10$; opens upward

11. $f(x) = -(x + 8)^2 - 3$; vertex: (−8, −3); axis of symmetry: $x = -8$; opens downward

12. $f(x) = 5(x - 2)^2 + 11$; vertex: (2, 11); axis of symmetry: $x = 2$; opens upward

13. $f(x) = -2(x + 3)^2 + 23$; vertex: (−3, 23); axis of symmetry: $x = -3$; opens downward

14. Both are correct. The standard form of the equation is $f(x) = 1x^2 + 0x + (-5)$, which simplifies to $f(x) = x^2 - 5$. The vertex form of the equation is $f(x) = 1(x - 0)^2 + (-5)$, which also simplifies to $f(x) = x^2 - 5$.

ADAPT

Check students' answers to the Lesson Practice to ensure that they understand how to write the equation of a parabola in vertex form by completing the square. Additionally, students should have a clear understanding that vertex form reveals transformations better than standard form. Students can create practice problems for each other by beginning with the equation of a parabola in vertex form and then changing it to standard form.

My Notes

c. What is the axis of symmetry of the function's graph? How do you know?
$x = 4$. Sample explanation: The axis of symmetry of the graph of a quadratic function is a vertical line through the vertex.

d. Does the graph of the function open upward or downward? How do you know?
Upward. Sample explanation: The value of *a* in vertex form is positive, so the graph opens upward.

ACADEMIC VOCABULARY

An ***advantage*** is a benefit or a desirable feature.

A ***disadvantage*** is an undesirable feature.

Check Your Understanding

3. Write a set of instructions for a student who is absent explaining how to write the function $f(x) = x^2 + 6x + 11$ in vertex form.

4. What are some ***advantages*** of the vertex form of a quadratic function compared to the standard form?

5. A student is writing $f(x) = 4x^2 - 8x + 8$ in vertex form. What number should she write in the first box to complete the square inside the parentheses? What number should she write in the second box to keep the expression on the right side of the equation balanced? Explain.

$$f(x) = 4(x^2 - 2x + \square) - \square + 8$$

LESSON 11-3 PRACTICE

Write each function in vertex form. Then describe the transformation(s) from the parent function and use the transformations to graph the function.

6. $g(x) = x^2 + 6x + 5$

7. $h(x) = x^2 - 8x + 17$

8. $j(x) = 2x^2 + 4x + 5$

9. $k(x) = -3x^2 + 12x - 7$

Write each function in vertex form. Then identify the vertex and axis of symmetry of the function's graph, and tell which direction the graph opens.

10. $f(x) = x^2 - 20x + 107$

11. $f(x) = -x^2 - 16x - 67$

12. $f(x) = 5x^2 - 20x + 31$

13. $f(x) = -2x^2 - 12x + 5$

14. **Critique the reasoning of others.** Rebecca says that the function $f(x) = x^2 - 5$ is written in standard form. Lane says that the function is written in vertex form. Who is correct? Explain.

ACTIVITY 11 PRACTICE

Write your answers on notebook paper.
Show your work.

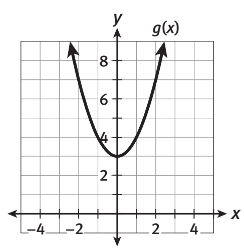
Lesson 11-1

For each function, identify all transformations of the function $f(x) = x^2$. Then graph the function.

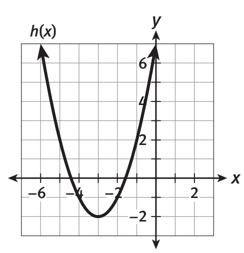
1. $g(x) = x^2 + 1$
2. $g(x) = (x - 4)^2$
3. $g(x) = (x + 2)^2 + 3$
4. $g(x) = (x - 3)^2 - 4$

Each function graphed below is a translation of $f(x) = x^2$. Describe the transformation. Then write the equation of the transformed function.

5.



6.



Use transformations of the parent quadratic function to determine the vertex and axis of symmetry of the graph of each function.

7. $g(x) = (x - 8)^2$
8. $g(x) = (x + 6)^2 - 4$

Write a quadratic function $g(x)$ that represents each transformation of the function $f(x) = x^2$.

9. translate 6 units right
10. translate 10 units down
11. translate 9 units right and 6 units up
12. translate 4 units left and 8 units down
13. The function $g(x)$ is a translation of $f(x) = x^2$. The vertex of the graph of $g(x)$ is $(-4, 7)$. What is the equation of $g(x)$? Explain your answer.

Lesson 11-2

For each function, identify all transformations of the function $f(x) = x^2$. Then graph the function.

14. $g(x) = -\frac{1}{3}x^2$
15. $g(x) = \frac{1}{5}x^2$
16. $g(x) = \frac{1}{2}(x - 3)^2$
17. $g(x) = -2(x + 3)^2 + 1$
18. $g(x) = -3(x + 2)^2 - 5$

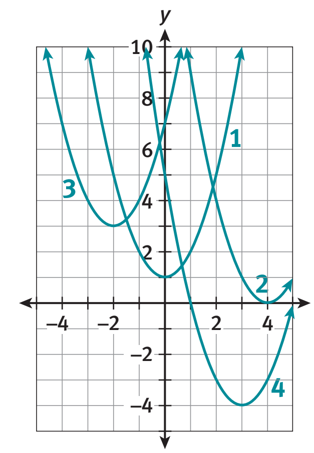
Write a quadratic function $g(x)$ that represents each transformation of the function $f(x) = x^2$.

19. shrink horizontally by a factor of $\frac{1}{4}$
20. stretch vertically by a factor of 8
21. shrink vertically by a factor of $\frac{1}{3}$, translate 6 units up
22. translate 1 unit right, stretch vertically by a factor of $\frac{3}{2}$, reflect over the x-axis, translate 7 units up

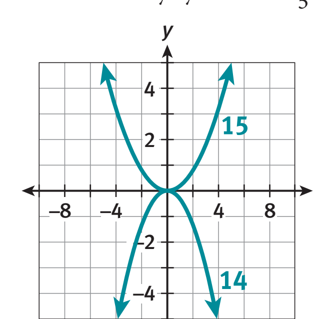
ACTIVITY 11 Continued

ACTIVITY PRACTICE

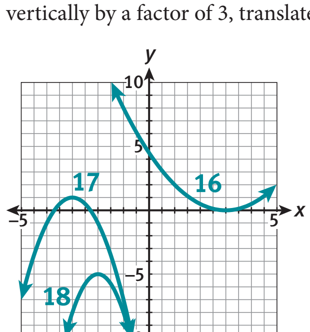
1. Translate 1 unit up.
2. Translate 4 units right.
3. Translate 2 units left and 3 units up.
4. Translate 3 units right and 4 units down.



5. Translate 3 units up; $g(x) = x^2 + 3$.
6. Translate 3 units left and 2 units down; $h(x) = (x + 3)^2 - 2$.
7. vertex: (8, 0); axis of symmetry: $x = 8$
8. vertex: $(-6, -4)$; axis of symmetry: $x = -6$
9. $g(x) = (x - 6)^2$
10. $g(x) = x^2 - 10$
11. $g(x) = (x - 9)^2 + 6$
12. $g(x) = (x + 4)^2 - 8$
13. $g(x) = (x + 4)^2 + 7$. Sample explanation: The coordinates of the vertex show that $g(x)$ is a translation of $f(x)$ 4 units to the left and 7 units up.
14. Shrink vertically by a factor of $\frac{1}{3}$. and reflect over the x-axis
15. Shrink vertically by a factor of $\frac{1}{5}$.



16. Translate 3 units to the right, shrink vertically by a factor of $\frac{1}{2}$.
17. Translate 3 units left, reflect over the x-axis, stretch vertically by a factor of 2, translate 1 unit up.
18. Translate 2 units left, reflect over the x-axis, stretch vertically by a factor of 3, translate 5 units down.



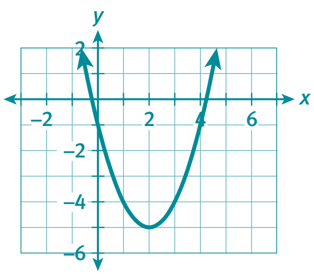
19. $g(x) = (4x)^2$
20. $g(x) = 8x^2$
21. $g(x) = \frac{1}{3}x^2 + 6$
22. $g(x) = -\frac{3}{2}(x - 1)^2 + 7$

23. vertical stretch by a factor of 3 and reflect over the x-axis; $g(x) = -3x^2$

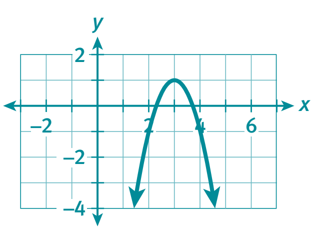
24. horizontal stretch by a factor of 3; $h(x) = \left(\frac{1}{3}x^2\right)$ (or vertical shrink by factor of 9; $h(x) = \frac{1}{9}x^2$)

25. D

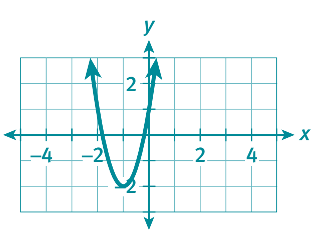
26. $g(x) = (x - 2)^2 - 5$; Translate 2 units right and 5 units down.



27. $g(x) = -2(x - 3)^2 + 1$; Translate 3 units right, reflect over the x-axis, vertically stretch by a factor of 2 and translate 1 unit up.



28. $g(x) = 3(x + 1)^2 - 2$; Translate 1 unit left, vertically stretch by a factor of 3 and translate 2 units down.



29. $f(x) = (x - 8)^2 + 7$; vertex: $(8, 7)$; axis of symmetry: $x = 8$; opens upward

30. $f(x) = 2(x + 9)^2 - 20$; vertex: $(-9, -20)$; axis of symmetry: $x = -9$; opens upward

31. $f(x) = -3(x - 1)^2 + 12$; vertex: $(1, 12)$; axis of symmetry: $x = 1$; opens downward

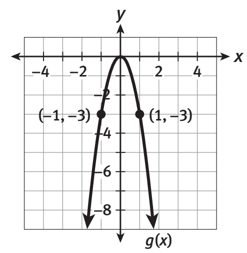
32. $f(x) = (x - 1)^2 + 4$; vertex: $(1, 4)$; axis of symmetry: $x = 1$; opens upward

ADDITIONAL PRACTICE

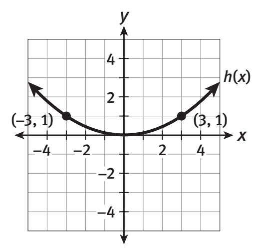
If students need more practice on the concepts in this activity, see the Teacher Resources at SpringBoard Digital for additional practice problems.

Each function graphed below is a transformation of $f(x) = x^2$. Describe the transformation. Then write the equation of the transformed function.

23.



24.



25. Which of these functions has the widest graph when they are graphed on the same coordinate plane?

A. $f(x) = -2x^2$ **B.** $f(x) = 5x^2$

C. $f(x) = \frac{1}{2}x^2$ **D.** $f(x) = -\frac{1}{5}x^2$

Lesson 11-3

Write each function in vertex form. Then describe the transformation(s) from the parent function and use the transformations to graph the function.

26. $g(x) = x^2 - 4x - 1$

27. $g(x) = -2x^2 + 12x - 17$

28. $g(x) = 3x^2 + 6x + 1$

Write each function in vertex form. Then identify the vertex and axis of symmetry of the function's graph, and tell which direction the graph opens.

29. $f(x) = x^2 - 16x + 71$

30. $f(x) = 2x^2 + 36x + 142$

31. $f(x) = -3x^2 + 6x + 9$

32. $f(x) = x^2 - 2x + 5$

33. The function $h(t) = -16t^2 + 22t + 4$ models the height h in feet of a football t seconds after it is thrown.

a. Write the function in vertex form.

b. To the nearest foot, what is the greatest height that the football reaches? Explain your answer.

c. To the nearest tenth of a second, how long after the football is thrown does it reach its greatest height? Explain your answer.

34. Which function has a vertex to the right of the y-axis?

A. $f(x) = -x^2 - 10x - 29$

B. $f(x) = x^2 - 12x + 40$

C. $f(x) = x^2 + 2x - 5$

D. $f(x) = x^2 + 6x + 2$

MATHEMATICAL PRACTICES

Construct Viable Arguments and Critique the Reasoning of Others

35. A student claims that the function $g(x) = -x^2 - 5$ has no real zeros. As evidence, she claims that the graph of $g(x)$ opens downward and its vertex is $(0, -5)$, which means that the graph never crosses the x-axis. Is the student's argument valid? Support your answer.

33. a. $h(t) = -16\left(t - \frac{11}{16}\right)^2 + \frac{185}{16}$

b. 12 ft. Sample explanation: The vertex form shows that the graph of the function opens downward and its vertex is $\left(\frac{11}{16}, \frac{185}{16}\right)$. The maximum value of the function is $\frac{185}{16} = 11\frac{9}{16}$, or about 12 ft.

c. 0.7 s. Sample explanation: The maximum value of the function is $\left(\frac{11}{16}, \frac{185}{16}\right)$. The maximum height of $\frac{185}{16}$ ft occurs when $t = \frac{11}{16}$ s, or about 0.7 s.

34. B

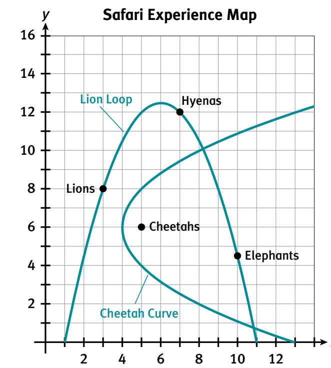
35. Yes, the argument is valid. The graph of $g(x)$ is a reflection of $f(x) = x^2$ over the x-axis followed by a translation 5 units down. The reflection over the x-axis results in the graph of $g(x)$ opening downward, which means that $g(x)$ has a maximum value at its vertex. The vertex form of the equation is $g(x) = -(x - 0)^2 + (-5)$, confirming that the vertex is $(0, -5)$. The greatest value of $g(x)$ is -5, which means that there is no real value of x for which $g(x) = 0$.

Writing and Transforming Quadratic Functions

THE SAFARI EXPERIENCE

Embedded Assessment 2
Use after Activity 11

A zoo is constructing a new exhibit of African animals called the Safari Experience. A path called the Lion Loop will run through the exhibit. The Lion Loop will have the shape of a parabola and will pass through these points shown on the map: (3, 8) near the lions, (7, 12) near the hyenas, and (10, 4.5) near the elephants.



1. Write the standard form of the quadratic function that passes through the points (3, 8), (7, 12), and (10, 4.5). This function models the Lion Loop on the map.

2. A lemonade stand will be positioned at the vertex of the parabola formed by the Lion Loop.
   a. Write the equation that models the Lion Loop in vertex form, $y = a(x - h)^2 + k$.
   b. What are the map coordinates of the lemonade stand? Explain how you know.

3. A graphic artist needs to draw the Lion Loop on the map.
   a. Provide instructions for the artist that describe the shape of the Lion Loop as a set of transformations of the graph of $f(x) = x^2$.
   b. Use the transformations of $f(x)$ to draw the Lion Loop on the map.

4. The Safari Experience will also have a second path called the Cheetah Curve. This path will also be in the shape of a parabola. It will open to the right and have its focus at the cheetah exhibit at map coordinates (5, 6).
   a. Choose a vertex for the Cheetah Curve. Explain why the coordinates you chose for the vertex are appropriate.
   b. Use the focus and the vertex to write the equation that models the Cheetah Curve.
   c. What are the directrix and the axis of symmetry of the parabola that models the Cheetah Curve?
   d. Draw and label the Cheetah Curve on the map.

Embedded Assessment 2

Assessment Focus

- Standard form of a parabola
- Vertex form of a parabola
- Transformations
- Directrix
- Focus
- Axis of symmetry

Answer Key

1. $y = -0.5x^2 + 6x - 5.5$

2. a. $y = -0.5(x - 6)^2 + 12.5$

b. (6, 12.5); Sample explanation: The vertex form of the equation shows that the value of h is 6 and the value of k is 12.5, so the vertex (h, k) is (6, 12.5).

3. a. Translate the graph of $f(x) = x^2$ by 6 units to the right. Then reflect it over the x-axis and vertically shrink it by a factor of 0.5. Finally, translate it 12.5 units up.

b. See graph on student page.

4. a. Answers will vary, but the vertex should have an x-coordinate less than 5 and a y-coordinate of 6. Sample answer: (4, 6); The parabola opens to the right, so the vertex must be directly to the right of the focus. The point I picked for the vertex is appropriate because it is 1 unit to the left of the focus.

b. Answers will vary, depending on the answer to part a, but should have the form $x = \frac{1}{4p}(y - 6)^2 + h$.

Sample answer: $x = \frac{1}{4}(y - 6)^2 + 4$

c. Answers for directrix will vary, depending on the answer to part a, but should be a vertical line to the left of $x = 5$. Sample answer: $x = 3$

Axis of symmetry: $y = 6$

d. Answers will vary, depending on the equation in part b, but students should draw a parabola that opens to the right with a focus at (5, 6). See graph on student page for a sample answer.

Teacher to Teacher

Making a chart of information will help students decipher what information they know and what they need to know to write an equation of a parabola. Encourage them to make a chart with the following columns: vertex, axis of symmetry, directrix and focus.

Common Core State Standards for Embedded Assessment 2

HSA-CED.A.1	Create equations and inequalities in one variable and use them to solve problems. Include equations arising from linear functions.
HSA-CED.A.2	Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.
HSA-IF.C.8	Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.
HSF-BF.B.3	Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $kf(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology.

Teacher to Teacher

You may wish to read through the scoring guide with students and discuss the differences in the expectations at each level. Check that students understand the terms used.

Unpacking Embedded Assessment 3

Once students have completed this Embedded Assessment, turn to Embedded Assessment 3 and unpack it with them. Use a graphic organizer to help students understand the concepts they will need to know to be successful on Embedded Assessment 3.

Embedded Assessment 2
Use after Activity 11

Writing and Transforming Quadratic Functions

THE SAFARI EXPERIENCE

Scoring Guide	Exemplary	Proficient	Emerging	Incomplete
	The solution demonstrates these characteristics:			
Mathematics Knowledge and Thinking (Items 1, 2, 3a, 4a-c)	• Effective understanding of quadratic functions as transformations of $f(x) = x^2$ • Clear and accurate understanding of how to write a quadratic function in standard form given three points on its graph • Clear and accurate understanding of how to transform a quadratic function from standard to vertex form • Clear and accurate understanding of how to identify key features of a graph of a parabola and how they relate to the equation for a parabola	• Adequate understanding of quadratic functions as transformations of $f(x) = x^2$ • Largely correct understanding of how to write a quadratic function in standard form given three points on its graph • Largely correct understanding of how to transform a quadratic function from standard to vertex form • Largely correct understanding of how to identify key features of a graph of a parabola and how they relate to the equation for a parabola	• Partial understanding of quadratic functions as transformations of $f(x) = x^2$ • Partial understanding of how to write a quadratic function in standard form given three points on its graph • Difficulty with transforming a quadratic function from standard to vertex form • Partial understanding of how to identify key features of a graph of a parabola and how they relate to the equation for a parabola	• Inaccurate or incomplete understanding of quadratic functions as transformations of $f(x) = x^2$ • Little or no understanding of how to write a quadratic function in standard form given three points on its graph • Little or no understanding of how to transform a quadratic function from standard to vertex form • Little or no understanding of how to identify key features of a graph of a parabola and how they relate to the equation for a parabola
Problem Solving (Items 1, 2b, 4b)	• An appropriate and efficient strategy that results in a correct answer	• A strategy that may include unnecessary steps but results in a correct answer	• A strategy that results in some incorrect answers	• No clear strategy when solving problems
Mathematical Modeling / Representations (Items 1, 2b, 3b, 4b, 4d)	• Effective understanding of how to model real-world scenarios with quadratic functions and parabolas and interpret their key features • Clear and accurate understanding of how to graph quadratic functions using transformations, and how to graph parabolas	• Adequate understanding of how to model real-world scenarios with quadratic functions and parabolas and interpret their key features • Largely correct understanding of how to graph quadratic functions using transformations, and how to graph parabolas	• Partial understanding of how to model real-world scenarios with quadratic functions and parabolas and interpret their key features • Some difficulty with understanding how to graph quadratic functions using transformations and with graphing parabolas	• Little or no understanding of how to model real-world scenarios with quadratic functions and parabolas and interpret their key features • Inaccurate or incomplete understanding of how to graph quadratic functions using transformations, and how to graph parabolas
Reasoning and Communication (Items 2b, 3a, 4a)	• Precise use of appropriate math terms and language to describe how to graph a quadratic function as a transformation of $f(x) = x^2$ • Precise use of appropriate math terms and language to explain how features of a graph relate to a real-world scenario	• Adequate descriptions of how to graph a quadratic function as a transformation of $f(x) = x^2$ • Adequate explanation of how features of a graph relate to a real-world scenario	• Misleading or confusing descriptions of how to graph a quadratic function as a transformation of $f(x) = x^2$ • Partially correct explanation of how features of a graph relate to a real-world scenario	• Incomplete or inaccurate descriptions of how to graph a quadratic function as a transformation of $f(x) = x^2$ • Incorrect or incomplete explanation of how features of a graph relate to a real-world scenario

Graphing Quadratics and Quadratic Inequalities

ACTIVITY 12

Calendar Art

Lesson 12-1 Key Features of Quadratic Functions

My Notes

Learning Targets:

- Write a quadratic function from a verbal description.
- Identify and interpret key features of the graph of a quadratic function.

SUGGESTED LEARNING STRATEGIES: Marking the Text, Paraphrasing, Create Representations, Quickwrite, Self Revision/Peer Revision

Ms. Picasso, sponsor for her school's art club, sells calendars featuring student artwork to raise money for art supplies. A local print shop sponsors the calendar sale and donates the printing and supplies. From past experience, Ms. Picasso knows that she can sell 150 calendars for $3.00 each. She considers raising the price to try to increase the profit that the club can earn from the sale. However, she realizes that by raising the price, the club will sell fewer than 150 calendars.

1. If Ms. Picasso raises the price of the calendar by x dollars, write an expression for the price of one calendar.
$3 + x$

2. In previous years, Ms. Picasso found that for each $0.40 increase in price, the number of calendars sold decreased by 10. Complete the table below to show that relationship between the price increase and the number of calendars sold.

Increase in price ($), x	Number of calendars sold
0.00	150
0.40	140
0.80	130
1.20	120

3. Model with mathematics. Use the data in the table to write an expression that models the number of calendars sold in terms of x, the price increase.
$150 - 25x$

MATH TIP

If the value of one quantity decreases by a constant amount as another quantity increases by a constant amount, the relationship between the quantities is linear.

4. Write a function that models $A(x)$, the amount of money raised selling calendars when the price is increased x dollars.
$A(x) = (3 + x)(150 - 25x)$

ACTIVITY 12

Guided

Activity Standards Focus

In Activity 12, students graph quadratic equations and quadratic inequalities. They write quadratic functions from verbal descriptions and identify and interpret key features of those functions. They also graph quadratic inequalities and use those graphs to determine solutions to the quadratic inequalities. Throughout this activity, have students discuss the key features of quadratic functions and discuss how those key features help them graph the functions.

Lesson 12-1

PLAN

Pacing: 1 class period

Chunking the Lesson

#1–2 #3–5 #6
#7–9 #10
Check Your Understanding
Lesson Practice

TEACH

Bell-Ringer Activity

Have students translate the word problems to an algebraic equation and solve.

1. Veronica receives a $15 commission for each membership she sells. One week Veronica received $165 in commissions. How many memberships did she sell?
[$15m = 165$; 11]

2. Matt and Samantha work for two different car-detailing companies. Each day that Matt works, he receives a flat rate of $50 plus an additional $20 per car. Each day Samantha works, she earns a flat rate of $30 plus an additional $25 per car. One day they earned the same amount of money for detailing the same number of cars. How many cars did each of them detail?
[$50 + 20c = 30 + 25c$; 4]

1–2 Activating Prior Knowledge, Create Representations Students will likely have little difficulty with Item 1, but Item 2 will require students to create a linear expression. This is a good opportunity for formative assessment of students' understanding of linear relationships.

3–5 Create Representations In Item 4, students will need to multiply the two expressions in Items 1 and 3 to develop the function. Students should be able to write the function in standard form, again providing formative assessment.

Common Core State Standards for Activity 12

HSF-IF.B.4	For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship.
HSF-IF.B.5	Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes.

6 Create Representations Students may require assistance in graphing the equation, so a graphing calculator is a viable option for this item. Be certain that the graph is accurate with regard to the intercepts and vertex—students may need help in finding the coordinates of the vertex.

7–9 Quickwrite, Self Revision/Peer Revision Some students may address the concept of a maximum informally by referring to a maximum point. The vertex is related to the maximum of the function, but the maximum value is the y-coordinate of the vertex. The x-coordinate is merely the domain value at which this maximum value occurs.

Be sure to emphasize that there is a unit associated with the maximum—the dollars raised by selling calendars.

CONNECT TO AP

Understanding the units in problem situations is especially important in AP Calculus and in helping students achieve success in the free-response items on the AP test.

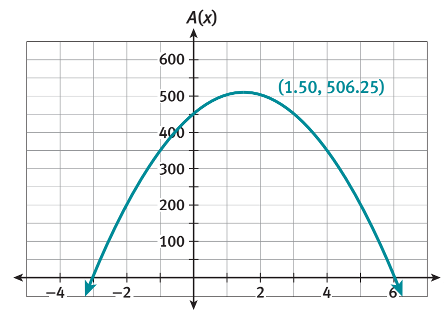
My Notes

MATH TIP

A quadratic function in *standard form* is written as $f(x) = ax^2 + bx + c$.

5. Write your function $A(x)$ in standard form. Identify the constants a, b, and c.
$A(x) = -25x^2 + 75x + 450$; $a = -25$, $b = 75$, $c = 450$

6. Graph $A(x)$ on the coordinate grid.



7. a. For what values of x does the value of $A(x)$ increase as you move from left to right on the graph?
The value of $A(x)$ increases for values of x less than 1.5.

b. For what values of x does the value of $A(x)$ decrease as you move from left to right on the graph?
The value of $A(x)$ decreases for values of x greater than 1.5.

8. Reason quantitatively. Based on the model, what is the maximum amount of money that can be earned? What is the increase in price of a calendar that will yield that maximum amount of money?
Maximum amount of money that can be earned is $506.25, and the increase in price that yields this maximum is $1.50.

Lesson 12-1
Key Features of Quadratic Functions

9. a. What feature of the graph gives the information that you used to answer Item 8?

Answers may vary but must identify the "highest point" on the graph of the function, the vertex.

b. How does this feature relate to the intervals of x for which $A(x)$ is increasing and decreasing?

The vertex (or highest point) separates the two intervals. From left to right on the graph, the value of $A(x)$ increases until it reaches the vertex and then decreases.

The point that represents the maximum value of $A(x)$ is the *vertex* of this parabola. The x-coordinate of the vertex of the graph of $f(x) = ax^2 + bx + c$ can be found using the formula $x = -\frac{b}{2a}$.

10. Use this formula to find the x-coordinate of the vertex of $A(x)$.

$-\frac{75}{2(-25)} = \frac{-75}{-50} = 1.5$

My Notes

MATH TIP

Substitute the x-coordinate of the vertex into the quadratic equation to find the y-coordinate of the vertex.

Check Your Understanding

11. Look back at the expression you wrote for $A(x)$ in Item 4. Explain what each part of the expression equal to $A(x)$ represents.

12. Is the vertex of the graph of a quadratic function always the highest point? Explain.

13. The graph of a quadratic function $f(x)$ opens upward, and its vertex is $(-2, 5)$. For what values of x is the value of $f(x)$ increasing? For what values of x is the value of $f(x)$ decreasing? Explain your answers.

14. Construct viable arguments. Suppose you are asked to find the vertex of the graph of $f(x) = -3(x - 4)^2 + 1$. Which method would you use? Explain why you would choose that method.

ACTIVITY 12 Continued

10 Create Representations Given the formula for the x-coordinate of the vertex, students use it to confirm their findings from Items 8 and 9.

Check Your Understanding

Debrief students' answers to these items to ensure that they understand concepts related to the vertex of a quadratic function.

Answers

11. Sample answer: The expression consists of 2 factors. The first factor represents the cost of 1 calendar, and the second factor represents the number of calendars sold. Their product represents the total amount earned from selling the calendars. The cost of 1 calendar is equal to the original price, \$3, plus the increase in price, x. The number of calendars sold is given by $150 - 25x$, where 150 represents the number sold with no price increase and 25 represents the decrease in the number sold per dollar increase in price.

12. No. The vertex is only the highest point if the graph of the quadratic function opens downward. If the graph opens upward, the vertex is the lowest point.

13. The value of $f(x)$ decreases for $x < -2$ and increases for $x > -2$. Sample explanation: If the graph opens upward, the vertex is the lowest point. So, the value of $f(x)$ decreases as the graph moves from left to right toward the vertex. The x-coordinate of the vertex is -2, so the function decreases for $x < -2$. The value of $f(x)$ increases as the graph moves from left to right away from the vertex, so the function increases for $x > -2$.

14. Sample answer: The function is already written in vertex form $f(x) = a(x - h)^2 + k$, so I would use the values of h and k to find the vertex. The vertex (h, k) is $(4, 1)$. I would use this method because it doesn't involve graphing the function or writing the equation in a different form, both of which would require more steps.

ASSESS

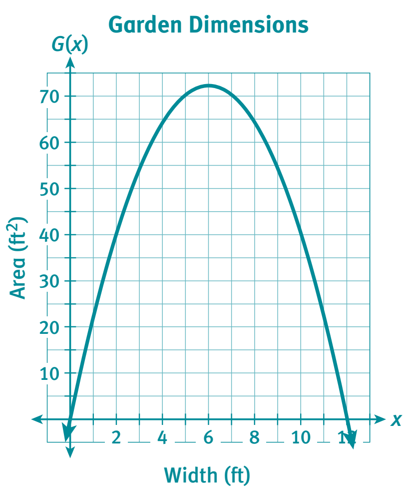
Students' answers to Lesson Practice problems will provide you with a formative assessment of their understanding of the lesson concepts and their ability to apply their learning.

See the Activity Practice for additional problems for this lesson. You may assign the problems here or use them as a culmination for the activity.

LESSON 12-1 PRACTICE

15. Sample answer: The total length of fencing, 24 ft, must equal twice the width of the garden plus the length. If x represents the width in ft and l represents the length in ft, then $2x + l = 24$. Solving for l shows that $l = 24 - 2x$.

16. $G(x) = -2x^2 + 24x$



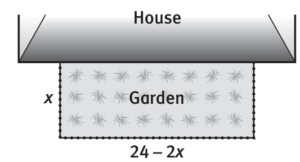
17. (6, 72); The x-coordinate of 6 represents the width in feet that gives the greatest area. The y-coordinate of 72 represents the greatest area in square feet.

ADAPT

Check students' answers to the Lesson Practice to ensure that they understand how to determine each of the important characteristics of quadratic functions when the function is presented in standard form. Students should also be able to use a quadratic model to answer questions both algebraically and graphically. As reinforcement, have students create a guide to quadratic functions that compares and contrasts finding characteristics of a quadratic function written in vertex form and written in standard form.

LESSON 12-1 PRACTICE

Mr. Picasso would like to create a small rectangular vegetable garden adjacent to his house. He has 24 ft of fencing to put around three sides of the garden.



15. **Construct viable arguments.** Explain why $24 - 2x$ is an appropriate expression for the length of the garden in feet given that the width of the garden is x ft.

16. Write the standard form of a quadratic function $G(x)$ that gives the area of the garden in square feet in terms of x. Then graph $G(x)$.

17. What is the vertex of the graph of $G(x)$? What do the coordinates of the vertex represent in this situation?

18. **Reason quantitatively.** What are the dimensions of the garden that yield that maximum area? Explain your answer.

Write each quadratic function in standard form and identify the vertex.

19. $f(x) = (3x - 6)(x + 4)$

20. $f(x) = 2(x - 6)(20 - 3x)$

18. 6 ft by 12 ft; The x-coordinate of the vertex, 6, is the width that results in the greatest area. Substituting this value of x into the expression for the length gives the length that results in the greatest area: $24 - 2(6) = 12$ ft.

19. $f(x) = 3x^2 + 6x - 24$; $(-1, -27)$

20. $f(x) = -6x^2 + 76x - 240$; $\left(\frac{19}{3}, \frac{2}{3}\right)$

Lesson 12-2
More Key Features of Quadratic Functions

Learning Targets:
- Write a quadratic function from a verbal description.
- Identify and interpret key features of the graph of a quadratic function.

SUGGESTED LEARNING STRATEGIES: Interactive Word Wall, Quickwrite, Think Aloud, Discussion Groups, Self Revision/Peer Revision

An intercept occurs at the point of intersection of a graph and one of the axes. For a function f, an x-intercept is a value n for which $f(n) = 0$. The y-intercept is the value of $f(0)$. Use the graph that you made in Item 6 in the previous lesson for Items 1 and 2 below.

1. What is the y-intercept of the graph of $A(x)$? What is the significance of the y-intercept in terms the calendar problem?
The y-intercept of the graph is 450. This represents the amount earned if the price is increased $0.

2. Make sense of problems. What are the x-intercepts of the graph of $A(x)$? What is the significance of each x-intercept in terms of the calendar problem?
The x-intercepts are −3 and 6. −3 represents a decrease in price of $3 that will yield no profit (the calendars are free). 6 represents an increase of $6 in price that will yield no profit, meaning no calendars sold.

3. The x-intercepts of the graph of $f(x) = ax^2 + bx + c$ can be found by solving the equation $ax^2 + bx + c = 0$. Solve the equation $A(x) = 0$ to verify the x-intercepts of the graph.
$-25x^2 + 75x + 450 = 0$ factors to $-25(x + 3)(x - 6) = 0$. The solutions are $x = -3, x = 6$.

4. a. Recall that x represents the increase in the price of the calendars. Explain what negative values of x represent in this situation.
A negative value of x would indicate a decrease in the price of the calendars. For example, an x-value of −1 represents a $1 decrease in the price of the calendars.

b. Recall that $A(x)$ represents the amount of money raised from selling the calendars. Explain what negative values of $A(x)$ represent in this situation.
A negative value of $A(x)$ would represent a loss of money from the calendar sales. For example, a value of −1 for $A(x)$ would indicate that the club lost $1 by selling the calendars. However, this value occurs only when the price of the calendar is reduced below 0, which does not make sense.

My Notes

MATH TIP
As with graphs of linear functions, graphs of quadratic functions have intercepts where the graph intersects one of the axes.

An ***x*-intercept** is the x-coordinate of a point where a graph intersects the x-axis. Quadratic functions can have 0, 1, or 2 x-intercepts.

A ***y*-intercept** is the y-coordinate of a point where a graph intersects the y-axis. A quadratic function will only have one y-intercept.

4–6 Chunking the Activity, Activating Prior Knowledge, Group Presentation For Items 4–6, place students in small groups of varying abilities to provide them with opportunities to further explore the interpretations of the graphs together. Item 4 addresses what it means to have a negative x-value and a negative $A(x)$-value. For part b, ask students to determine whether there is a price increase for the calendars that would also result in a negative value of $A(x)$. They should find that an increase of more than $6 would cause this.

Lesson 12-2

PLAN
Pacing: 1 class period

Chunking the Lesson

#1–2 #3

#4–6 #7

Check Your Understanding

Lesson Practice

TEACH
Bell-Ringer Activity

Have students find the x-intercepts and y-intercepts of the following linear equations. This will help prepare them for finding intercepts of quadratic equations.

1. $x + 2y = 8$
   [x-intercept = 8; y-intercept = 4]
2. $2x - 6y = 12$
   [x-intercept = 6; y-intercept = −2]
3. $-5x + y = 10$
   [x-intercept = −2; y-intercept = 10]

Developing Math Language
When written as ordered pairs, x-intercepts represent the value of the equation when $y = 0$. Therefore, x-intercepts are always of the form $(x, 0)$ when written as an ordered pair. When written as ordered pairs, y-intercepts represent the value of the equation when $x = 0$. Therefore, y-intercepts are always of the form $(0, y)$, when written as an ordered pair.

Additionally, x-intercepts and y-intercepts can both be written in function notation. In function notation, the x-intercept for a function f is a value for which $f(x) = 0$. The y-intercept is the value of $f(0)$.

1–2 Create Representations, Quickwrite While students may find the intercepts easily, they may have difficulty with the interpretations in the problem context. Be sure that students recognize that the negative x-intercept actually represents a decrease in price.

3 Create Representations The intent of Item 3 is that students find the x-intercepts algebraically to verify the answer found in Item 2.

4–6 (continued) Item 5 asks students to find a *reasonable* domain for the function, where the graph is above a zero profit. Item 6 asks students to find a *reasonable* range with the assumption that a profit is made. Both Items 5 and 6 ask students to use prior knowledge by providing answers in inequality notation, interval notation, and set notation. After they have had some time to collaborate, have students present and explain their solutions to the class.

Note that the wording in Item 6a is "assuming that the club makes a profit" which excludes a profit of 0. Later, in Activity Practice Item 14, the wording is "assuming that the club does not want to lose money" which does *not* exclude a profit of 0.

ELL Support

Explain to students that the use of the word *reasonable*, when referring to the domain and range in Items 5 and 6, means "sensible." In other words, it does not make sense to include a number of calendars in the domain, where the profit is ≤ 0. That is why it is between the x-intercepts (noninclusive). It does not make sense to include anything other than a positive range to represent profit; therefore, the range spans from anything greater than zero up to the maximum point on the parabola, 506.25.

7 Activating Prior Knowledge, Quickwrite, Debriefing Prior knowledge of line symmetry is necessary to answer this item. Students should recognize that the x-intercepts are equidistant from the axis of symmetry.

Check Your Understanding

Debrief students' answers to these items to ensure that they understand concepts related to intercepts of quadratic functions.

Answers

8. Sample answer: The amount the club will make is the product of the cost per calendar and the number of calendars sold. Both of these factors depend on x, the increase in price of the calendars. The product of 2 factors that contain x will have an x^2-term. So, the amount the club will make is a quadratic function in terms of x.

My Notes

MATH TIP

The reasonable domain and range of a function are the values in the domain and range of the function that make sense in a given real-world situation.

WRITING MATH

You can write a domain of $4 < x \le 2$ in interval notation as (4, 2] and in set notation as $\{x \mid x \in \mathbb{R}, 4 < x \le 2\}$.

MATH TIP

The vertical line $x = -\frac{b}{2a}$ is the axis of symmetry for the graph of the function $f(x) = ax^2 + bx + c$.

5. a. Reason quantitatively. What is a reasonable domain of $A(x)$, assuming that the club makes a profit from the calendar sales? Write the domain as an inequality, in interval notation, and in set notation.
$-3 < x < 6$; $(-3, 6)$; $\{x \mid x \in \mathbb{R}, -3 < x < 6\}$

b. Explain how you determined the reasonable domain.
A positive value of $A(x)$ represents a profit for the club, so the reasonable domain includes only the values of x for which $A(x)$ is positive. $A(x)$ is positive only when x is greater than -3 and less than 6.

6. a. What is a reasonable range of $A(x)$, assuming that the club makes a profit from the calendar sales? Write the range as an inequality, in interval notation, and in set notation.
$0 < y \le 506.25$; $(0, 506.25]$; $\{y \mid y \in \mathbb{R}, 0 < y \le 506.25\}$

b. Explain how you determined the reasonable range.
A positive value of $A(x)$ represents a profit for the club, so the value of $A(x)$ must be greater than 0. The graph and the function rule show that the maximum value of $A(x)$ is 506.25. Thus, the reasonable range includes values greater than 0 and no more than 506.25.

7. What is the average of the x-intercepts in Item 2? How does this relate to the symmetry of a parabola?
The average of 6 and -3 is 1.5, the x-coordinate of the vertex. The axis of symmetry is a vertical line through the vertex. Therefore, a point on one side of the axis of symmetry will have a corresponding point on the other side the same distance away. This is true of the x-intercepts.

Lesson 12-2
More Key Features of Quadratic Functions

My Notes

Check Your Understanding

8. Construct viable arguments. Explain why a quadratic function is an appropriate model for the amount the club will make from selling calendars.

9. Can a function have more than one y-intercept? Explain.

10. Do all quadratic functions have two x-intercepts? Explain.

11. Reason abstractly. Explain how the reasonable domain of a quadratic function helps to determine its reasonable range.

LESSON 12-2 PRACTICE

Ms. Picasso is also considering having the students in the art club make and sell candles to raise money for supplies. The function $P(x) = -20x^2 + 320x - 780$ models the profit the club would make by selling the candles for x dollars each.

12. What is the y-intercept of the graph of $P(x)$, and what is its significance in this situation?

13. What are the x-intercepts of the graph of $P(x)$, and what is their significance in this situation?

14. Give the reasonable domain and range of $P(x)$, assuming that the club does not want to lose money by selling the candles. Explain how you determined the reasonable domain and range.

15. Make sense of problems. What selling price for the candles would maximize the club's profit? Explain your answer.

Identify the x- and y-intercepts of each function.

16. $f(x) = x^2 + 11x + 30$

17. $f(x) = 4x^2 + 14x - 8$

CONNECT TO TECHNOLOGY

When answering Items 12–15, it may help you to view a graph of the function on a graphing calculator.

Answers

9. No. Sample explanation: If a graph of a relationship has more than one y-intercept, then the vertical line $x = 0$ would intersect the graph at more than one point. If the graph of a relationship fails the vertical line test, then it is not a function.

10. No. If the vertex of the graph of a quadratic function is on the x-axis, then the function has only one x-intercept. If the graph of a quadratic function opens upward and its vertex is above the x-axis, then the function has no x-intercepts. Similarly, if the graph of a quadratic function opens downward and its vertex is below the x-axis, then the function has no x-intercepts.

11. The reasonable domain includes only the values of x that make sense as inputs for the quadratic function in the given situation. The reasonable domain restricts the range to the values of the function for those values of x. (Note that there may be restrictions on the reasonable range other than those having to do with the domain.)

ACTIVITY 12 Continued

ASSESS

Students' answers to Lesson Practice problems will provide you with a formative assessment of their understanding of the lesson concepts and their ability to apply their learning.

See the Activity Practice for additional problems for this lesson. You may assign the problems here or use them as a culmination for the activity.

LESSON 12-2 PRACTICE

12. -780; The y-intercept represents the profit the club would make for selling the candles for \$0 each. The y-intercept is negative, which indicates a loss of money. The club would lose \$780 if it gave the candles away for free.

13. 3 and 13; The x-intercepts represent selling prices that would result in a profit of \$0. The club would make no profit (or break even) if it were to sell the candles for \$3 or for \$13.

14. Reasonable domain: $3 \le x \le 13$; reasonable range: $0 \le y \le 500$. Sample explanation: A graph of $P(x)$ shows that the club's profit is greater than or equal to \$0 when the selling price x is between \$3 and \$13, so the reasonable domain is $3 \le x \le 13$. The graph also shows that the maximum value of the club's profit is \$500. Because the profit must be greater than or equal to \$0, the reasonable range of the function is $0 \le y \le 500$.

15. \$8; The graph of $P(x)$ opens downward and its vertex is (8, 500). The vertex indicates that the club will make a maximum profit of \$500 by selling the candles for \$8 each.

16. y-intercept: 30; x-intercepts: -6 and -5

17. y-intercept: -8; x-intercepts: -4 and $\frac{1}{2}$

ADAPT

Check students' answers to the Lesson Practice to ensure that they understand how the graph of a quadratic function is related to the discriminant of the related quadratic equation. Provide extra practice for students by asking students to find the equation of a downward opening parabola with its vertex in the fourth quadrant, to find the equation of an upward opening parabola with one positive rational zero, and to find the equation of a parabola with two irrational zeros (one on either side of the y-axis).

Lesson 12-3

PLAN

Pacing: 1 class period

Chunking the Lesson

Example A #1–4
Check Your Understanding
Lesson Practice

TEACH

Bell-Ringer Activity

In order to help students with Example A, review the following key elements that describe graphs of quadratic functions in the form $f(x) = ax^2 + bx + c$.

- If $a > 0$, the graph opens upward.
- If $a < 0$, the graph opens downward.
- If $|a| > 1$, it will be narrower than the parent function of $y = x^2$.
- If $|a| < 1$, it will be wider than the parent function of $y = x^2$.
- The axis of symmetry is $x = -\frac{b}{2a}$.
- The vertex has an x-coordinate of $-\frac{b}{2a}$.
- The y-intercept is c. Therefore, the point $(0, c)$ is on the parabola.

Example A Create Representations, Group Presentation, Debriefing As shown by the items in Try These A, functions may have irrational x-intercepts or no x-intercepts. This may initiate discussion that will enable students to make some connections to prior learning regarding discriminants of a quadratic equation. Thorough debriefing and group presentations should follow this Example and Try These items. Students should find that the graph in Try These Item d has no x-intercepts because $f(x) = 0$ has no real solutions. Because of this, students may not immediately see how to draw the parabola, because they have only two points. Use your questioning skills to help them realize that another point can be found by reflecting the point containing the y-intercept over the axis of symmetry.

My Notes

Learning Targets:

- Identify key features of a quadratic function from an equation written in standard form.
- Use key features to graph a quadratic function.

SUGGESTED LEARNING STRATEGIES: Note Taking, Create Representations, Group Presentation, Identify a Subtask, Quickwrite

Example A

For the quadratic function $f(x) = 2x^2 - 9x + 4$, identify the vertex, the y-intercept, x-intercept(s), and the axis of symmetry. Graph the function.

Identify a, b, and c. $a = 2, b = -9, c = 4$

Vertex

Use $-\frac{b}{2a}$ to find the x-coordinate of the vertex. $-\frac{(-9)}{2(2)} = \frac{9}{4}$; $f\left(\frac{9}{4}\right) = -\frac{49}{8}$

Then use $f\left(-\frac{b}{2a}\right)$ to find the y-coordinate. vertex: $\left(\frac{9}{4}, -\frac{49}{8}\right)$

y-intercept

Evaluate $f(x)$ at $x = 0$. $f(0) = 4$, so y-intercept is 4.

x-intercepts

Let $f(x) = 0$. $2x^2 - 9x + 4 = 0$

Then solve for x by factoring or by using the Quadratic Formula. $x = \frac{1}{2}$ and $x = 4$ are solutions, so x-intercepts are $\frac{1}{2}$ and 4.

Axis of Symmetry

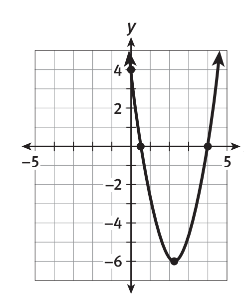
Find the vertical line through the vertex, $x = -\frac{b}{2a}$. $x = \frac{9}{4}$

Graph

Graph the points identified above: vertex, point on y-axis, points on x-axis.

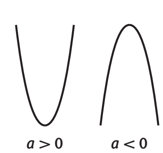
Then draw the smooth curve of a parabola through the points.

The y-coordinate of the vertex represents the minimum value of the function. The minimum value is $-\frac{49}{8}$.



MATH TIP

The graph of the function $f(x) = ax^2 + bx + c$ will open upward if $a > 0$ and will open downward if $a < 0$.



If the parabola opens up, then the y-coordinate of the vertex is the *minimum* value of the function. If it opens down, the y-coordinate of the vertex is the *maximum* value of the function.

My Notes

Try These A

For each quadratic function, identify the vertex, the y-intercept, the x-intercept(s), and the axis of symmetry. Then graph the function and classify the vertex as a maximum or minimum.

a. $f(x) = x^2 - 4x - 5$

vertex: $(2, -9)$

y-intercept: -5

x-intercepts: $-1, 5$

axis of symmetry: $x = 2$

vertex is a minimum

b. $f(x) = -3x^2 + 8x + 16$

vertex: $\left(\frac{4}{3}, \frac{64}{3}\right)$

y-intercept: 16

x-intercepts: $-\frac{4}{3}, 4$

axis of symmetry: $x = \frac{4}{3}$

vertex is a maximum

c. $f(x) = 2x^2 + 8x + 3$

vertex: $(-2, -5)$

y-intercept: 3

x-intercepts: $-2 - \frac{1}{2}\sqrt{10}, -2 + \frac{1}{2}\sqrt{10}$

axis of symmetry: $x = -2$

vertex is a minimum

d. $f(x) = -x^2 + 4x - 7$

vertex: $(2, -3)$

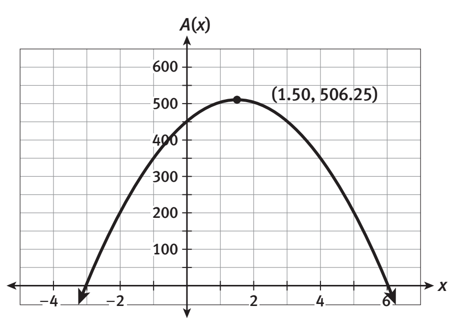
y-intercept: -7

x-intercepts: none

axis of symmetry: $x = 2$

vertex is a maximum

Consider the calendar fund-raising function from Lesson 12-1, Item 5, $A(x) = -25x^2 + 75x + 450$, whose graph is below.



1. **Make sense of problems.** Suppose that Ms. Picasso raises \$450 in the calendar sale. By how much did she increase the price? Explain your answer graphically and algebraically.

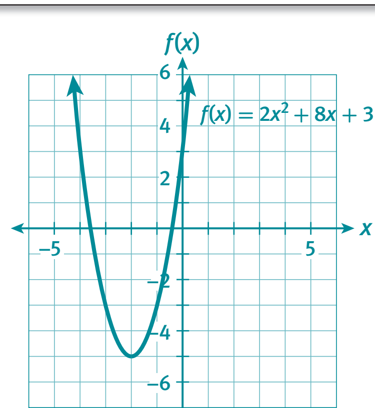
   Price increase is either \$0 (represented by the y-intercept) or \$3. Algebraically, the solutions are found by solving the equation $450 = -25x^2 + 75x + 450$. Graphically, they are the x-coordinates of the two points on the graph that have a y-coordinate of 450.

MATH TIP

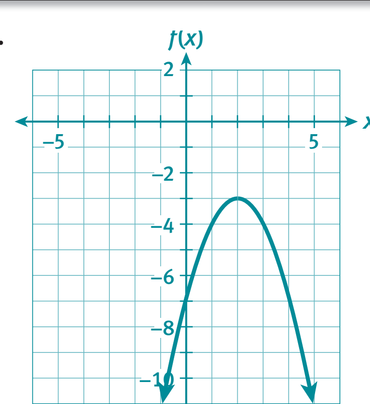
Quadratic equations may be solved by algebraic methods such as factoring or the Quadratic Formula.

An equation can be solved on a graphing calculator by entering each side of the equation as a function, graphing both functions, and finding the points of intersection. The x-coordinates of the intersection points are the solutions.

c.



d.

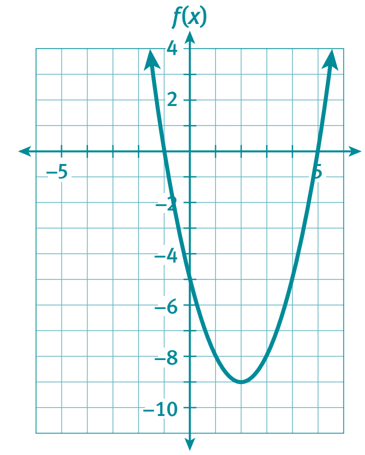


ACTIVITY 12 Continued

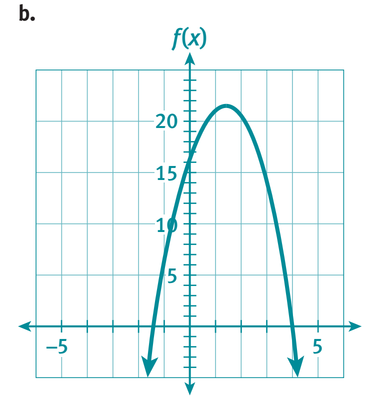
Try These A

Answers

a.



b.



Differentiating Instruction

Some students may confuse the maximum or minimum value of the quadratic function with the x-coordinate of the vertex. Emphasize that the maximum or minimum value is actually the y-coordinate that corresponds with the x-coordinate of $-\frac{b}{2a}$.

1–4 Identify a Subtask, Create Representations, Quickwrite, Debriefing At first, for Item 1, students may think that the \$450 amount is only possible with a \$0 increase. Guide students toward a graphic solution to point out that there are two possible solutions.

1–4 (continued) Item 2 provides students an opportunity to identify situations where there are no solutions both graphically and algebraically. Item 3 provides students an opportunity to identify situations where there is one solution both graphically and algebraically. Item 4 provides students an opportunity to identify situations where there are two solutions both graphically and algebraically.

My Notes

2. Suppose Ms. Picasso wants to raise \$600. Describe why this is not possible, both graphically and algebraically.
Raising \$600 is not possible because solutions to the equation $600 = -25x^2 + 75x + 450$ are complex. Graphically, no point on the graph has a *y*-coordinate of 600.

3. In Lesson 12-1, Item 8, you found that the maximum amount of money that could be raised was \$506.25. Explain both graphically and algebraically why this is true for only one possible price increase.
Solving the equation $506.25 = -25x^2 + 75x + 450$ yields $(2x - 3)^2 = 0$ with only one solution, $x = 1.5$. Graphically, there is only one point on the graph with this *x*-value.

4. Reason quantitatively. What price increase would yield \$500 in the calendar sale? Explain how you determined your solution.
The price increase that will yield \$500 in the calendar sale is either \$1 or \$2. This can be solved either algebraically or graphically.

Lesson 12-3
Graphing Quadratic Functions

Check Your Understanding

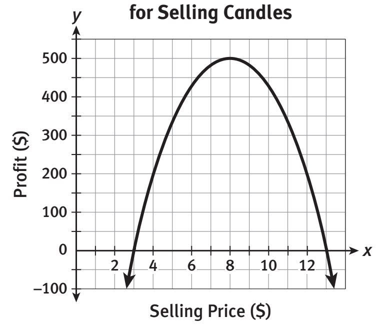
5. **Make use of structure.** If you are given the equation of a quadratic function in standard form, how can you determine whether the function has a minimum or maximum?
6. Explain how to find the x-intercepts of the quadratic function $f(x) = x^2 + 17x + 72$ without graphing the function.
7. Explain the relationships among these features of the graph of a quadratic function: the vertex, the axis of symmetry, and the minimum or maximum value.

My Notes

LESSON 12-3 PRACTICE

Recall that the function $P(x) = -20x^2 + 320x - 780$ models the profit the art club would make by selling candles for x dollars each. The graph of the function is below.

Profit Model for Selling Candles



8. Based on the model, what selling price(s) would result in a profit of \$320? Explain how you determined your answer.
9. **Construct viable arguments.** Could the club make \$600 in profit by selling candles? Justify your answer both graphically and algebraically.
10. If the club sells the candles for \$6 each, how much profit can it expect to make? Explain how you determined your answer.

For each function, identify the vertex, y-intercept, x-intercept(s), and axis of symmetry. Graph the function. Identify whether the function has a maximum or minimum and give its value.

11. $f(x) = -x^2 + x + 12$
12. $g(x) = 2x^2 - 11x + 15$

LESSON 12-3 PRACTICE

8. \$5 and \$11; Sample explanation: Set $P(x)$ equal to 320: $320 = -20x^2 + 320x - 780$. Subtract 320 from both sides to get $0 = -20x^2 + 320x - 1100$. Factor: $0 = -20(x - 5)(x - 11)$. So, $x = 5$ or $x = 11$, which means that a selling price of \$5 or \$11 will result in a profit of \$320.
9. No. Graphically: The graph shows that the vertex of the profit function is (8, 500), so the maximum profit the club can earn is \$500. Algebraically: Set $P(x)$ equal to 600: $600 = -20x^2 + 320x - 780$. Subtract 600 from both sides to write the equation in standard form: $0 = -20x^2 + 320x - 1380$. Use the Quadratic Formula to solve for x, which shows that $x = 8 \pm i\sqrt{5}$. Because the equation has complex solutions, there is no real value of x that results in a profit of \$600.
10. \$420. Sample explanation: Evaluate $P(x)$ for $x = 6$: $P(6) = -20(62) + 320(6) - 780 = 420$. The club can expect to make a profit of \$420 if it sells the candles for \$6 each.
11. Vertex is $\left(\frac{1}{2}, \frac{49}{4}\right)$; y-intercept is 12; x-intercepts are -3 and 4; axis of symmetry is $x = \frac{1}{2}$; maximum value is $\frac{49}{4}$. Check students' graphs.
12. Vertex is $\left(\frac{11}{4}, -\frac{1}{8}\right)$; y-intercept is 15; x-intercepts are $\frac{5}{2}$ and 3; axis of symmetry is $x = \frac{11}{4}$; minimum value is $-\frac{1}{8}$. Check students' graphs.

ACTIVITY 12 Continued

Check Your Understanding

Debrief students' answers to these items to ensure that they understand concepts related to key features of quadratic functions.

Answers

5. Look at the coefficient of the x^2-term. If the coefficient is positive, the graph of the function opens upward, and the function has a minimum. If the coefficient is negative, the graph of the function opens downward, and the function has a maximum.
6. Set $f(x) = 0$. Then solve the resulting equation, $0 = x^2 + 17x + 72$, for x. The right side of the equation can be factored: $0 = (x + 9)(x + 8)$, and the solutions are $x = -9$ and $x = -8$, which means that the x-intercepts of the function are -9 and -8.
7. The vertex of a quadratic function $f(x) = ax^2 + bx + c$ is given by $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$. The axis of symmetry is the vertical line through the vertex, so the x-coordinate of the vertex can be used to determine the equation of the axis of symmetry. Thus, the axis of symmetry is the line $x = -\frac{b}{2a}$. The minimum or maximum value is the value of the function at the vertex, given by the y-coordinate of the vertex. Thus, the minimum or maximum value is $f\left(-\frac{b}{2a}\right)$.

ASSESS

Students' answers to Lesson Practice problems will provide you with a formative assessment of their understanding of the lesson concepts and their ability to apply their learning.

See the Activity Practice for additional problems for this lesson. You may assign the problems here or use them as a culmination for the activity.

ADAPT

Check students' answers to the Lesson Practice to ensure that they understand how to graph a quadratic function. Remind students to first find the vertex and intercepts, and determine if the vertex is a maximum or a minimum. Students should also be able to use the graph to answer questions about the function. For students who need extra help, pair them with a partner who is proficient at the task to complete additional practice problems.

Lesson 12-4

PLAN

Pacing: 1 class period

Chunking the Lesson

Discussion: the Discriminant
Check Your Understanding
Lesson Practice

TEACH

Bell-Ringer Activity

Review the *discriminant*, the expression $b^2 - 4ac$ under the radical symbol of the quadratic formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, as a means of determining the number and nature of the solution(s) to its corresponding equation $ax^2 + bx + c = 0$. This will refresh students' memory of Activity 9 prior to extending the use of the discriminant to x-intercepts in this lesson.

Have students find the discriminant of the following quadratic equations and state the number and nature of their solutions.

1. $x^2 + 5x - 3 = 0$ [*discriminant* $= 37$; $37 > 0$, *therefore the solutions are real; furthermore, since 37 is not a perfect square, these real solutions will be irrational.*]
2. $4x^2 - 12x + 9 = 0$ [*discriminant* $= 0$; *There is one real, rational solution.*]
3. $x^2 + 5x + 8 = 0$ [*discriminant* $= -7$; $-7 < 0$, *therefore the solutions are imaginary complex conjugates.*]

Discuss results with students as a springboard into this lesson.

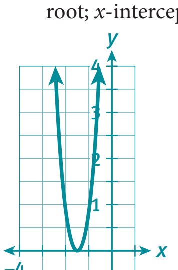
Activating Prior Knowledge, Debriefing The concept of discriminants and their relation to the nature of the roots of a quadratic equation was discussed previously in Activity 9. In this table, those same concepts are connected to the x-intercepts of the graphs of the related quadratic function.

Check Your Understanding

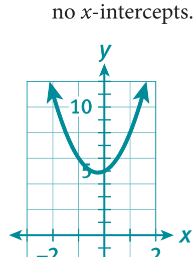
Debrief students' answers to these items to ensure that they understand concepts related to the discriminant and the nature of the solutions of quadratic equations.

Answers

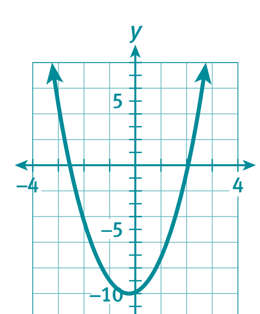
1. Discriminant is 0; 1 real (double) root; x-intercept is -1.5.



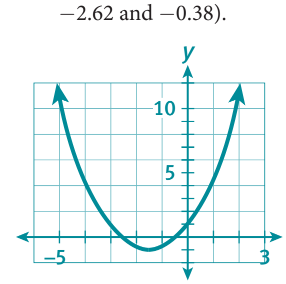
2. Discriminant is -39; no real roots (complex conjugates); no x-intercepts.



3. Discriminant is 81; 2 real, rational roots; x-intercepts are -2.5 and 2.



4. Discriminant is 5; 2 real, irrational roots; x-intercepts are $-\frac{3}{2} \pm \frac{\sqrt{5}}{2}$ (approximately -2.62 and -0.38).



My Notes

MATH TIP

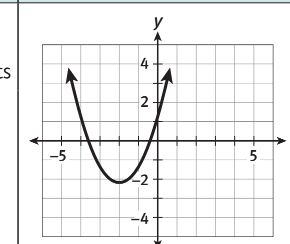
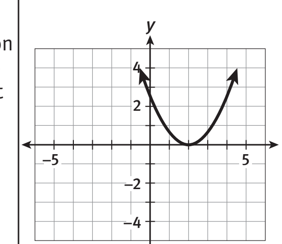
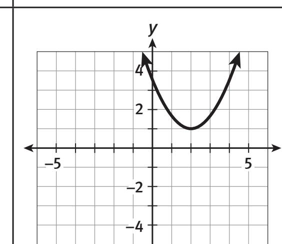
The *x-intercepts* of a quadratic function $y = ax^2 + bx + c$ are the ***zeros*** of the function. The solutions of a quadratic equation $ax^2 + bx + c = 0$ are the ***roots*** of the equation.

Learning Targets:

- Use the discriminant to determine the nature of the solutions of a quadratic equation.
- Use the discriminant to help graph a quadratic function.

SUGGESTED LEARNING STRATEGIES: Summarizing, Note Taking, Create Representations, Quickwrite, Self Revision/Peer Revision

The discriminant of a quadratic equation $ax^2 + bx + c = 0$ can determine not only the nature of the solutions of the equation, but also the number of x-intercepts of its related function $f(x) = ax^2 + bx + c$.

Discriminant of $ax^2 + bx + c = 0$	Solutions and x-intercepts	Sample Graph of $f(x) = ax^2 + bx + c$
$b^2 - 4ac > 0$ If $b^2 - 4ac$ is: • a perfect square • not a perfect square	• Two real solutions • Two x-intercepts • roots are rational • roots are irrational	
$b^2 - 4ac = 0$	• One real, rational solution (a double root) • One x-intercept	
$b^2 - 4ac < 0$	• Two complex conjugate solutions • No x-intercepts	

My Notes

Check Your Understanding

For each equation, find the value of the discriminant and describe the nature of the solutions. Then graph the related function and find the x-intercepts.

1. $4x^2 + 12x + 9 = 0$

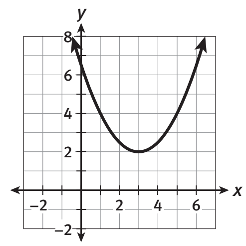
2. $2x^2 + x + 5 = 0$

3. $2x^2 + x - 10 = 0$

4. $x^2 + 3x + 1 = 0$

5. **Reason abstractly.** How can calculating the discriminant help you decide whether to use factoring to solve a quadratic equation?

6. The graph of a quadratic function $f(x)$ is shown at right. Based on the graph, what can you conclude about the value of the discriminant and the nature of the solutions of the related quadratic equation? Explain.



LESSON 12-4 PRACTICE

7. A quadratic equation has two rational solutions. How many x-intercepts does the graph of the related quadratic function have? Explain your answer.

8. **Make sense of problems.** The graph of a quadratic function has one x-intercept. What can you conclude about the value of the discriminant of the related quadratic equation? Explain your reasoning.

9. A quadratic equation has two irrational roots. What can you conclude about the value of the discriminant of the equation?

For each equation, find the value of the discriminant and describe the nature of the solutions. Then graph the related function and find the x-intercepts.

10. $x^2 - 4x + 1 = 0$

11. $x^2 - 6x + 15 = 0$

12. $4x^2 + 4x + 1 = 0$

13. $x^2 - 2x - 15 = 0$

5. If the discriminant is both positive and a perfect square, then the roots of the equation are rational, so the equation can be solved by factoring.

6. The graph of the function has no x-intercepts, so the solutions of the related quadratic equation are complex conjugates and the discriminant is negative.

ASSESS

Students' answers to Lesson Practice problems will provide you with a formative assessment of their understanding of the lesson concepts and their ability to apply their learning.

See the Activity Practice for additional problems for this lesson. You may assign the problems here or use them as a culmination for the activity.

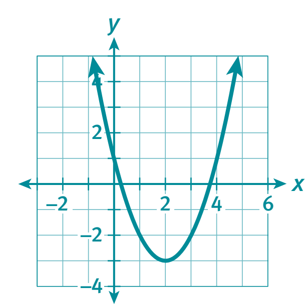
LESSON 12-4 PRACTICE

7. 2 x-intercepts; The graph of a quadratic function has an x-intercept for each real solution of the related quadratic equation. Rational solutions are real, so if a quadratic equation has 2 rational solutions, the graph of its related function will have 2 x-intercepts.

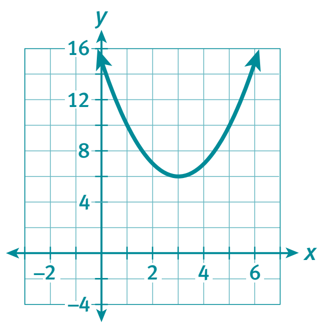
8. The discriminant is 0. If the graph of a quadratic function has 1 x-intercept, then its related quadratic equation has 1 real solution. If a quadratic equation has 1 real solution, then its discriminant is 0.

9. The discriminant is positive and not a perfect square.

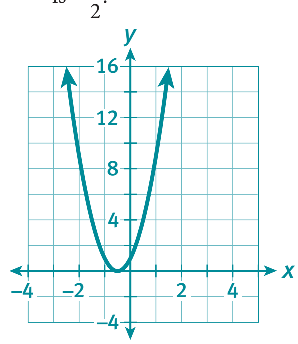
10. Discriminant is 12; 2 real, irrational roots; x-intercepts are $2 \pm \sqrt{3}$ (approximately 0.27 and 3.73).



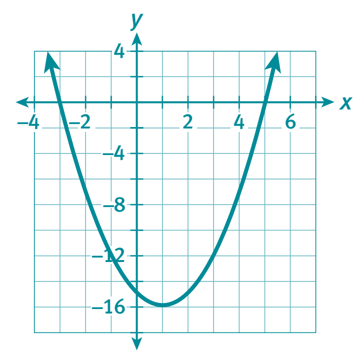
11. Discriminant is -24; 2 complex conjugate roots; no x-intercepts.



12. Discriminant is 0; 1 real, rational root (a double root); x-intercept is $-\frac{1}{2}$.



13. Discriminant is 64; 2 rational roots; x-intercepts are -3 and 5.



ADAPT

Check students' answers to the Lesson Practice to ensure that they understand how to use the discriminant to determine the nature of the roots of an equation. Students can make a graphic organizer to display the information and refer to the organizer until they are proficient in the concept.

Lesson 12-5

PLAN

Pacing: 1 class period

Chunking the Lesson

Example A
Check Your Understanding
Lesson Practice

TEACH

Bell-Ringer Activity

Remind students that the solutions of an inequality in two variables are all the ordered pairs that make the inequality a true statement. Have the students prepare for quadratic inequalities by completing the following items regarding the linear inequality $3x + 2y > 15$.

1. Is (2, 5) a solution? [*yes*]
2. Will its graph be a solid or dotted line? [*dotted*]
3. Will the shaded region be above or below the line? [*above*]

Example A Activating Prior Knowledge, Create Representations, Debriefing When solving quadratic inequalities, it may be helpful to recall and compare the process to the process for solving linear inequalities. You can reinforce the reason for solving by graphing by choosing several points from the shaded area and testing them in the inequality.

My Notes

Learning Targets:

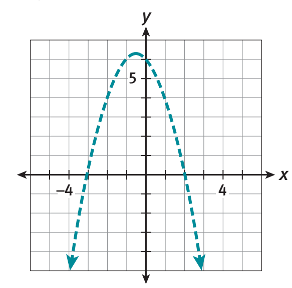
- Graph a quadratic inequality in two variables.
- Determine the solutions to a quadratic inequality by graphing.

SUGGESTED LEARNING STRATEGIES: Marking the Text, Create Representations, Guess and Check, Think-Pair-Share, Quickwrite

The solutions to quadratic inequalities of the form $y > ax^2 + bx + c$ or $y < ax^2 + bx + c$ can be most easily described using a graph. An important part of solving these inequalities is graphing the related quadratic functions.

Example A

Solve $y > -x^2 - x + 6$.



Graph the related quadratic function $y = -x^2 - x + 6$.

If the inequality symbol is $>$ or $<$, use a *dotted curve*.

If the symbol is $\geq$ or $\leq$, then use a *solid curve*.

This curve divides the plane into two regions.

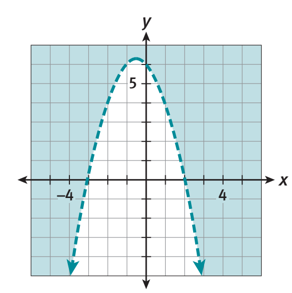
Test (0, 0) in $y > -x^2 - x + 6$.

$0 > -0^2 - 0 + 6$

$0 > 6$ is a false statement.

Choose a point on the plane, but not on the curve, to test.

(0, 0) is an easy point to use, if possible.



If the statement is true, shade the region that contains the point. If it is false, shade the other region.

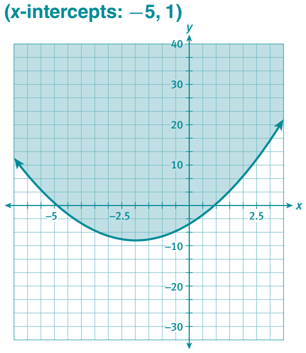
The shaded region represents all solutions to the quadratic inequality.

Lesson 12-5
Graphing Quadratic Inequalities

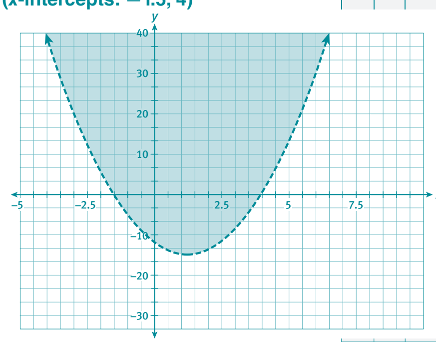
Try These A

Solve each inequality by graphing.

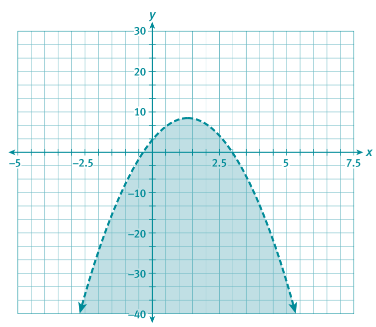
a. $y \geq x^2 + 4x - 5$
(x-intercepts: $-5, 1$)



b. $y > 2x^2 - 5x - 12$
(x-intercepts: $-1.5, 4$)



c. $y < -3x^2 + 8x + 3$

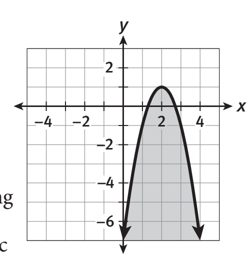


(x-intercepts: $-\frac{1}{3}, 3$)

My Notes

Check Your Understanding

1. The solutions of which inequality are shown in the graph?



A. $y \leq -2x^2 + 8x - 7$
B. $y \geq -2x^2 + 8x - 7$
C. $y \leq 2x^2 - 8x - 7$
D. $y \geq 2x^2 - 8x - 7$

2. Reason abstractly. How does graphing a quadratic inequality in two variables differ from graphing the related quadratic function?

3. Graph the quadratic inequality $y \geq -x^2 - 6x - 13$. Then state whether each ordered pair is a solution of the inequality.

a. $(-1, -6)$ **b.** $(-4, -8)$ **c.** $(-6, -10)$ **d.** $(-2, -5)$

ACTIVITY 12 Continued

Differentiating Instruction

Students should use the same basic steps to graph quadratic inequalities that they used for linear inequalities.

- Graph the equation.
- Use a dotted line for $<$ or $>$ symbols.
- Use a solid line for $\leq$ or $\geq$ symbols.
- Test a point in the coordinate plane that is *not* on the curve of the parabola. If the point you select is a solution, shade the region accordingly.

Hint: it may be helpful to get in the habit of always choosing a test point from inside the parabola. In the event the test point is a solution, automatically shade inside the parabola. On the other hand, if the test point is not a solution, automatically shade outside the parabola.

Check Your Understanding

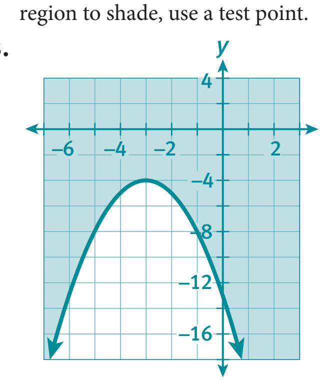
Debrief students' answers to these items to ensure that they understand concepts related to solving quadratic inequalities.

Answers

1. A

2. Sample answer: To graph a quadratic inequality in 2 variables, start by graphing the related quadratic function, but use a dotted curve for the parabola if the inequality symbol is $<$ or $>$. Otherwise, use a solid curve. You must also shade the region inside the parabola or outside the parabola when graphing a quadratic inequality. To decide which region to shade, use a test point.

3.



a. solution
b. not a solution
c. solution
d. solution

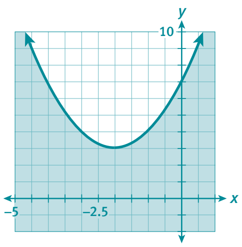
ASSESS

Students' answers to Lesson Practice problems will provide you with a formative assessment of their understanding of the lesson concepts and their ability to apply their learning.

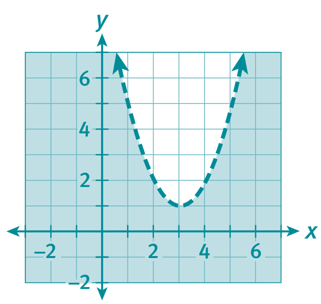
See the Activity Practice for additional problems for this lesson. You may assign the problems here or use them as a culmination for the activity.

LESSON 12-5 PRACTICE

4.



5.



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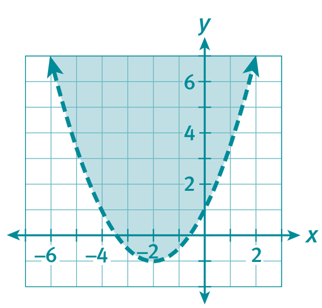

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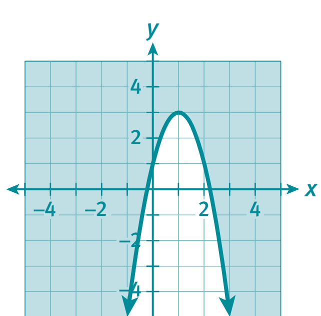


ADAPT

Check students' answers to the Lesson Practice to ensure that they understand how to solve a quadratic inequality. Encourage students to check several test points to make sure they shaded the correct region. They should check a test point in the solution region, outside the solution region, and on the boundary.

My Notes

LESSON 12-5 PRACTICE

Graph each inequality.

4. $y \le x^2 + 4x + 7$

5. $y < x^2 - 6x + 10$

6. $y > \frac{1}{2}x^2 + 2x + 1$

7. $y \ge -2x^2 + 4x + 1$

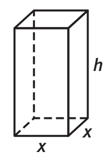
8. Construct viable arguments. Give the coordinates of two points that are solutions of the inequality $y \le x^2 - 6x + 4$ and the coordinates of two points that are not solutions of the inequality. Explain how you found your answers.

9. Model with mathematics. The students in Ms. Picasso's art club decide to sell candles in the shape of square prisms. The height of each candle will be no more than 10 cm. Write an inequality to model the possible volumes in cubic centimeters of a candle with a base side length of x cm.

10. Make sense of problems. Brendan has 400 cm^3 of wax. Can he make a candle with a base side length of 6 cm that will use all of the wax if the height is limited to 10 cm? Explain your answer using your inequality from Item 9.

CONNECT TO GEOMETRY

A square prism has two square bases. The volume of a square prism is equal to the area of one of its bases times its height.



8. Sample answer: Solutions: (0, 0) and (3, −6); not solutions: (3, 0) and (1, 5). Sample explanation: On a graph of the inequality, if a point is in the shaded region, it is a solution; if it is in the unshaded region, it is not a solution.

9. $y \le 10x^2$. Sample explanation: The volume y is determined by multiplying the height by the area of the base, x^2. The maximum height is 10 cm, so the volume is modeled by $y \le 10x^2$.

10. No; Sample explanation: Graph the inequality that represents the possible volumes and observe that (6, 400) is not a solution of the inequality. So, Brendan cannot make a candle with a base side length of 6 cm that will use 400 cm^3 of wax.

ACTIVITY 12 PRACTICE

Write your answers on notebook paper. Show your work.

Lesson 12-1

The cost of tickets to a whale-watching tour depends on the number of people in the group. For each additional person, the cost per ticket decreases by \$1. For a group with only two people, the cost per ticket is \$44. Use this information for Items 1–7.

1. Complete the table below to show the relationship between the number of people in a group and the cost per ticket.

Number of People	Cost per Ticket (\$)
2	
3	
4	
5	

2. Use the data in the table to write an expression that models the cost per ticket in terms of x, the number of people in a group.

3. Write a quadratic function in standard form that models $T(x)$, the total cost of the tickets for a group with x people.

4. Graph $T(x)$ on a coordinate grid.

5. **a.** For what values of x does the value of $T(x)$ increase as you move from left to right on the graph?

b. For what values of x does the value of $T(x)$ decrease as you move from left to right on the graph?

6. What is the vertex of the graph of $T(x)$? What do the coordinates of the vertex represent in this situation?

7. Groups on the tour are limited to a maximum size of 20 people. What is the total cost of the tickets for a group of 20 people? Explain how you found your answer.

Write each quadratic function in standard form and identify the vertex.

8. $f(x) = (4x - 4)(x + 5)$

9. $f(x) = 4(x + 8)(10 - x)$

Lesson 12-2

Mr. Gonzales would like to create a playground in his backyard. He has 20 ft of fencing to enclose the play area. Use this information for Items 10–13.

10. Write a quadratic function in standard form that models $f(x)$, the total area of the playground in square feet in terms of its width x in feet. Then graph $f(x)$.

11. Write the x- and y-intercepts of $f(x)$ and interpret them in terms of the problem.

12. Give the reasonable domain and range of $f(x)$ as inequalities, in interval notation, and in set notation. Explain how you determined the reasonable domain and range.

13. What is the maximum area for the playground? What are the dimensions of the playground with the maximum area?

Identify the x- and y-intercepts of each function.

14. $f(x) = x^2 + 3x - 28$

15. $f(x) = 2x^2 + 13x + 15$

Lesson 12-3

For each function, identify the vertex, y-intercept, x-intercept(s), and axis of symmetry. Identify whether the function has a maximum or minimum and give its value.

16. $f(x) = -x^2 + 4x + 5$

17. $f(x) = 2x^2 - 12x + 13$

18. $f(x) = -3x^2 + 12x - 9$

ACTIVITY PRACTICE

1.

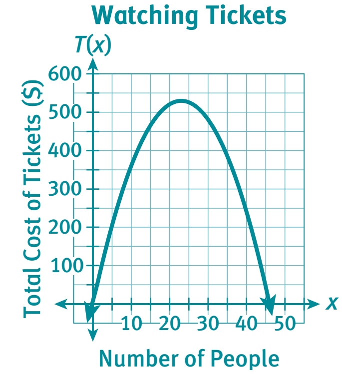
Number of People	Cost per Ticket (\$)
2	44
3	43
4	42
5	41

2. $46 - x$

3. $T(x) = -x^2 + 46x$

4.

Cost of Whale Watching Tickets



5. **a.** $x < 23$ **b.** $x > 23$

6. (23, 529); The total cost of tickets for a group of 23 people would be \$529.

7. \$520; Sample explanation: I evaluated $T(x)$ for $x = 20$: $T(20) = -20^2 + 46(20) = 520$.

8. $f(x) = 4x^2 + 16x - 20$; $(-2, -36)$

9. $f(x) = -4x^2 + 8x + 320$; $(1, 324)$

10. $f(x) = -x^2 + 10x$; Check students' graphs. Graphs should show a parabola with vertex at (5, 25) and x-intercepts at 0 and 10.

11. The x-intercepts of 0 and 10 represent the widths of the playground in feet that result in an area of 0 ft^2. The y-intercept of 0 represents an area of 0 ft^2 when the width is 0 ft.

12. Reasonable domain: $0 < x < 10$, $(0, 10)$, $\{x \mid x \in \mathbb{R}, 0 < x < 10\}$; reasonable range: $0 < y \le 25$, $(0, 25]$, $\{y \mid y \in \mathbb{R}, 0 < y \le 25\}$; Sample explanation: The area of the playground must be positive, and a graph shows that $f(x)$ is positive when its width x in feet is between 0 and 10, so the reasonable domain is $0 < x < 10$. The graph also shows that the maximum value of the area is 25 ft^2. So, the reasonable range of the function is $0 < y \le 25$.

13. maximum area: 25 ft^2; length = width = 5 ft

14. x-intercepts: -7 and 4; y-intercept: -28

15. x-intercepts: -5 and $-\frac{3}{2}$; y-intercept: 15

16. Vertex is (2, 9); y-intercept is 5; x-intercepts are -1 and 5; axis of symmetry is $x = 2$; maximum value is 9.

17. Vertex is $(3, -5)$; y-intercept is 13; x-intercepts are $3 \pm \frac{\sqrt{10}}{2}$; axis of symmetry is $x = 3$; minimum value is -5.

18. Vertex is (2, 3); y-intercept is -9; x-intercepts are 1 and 3; axis of symmetry is $x = 2$; maximum value is 3.

19. Sample explanation: At $x = 0$: $f(0) = 0^2 - 3(0) - 18 = -18$. The y-intercept is -18.

20. 0.8 s and 2.2 s after the arrow is shot; Sample explanation: Set $h(t)$ equal to 10: $10 = -5t^2 + 15t + 1$. Subtract 10 from both sides: $0 = -5t^2 + 15t - 9$. Then use the Quadratic Formula to solve for t.

21. Yes. Graphically: The parabola intersects the horizontal line $y = 12$ at 2 points in Quadrant I, representing 2 times at which the arrow has a height of 12 m. Algebraically: Set $h(t)$ equal to 12: $12 = -5t^2 + 15t + 1$. Subtract 12 from both sides: $0 = -5t^2 + 15t - 11$. Then use the Quadratic Formula to solve for t: $t \approx 1.3$ or $t \approx 1.7$. These are the 2 values of t for which the height of the arrow will be 12 m.

22. discriminant: 49; 2 rational roots; x-intercepts: $-\frac{1}{2}$ and 3

23. discriminant: -23; 2 complex conjugate roots; no x-intercepts

24. discriminant: 0; 1 real, rational root; x-intercept: $-\frac{1}{2}$

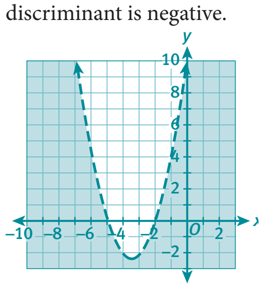
25. discriminant: 12; 2 real, irrational roots; x-intercepts: $-\frac{3}{2} \pm \frac{\sqrt{3}}{2}$

26. D

27. 1 x-intercept; The graph of a quadratic function has an x-intercept for each real solution, so if a quadratic equation has 1 rational solution, the graph of its related function will have 1 x-intercept.

28. The discriminant is negative. If the graph of a quadratic function has no x-intercepts, then its related quadratic equation has 2 complex conjugate solutions and its discriminant is negative.

29.



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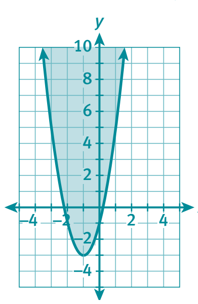

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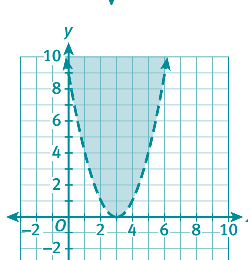


ADDITIONAL PRACTICE

If students need more practice on the concepts in this activity, see the Teacher Resources at SpringBoard Digital for additional practice problems.

ACTIVITY 12 *continued*

Graphing Quadratics and Quadratic Inequalities

Calendar Art

19. Explain how to find the y-intercept of the quadratic function $f(x) = x^2 - 3x - 18$ without graphing the function.

The function $h(t) = -5t^2 + 15t + 1$ models the height in meters of an arrow t seconds after it is shot. Use this information for Items 20 and 21.

20. Based on the model, when will the arrow have a height of 10 m? Round times to the nearest tenth of a second. Explain how you determined your answer.

21. Does the arrow reach a height of 12 m? Justify your answer both graphically and algebraically.

Lesson 12-4

For each equation, find the value of the discriminant and describe the nature of the solutions. Then find the x-intercepts.

22. $2x^2 - 5x - 3 = 0$

23. $3x^2 + x + 2 = 0$

24. $4x^2 + 4x + 1 = 0$

25. $2x^2 + 6x + 3 = 0$

26. A quadratic equation has two distinct rational roots. Which one of the following could be the discriminant of the equation?

A. -6 **B.** 0
C. 20 **D.** 64

27. A quadratic equation has one distinct rational solution. How many x-intercepts does the graph of the related quadratic function have? Explain your answer.

28. The graph of a quadratic function has no x-intercepts. What can you conclude about the value of the discriminant of the related quadratic equation? Explain your reasoning.

Lesson 12-5

Graph each quadratic inequality.

29. $y < x^2 + 7x + 10$

30. $y \geq 2x^2 + 4x - 1$

31. $y > x^2 - 6x + 9$

32. $y \leq -x^2 + 3x + 4$

33. Which of the following is a solution of the inequality $y > -x^2 - 8x - 12$?

A. $(-6, 0)$ **B.** $(-4, -2)$
C. $(-3, 1)$ **D.** $(-2, 4)$

The time in minutes a factory needs to make x cell phone parts in a single day is modeled by the inequality $y \leq -0.0005x^2 + x + 20$, for the domain $0 \leq x \leq 1000$. Use this information for Items 34–36.

34. a. Is the ordered pair (200, 100) a solution of the inequality? How do you know?
b. What does the ordered pair (200, 100) represent in this situation?

35. What is the longest it will take the factory to make 600 cell phone parts? Explain how you determined your answer.

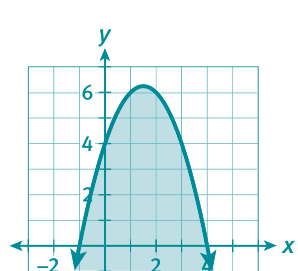
36. Can the factory complete an order for 300 parts in 4 hours? Explain.

37. Give the coordinates of two points that are solutions of the inequality $y \leq x^2 - 3x - 10$ and the coordinates of two points that are not solutions of the inequality. Explain how you found your answers.

MATHEMATICAL PRACTICES
Look for and Make Use of Structure

38. Describe the relationship between solving a quadratic equation and graphing the related quadratic function.

32.



33. D

34. a. Yes. Sample explanation: (200, 100) is a solution of the inequality.
b. The company can make 200 cell phone parts in 100 min.

35. 440 minutes (or 7 hours 20 minutes); Sample explanation: When $x = 600$, the solution with the greatest y-value is (600, 440).

36. Yes; Sample explanation: 4 hours is equal to 240 minutes. (300, 240) is a solution of the inequality.

37. Sample answer: Solutions: $(-2, -5)$ and $(5, -5)$; not solutions: (0, 0) and (2, 5); Sample explanation: Points in the shaded region of the graph of the inequality represent solutions, and points in the unshaded region are not solutions.

38. Sample answer: When you solve a quadratic equation, the values of any real solutions are equal to the x-intercepts of the graph of the related quadratic function.

Systems of Linear and Nonlinear Equations

ACTIVITY 13

Supply and Demand

Lesson 13-1 Solving a System Graphically

My Notes

Learning Targets:

- Use graphing to solve a system consisting of a linear and a nonlinear equation.
- Interpret the solutions of a system of equations.

SUGGESTED LEARNING STRATEGIES: Close Reading, Think Aloud, Discussion Groups, Create Representations, Look for a Pattern

The owner of Salon Ultra Blue is working with a pricing consultant to determine the best price to charge for a basic haircut. The consultant knows that, in general, as the price of a haircut at a salon goes down, demand for haircuts at the salon goes up. In other words, if Salon Ultra Blue decreases its prices, more customers will want to get their hair cut there.

Based on the consultant's research, customers will demand 250 haircuts per week if the price per haircut is \$20. For each \$5 increase in price, the demand will decrease by 25 haircuts per week.

1. Let the function $f(x)$ model the quantity of haircuts demanded by customers when the price of haircuts is x dollars.

a. Reason quantitatively. What type of function is $f(x)$? How do you know?

The function is linear. Sample explanation: The function has a constant rate of change. For each increase of 5 in the value of x, the value of $f(x)$ decreases by 25.

b. Write the equation of $f(x)$.

$f(x) = -5x + 350$ or equivalent

The price of a haircut not only affects demand, but also affects supply. As the price charged for a haircut increases, cutting hair becomes more profitable. More stylists will want to work at the salon, and they will be willing to work longer hours to provide more haircuts.

CONNECT TO ECONOMICS

In economics, *demand* is the quantity of an item that customers are willing to buy at a particular price. The *law of demand* states that as the price of an item decreases, the demand for the item tends to increase.

CONNECT TO ECONOMICS

Supply is the quantity of an item that businesses are willing to sell at a particular price. The *law of supply* states that as the price of an item increases, the supply of the item tends to increase.

Common Core State Standards for Activity 13

HSA-REI.D.11 Explain why the x-coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where $f(x)$ and/or $g(x)$ are linear, polynomial, absolute value.

ACTIVITY 13

Guided

Activity Standards Focus

In Activity 13, students solve systems of equations that include a linear and nonlinear equation. First they look at solutions graphically and then transition to algebraic solution methods. Throughout this activity, emphasize whether solutions are reasonable.

Lesson 13-1

PLAN

Pacing: 1 class period

Chunking the Lesson

#1 #2–3 #4–5
#6–7 #8 #9–12
#13–15
Check Your Understanding
Lesson Practice

TEACH

Bell-Ringer Activity

Remind students that the way that they can determine whether a function is linear or nonlinear (without the benefit of the actual equation) is by its rate of change. If the rate of change is constant (slope), then the graph of the function is a line. If the rate of change is not constant, then the graph of the function is nonlinear. Have students identify which functions below are linear.

1. $y = x^2 + 7$ [*not linear*]

2. $y = 3x - 1$ [*linear*]

3. $y = 12$ [*linear*]

1 Debriefing Help students with Item 1b. Explain to them that since it is known that this is a linear function, they can write its corresponding equation if they have a minimum of two points through which the line passes. Since the problem states 250 haircuts are demanded at a price of \$20, the point (20, 250) is on the line. Since the problem also states that for each price increase of \$5, the number of haircuts decreases by 25, they can use this information to find a second point on the line, such as (25, 225). Using these two points, find the slope, $m = -5$. Using the point-slope form of a line, $y - 250 = -5(x - 20)$. Simplify to $y = -5x + 350$, or $f(x) = -5x + 350$.

2–3 Predict and Confirm, Activating Prior Knowledge, Debriefing Have students predict what they think the function in Item 2 might look like. The easiest way to find the equation in Item 2 is to use a graphing calculator. In doing so, students will enter all data from the table into two lists and utilize the quadratic regression function. For additional technology resources, visit SpringBoard Digital.
If a student chooses not to use a graphing calculator, then he or she will have to write three equations in three variables with 3 out of the 4 ordered pairs from the table. Then they can proceed by using the substitution and elimination methods to simultaneously solve a system of 3 equations in 3 variables. Caution students that there are many places to make an error in this process. In Item 3, students simply have to write their functions from Items 1 and 2 as equations in terms of y.

4–5 Chunking the Activity, Activating Prior Knowledge, Discussion Groups, Group Presentation For Item 4, have students work with a partner to graph both of these equations on a coordinate plane. (You may need to help get them off to a good start, by giving them some minimum and maximum values with which to label the axes).

Differentiating Instruction

Some students may be struggling because they have learned several ways to graph quadratic functions. Since this graph is given in standard form, it will probably be easier for them to use the key elements of a quadratic equation in standard form. Some students may need a quick review of the following:

- Whether the graph opens up or down, based upon the value of a.
- Whether the graph is narrower or wider than $y = x^2$, based on the value of a.
- The axis of symmetry is $x = -\frac{b}{2a}$, and the vertex has an x-coordinate of $-\frac{b}{2a}$.
- The y-intercept is c, so the point $(0, c)$ is on the parabola.

My Notes

The consultant gathered the following data on how the price of haircuts affects the number of haircuts the stylists are willing to supply each week.

Supply of Haircuts

Price per Haircut ($)	Number of Haircuts Available per Week
20	15
30	55
40	115
50	195

CONNECT TO TECHNOLOGY

One way to write the equation of the quadratic function is to perform a quadratic regression on the data in the table. See Activity 10 for more information.

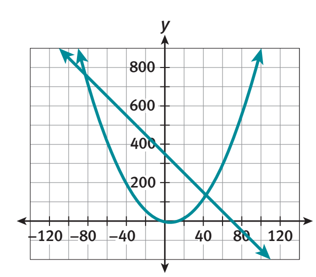
2. The relationship shown in the table is quadratic. Write the equation of a quadratic function $g(x)$ that models the quantity of haircuts the stylists are willing to supply when the price of haircuts is x dollars.

$g(x) = \frac{1}{10}x^2 - x - 5$

3. **Model with mathematics.** Write a system of two equations in two variables for the demand and supply functions. In each equation, let y represent the quantity of haircuts and x represent the price in dollars per haircut.

$$\begin{cases} y = -5x + 350 \\ y = \frac{1}{10}x^2 - x - 5 \end{cases}$$

4. Graph the system on the coordinate plane.



Lesson 13-1
Solving a System Graphically

My Notes

5. Explain how you determine the location of the solutions on the graph in Item 4.
The intersection points of the graphs of the equations represent the solutions of the system.

6. Explain the relationship of the solution to the demand function $f(x)$ and the supply function $g(x)$.
The *x*-coordinates of the solutions represent prices at which the demand for haircuts, *f*(*x*), is equal to the supply of haircuts, *g*(*x*).

7. Use the graph to approximate the solutions of the system of equations.
Answers may vary but should be close to (−80, 750) and (40, 150).

Now use a graphing calculator to make better approximations of the solutions of the system of equations. First, enter the equations from the system as Y1 and Y2.

8. Use appropriate tools strategically. Now view a table showing values of X, Y1, and Y2.

a. How can you approximate solutions of a system of two equations in two variables by using a table of values on a graphing calculator?
Look for values of X for which Y1 ≈ Y2.

b. Use the table to approximate the solutions of the system. Find the coordinates of the solutions to the nearest integer.
(−83, 764) and (43, 136)

TECHNOLOGY TIP

You can change the table settings on a graphing calculator by pressing [2nd] and then the key with TblSet printed above it. The table start setting (TblStart) lets you change the first value of X displayed in the table. The table step setting (ΔTbl) lets you adjust the change in X between rows of the table.

4–5 (continued) For Item 5, explain to students that this is the same concept as when they found solutions to a system of linear equations. The only difference is that now one of the graphs is a parabola. Allow time to discuss the results of these items as a class.

6–7 Debriefing The x-axis represents the price of haircuts. The function $f(x)$ represents the demand of haircuts for a week, based upon the price. The function $g(x)$ represents the supply of haircuts for a week, based upon the price. Therefore, the x-values of the points where these two graphs intersect represent the two prices at which the demand and supply are equal. Have students round to the nearest 10 when they approximate solutions.

8 Predict and Confirm, Debriefing The students will take what they already know about the graphs of these two functions to find two ordered pairs where the y-values are closest to each other. When using the graphing calculator to view the table of values corresponding to these two functions, students should know that in the TblSet feature, the smaller the value of (ΔTbl), the more accurate their solutions will be. Since Item 8b is asking for coordinates to the nearest integer, students should set their (ΔTbl) to a value of 1. Based upon the graph, students should know approximately where they need to scroll within the list of values in order to find the integer solutions.

CONNECT TO AP

Note that in the Technology Tip there is a reference made to the table step setting on the graphing calculator as (ΔTbl). It further explains that this function allows you to adjust the change in X between rows of the table. The Greek letter delta (Δ) is frequently used in calculus. It represents the phrase "the change in." For example, Δx is interpreted as "the change in the value of the variable x."

9–12 Think-Pair-Share, Levels of Questions, Debriefing Have students individually answer Items 9–12. Emphasize the importance of thinking about the solutions. Are they reasonable? Do they make sense? To do this, students need to focus on what the axes represent. Remind students that the *x*-axis is the price of a haircut, and the *y*-axis is the number of haircuts within a week. Would it make sense for the cost of a haircut to be less than zero dollars? What does the point of intersection of these two graphs actually represent? Does the point of intersection seem reasonable? Does its *x*-coordinate represent a realistic price for a haircut? Does its *y*-coordinate represent a realistic number of haircuts that a salon could complete in one week? After students answer these items, have them confer with a partner, sharing results as a class.

Technology Tip

For those students still having trouble finding the intersection of these two graphs using their graphing calculator, tell them to press the [2nd] and [TRACE] buttons, access the [CALC] function, and select option 5 (intersect). Once they do this, a small icon will appear on the graph. Now they should use the arrow keys to move the cursor to where the graphs intersect. Once the cursor is blinking close to the point of intersection, they should press the [ENTER] key 3 times. This will give them the exact point of intersection out to five decimal places. Remind students that they will have to perform this process twice, as there are two places where the graphs intersect.

For additional technology resources, visit SpringBoard Digital.

My Notes

TECHNOLOGY TIP

To use the intersect feature on a graphing calculator, access the calculate menu by pressing [2nd] and then the key with Calc printed above it. Next, select **5: Intersect**, and then follow the instructions.

9. Next, view a graph of the system of equations on the graphing calculator. Adjust the viewing window as needed so that the intersection points of the graphs of the equations are visible. Then use the intersect feature to approximate the solutions of the system of equations.
(−82.84903, 764.24513) and (42.849025, 135.75487)

10. Explain why one of the solutions you found in Item 9 does not make sense in the context of the supply and demand functions for haircuts at the salon.
The variable *x* represents the price in dollars of a haircut, so it does not make sense in this situation for *x* to be negative. Therefore, the solution with the negative *x*-value should be ignored.

11. Make sense of problems. Interpret the remaining solution in the context of the situation.
The *x*-coordinate of the remaining solution shows that when haircuts are priced at about $42.85, the number of haircuts demanded by customers will equal the number of haircuts that the stylists are willing to supply. The *y*-coordinate of the solution shows that this number of haircuts is about 136 per week.

12. Explain why the solution you described in Item 11 is reasonable.
Sample answer: When I substitute 42.85 for *x* in each equation in the system, I get a value of *y* that is approximately equal to 136. In addition, a price of $42.85 for a haircut seems realistic. It also seems reasonable that a salon could give 136 haircuts in a week.

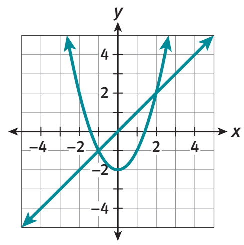
13. The pricing consultant recommends that Salon Ultra Blue price its haircuts so that the weekly demand is equal to the weekly supply. Based on this recommendation, how much should the salon charge for a basic haircut?
$42.85

Lesson 13-1
Solving a System Graphically

14. Model with mathematics. Graph each system of one linear equation and one quadratic equation. For each system, list the number of real solutions.

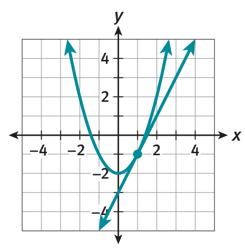
a. $\begin{cases} y = x \\ y = x^2 - 2 \end{cases}$

2 real solutions



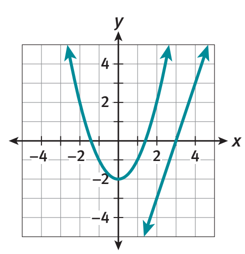
b. $\begin{cases} y = 2x - 3 \\ y = x^2 - 2 \end{cases}$

1 real solution



c. $\begin{cases} y = 3x - 9 \\ y = x^2 - 2 \end{cases}$

0 real solutions



15. Make a conjecture about the possible number of real solutions of a system of two equations that includes one linear equation and one quadratic equation.

A system of one linear equation and one quadratic equation may have 0, 1, or 2 real solutions.

My Notes

13–15 Predict and Confirm, Activating Prior Knowledge, Debriefing Prior to graphing the equations in Item 14, have students make a prediction with rough sketches on scrap paper as to how many different ways a line and a parabola can intersect. Also point out that in Item 14, the quadratic function is the same in all three examples. Remind students when graphing this quadratic equation not to overcomplicate things. The equation of $y = x^2 - 2$ is a simple shift from the parent function of $y = x^2$. You can let students check their graphs by using a graphing calculator. The coordinates of the points of intersection are not as important in this problem as the number of times the lines and parabolas intersect.

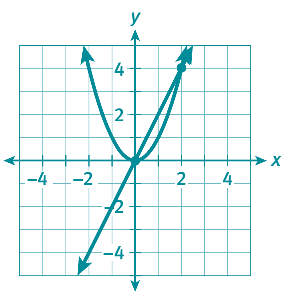
Check Your Understanding (p. 216)
Debrief students' answers to these items to ensure that they understand concepts related to solutions of a system of one linear equation and one quadratic equation.

Answers

16. The x-coordinate of the solution represents a price in dollars, so this value should be rounded to two decimal places (or to the nearest cent). The y-coordinate of the solution represents a number of haircuts. Because only a whole number of haircuts makes sense, the y-coordinate should be rounded to the nearest whole number.

17. No. A system of two equations in two variables has infinitely many solutions only if the graph of each equation is the same. The graph of a linear equation is a line and the graph of a quadratic equation is a parabola, so the equations in a system of one linear equation and one quadratic equation cannot have the same graph.

18. Answers may vary, but should represent a system of one linear equation and one quadratic equation whose graph has two intersection points on the same side of the vertex of the parabola, or one intersection point at the vertex and one intersection point elsewhere on the parabola. Sample answer: The graph of the system $\begin{cases} y = 2x \\ y = x^2 \end{cases}$ has one intersection point at the vertex of the parabola and one intersection point to the right of the vertex.



19. Sample answer: A graph allows you to identify how many real solutions the system has. It also lets you quickly estimate the coordinates of the real solutions. A table will give more exact values for the real solutions.

ASSESS

Students' answers to Lesson Practice problems will provide you with a formative assessment of their understanding of the lesson concepts and their ability to apply their learning.

See the Activity Practice for additional problems for this lesson. You may assign the problems here or use them as a culmination for the activity.

ADAPT

Check students' answers to the Lesson Practice to ensure that they understand how to model a problem using a system of one linear and one quadratic equation. Additionally, students should be able to solve the system and interpret the solutions. As additional practice, have each student make up three systems of equations: one that has no solution, one that has two solutions, and one that has exactly one solution. Students should then trade their systems with a partner to practice the solution process.

My Notes

ACADEMIC VOCABULARY

A ***counterexample*** is an example that demonstrates that a statement is not true.

Check Your Understanding

16. When interpreting the solution of the system in Item 11, how did you decide how to round the *x*- and *y*-coordinates of the solution?

17. Construct viable arguments. Can a system of a linear equation and a quadratic equation have infinitely many solutions? Explain your reasoning.

18. A student claims that if a system of a linear equation and a quadratic equation has two real solutions, then a graph of the system will have one intersection point to the left of the vertex of the parabola and one intersection point to the right of the vertex. Provide a ***counterexample*** to show that the student's claim is not correct.

19. Compare and contrast using a graph and a table to approximate the solution of a system of one linear equation and one quadratic equation.

LESSON 13-1 PRACTICE

The owner of Salon Ultra Blue also wants to set the price for styling hair for weddings, proms, and other formal events.

20. Make sense of problems. Based on the pricing consultant's research, customers will demand 34 formal hairstyles per week if the price per hairstyle is \$40. For each \$10 increase in price, the demand will decrease by 4 hairstyles per week. Write a linear function $f(x)$ that models the quantity of formal hairstyles demanded by customers when the price of the hairstyles is x dollars.

21. The table shows how the price of formal hairstyles affects the number the stylists are willing to supply each week. Write the equation of a quadratic function $g(x)$ that models the quantity of formal hairstyles the stylists are willing to supply when the price of hairstyles is x dollars.

Supply of Formal Hairstyles

Price per Hairstyle (\$)	Number Available per Week
40	3
50	9
60	17

22. Model with mathematics. Write a system of two equations in two variables for the demand and supply functions. In each equation, let y represent the quantity of formal hairstyles and x represent the price in dollars per hairstyle.

23. Approximate the solutions of the system by using a graph or table.

24. How much should the salon charge for a formal hairstyle so that the weekly demand is equal to the weekly supply? Explain how you determined your answer.

25. Explain why your answer to Item 24 is reasonable.

LESSON 13-1 PRACTICE

20. $f(x) = -\frac{2}{5}x + 50$

21. $g(x) = \frac{1}{100}x^2 - \frac{3}{10}x - 1$

22. $\begin{cases} y = -\frac{2}{5}x + 50 \\ y = \frac{1}{100}x^2 - \frac{3}{10}x - 1 \end{cases}$

23. Approximations should be close to (−76.59, 80.64) and (66.59, 23.36).

24. \$66.59; Sample explanation: The solution with a negative value for x, the price in dollars, does not make sense in this situation and should be ignored. The x-coordinate of the remaining solution shows that when formal hairstyles are priced at about \$66.59, the number of formal hairstyles demanded by customers will equal the number that the stylists are willing to supply.

25. Sample answer: When I substitute 66.59 for x into each equation in the system, I get a value of y that is approximately equal to 23. In addition, a price of \$66.59 for a formal hairstyle seems realistic.

Lesson 13-2
Solving a System Algebraically

ACTIVITY 13
continued

My Notes

Learning Targets:
- Use substitution to solve a system consisting of a linear and nonlinear equation.
- Determine when a system consisting of a linear and nonlinear equation has no solution.

SUGGESTED LEARNING STRATEGIES: Summarizing, Identify a Subtask, Think-Pair-Share, Drafting, Self Revision/Peer Revision

In the last lesson, you approximated the solutions to systems of one linear equation and one quadratic equation by using tables and graphs. You can also solve such systems algebraically, just as you did when solving systems of two linear equations.

Example A

The following system represents the supply and demand functions for basic haircuts at Salon Ultra Blue, where y is the quantity of haircuts demanded or supplied when the price of haircuts is x dollars. Solve this system algebraically to find the price at which the supply of haircuts equals the demand.

$$\begin{cases} y = -5x + 350 \\ y = \frac{1}{10}x^2 - x - 5 \end{cases}$$

Step 1: Use substitution to solve for x.

$y = -5x + 350$	The first equation is solved for y.
$-5x + 350 = \frac{1}{10}x^2 - x - 5$	Substitute for y in the second equation.
$0 = \frac{1}{10}x^2 + 4x - 355$	Write the equation in standard form.
$0 = x^2 + 40x - 3550$	Multiply both sides by 10 to eliminate the fraction.
$x = \frac{-40 \pm \sqrt{40^2 - 4(1)(-3550)}}{2(1)}$	Use the Quadratic Formula.
$x = -20 \pm 5\sqrt{158}$	
$x \approx -82.85$ or $x \approx 42.85$	

Step 2: Substitute each value of x into one of the original equations to find the corresponding value of y.

$y = -5x + 350$	$y = -5x + 350$
$y \approx -5(-82.85) + 350$	$y \approx -5(42.85) + 350$
$y \approx 764$	$y \approx 136$

MATH TIP

In this example, the exact values of x are irrational. Because x represents a price in dollars, use a calculator to find rational approximations of x to two decimal places.

Lesson 13-2

PLAN

Pacing: 1 class period

Chunking the Lesson

Example A #1–3
Check Your Understanding
Lesson Practice

TEACH

Bell-Ringer Activity

Have students solve each of the systems of linear equations by using the substitution method to help prepare them for solving a system of linear and nonlinear equations.

1. $\begin{cases} x + y = 10 \\ x = y + 8 \end{cases}$ [(9, 1)]
2. $\begin{cases} a - b = 32 \\ 3a - 8b = 6 \end{cases}$ [(50, 18)]
3. $\begin{cases} x + y = 8 \\ -x - y = -8 \end{cases}$ [*infinitely many solutions*]

Differentiating Instruction

Compare and contrast the possible number of real solutions between a system of two linear equations and a system of two equations that includes one linear equation and one quadratic equation.

Two linear equations
- may have one solution, because they intersect at one point;
- may have no solution, because they do not intersect (parallel lines);
- may intersect and have infinitely many solutions (same line).

One linear and one quadratic equation
- may have one real solution, because they intersect at one point where the line touches the parabola
- may have no solution, because they do not intersect
- may have two real solutions, because the line intersects the parabola twice

Example A Activating Prior Knowledge, Debriefing Explain that the benefit of multiplying both sides of the equation by 10 is to eliminate using fractions in the quadratic formula. This can get very messy and cause unnecessary arithmetic errors. Remind students that if there were more than one fraction present, they would multiply both sides of the equation by the least common denominator.

1–3 Activating Prior Knowledge, Chunking the Activity, Debriefing Be sure students understand that if they see a discriminant with a negative value, the solutions will be complex. Remind them that there are no "real" solutions, because while all real numbers are complex, all complex numbers are not real. Complex numbers consist of all the real numbers plus the imaginary numbers. When linear-quadratic solutions are complex, it is an indication that the graphs of the two equations do not intersect.

ELL Support

The meaning of the word *complex* that most students are familiar with is "complicated" or "difficult to understand." However, when it comes to mathematics, *complex* means "composed" or "made up of parts joined together." A complex number is made up of a real part and an imaginary part joined together, either of which can be zero.

Examples:

1. $5 + 7i$ Real part is 5; imaginary part is $7i$.
2. 9 Real part is 9; imaginary part is 0.
3. $3i$ Real part is 0; imaginary part is $3i$.

Ask students if they can think of any other real-life situations, aside from mathematics, where the word *complex* is used to mean "joining parts together."

Sample answer: In an apartment complex; because they are made up of apartments joined together

My Notes

Step 3: Write the solutions as ordered pairs.

The solutions are approximately $(-82.85, 764)$ and $(42.85, 136)$. Ignore the first solution because a negative value of x does not make sense in this situation.

Solution: The price at which the supply of haircuts equals the demand is \$42.85. At this price, customers will demand 136 haircuts, and the stylists will supply them.

Try These A

Solve each system algebraically. Check your answers by substituting each solution into one of the original equations. Show your work.

a. $\begin{cases} y = -2x - 7 \\ y = -2x^2 + 4x + 1 \end{cases}$

$(-1, -5), (4, -15)$

b. $\begin{cases} y = x^2 + 6x + 5 \\ y = 2x + 1 \end{cases}$

$(-2, -3)$

c. $\begin{cases} y = \frac{1}{2}(x + 4)^2 + 5 \\ y = \frac{17}{2} - x \end{cases}$

$\left(-1, \frac{19}{2}\right), \left(-9, \frac{35}{2}\right)$

d. $\begin{cases} y = -4x^2 + 5x - 8 \\ y = -3x - 24 \end{cases}$

$\left(1 - \sqrt{5}, -27 + 3\sqrt{5}\right), \left(1 + \sqrt{5}, -27 - 3\sqrt{5}\right)$

1. Use substitution to solve the following system of equations. Show your work.

$$\begin{cases} y = 4x + 24 \\ y = -x^2 + 18x - 29 \end{cases}$$

$(7 - 2i, 52 - 8i), (7 + 2i, 52 + 8i)$

Lesson 13-2
Solving a System Algebraically

My Notes

2. Describe the solutions of the system of equations from Item 1.
The x- and y-coordinates of each of the two solutions are complex numbers. The system of equations has no real solutions.

3. **Use appropriate tools strategically.** Confirm that the system of equations from Item 1 has no real solutions by graphing the system on a graphing calculator. How does the graph show that the system has no real solutions?
It shows that the graphs of the two equations do not intersect. So, there is no real value of x for which the y-values of the two equations are equal.

Check Your Understanding

4. How does solving a system of one linear equation and one quadratic equation by substitution differ from solving a system of two linear equations by substitution?
5. **Reason abstractly.** What is an advantage of solving a system of one linear equation and one quadratic equation algebraically rather than by graphing or using a table of values?
6. Write a journal entry in which you explain step by step how to solve the following system by using substitution.

$$\begin{cases} y = 2x^2 - 3x + 6 \\ y = -2x + 9 \end{cases}$$

7. Could you solve the system in Item 6 by using elimination rather than substitution? Explain.
8. Explain how you could use the discriminant of a quadratic equation to determine how many real solutions the following system has.

$$\begin{cases} y = 4x - 21 \\ y = x^2 - 4x - 5 \end{cases}$$

MATH TIP

To review solving a system of equations by elimination, see Activity 3.

ACTIVITY 13 Continued

Check Your Understanding

Debrief students' answers to these items to ensure that they understand concepts related to solving a system of one linear equation and one quadratic equation algebraically.

Answers

4. Sample answer: When solving a system of two linear equations, you end up with a linear equation after you solve one equation for y and substitute that expression for y into the other equation. When solving a system of one linear equation and one quadratic equation, you end up with a quadratic equation after you solve one equation for y and substitute that expression for y into the other equation.
5. Sample answer: When you solve the system algebraically, you can find the exact values of the coordinates of the solutions. When you solve the system by graphing or using a table of values, you may only be able to approximate the coordinates of the solutions.
6. See below.
7. Yes. Sample explanation: If you subtract the second equation in the system from the first equation, the variable y is eliminated, leaving the equation $0 = 2x^2 - x - 3$. You can then solve this equation for x, which will give the x-values of the solutions of the system of equations.
8. Use substitution to find that $4x - 21 = x^2 - 4x - 5$. Then write this equation in standard form: $0 = x^2 - 8x + 16$. The discriminant of this quadratic equation is $(-8)^2 - 4(1)(16) = 0$. A discriminant of 0 means that the equation $0 = x^2 - 8x + 16$ has only one real solution. The system of equations also has only one real solution.

6. Sample answer:
First use substitution to solve for x.

$y = 2x^2 - 3x + 6$	The first equation is solved for y.
$2x^2 - 3x + 6 = -2x + 9$	Substitute for y in the second equation.
$2x^2 - x - 3 = 0$	Write the equation in standard form.
$(2x - 3)(x + 1) = 0$	Factor the left side.
$x = \left(\frac{3}{2}\right)$ or $x = -1$	Solve for x.

Substitute each value of x into one of the original equations to find the corresponding value of y.

$y = -2x + 9$	$y = -2x + 9$	Use the first original equation.
$y = -2\left(\frac{3}{2}\right) + 9$	$y = -2(-1) + 9$	Substitute $\frac{3}{2}$ for x and -1 for x.
$y = 6$	$y = 11$	Simplify.

Write the solutions as ordered pairs.

The solutions are $\left(\frac{3}{2}, 6\right)$ and $(-1, 11)$.

ASSESS

Students' answers to Lesson Practice problems will provide you with a formative assessment of their understanding of the lesson concepts and their ability to apply their learning. See the Activity Practice for additional problems for this lesson. You may assign the problems here or use them as a culmination for the activity.

LESSON 13-2 PRACTICE

9. $(-5, 7), (-2, -2)$

10. $(7, -12)$

11. $(2, -10), \left(25, \frac{29}{4}\right)$

12. no real solutions

13. $(-50, 168), \left(\frac{220}{3}, \frac{208}{3}\right)$ or approximately $(73.33, 69)$

14. $73.33; Sample explanation: The solution with the negative x-value, representing the price in dollars, does not make sense in this situation and should be ignored. The x-coordinate of the remaining solution shows that when hair highlights are priced at about $73.33, the number of hair highlights demanded by customers will equal the number that the stylists are willing to supply.

15. Sample answer: When I substitute 73.33 for x into each equation in the system, I get a value of y that is approximately equal to 69. In addition, a price of $73.33 for hair highlights seems realistic.

ADAPT

Check students' answers to the Lesson Practice to ensure that they understand how to solve a system of one linear and one quadratic equation algebraically. Students should be cognizant of the possible numbers of solutions prior to beginning the algebraic solution process. Some students may benefit from making a graphic organizer that shows the graphical and algebraic solutions for each of the three possible numbers of solutions.

My Notes

LESSON 13-2 PRACTICE

Find the real solutions of each system algebraically. Show your work.

9. $\begin{cases} y = -3x - 8 \\ y = x^2 + 4x + 2 \end{cases}$

10. $\begin{cases} y = -2x^2 + 16x - 26 \\ y = 72 - 12x \end{cases}$

11. $\begin{cases} y = \frac{1}{4}x^2 - 6x + 1 \\ y = \frac{3}{4}x - \frac{23}{2} \end{cases}$

12. $\begin{cases} y = (x-5)^2 - 3 \\ y = -2x - 3 \end{cases}$

The owner of Salon Ultra Blue is setting the price for hair highlights. The following system represents the demand and supply functions for hair highlights, where y is the quantity demanded or supplied per week for a given price x in dollars.

$$\begin{cases} y = -0.8x + 128 \\ y = 0.03x^2 - 1.5x + 18 \end{cases}$$

13. Use substitution to solve the system of equations.

14. **Attend to precision.** How much should the salon charge for hair highlights so that the weekly demand is equal to the weekly supply? Explain how you determined your answer.

15. Explain why your answer to Item 14 is reasonable.

Systems of Linear and Nonlinear Equations
Supply and Demand

ACTIVITY 13 PRACTICE
Write your answers on notebook paper.
Show your work.

Lesson 13-1

Lori was partway up an escalator when her friend Evie realized that she had Lori's keys. Evie, who was still on the ground floor, tossed the keys up to Lori. The function $f(x) = -16x^2 + 25x + 5$ models the height in feet of the keys x seconds after they were thrown. Use this information for Items 1–5.

1. When the keys are thrown, Lori's hands are 9 ft above ground level and moving upward at a rate of 0.75 ft/s. Write the equation of a function $g(x)$ that gives the height of Lori's hands compared to ground level x seconds after the keys are thrown.
2. Write the functions $f(x)$ and $g(x)$ as a system of two equations in two variables. In each equation, let y represent height in feet and x represent time in seconds.
3. Graph the system of equations, and use the graph to approximate the solutions of the system.
4. How long after the keys are thrown will Lori be able to catch them? Assume that Lori can catch the keys when they are at the same height as her hands. Explain how you determined your answer.
5. Explain why your answer to Item 4 is reasonable.

Solve each system by using a graph or table (answers will be approximate).

6. $\begin{cases} y = 10 - 2x \\ y = x^2 - 12x + 31 \end{cases}$

7. $\begin{cases} y = 5x + 39 \\ y = x^2 + 14x + 52 \end{cases}$

8. $\begin{cases} y = -2(x-3)^2 + 9 \\ y = -4x + 3 \end{cases}$

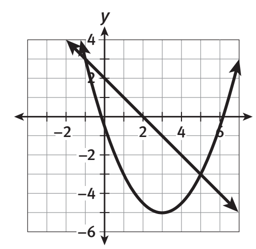
Use a graph to determine the number of real solutions of each system.

9. $\begin{cases} y = 3x^2 + 6x + 4 \\ y = 0.5x + 8 \end{cases}$

10. $\begin{cases} y = -2x^2 + 8x - 10 \\ y = -2x + 4 \end{cases}$

11. $\begin{cases} y = 24 - 4x \\ y = x^2 - 12x + 40 \end{cases}$

12. Which ordered pair is a solution of the system of equations graphed below?



A. $(-3, 5)$ **B.** $(-1, 3)$
C. $(2, 0)$ **D.** $(3, -5)$

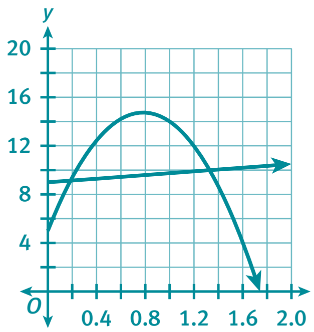
A parallelogram has a height of x cm. The length of its base is 4 cm greater than its height. A triangle has the same height as the parallelogram. The length of the triangle's base is 20 cm.

13. Write a system of two equations in two variables that can be used to determine the values of x for which the parallelogram and the triangle have the same area.
14. Solve the system by using a graph or table.
15. Interpret the solutions of the system in the context of the situation.

ACTIVITY 13 Continued

ACTIVITY PRACTICE

1. $g(x) = 0.75x + 9$
2. $\begin{cases} y = -16x^2 + 25x + 5 \\ y = 0.75x + 9 \end{cases}$
3. Solutions are approximately (0.2, 9.1) and (1.3, 10.0).



4. Lori has two chances to catch the keys: about 0.2 s after they are thrown and about 1.3 s after they are thrown. The x-values of the solutions of the system represent how long after the keys are thrown that they will be at the same height as Lori's hands.
5. Sample answer: When I substitute 0.2 for x into each equation in the system, I get a value of y that is approximately equal to 9.1. When I substitute 1.3 for x into each equation in the system, I get a value of y that is approximately equal to 10.0. In addition, it makes sense that Lori will have two chances to catch the keys: once when they are on their way up and once when they are on their way down.
6. (3, 4), (7, −4)
7. approximately (−7.2, 3.0), (−1.8, 30.0)
8. approximately (0.8, −0.4), (7.2, −25.6)
9. 2 real solutions
10. no real solutions
11. 1 real solution
12. B
13. $\begin{cases} y = (x+4)x \\ y = \frac{1}{2}(20)x \end{cases}$ or equivalent
14. (0, 0), (6, 60)
15. It does not make sense for a parallelogram or a triangle to have a height of 0 cm, so the solution (0, 0) can be ignored. The solution (6, 60) shows that the parallelogram and the triangle have the same area when the height of each is 6 cm. The area of both the parallelogram and the triangle when their height is 6 cm is 60 cm^2.

ACTIVITY PRACTICE

16. $(-3, -10), (0, -7)$
17. $(5, 16)$
18. $(5, -1), (1, -25)$
19. $(-8, 3), (1, -1.5)$
20. $\begin{cases} y = x^2 - 2x - 4 \\ y = -4x - 5 \end{cases}$
21. $(-1, -1)$
22. The solution of the system indicates that the paths of the boats will cross at one point, represented by $(-1, -1)$ on the map.
23. A
24. Sample answer: Use substitution to solve the system. Substitute the expression for y from the first equation into the second equation: $-x^2 + 4x = 3x + 5$. Write the equation in standard form: $0 = x^2 - x + 5$. Use the Quadratic Formula to solve for x: $x = \frac{1}{2} \pm \frac{i\sqrt{19}}{2}$. The values of x are complex conjugates, so the system of equations has no real solutions.
25. $f(x) = 0.75x$
26. $g(x) = 0.015x^2 + 0.15x$
27. $\begin{cases} y = 0.75x \\ y = 0.015x^2 + 0.15x \end{cases}$
28. $(0, 0), (40, 30)$
29. If the length of the longest side is 40 in., the charge for nonglare glass will be the same as the charge for regular glass. This charge will be \$30. Sample explanation: The x-coordinates of the solutions of the system represent lengths for which the charges for the two types of glass will be equal. Because the length of a piece of glass must be greater than 0 in., the solution (0, 0) can be ignored. The solution is (40, 30), meaning when the longest side is 40 in., the charge for both types of glass will be \$30.
30. $\begin{cases} y = 200 + 8x - 0.01x^2 \\ y = 18x \end{cases}$; approximately $(-1020, -18{,}353)$ and $(20, 353)$; The solutions indicate the number of magnet sets for which Austin's cost of making the magnets will equal his income from selling them. It does not make sense for Austin to make a negative number of magnet sets, so the solution with a negative x-value can be ignored. The solution (20, 353) shows that if Austin makes and sells approximately 20 magnet sets, his cost of making the sets and his income from selling the sets both are about \$353.

Systems of Linear and Nonlinear Equations
Supply and Demand

Lesson 13-2

Solve each system algebraically. Check your answers by substituting each solution into one of the original equations. Show your work.

16. $\begin{cases} y = x - 7 \\ y = -x^2 - 2x - 7 \end{cases}$

17. $\begin{cases} y = 2x^2 - 12x + 26 \\ y = 8x - 24 \end{cases}$

18. $\begin{cases} y = -3(x - 4)^2 + 2 \\ y = 6x - 31 \end{cases}$

19. $\begin{cases} y = -0.5x - 1 \\ y = 0.5x^2 + 3x - 5 \end{cases}$

A map of a harbor is laid out on a coordinate grid, with the origin marking a buoy at the center of the harbor. A fishing boat is following a path that can be represented on the map by the equation $y = x^2 - 2x - 4$. A ferry is following a linear path that passes through the points $(-3, 7)$ and $(0, -5)$ when represented on the map. Use this information for Items 20–22.

20. Write a system of equations that can be used to determine whether the paths of the boats will cross.

21. Use substitution to solve the system.

22. Interpret the solution(s) of the system in the context of the situation.

23. How many real solutions does the following system have?

$$\begin{cases} y = -x^2 + 4x \\ y = 3x + 5 \end{cases}$$

A. none
B. one
C. two
D. infinitely many

24. Explain how you can support your answer to Item 23 algebraically.

A picture-framing company sells two types of glass: regular and nonglare. For a piece of nonglare glass, the charge is equal to the length of the longest side in inches multiplied by the rate \$0.75 per inch. The table shows the charge for several sizes of regular glass.

Charge for Regular Glass

Length of Longest Side (in.)	Charge (\$)
12	3.96
18	7.56
24	12.24

25. Write a linear function $f(x)$ that gives the charge in dollars for a piece of nonglare glass whose longest side measures x inches.

26. Write a quadratic function $g(x)$ that gives the charge in dollars for a piece of regular glass whose longest side measures x inches.

27. Write the functions $f(x)$ and $g(x)$ as a system of equations in terms of y, the charge in dollars for a piece of glass, and x, the length of the longest side in inches.

28. Solve the system by using substitution.

29. For what length will the charge for nonglare glass be the same as the charge for regular glass? What will the charge be? Explain your answers.

MATHEMATICAL PRACTICES
Reason Abstractly and Quantitatively

30. Austin sells sets of magnets online. His cost in dollars of making the magnets is given by $f(x) = 200 + 8x - 0.01x^2$, where x is the number of magnet sets he makes. His income in dollars from selling the magnets is given by $g(x) = 18x$, where x is the number of magnet sets he sells. Write and solve the system, and then explain what the solution(s) mean in the context of the situation.

ADDITIONAL PRACTICE

If students need more practice on the concepts in this activity, see the Teacher Resources at SpringBoard Digital for additional practice problems.

Graphing Quadratic Functions and Solving Systems

THE GREEN MONSTER

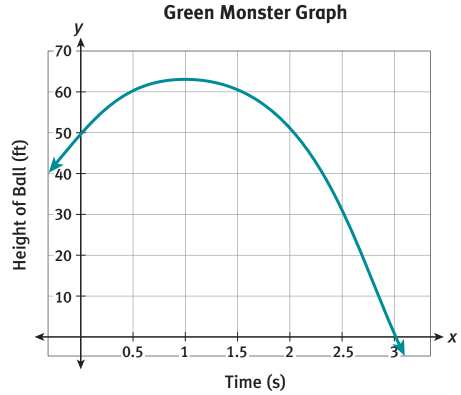
Embedded Assessment 3

Use after Activity 13

During a Boston Red Sox baseball game at Fenway Park, the opposing team hit a home run over the left field wall. An unhappy Red Sox fan caught the ball and threw it back onto the field. The height of the ball, $h(t)$, in feet, t seconds after the fan threw the baseball, is given by the function $h(t) = -16t^2 + 32t + 48$.

1. Graph the equation on the coordinate grid below.

Green Monster Graph



CONNECT TO HISTORY

The left field wall in Fenway Park is called the Green Monster, a reference to its unusual height.

2. Find each measurement value described below. Then tell how each value relates to the graph.
   a. At what height was the fan when he threw the ball?
   b. What was the maximum height of the ball after the fan threw it?
   c. When did the ball hit the field?

3. What are the reasonable domain and reasonable range of $h(t)$? Explain how you determined your answers.

4. Does the baseball reach a height of 65 ft? Explain your answer both graphically and algebraically.

5. Each baseball team in a minor league plays each other team three times during the regular season.
   a. The table shows the relationship between the number of teams in a baseball league and the total number of games required for each team to play each of the other teams three times. Write a quadratic equation that models the data in the table.

Number of Teams, x	Number of Games, y
2	3
3	9
4	18
5	30

   b. Last season, the total number of games played in the regular season was 35 more than 10 times the number of teams. Use this information to write a linear equation that gives the number of regular games y in terms of the number of teams x.
   c. Write a system of equations using the quadratic equation from part a and the linear equation from part b. Then solve the system and interpret the solutions.

Common Core State Standards for Embedded Assessment 3

HSA-REI.D.11	Explain why the x-coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where $f(x)$ and/or $g(x)$ are linear, polynomial, absolute value.
HSF-IF.B.4	For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship.
HSF-IF.B.5	Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes.

Embedded Assessment 3

Assessment Focus

- Graph of a parabola
- Maximum of a parabola
- Domain and range of quadratic functions
- System of equations with a linear equation and quadratic equation

Teacher to Teacher

The Green Monster is 37 feet tall. Prior to the 2003 baseball season, seats were constructed on top of the wall. Therefore, it is reasonable that the fan who caught the ball is at a location that is higher than the top of the wall.

Answer Key

2. a. 48 ft; The y-intercept represents the height of the ball when it was thrown.
   b. 64 ft; The y-coordinate of the vertex of the graph represents the maximum height of the ball.
   c. 3 s after the ball was thrown; The positive x-intercept indicates when the height of the ball is 0 ft. In other words, it shows the time at which the ball hits the ground.

3. Negative values for the time do not make sense, and the ball hits the ground after 3 s, so a reasonable domain for the function is $0 \le t \le 3$. Negative values for the height do not make sense, and the maximum height the ball reaches is 64 ft, so a reasonable range is $0 \le h \le 64$.

4. No. Graphically: The vertex of the graph is (1, 64), so the maximum height of the ball is 64 ft. Algebraically: Set $h(t)$ equal to 65: $65 = -16t^2 + 32t + 48$. Subtract 65 from both sides to write the equation in standard form: $0 = -16t^2 + 32t - 17$. The Quadratic Formula shows that $t = 1 \pm \frac{i}{4}$. Because the equation has complex solutions, there is no real value of t that results in a height of 65 ft.

5. a. $y = 1.5x^2 - 1.5x$
   b. $y = 10x + 35$
   c. $\begin{cases} y = 1.5x^2 - 1.5x \\ y = 10x + 35 \end{cases}$; $\left(-\frac{7}{3}, \frac{35}{3}\right)$ and (10, 135); The x-coordinates of the solutions represent the number of teams in the league. Only a positive, whole number of teams makes sense, so the solution $\left(-\frac{7}{3}, \frac{35}{3}\right)$ can be ignored. The solution (10, 135) indicates that the league has 10 teams and the teams played a total of 135 games.

Teacher to Teacher

You may wish to read through the scoring guide with students and discuss the differences in the expectations at each level. Check that students understand the terms used.

Embedded Assessment 3
Use after Activity 13

Graphing Quadratic Functions and Solving Systems

THE GREEN MONSTER

Scoring Guide	Exemplary	Proficient	Emerging	Incomplete
	The solution demonstrates these characteristics:			
Mathematics Knowledge and Thinking (Items 2, 4, 5)	• Effective understanding of how to solve quadratic equations and systems of equations • Clear and accurate understanding of how to write linear and quadratic models from verbal descriptions or tables of data • Clear and accurate understanding of how to use an equation or graph to identify key features of a quadratic function	• Adequate understanding of how to solve quadratic equations and systems of equations • Largely correct understanding of how to write linear and quadratic models from verbal descriptions or tables of data • Largely correct understanding of how to use an equation or graph to identify key features of a quadratic function	• Partial understanding of how to solve quadratic equations and systems of equations • Partial understanding of how to write linear and quadratic models from verbal descriptions or tables of data • Difficulty with using an equation or graph to identify key features of a quadratic function	• Inaccurate or incomplete understanding of how to solve quadratic equations and systems of equations • Little or no understanding of how to write linear and quadratic models from verbal descriptions or tables of data • Little or no understanding of how to use an equation or graph to identify key features of a quadratic function
Problem Solving (Items 2, 4, 5c)	• An appropriate and efficient strategy that results in a correct answer	• A strategy that may include unnecessary steps but results in a correct answer	• A strategy that results in some incorrect answers	• No clear strategy when solving problems
Mathematical Modeling / Representations (Items 1, 2, 3, 4, 5)	• Effective understanding of how to interpret solutions to a system of equations that represents a real-world scenario • Clear and accurate understanding of how to model real-world scenarios with quadratic and linear functions, including reasonable domain and range • Clear and accurate understanding of how to graph and interpret key features of a quadratic function that represents a real-world scenario	• Adequate understanding of how to interpret solutions to a system of equations that represents a real-world scenario • Largely correct understanding of how to model real-world scenarios with quadratic and linear functions, including reasonable domain and range • Largely correct understanding of how to graph and interpret key features of a quadratic function that represents a real-world scenario	• Partial understanding of how to interpret solutions to a system of equations that represents a real-world scenario • Some difficulty with modeling real-world scenarios with quadratic and linear functions, including reasonable domain and range • Some difficulty with graphing and interpreting key features of a quadratic function that represents a real-world scenario	• Little or no understanding of how to interpret solutions to a system of equations that represents a real-world scenario • Inaccurate or incomplete understanding of how to model real-world scenarios with quadratic and linear functions, including reasonable domain and range • Inaccurate or incomplete understanding of how to graph and interpret key features of a quadratic function that represents a real-world scenario
Reasoning and Communication (Items 2, 3, 4)	• Precise use of appropriate math terms and language to relate the features of a quadratic model, including reasonable domain and range, to a real-world scenario • Clear and accurate use of mathematical work to explain whether or not the height could reach 65 feet	• Adequate explanations to relate the features of a quadratic model, including reasonable domain and range, to a real-world scenario • Correct use of mathematical work to explain whether or not the height could reach 65 feet	• Misleading or confusing explanations to relate the features of a quadratic model, including reasonable domain and range, to a real-world scenario • Partially correct explanation of whether or not the height could reach 65 feet	• Incomplete or inaccurate explanations to relate the features of a quadratic model, including reasonable domain and range, to a real-world scenario • Incorrect or incomplete explanation of whether or not the height could reach 65 feet