

## UNIT 2: SIMILARITY, CONGRUENCE, AND PROOFS

This unit introduces the concepts of similarity and congruence. The definition of similarity is explored through dilation transformations. The concept of scale factor with respect to dilations allows figures to be enlarged or reduced. Rigid motions lead to the definition of congruence. Once congruence is established, various congruence criteria (e.g., ASA, SSS, SAS) can be explored. Once similarity is established, various similarity criteria (e.g., AA) can be explored. These criteria, along with other postulates and definitions, provide a framework to be able to prove various geometric proofs. In this unit, various geometric figures are constructed. These topics allow students a deeper understanding of formal reasoning, which will be beneficial throughout the remainder of Analytic Geometry. Students are asked to prove theorems about parallelograms. Theorems include opposite sides are congruent, opposite angles are congruent, the diagonals of a parallelogram bisect each other, and, conversely, rectangles are parallelograms with congruent diagonals. The method for proving is not specified, so it could be done by using knowledge of congruency and establishing a formalized proof, it could be proven by constructions, or it could be proved algebraically by using the coordinate plane.

### Understand Similarity in Terms of Similarity Transformations

**MGSE9-12.G.SRT.1** Verify experimentally the properties of dilations given by a center and a scale factor.

- a. The dilation of a line not passing through the center of the dilation results in a parallel line and leaves a line passing through the center unchanged.
- b. The dilation of a line segment is longer or shorter according to the ratio given by the scale factor.

**MGSE9-12.G.SRT.2** Given two figures, use the definition of similarity in terms of similarity transformations to decide if they are similar; explain, using similarity transformations, the meaning of similarity for triangles as the equality of all corresponding pairs of angles and the proportionality of all corresponding pairs of sides.

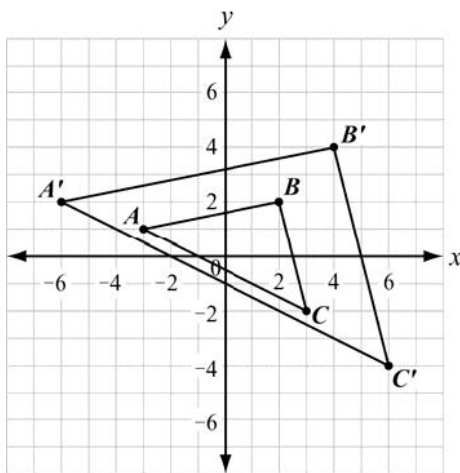
**MGSE9-12.G.SRT.3** Use the properties of similarity transformations to establish the AA criterion for two triangles to be similar.

### KEY IDEAS

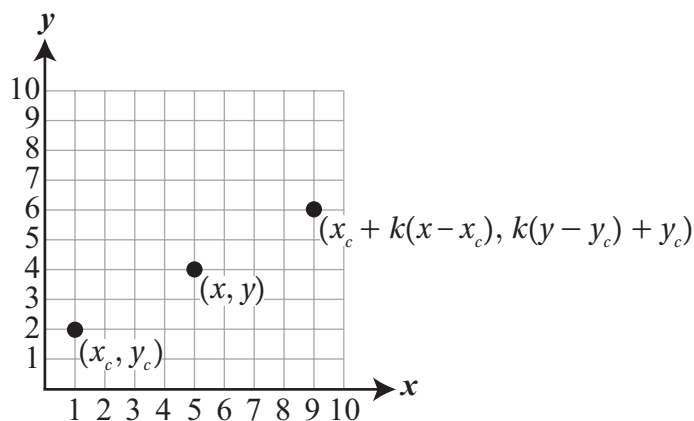
1. A **dilation** is a transformation that changes the size of a figure, but not the shape, based on a ratio given by a **scale factor** with respect to a fixed point called the **center**. When the scale factor is greater than 1, the figure is made larger. When the scale factor is between 0 and 1, the figure is made smaller. When the scale factor is 1, the figure does not change. When the center of dilation is the origin, you can multiply each coordinate of the original figure, or **pre-image**, by the scale factor to find the coordinates of the dilated figure, or **image**.

**Example:**

The diagram below shows  $\triangle ABC$  dilated about the origin with a scale factor of 2 to create  $\triangle A'B'C'$ .



2. When the center of dilation is not the origin, you can use a rule that is derived from shifting the center of dilation, multiplying the shifted coordinates by the scale factor, and then shifting the center of dilation back to its original location. For a point  $(x, y)$  and a center of dilation  $(x_c, y_c)$ , the rule for finding the coordinates of the dilated point with a scale factor of  $k$  is  $(x_c + k(x - x_c), k(y - y_c) + y_c)$ .



When a figure is transformed under a dilation, the **corresponding angles** of the pre-image and the image have equal measures.

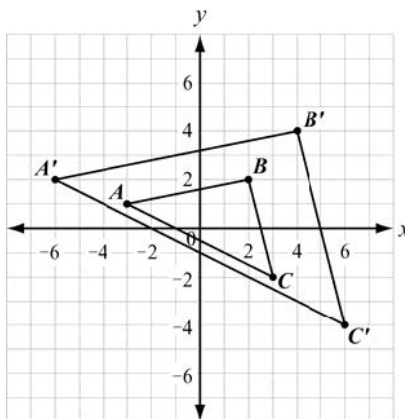
For  $\triangle ABC$  and  $\triangle A'B'C'$  on the next page,  $\angle A \cong \angle A'$ ,  $\angle B \cong \angle B'$ , and  $\angle C \cong \angle C'$ .

When a figure is transformed under a dilation, the **corresponding sides** of the pre-image and the image are proportional.

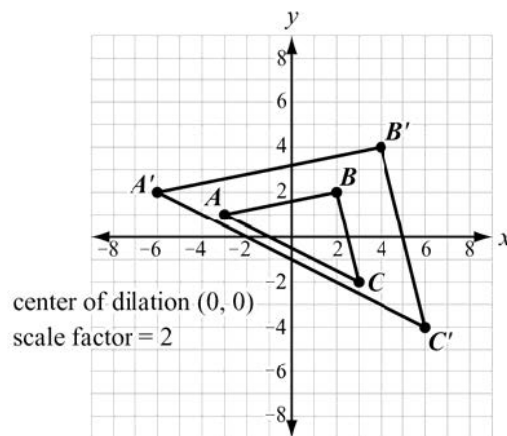
For  $\triangle ABC$  and  $\triangle A'B'C'$  on the next page,  $\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{AC}{A'C'}$ .

**So, when a figure is under a dilation transformation, the pre-image and the image are similar.**

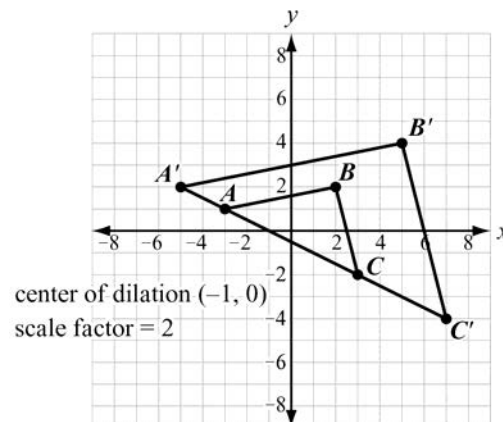
For  $\triangle ABC$  and  $\triangle A'B'C'$  below,  $\triangle ABC \sim \triangle A'B'C'$ .



3. When a figure is dilated, a segment of the pre-image that does not pass through the center of dilation is parallel to its image. In the figure below,  $\overline{AC} \parallel \overline{A'C'}$  since neither segment passes through the center of dilation. The same is true about  $\overline{AB}$  and  $\overline{A'B'}$  as well as  $\overline{BC}$  and  $\overline{B'C'}$ .



When the segment of a figure does pass through the center of dilation, the segment of the pre-image and image are on the same line. In the figure below, the center of dilation is on  $\overline{AC}$ , so  $\overline{AC}$  and  $\overline{A'C'}$  are on the same line.

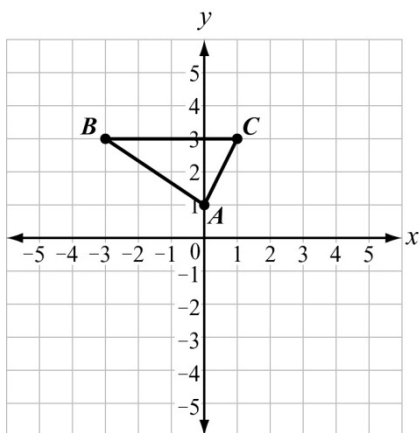


## REVIEW EXAMPLES

1. Draw a triangle with vertices at  $A(0, 1)$ ,  $B(-3, 3)$ , and  $C(1, 3)$ . Dilate the triangle using a scale factor of 1.5 and a center of  $(0, 0)$ . Sketch and name the dilated triangle  $A'B'C'$ .

**Solution:**

Plot points  $A(0, 1)$ ,  $B(-3, 3)$ , and  $C(1, 3)$ . Draw  $\overline{AB}$ ,  $\overline{AC}$ , and  $\overline{BC}$ .



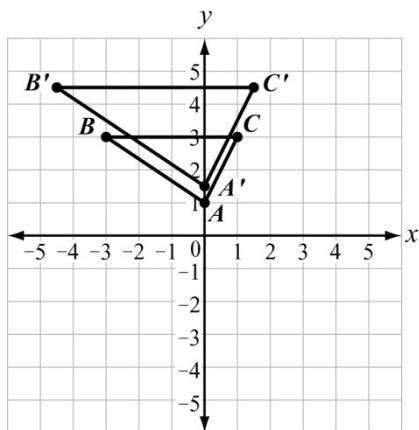
The center of dilation is the origin, so to find the coordinates of the image, multiply the coordinates of the pre-image by the scale factor 1.5.

$$\text{Point } A': (1.5 \cdot 0, 1.5 \cdot 1) = (0, 1.5)$$

$$\text{Point } B': (1.5 \cdot (-3), 1.5 \cdot 3) = (-4.5, 4.5)$$

$$\text{Point } C': (1.5 \cdot 1, 1.5 \cdot 3) = (1.5, 4.5)$$

Plot points  $A'(0, 1.5)$ ,  $B'(-4.5, 4.5)$ , and  $C'(1.5, 4.5)$ . Draw  $\overline{A'B'}$ ,  $\overline{A'C'}$ , and  $\overline{B'C'}$ .



**Note:** Since no part of the pre-image passes through the center of dilation,  $\overline{BC} \parallel \overline{B'C'}$ ,  $\overline{AB} \parallel \overline{A'B'}$ , and  $\overline{AC} \parallel \overline{A'C'}$ .

2. Line segment  $CD$  is 5 inches long. If line segment  $CD$  is dilated to form line segment  $C'D'$  with a scale factor of 0.6, what is the length of line segment  $C'D'$ ?

**Solution:**

The ratio of the length of the image and the pre-image is equal to the scale factor.

$$\frac{C'D'}{CD} = 0.6$$

Substitute 5 for  $CD$ .

$$\frac{C'D'}{5} = 0.6$$

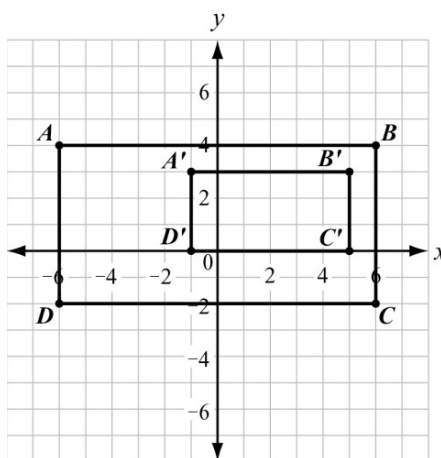
Solve for  $C'D'$ .

$$C'D' = 0.6 \cdot 5$$

$$C'D' = 3$$

The length of line segment  $C'D'$  is 3 inches.

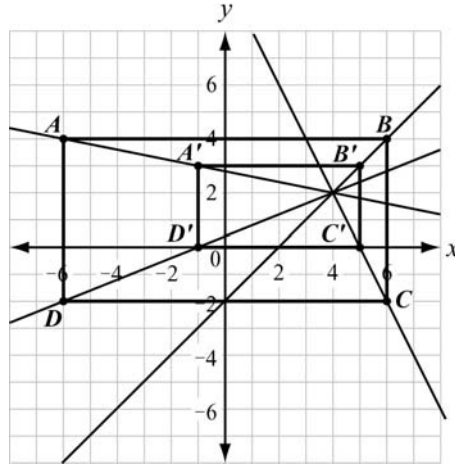
3. Figure  $A'B'C'D'$  is a dilation of figure  $ABCD$ .



- Determine the center of dilation.
- Determine the scale factor of the dilation.
- What is the relationship between the sides of the pre-image and the corresponding sides of the image?

**Solution:**

- a. To find the center of dilation, draw lines connecting each corresponding vertex from the pre-image to the image. The lines meet at the center of dilation.



The center of dilation is (4, 2).

- b. Find the ratios of the lengths of the corresponding sides.

$$\frac{A'B'}{AB} = \frac{6}{12} = \frac{1}{2}$$

$$\frac{B'C'}{BC} = \frac{3}{6} = \frac{1}{2}$$

$$\frac{C'D'}{CD} = \frac{6}{12} = \frac{1}{2}$$

$$\frac{A'D'}{AD} = \frac{3}{6} = \frac{1}{2}$$

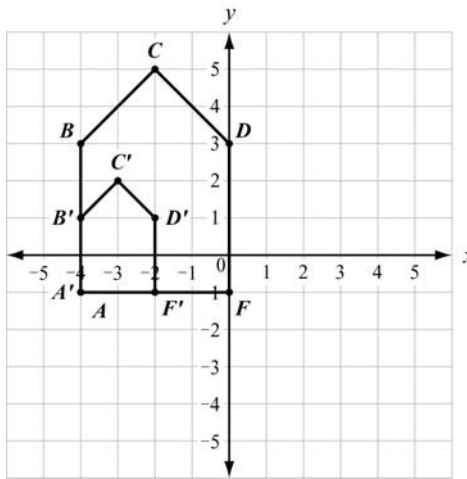
The ratio for each pair of corresponding sides is  $\frac{1}{2}$ , so the scale factor is  $\frac{1}{2}$ .

- c. Each side of the image is parallel to the corresponding side of its pre-image and is  $\frac{1}{2}$  the length.

**Note:** Lines connecting corresponding points pass through the center of dilation.

**SAMPLE ITEMS**

1. Figure  $A'B'C'D'F'$  is a dilation of figure  $ABCDF$  by a scale factor of  $\frac{1}{2}$ . The dilation is centered at  $(-4, -1)$ .



Which statement is true?

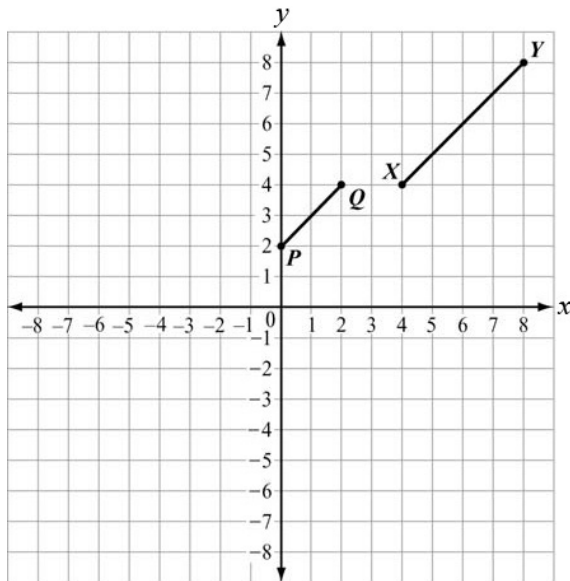
- A.  $\frac{AB}{A'B'} = \frac{B'C'}{BC}$
- B.  $\frac{AB}{A'B'} = \frac{BC}{B'C'}$
- C.  $\frac{AB}{A'B'} = \frac{BC}{D'F'}$
- D.  $\frac{AB}{A'B'} = \frac{D'F'}{BC}$

**Correct Answer: B**

2. Which transformation results in a figure that is similar to the original figure but has a greater area?
- A. a dilation of  $\triangle QRS$  by a scale factor of 0.25
  - B. a dilation of  $\triangle QRS$  by a scale factor of 0.5
  - C. a dilation of  $\triangle QRS$  by a scale factor of 1
  - D. a dilation of  $\triangle QRS$  by a scale factor of 2

**Correct Answer: D**

3. In the coordinate plane, segment  $\overline{PQ}$  is the result of a dilation of segment  $\overline{XY}$  by a scale factor of  $\frac{1}{2}$ .



Which point is the center of dilation?

- A.  $(-4, 0)$
- B.  $(0, -4)$
- C.  $(0, 4)$
- D.  $(4, 0)$

**Correct Answer:** A

**Note:** Draw lines connecting corresponding points to determine the point of intersection (center of dilation).



## Prove Theorems Involving Similarity

**MGSE9-12.G.SRT.4** Prove theorems about triangles. Theorems include: a line parallel to one side of a triangle divides the other two proportionally, (and its converse); the Pythagorean Theorem using triangle similarity.

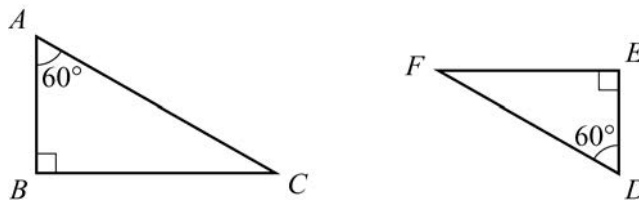
**MGSE9-12.G.SRT.5** Use congruence and similarity criteria for triangles to solve problems and to prove relationships in geometric figures.

### KEY IDEAS

1. When proving that two triangles are similar, it is sufficient to show that two pairs of corresponding angles of the triangles are congruent. This is called **Angle-Angle (AA) Similarity**.

#### Example:

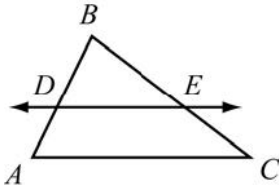
The triangles below are similar by AA Similarity because each triangle has a  $60^\circ$  angle and a  $90^\circ$  angle. The similarity statement is written as  $\triangle ABC \sim \triangle DEF$ , and the order in which the vertices are written indicates which angles/sides correspond to each other.



2. When a triangle is dilated, the pre-image and the image are similar triangles. There are three cases of triangles being dilated:
  - The image is congruent to the pre-image (scale factor of 1).
  - The image is smaller than the pre-image (scale factor between 0 and 1).
  - The image is larger than the pre-image (scale factor greater than 1).
3. When two triangles are **similar**, all corresponding pairs of angles are congruent.
4. When two triangles are **similar**, all corresponding pairs of sides are proportional.
5. When two triangles are **congruent**, the triangles are also similar.
6. A **two-column proof** is a series of statements and reasons often displayed in a chart that works from given information to the statement that needs to be proven. Reasons can be given information, can be based on definitions, or can be based on postulates or theorems.
7. A **paragraph proof** also uses a series of statements and reasons that work from given information to the statement that needs to be proven, but the information is presented as running text in paragraph form.

## REVIEW EXAMPLES

1. In the triangle shown,  $\overline{AC} \parallel \overline{DE}$ .

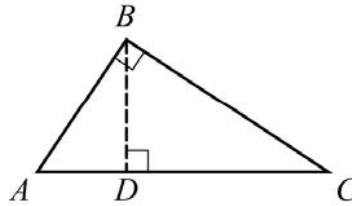


Prove that  $\overline{DE}$  divides  $\overline{AB}$  and  $\overline{CB}$  proportionally.

**Solution:**

Step	Statement	Justification
1	$\overline{AC} \parallel \overline{DE}$	Given
2	$\angle BDE \cong \angle BAC$	If two parallel lines are cut by a transversal, then corresponding angles are congruent.
3	$\angle DBE \cong \angle ABC$	Reflexive Property of Congruence because they are the same angle
4	$\triangle DBE \sim \triangle ABC$	Angle-Angle (AA) Similarity
5	$\frac{BA}{BD} = \frac{BC}{BE}$	Corresponding sides of similar triangles are proportional.
6	$BD + DA = BA$ $BE + EC = BC$	Segment Addition Postulate
7	$\frac{BD + DA}{BD} = \frac{BE + EC}{BE}$	Substitution
8	$\frac{BD}{BD} + \frac{DA}{BD} = \frac{BE}{BE} + \frac{EC}{BE}$	Rewrite each fraction as a sum of two fractions.
9	$1 + \frac{DA}{BD} = 1 + \frac{EC}{BE}$	Simplify
10	$\frac{DA}{BD} = \frac{EC}{BE}$	Subtraction Property of Equality
11	$\overline{DE}$ divides $\overline{AB}$ and $\overline{CB}$ proportionally.	Definition of proportionality

2. Gale is trying to prove the Pythagorean Theorem using similar triangles. Part of her proof is shown below.



Step	Statement	Justification
1	$\angle ABC \cong \angle BDC$	All right angles are congruent.
2	$\angle ACB \cong \angle BCD$	Reflexive Property of Congruence
3	$\triangle ABC \sim \triangle BDC$	Angle-Angle (AA) Similarity
4	$\frac{BC}{DC} = \frac{AC}{BC}$	Corresponding sides of similar triangles are proportional.
5	$BC^2 = AC \cdot DC$	In a proportion, the product of the means equals the product of the extremes.
6	$\angle ABC \cong \angle ADB$	All right angles are congruent.
7	$\angle BAC \cong \angle DAB$	Reflexive Property of Congruence
8	$\triangle ABC \sim \triangle ADB$	Angle-Angle (AA) Similarity
9	$\frac{AB}{AD} = \frac{AC}{AB}$	Corresponding sides of similar triangles are proportional.
10	$AB^2 = AC \cdot AD$	In a proportion, the product of the means equals the product of the extremes.

What should Gale do to finish her proof?

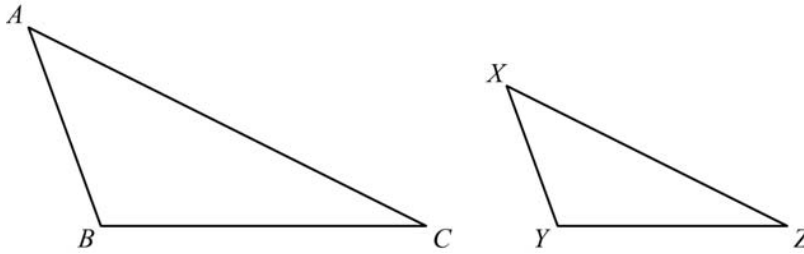
**Solution:**

Step	Statement	Justification
11	$AB^2 + BC^2 = AC \cdot AD + AC \cdot DC$	Addition Property of Equality
12	$AB^2 + BC^2 = AC(AD + DC)$	Distributive Property
13	$AC = AD + DC$	Segment Addition Postulate
14	$AB^2 + BC^2 = AC \cdot AC$	Substitution
15	$AB^2 + BC^2 = AC^2$	Definition of exponent

$AB^2 + BC^2 = AC^2$  is a statement of the Pythagorean Theorem, so Gale's proof is complete.

## SAMPLE ITEMS

1. In the triangles shown,  $\triangle ABC$  is dilated by a factor of  $\frac{2}{3}$  to form  $\triangle XYZ$ .

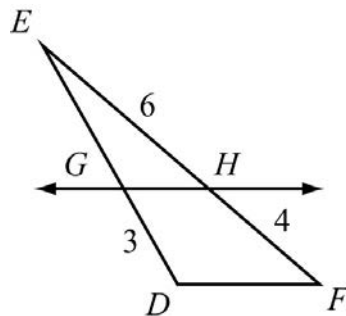


Given that  $m\angle A = 50^\circ$  and  $m\angle B = 100^\circ$ , what is  $m\angle Z$ ?

- A.  $15^\circ$
- B.  $25^\circ$
- C.  $30^\circ$
- D.  $50^\circ$

Correct Answer: C

2. In the triangle shown,  $\overline{GH} \parallel \overline{DF}$ .

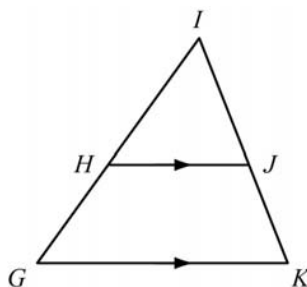


What is the length of  $\overline{GE}$ ?

- A. 2.0
- B. 4.5
- C. 7.5
- D. 8.0

Correct Answer: B

3. Use this triangle to answer the question.



This is a proof of the statement “If a line is parallel to one side of a triangle and intersects the other two sides at distinct points, then it separates these sides into segments of proportional lengths.”

Step	Statement	Justification
1	$\overline{GK}$ is parallel to $\overline{HJ}$ .	Given
2	$\angle HGK \cong \angle IHJ$ $\angle IKG \cong \angle IJH$	?
3	$\triangle GIK \sim \triangle HIJ$	AA Similarity
4	$\frac{IG}{IH} = \frac{IK}{IJ}$	Corresponding sides of similar triangles are proportional.
5	$\frac{HG + IH}{IH} = \frac{JK + IJ}{IJ}$	Segment Addition Postulate
6	$\frac{HG}{IH} = \frac{JK}{IJ}$	Subtraction Property of Equality

Which reason justifies Step 2?

- A. Alternate interior angles are congruent.
- B. Alternate exterior angles are congruent.
- C. Corresponding angles are congruent.
- D. Vertical angles are congruent.

Correct Answer: C

## Understand Congruence in Terms of Rigid Motions

**MGSE9-12.G.CO.6** Use geometric descriptions of rigid motions to transform figures and to predict the effect of a given rigid motion on a given figure; given two figures, use the definition of congruence in terms of rigid motions to decide if they are congruent.

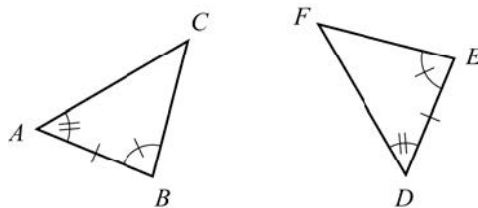
**MGSE9-12.G.CO.7** Use the definition of congruence in terms of rigid motions to show that two triangles are congruent if and only if corresponding pairs of sides and corresponding pairs of angles are congruent.

**MGSE9-12.G.CO.8** Explain how the criteria for triangle congruence (ASA, SAS, and SSS) follow from the definition of congruence in terms of rigid motions. (Extend to include HL and AAS.)

### KEY IDEAS

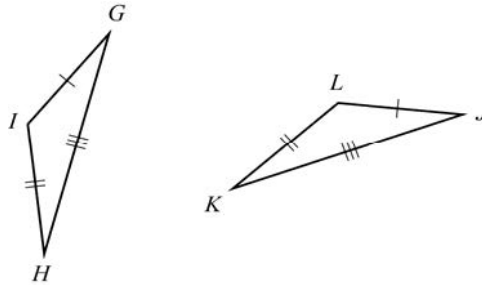
1. A **rigid motion** is a transformation of points in space consisting of a sequence of one or more translations, reflections, and/or rotations (in any order). This transformation leaves the size and shape of the original figure unchanged.
2. Two plane or solid figures are congruent if one can be obtained from the other by rigid motion (a sequence of rotations, reflections, and translations). **Congruent figures** have the same corresponding side lengths and the same corresponding angle measures as each other.
3. Two triangles are congruent if and only if their corresponding sides and corresponding angles are congruent. This is sometimes referred to as **CPCTC**, which means **C**orresponding **P**arts of **C**ongruent **T**riangles are **C**ongruent.
4. When given two congruent triangles, you can use a series of translations, reflections, and rotations to show the triangles are congruent.
5. You can use **ASA (Angle-Side-Angle)** to show two triangles are congruent. If two angles and the included side of a triangle are congruent to two angles and the included side of another triangle, then the triangles are congruent.

$\triangle ABC \cong \triangle DEF$  by ASA.



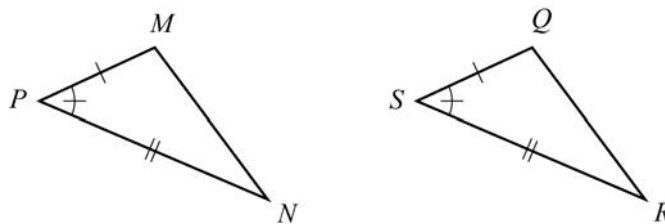
6. You can use **SSS (Side-Side-Side)** to show two triangles are congruent. If three sides of a triangle are congruent to three sides of another triangle, then the triangles are congruent.

$\triangle GIH \cong \triangle JLK$  by SSS.



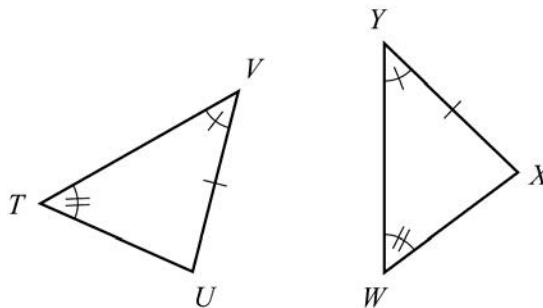
7. You can use **SAS (Side-Angle-Side)** to show two triangles are congruent. If two sides and the included angle of a triangle are congruent to two sides and the included angle of another triangle, then the triangles are congruent.

$\triangle MPN \cong \triangle QSR$  by SAS.





8. You can use **AAS (Angle-Angle-Side)** to show two triangles are congruent. If two angles and a non-included side of a triangle are congruent to two angles and the corresponding non-included side of another triangle, then the triangles are congruent.

$\triangle VTU \cong \triangle YWX$  by AAS.

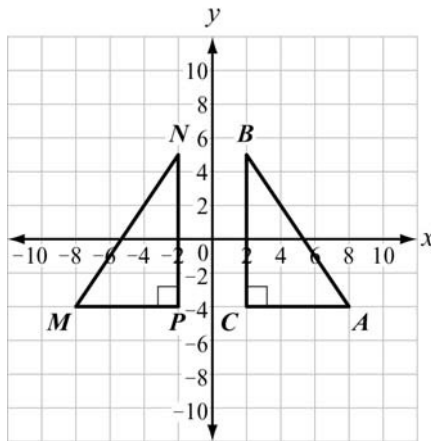


### Important Tips

-  If two sides and a **non-included angle** of one triangle are congruent to two sides and a non-included angle of a second triangle, the triangles are not necessarily congruent. Therefore, there is no way to show triangle congruency by Side-Side-Angle (SSA).
-  If two triangles have all three angles congruent to each other, the triangles are similar, but not necessarily congruent. Thus, you can show similarity by Angle-Angle-Angle (AAA), but you cannot show congruence by AAA.

## REVIEW EXAMPLES

1. Is  $\triangle ABC$  congruent to  $\triangle MNP$ ? Explain.



(scale unit = 2)

**Solution:**

$\overline{AC}$  corresponds to  $\overline{MP}$ . Both segments are 6 units long.  $\overline{BC}$  corresponds to  $\overline{NP}$ . Both segments are 9 units long. Angle  $C$  (the included angle of  $\overline{AC}$  and  $\overline{BC}$ ) corresponds to angle  $P$  (the included angle of  $\overline{MP}$  and  $\overline{NP}$ ). Both angles measure  $90^\circ$ . Because two sides and an included angle are congruent, the triangles are congruent by SAS.

Or,  $\triangle ABC$  is a reflection of  $\triangle MNP$  over the  $y$ -axis. This means that all of the corresponding sides and corresponding angles are congruent, so the triangles are congruent. (Reflections preserve angle measurement and lengths; therefore, corresponding angles and sides are congruent.)



2. Rectangle  $WXYZ$  has coordinates  $W(1, 2)$ ,  $X(3, 2)$ ,  $Y(3, -3)$ , and  $Z(1, -3)$ .
  - a. Graph the image of rectangle  $WXYZ$  after a rotation of  $90^\circ$  clockwise about the origin. Label the image  $W'X'Y'Z'$ .
  - b. Translate rectangle  $W'X'Y'Z'$  2 units left and 3 units up.
  - c. Is rectangle  $WXYZ$  congruent to rectangle  $W''X''Y''Z''$ ? Explain.

**Solution:**

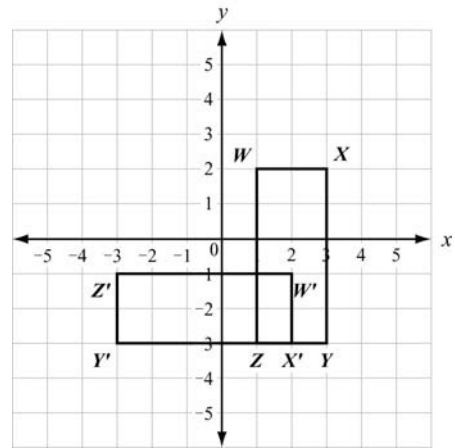
- a. For a  $90^\circ$  clockwise rotation about the origin, use the rule  $(x, y) \rightarrow (y, -x)$ .

$$W(1, 2) \rightarrow W'(2, -1)$$

$$X(3, 2) \rightarrow X'(2, -3)$$

$$Y(3, -3) \rightarrow Y'(-3, -3)$$

$$Z(1, -3) \rightarrow Z'(-3, -1)$$



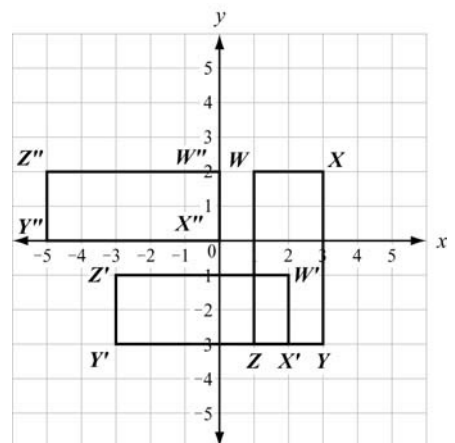
- b. To translate rectangle  $W'X'Y'Z'$  2 units left and 3 units up, use the rule  $(x, y) \rightarrow (x - 2, y + 3)$ .

$$W(2, -1) \rightarrow W''(0, 2)$$

$$X'(2, -3) \rightarrow X''(0, 0)$$

$$Y'(-3, -3) \rightarrow Y''(-5, 0)$$

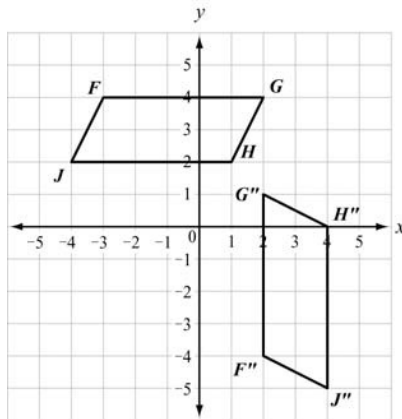
$$Z'(-3, -1) \rightarrow Z''(-5, 2)$$



- c. Rectangle  $W''X''Y''Z''$  is the result of a rotation and a translation of rectangle  $WXYZ$ . These are both rigid transformations, so the shape and the size of the original figure are unchanged. All of the corresponding parts of  $WXYZ$  and  $W''X''Y''Z''$  are congruent, so  $WXYZ$  and  $W''X''Y''Z''$  are congruent.

## SAMPLE ITEMS

1. Parallelogram  $FGHJ$  was translated 3 units down to form parallelogram  $F'G'H'J'$ . Parallelogram  $F'G'H'J'$  was then rotated  $90^\circ$  counterclockwise about point  $G'$  to obtain parallelogram  $F''G''H''J''$ .

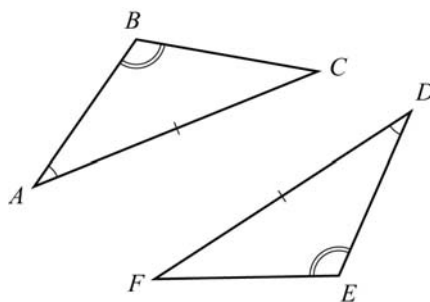


Which statement is true about parallelogram  $FGHJ$  and parallelogram  $F''G''H''J''$ ?

- A. The figures are both similar and congruent.
- B. The figures are neither similar nor congruent.
- C. The figures are similar but not congruent.
- D. The figures are congruent but not similar.

Correct Answer: A

2. Consider the triangles shown.

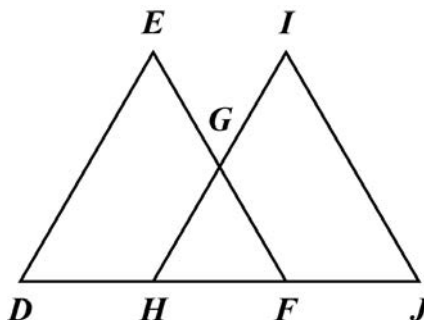


Which can be used to prove the triangles are congruent?

- A. SSS
- B. ASA
- C. SAS
- D. AAS

Correct Answer: D

3. In this diagram,  $\overline{DE} \cong \overline{JI}$  and  $\angle D \cong \angle J$ .



Which additional information is sufficient to prove that  $\triangle DEF$  is congruent to  $\triangle JIH$ ?

- A.  $\overline{ED} \cong \overline{IH}$
- B.  $\overline{DH} \cong \overline{JF}$
- C.  $\overline{HG} \cong \overline{GI}$
- D.  $\overline{HF} \cong \overline{JF}$

**Correct Answer:** B

## Prove Geometric Theorems

**MGSE9-12.G.CO.9** Prove theorems about lines and angles. Theorems include: vertical angles are congruent; when a transversal crosses parallel lines, alternate interior angles are congruent and corresponding angles are congruent; points on a perpendicular bisector of a line segment are exactly those equidistant from the segment's endpoints.

**MGSE9-12.G.CO.10** Prove theorems about triangles. Theorems include: measures of interior angles of a triangle sum to 180 degrees; base angles of isosceles triangles are congruent; the segment joining midpoints of two sides of a triangle is parallel to the third side and half the length; the medians of a triangle meet at a point.

**MGSE9-12.G.CO.11** Prove theorems about parallelograms. Theorems include: opposite sides are congruent, opposite angles are congruent, the diagonals of a parallelogram bisect each other, and conversely, rectangles are parallelograms with congruent diagonals.

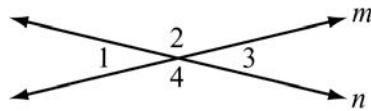
### KEY IDEAS

1. A **two-column proof** is a series of statements and reasons often displayed in a chart that works from given information to the statement that needs to be proven. Reasons can be given information, can be based on definitions, or can be based on postulates or theorems.
2. A **paragraph proof** also uses a series of statements and reasons that work from given information to the statement that needs to be proven, but the information is presented as running text in paragraph form.
3. It is important to plan a geometric proof logically. Think through what needs to be proven and decide how to get to that statement from the given information. Often a diagram or a flow chart will help to organize your thoughts.
4. An **auxiliary line** is a line drawn in a diagram that makes other figures, such as congruent triangles or angles formed by a transversal. Many times, an auxiliary line is needed to help complete a proof.
5. Once a theorem in geometry has been proven, that theorem can be used as a reason in future proofs.
6. Some important key ideas about lines and angles include the following:
  - **Vertical Angle Theorem:** Vertical angles are congruent.
  - **Alternate Interior Angles Theorem:** If two parallel lines are cut by a transversal, then alternate interior angles formed by the transversal are congruent.
  - **Corresponding Angles Postulate:** If two parallel lines are cut by a transversal, then corresponding angles formed by the transversal are congruent.
  - Points on a perpendicular bisector of a line segment are equidistant from both of the segment's endpoints.
7. Some important key ideas about triangles include the following:
  - **Triangle Angle-Sum Theorem:** The sum of the measures of the angles of a triangle is  $180^\circ$ .
  - **Isosceles Triangle Theorem:** If two sides of a triangle are congruent, then the angles opposite those sides are also congruent.

- **Triangle Midsegment Theorem:** If a segment joins the midpoints of two sides of a triangle, then the segment is parallel to the third side and half its length.
  - **Points of Concurrency:** incenter, centroid, orthocenter, and circumcenter
8. Some important key ideas about parallelograms include the following:
- Opposite sides are congruent and opposite angles are congruent.
  - The diagonals of a parallelogram bisect each other.
  - If the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.
  - A rectangle is a parallelogram with congruent diagonals.

## REVIEW EXAMPLES

1. In this diagram, line  $m$  intersects line  $n$ .



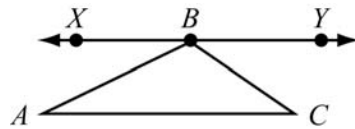
Write a two-column proof to show that vertical angles  $\angle 1$  and  $\angle 3$  are congruent.

**Solution:**

Construct a proof using intersecting lines.

Step	Statement	Justification
1	Line $m$ intersects line $n$ .	Given
2	$\angle 1$ and $\angle 2$ form a linear pair. $\angle 2$ and $\angle 3$ form a linear pair.	Definition of a linear pair
3	$m\angle 1 + m\angle 2 = 180^\circ$ $m\angle 2 + m\angle 3 = 180^\circ$	Angles that form a linear pair have measures that sum to $180^\circ$ .
4	$m\angle 1 + m\angle 2 = m\angle 2 + m\angle 3$	Substitution
5	$m\angle 1 = m\angle 3$	Subtraction Property of Equality
6	$\angle 1 \cong \angle 3$	Definition of congruent angles

2. In this diagram,  $\overline{XY}$  is parallel to  $\overline{AC}$ , and point  $B$  lies on  $\overline{XY}$ .



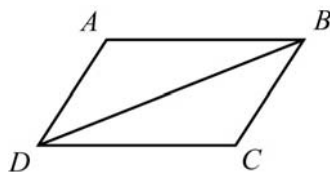
Write a paragraph to prove that the sum of the angles in a triangle is  $180^\circ$ .

**Solution:**

$\overline{AC}$  and  $\overline{XY}$  are parallel, so  $\overline{AB}$  is a transversal. The alternate interior angles formed by the transversal are congruent. So,  $m\angle A = m\angle ABX$ . Similarly,  $\overline{BC}$  is a transversal, so  $m\angle C = m\angle CBY$ . The sum of the angle measures that make a straight line is  $180^\circ$ .

So,  $m\angle ABX + m\angle ABC + m\angle CBY = 180^\circ$ . Now, substitute  $m\angle A$  for  $m\angle ABX$  and  $m\angle C$  for  $m\angle CBY$  to get  $m\angle A + m\angle ABC + m\angle C = 180^\circ$ .

3. In this diagram,  $ABCD$  is a parallelogram and  $\overline{BD}$  is a diagonal.



Write a two-column proof to show that  $\overline{AB}$  and  $\overline{CD}$  are congruent.

**Solution:**

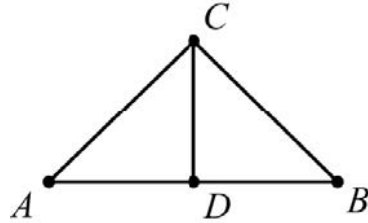
Construct a proof using properties of the parallelogram and its diagonal.

Step	Statement	Justification
1	$ABCD$ is a parallelogram.	Given
2	$\overline{BD}$ is a diagonal.	Given
3	$\overline{AB}$ is parallel to $\overline{DC}$ . $\overline{AD}$ is parallel to $\overline{BC}$ .	Definition of parallelogram
4	$\angle ABD \cong \angle CDB$ $\angle DBC \cong \angle BDA$	Alternate interior angles are congruent.
5	$\overline{BD} \cong \overline{BD}$	Reflexive Property of Congruence
6	$\triangle ADB \cong \triangle CBD$	ASA
7	$\overline{AB} \cong \overline{CD}$	CPCTC

**Note:** Corresponding parts of congruent triangles are congruent.

## SAMPLE ITEMS

1. In this diagram,  $\overline{CD}$  is the perpendicular bisector of  $\overline{AB}$ . The two-column proof shows that  $\overline{AC}$  is congruent to  $\overline{BC}$ .



Step	Statement	Justification
1	$\overline{CD}$ is the perpendicular bisector of $\overline{AB}$ .	Given
2	$\overline{AD} \cong \overline{BD}$	Definition of bisector
3	$\overline{CD} \cong \overline{CD}$	Reflexive Property of Congruence
4	$\angle ADC$ and $\angle BDC$ are right angles.	Definition of perpendicular lines
5	$\angle ADC \cong \angle BDC$	All right angles are congruent.
6	$\triangle ADC \cong \triangle BDC$	_____ ? _____
7	$\overline{AC} \cong \overline{BC}$	CPCTC

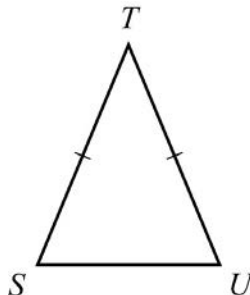
Which of the following would justify Step 6?

- A. AAS
- B. ASA
- C. SAS
- D. SSS

Correct Answer: C



2. In this diagram,  $\triangle STU$  is an isosceles triangle where  $\overline{ST}$  is congruent to  $\overline{UT}$ . The paragraph proof shows that  $\angle S$  is congruent to  $\angle U$ .



It is given that  $\overline{ST}$  is congruent to  $\overline{UT}$ . Draw  $\overline{TV}$  such that V is on  $\overline{SU}$  and  $\overline{TV}$  bisects  $\angle T$ . By the definition of an angle bisector,  $\angle STV$  is congruent to  $\angle UTV$ . By the Reflexive Property of Congruence,  $\overline{TV}$  is congruent to  $\overline{TV}$ . Triangle  $STV$  is congruent to triangle  $UTV$  by SAS.  $\angle S$  is congruent to  $\angle U$  by \_\_\_\_\_ ?

Which step is missing in the proof?

- A. CPCTC
- B. Reflexive Property of Congruence
- C. Definition of right angles
- D. Angle Congruence Postulate

Correct Answer: A


## Make Geometric Constructions

**MGSE9-12.G.CO.12** Make formal geometric constructions with a variety of tools and methods (compass and straightedge, string, reflective devices, paper folding, dynamic geometric software, etc.). Copying a segment; copying an angle; bisecting a segment; bisecting an angle; constructing perpendicular lines, including the perpendicular bisector of a line segment; and constructing a line parallel to a given line through a point not on the line.

**MGSE9-12.G.CO.13** Construct an equilateral triangle, a square, and a regular hexagon, each inscribed in a circle.

### KEY IDEAS

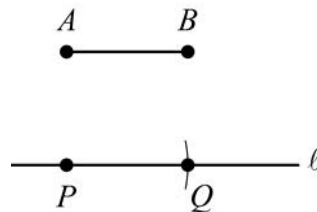
1. To **copy a segment**, follow the steps given:

Given:  $\overline{AB}$  

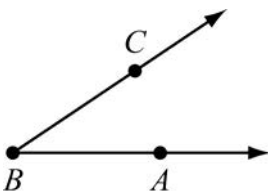
Construct:  $\overline{PQ}$  congruent to  $\overline{AB}$

Procedure:

1. Use a straightedge to draw a line,  $l$ .
2. Choose a point on line  $l$  and label it point  $P$ .
3. Place the compass point on point  $A$ .
4. Adjust the compass width to the length of  $\overline{AB}$ .
5. Without changing the compass, place the compass point on point  $P$  and draw an arc intersecting line  $l$ . Label the point of intersection as point  $Q$ .
6.  $\overline{PQ} \cong \overline{AB}$ .



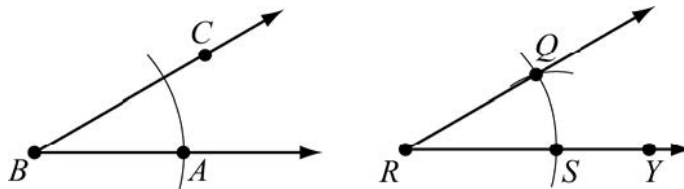
2. To **copy an angle**, follow the steps given:

Given:  $\angle ABC$  

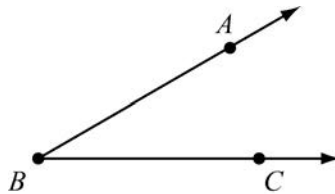
Construct:  $\angle QRY$  congruent to  $\angle ABC$

Procedure:

1. Draw a point  $R$  that will be the vertex of the new angle.
2. From point  $R$ , use a straightedge to draw  $\overline{RY}$ , which will become one side of the new angle.
3. Place the compass point on vertex  $B$  and draw an arc through point  $A$ .
4. Without changing the compass, place the compass point on point  $R$ , draw an arc intersecting  $\overline{RY}$ , and label the point of intersection point  $S$ .
5. Place the compass point on point  $A$  and adjust its width to where the arc intersects  $\overline{BC}$ .
6. Without changing the compass width, place the compass point on point  $S$  and draw another arc across the first arc. Label the point where both arcs intersect as point  $Q$ .
7. Use a straightedge to draw  $\overline{RQ}$ .
8.  $\angle QRY \cong \angle ABC$



3. To **bisect an angle**, follow the steps given:



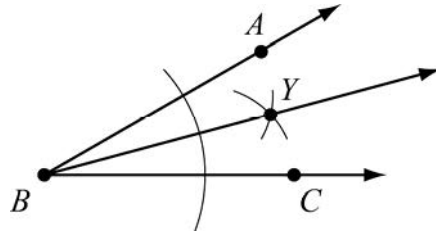
Given:  $\angle ABC$

Construct:  $\overline{BY}$ , the bisector of  $\angle ABC$

Procedure:


1. Place the compass point on vertex  $B$ .
2. Open the compass and draw an arc that crosses both sides of the angle.
3. Set the compass width to more than half the distance from point  $B$  to where the arc crosses  $\overline{BA}$ . Place the compass point where the arc crosses  $\overline{BA}$  and draw an arc in the angle's interior.
4. Without changing the compass width, place the compass point where the arc crosses  $\overline{BC}$  and draw an arc so that it crosses the previous arc. Label the intersection point  $Y$ .

5. Using a straightedge, draw a ray from vertex  $B$  through point  $Y$ .
6.  $\overline{BY}$  is the bisector of  $\angle ABC$ , and  $\angle ABY \cong \angle YBC$ .



4. To **construct a perpendicular bisector of a line segment**, follow the steps given:

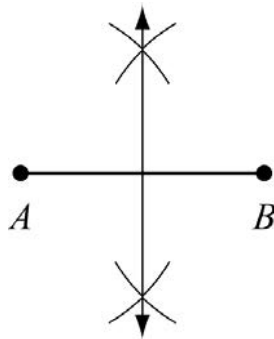
Given:  $\overline{AB}$



Construct: The perpendicular bisector of  $\overline{AB}$

Procedure:


1. Adjust the compass to a width greater than half the length of  $\overline{AB}$ .
2. Place the compass on point  $A$  and draw an arc passing above  $\overline{AB}$  and an arc passing below  $\overline{AB}$ .
3. Without changing the compass width, place the compass on point  $B$  and draw an arc passing above and an arc passing below  $\overline{AB}$ .
4. Use a straightedge to draw a line through the points of intersection of these arcs.
5. The segment is the perpendicular bisector of  $\overline{AB}$ .



**Note:** To bisect  $\overline{AB}$ , follow the same steps listed above to construct the perpendicular bisector. The point where the perpendicular bisector intersects  $\overline{AB}$  is the midpoint of  $\overline{AB}$ .

5. To **construct a line perpendicular to a given line through a point not on the line**, follow the steps given:

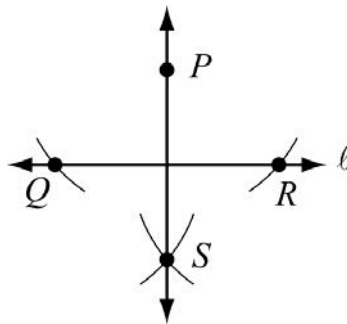
•  $P$

Given: Line  $l$  and point  $P$  that is not on line  $l$  

Construct: The line perpendicular to line  $l$  through point  $P$

Procedure:

1. Place the compass point on point  $P$ .
2. Open the compass to a distance that is wide enough to draw two arcs across line  $l$ , one on each side of point  $P$ . Label these points  $Q$  and  $R$ .
3. From points  $Q$  and  $R$ , draw arcs on the opposite side of line  $l$  from point  $P$  so that the arcs intersect. Label the intersection point  $S$ .
4. Using a straightedge, draw  $\overline{PS}$ .
5.  $\overline{PS} \perp \overline{QR}$ .



6. To **construct a line parallel to a given line through a point not on the line**, follow the steps given:

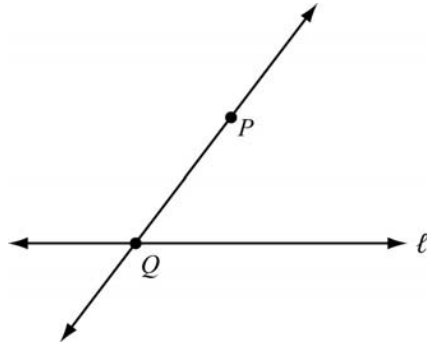
•  $P$

Given: Line  $l$  and point  $P$  that is not on line  $l$  

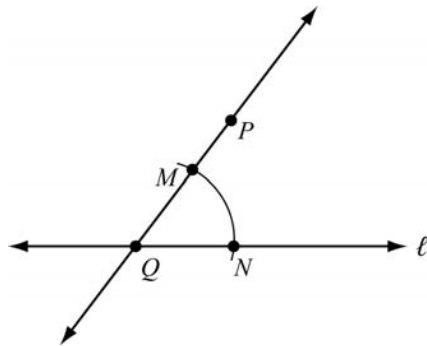
Construct: The line parallel to line  $l$  through point  $P$

Procedure:

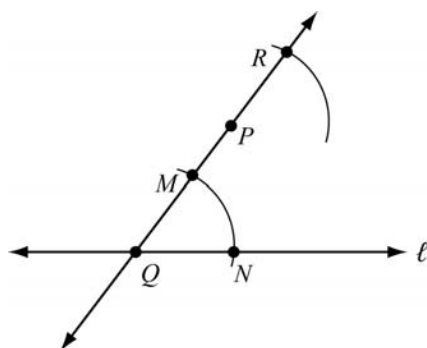
1. Draw a transversal line through point  $P$  crossing line  $l$  at a point. Label the point of intersection  $Q$ .



2. Open the compass to a width about half the distance from point  $P$  to point  $Q$ . Place the compass point on point  $Q$  and draw an arc that intersects both lines. Label the intersection of the arc and  $\overline{PQ}$  as point  $M$  and the intersection of the arc and line  $l$  as point  $N$ .

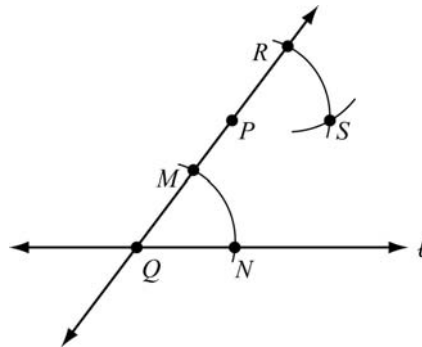


3. Without changing the compass width, place the compass point on point  $P$  and draw an arc that crosses  $\overline{PQ}$  above point  $P$ . Note that this arc must have the same orientation as the arc drawn from point  $M$  to point  $N$ . Label the point of intersection  $R$ .

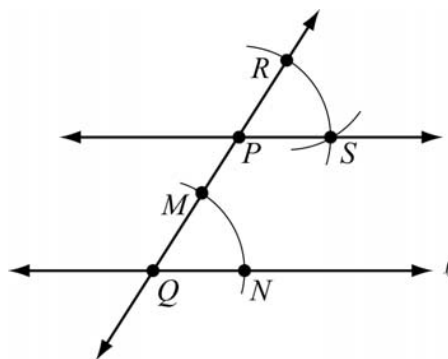


4. Set the compass width to the distance from point  $M$  to point  $N$ .

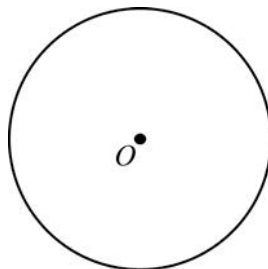
- Place the compass point on point  $R$  and draw an arc that crosses the upper arc. Label the point of intersection  $S$ .



- Using a straightedge, draw a line through points  $P$  and  $S$ .
- $\overline{PS} \parallel l$



- To **construct an equilateral triangle inscribed in a circle**, follow the steps given:



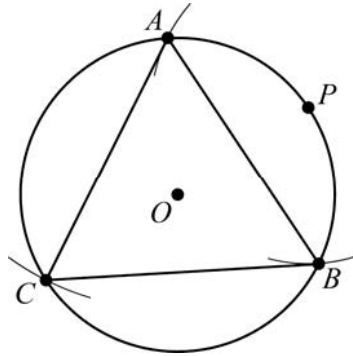
Given: Circle  $O$

Construct: Equilateral  $\triangle ABC$  inscribed in circle  $O$

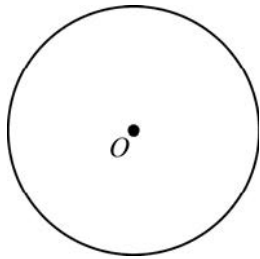
Procedure:

- Mark a point anywhere on the circle and label it point  $P$ .
- Open the compass to the radius of circle  $O$ .
- Place the compass point on point  $P$  and draw an arc that intersects the circle at two points. Label the points  $A$  and  $B$ .
- Using a straightedge, draw  $\overline{AB}$ .
- Open the compass to the length of  $\overline{AB}$ .
- Place the compass point on  $A$ . Draw an arc from point  $A$  that intersects the circle. Label this point  $C$ .

7. Using a straightedge, draw  $\overline{AC}$  and  $\overline{BC}$ .
8. Equilateral  $\triangle ABC$  is inscribed in circle  $O$ .



8. To **construct a square inscribed in a circle**, follow the steps given:

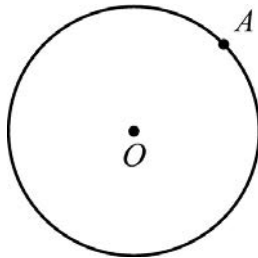


Given: Circle  $O$

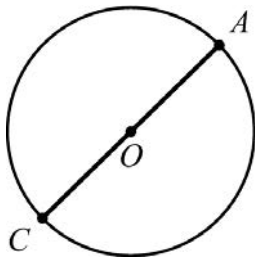
Construct: Square  $ABCD$  inscribed in circle  $O$

Procedure:

1. Mark a point anywhere on the circle and label it point  $A$ .

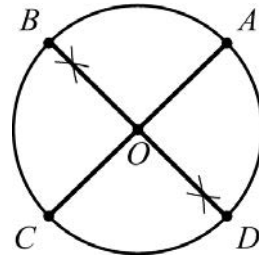


2. Using a straightedge, draw a diameter from point  $A$ . Label the other endpoint of the diameter as point  $C$ . This is diameter  $\overline{AC}$ .

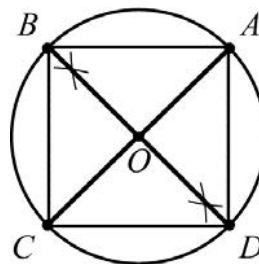




3. Construct a perpendicular bisector of  $\overline{AC}$  through the center of circle  $O$ . Label the points where it intersects the circle as point  $B$  and point  $D$ .

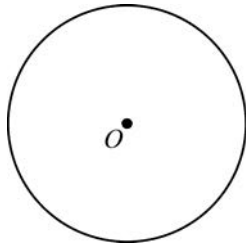


4. Using a straightedge, draw  $\overline{AB}$ ,  $\overline{BC}$ ,  $\overline{CD}$ , and  $\overline{AD}$ .



5. Square  $ABCD$  is inscribed in circle  $O$ .

9. To **construct a regular hexagon inscribed in a circle**, follow the steps given:

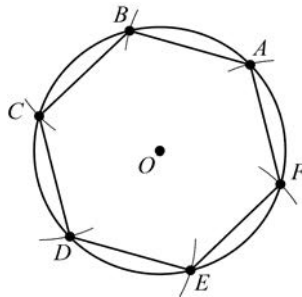


Given: Circle  $O$

Construct: Regular hexagon  $ABCDEF$  inscribed in circle  $O$

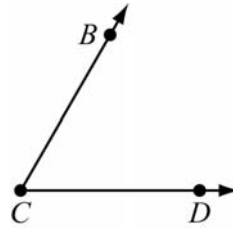
Procedure:

1. Mark a point anywhere on the circle and label it point  $A$ .
2. Open the compass to the radius of circle  $O$ .
3. Place the compass point on point  $A$  and draw an arc across the circle. Label this point  $B$ .
4. Without changing the width of the compass, place the compass point on point  $B$  and draw another arc across the circle. Label this point  $C$ .
5. Repeat this process from point  $C$  to a point  $D$ , from point  $D$  to a point  $E$ , and from point  $E$  to a point  $F$ .
6. Use a straightedge to draw  $\overline{AB}$ ,  $\overline{BC}$ ,  $\overline{CD}$ ,  $\overline{DE}$ ,  $\overline{EF}$ , and  $\overline{AF}$ .
7. Regular hexagon  $ABCDEF$  is inscribed in circle  $O$ .



## REVIEW EXAMPLES

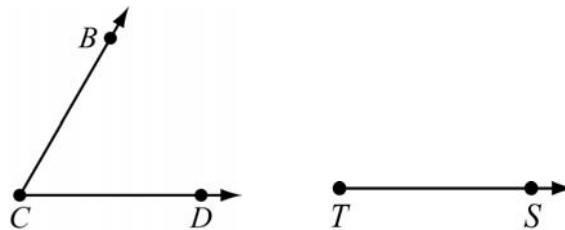
1. Allan drew angle  $BCD$ .



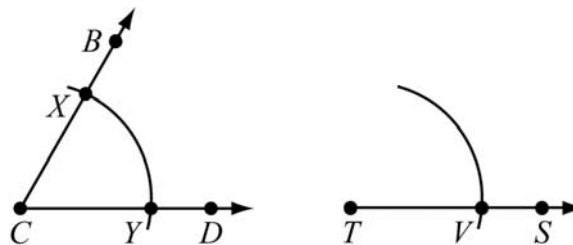
- Copy angle  $BCD$ . List the steps you used to copy the angle. Label the copied angle  $RTS$ .
- Without measuring the angles, how can you show they are congruent to one another?

**Solution:**

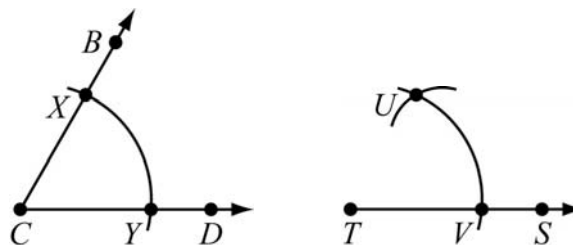
- a. Draw point  $T$ . Draw  $\overline{TS}$ .



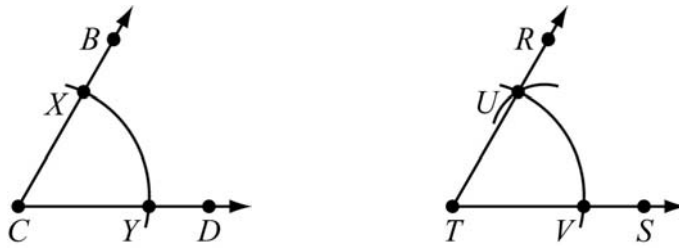
Place the point of a compass on point  $C$ . Draw an arc. Label the intersection points  $X$  and  $Y$ . Keep the compass width the same, and place the point of the compass on point  $T$ . Draw an arc and label the intersection point  $V$ .



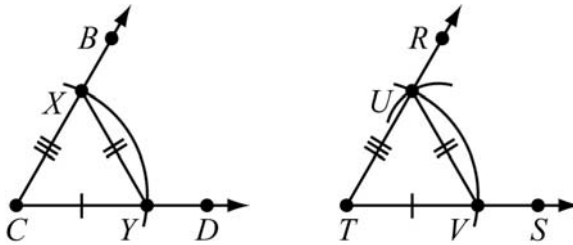
Place the point of the compass on point  $Y$  and adjust the width to point  $X$ . Then place the point of the compass on point  $V$  and draw an arc that intersects the first arc. Label the intersection point  $U$ .



Draw  $\overline{TU}$  and point  $R$  on  $\overline{TU}$ . Angle  $BCD$  has now been copied to form angle  $RTS$ .



- b. Connect points  $X$  and  $Y$  and points  $U$  and  $V$  to form  $\triangle XCY$  and  $\triangle UTV$ .  $\overline{CY}$  and  $\overline{TV}$ ,  $\overline{XY}$  and  $\overline{UV}$ , and  $\overline{CX}$  and  $\overline{TU}$  are congruent because they were drawn with the same compass width. So,  $\triangle XCY \cong \triangle UTV$  by SSS, and  $\angle C \cong \angle T$  because congruent parts of congruent triangles are congruent.

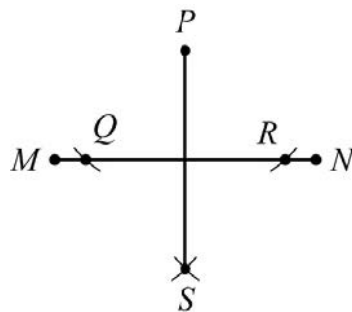


2. Construct a line segment perpendicular to  $\overline{MN}$  from a point not on  $\overline{MN}$ . Explain the steps you used to make your construction.



**Solution:**

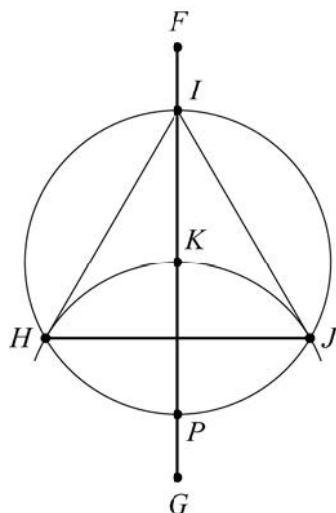
Draw a point  $P$  that is not on  $\overline{MN}$ . Place the compass point on point  $P$ . Draw an arc that intersects  $\overline{MN}$  at two points. Label the intersections points  $Q$  and  $R$ . Without changing the width of the compass, place the compass on point  $Q$  and draw an arc under  $\overline{MN}$ . Place the compass on point  $R$  and draw another arc under  $\overline{MN}$ . Label the intersection point  $S$ . Draw  $\overline{PS}$ . Segment  $PS$  is perpendicular to and bisects  $\overline{MN}$ .



3. Construct equilateral  $\triangle HIJ$  inscribed in circle  $K$ . Explain the steps you used to make your construction.

**Solution:**

(This is an alternate method from the method shown in Key Idea 7.) Draw circle  $K$ . Draw segment  $\overline{FG}$  through the center of circle  $K$ . Label the points where  $\overline{FG}$  intersects circle  $K$  as points  $I$  and  $P$ . Using the compass setting you used when drawing the circle, place a compass on point  $P$  and draw an arc passing through point  $K$ . Label the points where the arc intersects circle  $K$  as points  $H$  and  $J$ . Draw  $\overline{HJ}$ ,  $\overline{IJ}$ , and  $\overline{HI}$ . Triangle  $HIJ$  is an equilateral triangle inscribed in circle  $K$ .



## Use Coordinates to Prove Simple Geometric Theorems Algebraically

**MGSE9-12.G.GPE.4** Use coordinates to prove simple geometric theorems algebraically.

For example, prove or disprove that a figure defined by four given points in the coordinate plane is a rectangle; prove or disprove that the point  $(1, \sqrt{3})$  lies on the circle centered at the origin and containing the point  $(0, 2)$ .

(Focus on quadrilaterals, right triangles, and circles.)

### KEY IDEAS

- To prove properties about special parallelograms on a coordinate plane, you can use the midpoint, distance, and slope formulas:
  - The **midpoint formula** is  $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$ . This formula is used to find the coordinates of the midpoint of  $\overline{AB}$ , given  $A(x_1, y_1)$  and  $B(x_2, y_2)$ .
  - The **distance formula** is  $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ . This formula is used to find the length of  $\overline{AB}$ , given  $A(x_1, y_1)$  and  $B(x_2, y_2)$ .
  - The **slope formula** is  $m = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)$ . This formula is used to find the slope of a line or line segment, given any two points on the line or line segment  $A(x_1, y_1)$  and  $B(x_2, y_2)$ .
- You can use properties of quadrilaterals to help prove theorems, such as the following:
  - To prove a quadrilateral is a parallelogram, show that the opposite sides are parallel using slope.
  - To prove a quadrilateral is a rectangle, show that the opposite sides are parallel and the consecutive sides are perpendicular using slope.
  - To prove a quadrilateral is a rhombus, show that all four sides are congruent using the distance formula.
  - To prove a quadrilateral is a square, show that all four sides are congruent and consecutive sides are perpendicular using slope and the distance formula.

- You can also use diagonals of a quadrilateral to help prove theorems, such as the following:
  - To prove a quadrilateral is a parallelogram, show that its diagonals bisect each other using the midpoint formula.
  - To prove a quadrilateral is a rectangle, show that its diagonals bisect each other and are congruent using the midpoint and distance formulas.
  - To prove a quadrilateral is a rhombus, show that its diagonals bisect each other and are perpendicular using the midpoint and slope formulas.
  - To prove a quadrilateral is a square, show that its diagonals bisect each other, are congruent, and are perpendicular using the midpoint, distance, and slope formulas.

### Important Tips

- When using the formulas for midpoint, distance, and slope, the order of the points does not matter. You can use either point to be  $(x_1, y_1)$  and  $(x_2, y_2)$ , but be careful to always subtract in the same order.
- Parallel lines have the same slope. Perpendicular lines have slopes that are the negative reciprocal of each other.

### REVIEW EXAMPLE

- Quadrilateral  $ABCD$  has vertices  $A(-1, 3)$ ,  $B(3, 5)$ ,  $C(4, 3)$ , and  $D(0, 1)$ . Is  $ABCD$  a rectangle? Explain how you know.

#### Solution:

First determine whether the figure is a parallelogram. If the figure is a parallelogram, then the diagonals bisect each other. If the diagonals bisect each other, then the midpoints of the diagonals are the same point. Use the midpoint formula to determine the midpoints for each diagonal.

$$\text{Midpoint } \overline{AC} : \left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right) = \left( \frac{-1 + 4}{2}, \frac{3 + 3}{2} \right) = \left( \frac{3}{2}, \frac{6}{2} \right) = (1.5, 3)$$

$$\text{Midpoint } \overline{BD} : \left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right) = \left( \frac{3 + 0}{2}, \frac{5 + 1}{2} \right) = \left( \frac{3}{2}, \frac{6}{2} \right) = (1.5, 3)$$

The diagonals have the same midpoint; therefore, the diagonals bisect each other and the figure is a parallelogram.

A parallelogram with congruent diagonals is a rectangle. Determine whether the diagonals are congruent.

Use the distance formula to find the length of the diagonals.

$$AC = \sqrt{(4 - (-1))^2 + (3 - 3)^2} = \sqrt{(5)^2 + (0)^2} = \sqrt{25 + 0} = \sqrt{25} = 5$$

$$BD = \sqrt{(0 - 3)^2 + (1 - 5)^2} = \sqrt{(-3)^2 + (-4)^2} = \sqrt{9 + 16} = \sqrt{25} = 5$$

The diagonals are congruent because they have the same length.

The figure is a parallelogram with congruent diagonals, so the figure is a rectangle.

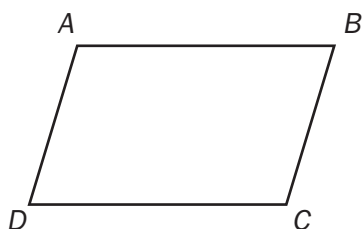
**SAMPLE ITEMS**

1. Which information is needed to show that a parallelogram is a rectangle?

- A. The diagonals bisect each other.
- B. The diagonals are congruent.
- C. The diagonals are congruent and perpendicular.
- D. The diagonals bisect each other and are perpendicular.

**Correct Answer:** B

2. Look at quadrilateral  $ABCD$ .



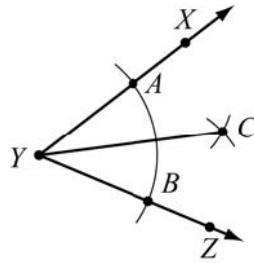
Which information is needed to show that quadrilateral  $ABCD$  is a parallelogram?

- A. Use the distance formula to show that diagonals  $AC$  and  $BD$  have the same length.
- B. Use the slope formula to show that segments  $AB$  and  $CD$  are perpendicular and segments  $AD$  and  $BC$  are perpendicular.
- C. Use the slope formula to show that segments  $AB$  and  $CD$  have the same slope and segments  $AD$  and  $BC$  have the same slope.
- D. Use the distance formula to show that segments  $AB$  and  $AD$  have the same length and segments  $CD$  and  $BC$  have the same length.

**Correct Answer:** C



3. Consider the construction of the angle bisector shown.



Which could have been the first step in creating this construction?

- A. Place the compass point on point  $A$  and draw an arc inside  $\angle Y$ .
- B. Place the compass point on point  $B$  and draw an arc inside  $\angle Y$ .
- C. Place the compass point on vertex  $Y$  and draw an arc that intersects  $\overline{YX}$  and  $\overline{YZ}$ .
- D. Place the compass point on vertex  $Y$  and draw an arc that intersects point  $C$ .

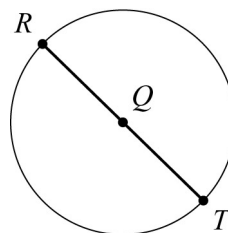
Correct Answer: C

4. Consider the beginning of a construction of a square inscribed in circle  $Q$ .

Step 1: Label point  $R$  on circle  $Q$ .

Step 2: Draw a diameter through  $R$  and  $Q$ .

Step 3: Label the point where the diameter intersects the circle as point  $T$ .



What is the next step in this construction?

- A. Draw radius  $\overline{SQ}$ .
- B. Label point  $S$  on circle  $Q$ .
- C. Construct a line segment parallel to  $\overline{RT}$ .
- D. Construct the perpendicular bisector of  $\overline{RT}$ .

Correct Answer: D