ASSIGNMENT PROBLEM

The assignment problem is a special case of transportation problem in which the objective is to assign ‘m’ jobs or workers to ‘n’ machines such that the cost incurred is minimized.

JOBS

1 2 -------- n

	1				
	2	**C11**	**C12**	**----------------**	**C1n**
	--	**C21**	**C22**	**----------------**	**C2n**
WORKERS	--	**-**	**-**		**-**
	--	**-**	**-**		**-**
	n	**-**	**-**		**-**
		Cn1	**Cn2**	**--------------**	**Cmn**

The element Cij represents the cost of assigning worker I to job (I,j= 1,2,---n). There is no loss in generality in assuming that the number of workers always equals the number of jobs because we can always add fictitious (untrue or fabricated) workers or fictitious jobs to effect this result.

The assignment model is actually a special case of the transportation model in which the workers represent the sources and the jobs represent the destinations.

The supply amount at each source and the demand amount at each destination exactly equal 1.

The cost of transporting workers I to job j is C_{ij} .

The assignment model can be solved directly as a regular transportation model.

The fact that all the supply and demand amounts equal 1 has led to the development of a simple solution algorithm called the Hungarian method.

Difference between transportation and Assignment problems		
Sl. No.	Transportation	Assignment
1	This problem contains specific demand and requirement in columns and rows	The demand and availability in each column or row is one
2	Total demand must be equal to the total availability	It is a square matrix. The no of rows must be equal to the no of columns.
3	The optimal solution involves the following conditions M+N-1 M $\longrightarrow$ rows N $\longrightarrow$ columns	The optimal solutions involves one assignment in each row and each column
4	There is no restriction in the number of allotments in any row or column	There should be only one allotment in each row and each column
5	It is a problem of allocating multiple resources to multiple markets	It is a problem of allocation resources to job j

Assignment Algorithm (Hungarian Method)

Step I :- Create Zero elements in the cost matrix by subtract the smallest element in each row column for the corresponding row and column.

Step II:- Drop the least number of horizontal and vertical lines so as to cover all zeros if the no of there lines are 'N'

i) If N = n (n=order of the square matrix) then an optimum assignment has been obtained

ii) If N<n proceeds to step III

Step III :- determine the smallest cost cell from among the uncrossed cells subtract. This cost from all the uncrossed cells and add the same to all those cells laying in the intersection of horizontal and vertical lines.

Step IV:- repeat steps II and III until N=n.

Step V:- examine the rows (column) successively until a row (column) with are zero is found enclose the zero in a square (0) and cancel out (~~0~~) any other zeros laying in the column (row) of the Matrix. Continue in this way until all the rim requirements are satisfied i.e N=n.

Step VI:- repeat step 5 successively one of the following arises.

i) No unmarked zero is left

ii) If more then one unmarked zeros in one column or row.

In case i) the algorithm stops

ii)Encircle one of the unmarked zeros arbitrary and mark a cross in the cells of remaining zeroes in it's row and column. Repeat the process until no unmarked zero is left in the cost matrix.

Step VII) we now have exactly one encircled zero in each row and each column of the cost matrix. The assignment schedule corresponding to there zeros is the optimum (maximal) assignment.

Note: the above procedure for assignment is Hungarian assignment method

Problem 1.

Three jobs A B C are to be assigned to three machines x Y Z. The processing costs are as given in the matrix shown below. Find the allocation which will minimize the overall processing cost.

Machines				
Jobs		X	Y	Z
	A	19	28	31
	B	11	17	16
	C	12	15	13

Solution:

Step 1: create zero in each row or column by subtracting by selecting least number in each row and column

Row Minimization

0	9	12
0	6	5
0	3	1

Column Minimization

0	6	11
0	3	4
0	0	0

Now draw Horizontal and vertical lines

0	6	11
0	3	4
0	0	0

Here, no of horizontal lines is one and vertical line is one

The order of matrix is 3 x 3, therefore, N≠ n

Now, in the uncrossed cell the least cost is selected and subtracted for the remaining uncrossed cell by the least value and for the intersection of the horizontal line and vertical line the least value should be added and the resulting matrix.

0	3	8
0	0	1
3	0	0

The above matrix has two horizontal line and one vertical line which satisfies our condition N= n

{0}	3	8
~~0~~	{0}	~~1~~
3	~~0~~	{0}

The assignment are A → X = 19

B → Y = 17

C → Z = 13

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Problem 2

Solve of the assignment problem

	I	II	III	IV	V
1	11	17	8	16	20
2	9	7	12	6	15
3	13	16	15	12	16
4	21	24	17	28	26
5	14	10	12	11	15

Solns:

Row Minimization

3	9	0	8	12
3	1	6	0	9
1	4	3	0	4
4	7	0	11	9
4	0	2	1	5

Column Minimization

2	9	0	8	8
2	1	6	0	1
0	4	3	0	0
3	7	0	11	5
3	0	2	1	1

N≠n

5≠4

2	9	0	8	8
2	1	6	0	1
0	4	3	0	0
3	7	0	11	5
3	0	2	1	1

The least value in the uncrossed cell is 1, it is subtracted for the uncrossed cell and added for intersection of the vertical line and horizontal.

1	9	0	8	7
1	1	6	0	0
0	4	3	0	0
2	7	0	11	4
2	0	2	1	0

Again N ≠ n   least value is one again

0	8	0	7	6
1	1	6	0	0
0	4	3	0	0
1	6	0	10	3
2	0	2	1	0

Here, it satisfies our condition N=n

Now, the assignment for the optimum table

[0]	8	~~0~~	7	6
1	1	6	[0]	~~0~~
~~0~~	4	3	~~0~~	[0]
1	6	[0]	10	3
2	[0]	2	1	~~0~~

Assignment

1 ⟶ I = 11

2 ⟶ IV = 6

3 ⟶ V = 16

4 ⟶ III = 17

5 ⟶ II = 10

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Problem 3

Using the following cost matrix, determine a) optimal job assignment b) the cost of assignments

		JOB				
MECHANIC		1	2	3	4	5
	A	10	3	3	2	8
	B	9	7	8	2	7
	C	7	5	6	2	4
	D	3	5	8	2	4
	E	9	10	9	6	10

Row Minimization

8	1	1	0	6
7	5	6	0	5
5	3	4	0	2
1	3	6	0	2
3	4	3	0	4

Column minimization

7	0	0	0	4
6	4	5	0	3
4	2	3	0	0
0	2	5	0	0
2	3	2	0	2

Draw the horizontal and vertical lines

7	0	0	0	4
6	4	5	0	3
4	2	3	0	0
0	2	5	0	0
2	3	2	0	2

Here, N≠n, 4≠ 5

Then, we have to select the least value in the uncrossed cell i.e 2 the result table.

9	0	0	2	6
6	2	2	0	3
4	0	1	0	0
0	0	3	0	0
2	1	0	0	2

N=n satisfies our condition, so optimal assignment can be done

9	[0]	~~0~~	2	6
6	2	2	[0]	3
4	~~0~~	1	~~0~~	[0]
[0]	~~0~~	3	~~0~~	~~0~~
2	1	[0]	~~0~~	2

A $\longrightarrow$ 2 = 3

B $\longrightarrow$ 4 = 2

C $\longrightarrow$ 5 = 4

D $\longrightarrow$ 1 = 3

E $\longrightarrow$ 3 = 9

21 the minimum cost is Rs. 21

Problem 4

Job shop needs to assign 4 jobs to 4 workers. The cost of performing a job is a function of the skills of the workers. Table summarizes the cost of the assignments. Worker1 cannot do job3, and worker 3 cannot do job 4. Determine the optimal assignment using the Hungarian method.

		Job			
		1	2	3	4
	1	Rs. 50	Rs.50	--	Rs.20
Worker	2	Rs.70	Rs.40	Rs.20	Rs.30
	3	Rs.90	Rs.30	Rs.50	--
	4	Rs.70	Rs.20	Rs60	Rs70

Row Minimization

30	30	--	0
50	20	0	10
60	0	20	--
50	0	40	50

Column minimization

0	30	--	0
20	20	0	10
30	0	20	--
20	0	40	50

N≠n

3≠4

The least value in the uncrossed cell is 10 and the resulting table will be as follows

~~0~~	~~40~~	~~--~~	~~0~~
~~10~~	~~20~~	~~0~~	~~0~~
20	~~0~~	20	--
10	~~0~~	40	40

N≠n

3=4

The least value in the uncrossed cell is 10

~~0~~	~~50~~	~~--~~	~~0~~
~~10~~	~~20~~	~~0~~	~~0~~
~~10~~	~~0~~	~~10~~	~~--~~
~~0~~	~~0~~	~~30~~	~~30~~

N=n

The given problem satisfies the condition, the assignment can be made for the optimal table.

~~0~~	50	--	[0]
10	20	[0]	~~0~~
10	[0]	10	--
[0]	~~0~~	30	30

1--------- 4 = 20

2--------- 3 = 20

3--------- 2 = 30

4--------- 1 = 70

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4----------1---------4, 2--------3--------2

There are two sequences in the given problem

Problem 5

A typical assignment problem, presented in the classic manner, is shown in Fig. Here there are five machines to be assigned to five jobs. The numbers in the matrix indicate the cost of doing each job with each machine. Jobs with costs of M are disallowed assignments. The problem is to find the minimum cost matching of machines to jobs.

	J1	J2	J3	J4	J5
M1	M	8	6	12	1
M2	15	12	7	M	10
M3	10	M	5	14	M
M4	12	M	12	16	15
M5	18	17	14	M	13

Fig 1 Matrix model of the assignment problem.

The network model is in shown in Fig.2. It is very similar to the transportation model except the external flows are all +1 or -1. The only relevant parameter for the assignment model is arc cost (not shown in the figure for clarity) ; all other parameters should be set to default values. The assignment network also has the bipartite structure.

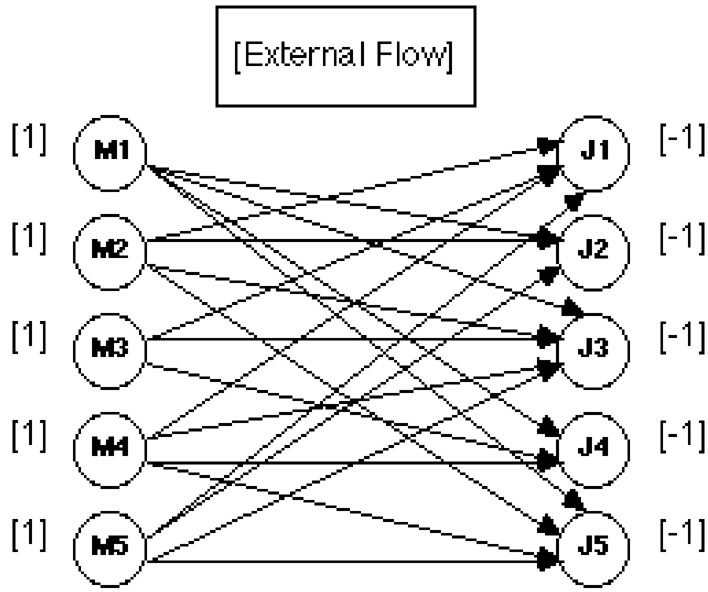

Figure 2. Network model of the assignment problem.

M	8	6	12	1
15	12	7	M	10
10	M	5	14	M
12	M	12	16	15
18	17	14	M	13

This is the given problem, using Hungarian method we solve the problem

Row minimization

M	7	5	11	0
8	5	0	M	3
5	M	0	9	M
0	M	0	4	3
5	4	1	M	0

Column minimization

M	3	5	7	0
8	1	0	M	3
5	M	0	5	M
0	M	0	0	3
5	0	1	M	0

M	3	5	7	0
8	1	0	M	3
5	M	0	5	M
0	M	0	0	3
5	0	1	M	0

N≠n, 4≠5 so select least value in

The uncrossed cell and subtract

M	2	5	6	0
7	0	0	M	3
4	M	0	4	M
0	M	1	0	3
2	0	1	M	0

N≠n, 4≠5 so select least value in The uncrossed cell and subtract

M	2	5	2	0
3	0	0	M	3
0	M	0	0	M
0	M	1	0	3
2	0	1	M	0

M	2	5	2	[0]
3	~~0~~	[0]	M	3
~~0~~	M	~~0~~	[0]	M
[0]	M	1	~~0~~	3
2	[0]	1	M	~~0~~

N=n and the assignment can be done

The solution to the assignment problem as shown in Fig. 3 has a total flow of 1 in every column and row, and is the assignment that minimizes total cost.

	J1	J2	J3	J4	J5
M1	0	0	0	0	1
M2	0	0	1	0	0
M3	0	0	0	1	0
M4	1	0	0	0	0
M5	0	1	0	0	0

Figure 3. Solution to the assignment Problem

Problem 6.

Four different jobs can be done on four different machines and take down time costs are prohibitively high for change overs. The matrix below gives the cost in rupees of producing job on machine j;

Jobs	Machine			
	M1	M2	M3	M4
J1	5	7	11	6
J2	8	5	9	6
J3	4	7	10	7
J4	10	4	8	3

How the jobs should be assigned to the various machines so that the total cost is minimized.

Row minimization

0	2	6	1
3	0	4	1
0	3	6	3
7	1	5	0

Column minimization

0	2	2	1
3	0	0	1
0	3	2	3
7	1	1	0

Draw the horizontal and vertical lines which covers max no of zeros

0	2	2	1
3	0	0	1
0	3	2	3
7	1	1	0

N≠n 3≠4

The least value is 1, the resulting table

0	1	1	1
4	0	0	2
0	2	1	3
7	0	0	0

N≠n

3≠4

The least value is 1, the resulting table

0	0	0	0
5	0	0	2
0	1	0	2
8	0	0	0

N=n

4=4 the assignment can be made for the above optimal table

~~0~~	~~0~~	~~0~~	[0]
5	[0]	0	2
[0]	1	~~0~~	2
8	~~0~~	[0]	~~0~~

J1 ----------M4 = 6

J2----------M2 = 5

J3----------M1 = 4

J4----------M3 = 8

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Alternate solution

[0]	~~0~~	~~0~~	~~0~~
5	[0]	~~0~~	2
~~0~~	1	[0]	2
8	~~0~~	~~0~~	[0]

J1 ----------M1 = 5

J2----------M2 = 5

J3----------M3 = 10

J4----------M4 = 3

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Problem 7

A company has 5 jobs tobe done the following matrix shows the return in Rs. of assigning ith machine (i=1,2,3,---5) to the jth job (j=1,2,3,----n). Assign the 5 jobs to the 5 machines so as to maximize the expected profit.

JOB						
Machine		1	2	3	4	5
	1	5	11	10	12	4
	2	2	4	6	3	5
	3	3	12	5	14	6
	4	6	14	4	11	7
	5	7	9	8	12	8

Since, the given problem is maximum

Step 1: to convert the problem to a minimum by multiply all elements C_{ij} Of the assignment matrix by -1

Then the given problem will in the form as shown below

JOB						
Machine		1	2	3	4	5
	1	-5	-11	-10	-12	-4
	2	-2	-4	-6	-3	-5
	3	-3	-12	-5	-14	-6
	4	-6	-14	-4	-11	-7
	5	-7	-9	-8	-12	-8

Step2: select the most –ve and subtract with other elements of the matrix minz= - (-maxZ)

In the matrix the most –ve value is -14. Using this value the matrix is subtracted and the resulting is the minimization matrix. This can be used for finding the optimal assignment table using usual procedure to solve the problem.

JOB						
Machine		1	2	3	4	5
	1	9	3	4	2	10
	2	12	10	8	11	9
	3	11	2	9	0	8
	4	8	0	10	3	7
	5	7	5	6	2	6

Example

C11= -5 – (-14) = 9 and continued for all other element

Step 3:- Using the above table i.e. minZ matrix table and all cost elements non-ve. The Hungarian method can be applied to find the optimal assignment problem.

Row minimization

7	1	2	0	8
4	2	0	3	1
11	2	9	0	8
8	0	10	3	7
5	3	4	0	4

Column minimization

3	~~1~~	2	~~0~~	7
~~0~~	~~2~~	~~0~~	~~3~~	~~0~~
7	~~2~~	9	~~0~~	7
4	~~0~~	10	~~3~~	6
1	~~3~~	4	~~0~~	3

N=3, n=5x5

N≠n, select the minimum value from the uncrossed cell and subtract for all the elements of uncrossed cell and add for the intersection of horizontal and vertical.

~~2~~	~~1~~	1	~~0~~	6
~~0~~	~~3~~	~~0~~	~~4~~	~~0~~
~~6~~	~~2~~	8	~~0~~	6
~~3~~	~~0~~	9	~~3~~	5
~~0~~	~~3~~	3	~~0~~	2

N=4 n=5x5

N≠n

The least value in the uncrossed cell is 1 again and subtracts using this value for all other elements and add for intersection of horizontal and vertical

~~2~~	~~1~~	~~0~~	~~0~~	~~5~~
~~1~~	~~4~~	~~0~~	~~5~~	~~0~~
~~6~~	~~2~~	~~7~~	~~0~~	~~5~~
~~3~~	~~0~~	~~8~~	~~3~~	~~4~~
~~0~~	~~3~~	~~2~~	~~0~~	~~1~~

N=n, for assignment the optimal table is obtained

2	1	[0]	~~0~~	5
1	4	~~0~~	5	[0]
6	2	7	[0]	5
3	[0]	8	3	4
[0]	3	2	~~0~~	1

Now the assignment is

1-----------3 = 10

2-----------5 = 5

3-----------4 = 14

4-----------2 = 14

5-----------1 = 7

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Problem 7

A marketing manager has 5 elements and there are 5 sales districts. Considering the capabilities of the salesmen and the nature of districts, the estimates made by the marketing manager for the sales per month (in 1000 rupees) for each salesman in each district would be as follows.

	A	B	C	D	E
1	32	38	40	28	40
2	40	24	28	21	36
3	41	27	33	30	37
4	22	38	41	36	36
5	29	33	40	35	39

Find the assignment of salesmen to the districts that will result in the maximum sale.

Solution:

The given problem is profit matrix. To maximize the profit, first we must convert it minimization. To convert to minimization we must select the maximum value of the matrix i.e., 41. This value is subtracted for all other elements in the matrix and the resulting matrix is minimization.

	A	B	C	D	E
1	9	3	1	13	1
2	1	17	13	20	5
3	0	14	8	11	4
4	19	3	0	5	5
5	12	8	1	6	2

Using above table we can solve the given problem by the Hungarian method.

Row minimization

8	2	0	12	0
0	16	12	19	4
0	14	8	11	4
19	3	0	5	5
11	7	0	5	1

Column minimization

8	0	0	7	0
0	14	12	14	4
0	12	8	6	4
19	1	0	0	5
11	5	0	0	1

Drawing the horizontal and vertical lines

N=4 n5x5

N≠n, now select the least value in the uncrossed cell and subtract to all the uncrossed cell and add to the intersection of horizontal and vertical line. The least value is 4

8	0	0	7	0
0	10	8	10	0
0	8	4	2	0
19	1	0	0	5
11	5	0	0	1

N=n, it satisfies our condition and now we can assign the workers to jobs using above table

8	[0]	~~0~~	7	~~0~~
[0]	10	8	10	~~0~~
~~0~~	8	4	2	[0]
19	1	[0]	~~0~~	5
11	5	~~0~~	[0]	1

1---------B = 38

2---------A = 40

3---------E = 37

4---------C = 41

5---------D = 35

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Problem 8

A company has four territories open and four salesmen available for assignment. The territories are not equally rich in their sales potential; it is estimated that a typical salesman operating in each territory would bring in the following annual sales:

Territory :	I	II	III	IV
Annual Sales (Rs)	60,000	50,000	40,000	30,000

Four salesmen are also considered to differ in chair ability: it is estimated that, working under the same conditions, their yearly sales would be proportionately as follow:

Salesmen:	A	B	C	D
Proportion:	7	5	5	4

If the criterion is maximum expected total sales, then including answer is to assign the best salesman to the richest territory, the next best salesman to the second richest, and so on, verify this answer by the assignment technique.

Solution:

Step 1 to construct the effectiveness of the matrix

By taking Rs. 10000/- as one unit and the sales proportion and the maximum sales matrix is obtained as follows:

		Sales in 10 thousand of rupees			
Sales Proportion		6	5	4	3
		I	II	III	IV
7	A	42	35	28	21
5	B	30	25	20	15
5	C	30	25	20	15
4	D	24	20	16	12

To find the value of c11= sales proportion X sales of territory

= 7X6=42

In the same it is continued for the remaining cells

Step 2: to convert the maximum sales matrix to minimum sales matrix

By simply multiplying each element of given matrix by -1. Thus resulting matrix becomes:

	I	II	III	IV
A	-42	-35	-28	-21
B	-30	-25	-20	-15
C	-30	-25	-20	-15
D	-24	-20	-16	-12

Step 3: select the most negative in the matrix i.e. is -42. With this element subtract all the elements in the matrix. The resulting is minimization table

	I	II	III	IV
A	0	7	14	21
B	12	17	22	27
C	12	17	22	27
D	18	22	26	30

Now, using the above table we can apply the Hungarian method to find the assignment for the given problem and the value should be taken from the original table since, it is a maximization problem

Row minimization

0	7	14	21
0	5	10	15
0	5	10	15
0	4	8	12

Column minimization

0	3	6	9
0	1	2	3
0	1	2	3
0	0	0	0

Draw horizontal and vertical lines

~~0~~	3	6	9
~~0~~	1	2	3
~~0~~	1	2	3
~~0~~	~~0~~	~~0~~	~~0~~

N≠n, 2≠4 so select the least value of the uncrossed cell and subtract

The least value is 1

0	2	5	8
0	0	1	2
0	0	1	2
0	0	0	0

N≠n, 3≠4 so select the least value of the uncrossed cell and subtract

The least value is 1

0	2	4	7
0	0	0	1
0	0	0	1
0	0	0	0

N=n, 4=4, the assignment of the given problem

[0]	2	4	7
~~0~~	[0]	~~0~~	1
~~0~~	~~0~~	[0]	1
~~0~~	~~0~~	~~0~~	[0]

A--------I

B-------II

C-------III

D------IV

Problem 9:

Alpha Corporation has four plants each of which can manufacture any one of four products production costs differ from one plant to another as do sales revenue. Given the revenue and cost data below, obtain which product each plant should produce to maximize profit.

	Sales revenue (Rs. 000s Product)			
Plant	1	2	3	4
A	50	68	49	62
B	60	70	51	74
C	55	67	53	70
D	58	65	54	69

	Production costs (Rs. 000s Product)			
Plant	1	2	3	4
A	49	60	45	61
B	55	63	45	69
C	52	62	49	68
D	55	64	48	66

Solution:

Now, we have found the profit matrix by using sales revenue and production cost.

Profit = sales - cost

Profit matrix

	1	2	3	4
A	1	8	4	1
B	5	7	6	5
C	3	5	4	2
D	3	1	6	3

Now we find the minimization matrix, by selecting the highest profit in the profit matrix i.e. 8 is subtract all the elements in the matrix and resulting will be the minimization matrix of the given problem.

7	0	4	7
3	1	2	3
5	3	4	6
5	7	2	5

Using Hungarian method

Row minimization

7	0	4	7
2	0	1	2
2	0	1	3
3	5	0	3

Column minimization

5	0	4	5
0	0	1	0
0	0	1	1
1	5	0	1

Draw the horizontal and vertical lines

5	0	4	5
0	0	1	0
0	0	1	1
1	5	0	1

N=n, the assignment can be done for the above table

5	[0]	4	5
~~0~~	~~0~~	1	[0]
[0]	~~0~~	1	1
1	5	[0]	1

A---------2

B---------4

C--------1

D--------3

Problem 11

An air-line operates seven days a week has time-table shown below. Crews must have a minimum layover (rest) time of 5 hrs, between flights. Obtain the pair of flights that minimimizes layover time away from home. For any given pair the crews will e based at the city that result in the smaller layover.

Delhi - Jaipur			Jaipur-Delhi		
Flight No.	Depart	Arrive	Flight No.	Depart	Arrive
1	7.00 AM	8.00 AM	101	8.00 AM	9.15 AM
2	8.00 AM	9.00 AM	102	8.30 AM	9.45 AM
3	1.30 PM	2.30 P.M	103	12.00 NOON	1.15 PM
4	6.30AM	7.30 PM	104	5.30 PM	6.45 PM

for each pair, mention the town where the crews should be based.

SOLUTION:

Step1 construct the table for layour times between flights when crew is based at Delhi, for simplicity consider 15 minutes = 1unit.

Table 1: layover times when crew based at Delhi

Flights	101	102	103	104
1	96	98	112	38
2	92	94	108	34
3	70	72	86	108
4	50	52	66	88

Since, the crew must have a minimum layover of 5 hrs between flights

The layover time between flights 1 and 101 will be 24 hrs (96 units)from 8.00 AM to 8.00 AM next day i.e flight 1 arrives jaipur at 8.00 am and leaves the jaipur 8.00 am next day because of minimum layover is 5 hrs between flights and other flights is there in between so flight will be there next day only.

Flight 1 to 102 will be (98units) 8.00 am arrives jaipur leaves jaipur 8.30 am next day= 24 hrs+30 minutes

Flight 1 to 103 will be (112 units) 8.00 am arrives jaipur leaves jaipur 12.00 noon next day= 24 hrs +4 hrs =112 units

Flight 1 to 104 will be (38 units)8.00 am arrives jaipur leaves jaipur 5.30 pm on the same day = 9 hrs +30 min = 38 mins

The layover time between Flight 2 to 101 will be (9.00 am arrival and depart from jaipur 8.00 am next day) =23 hrs = 92 units

Flight 2 to 102 will be (9.00 am arrives jaipur and depart from jaipur 8.30 am next day) = 23 hrs +30 minutes = 94 units

Flight 2 to 103 will be (9.00 am arrives jaipur and depart from jaipur 12.00 noon next day) = 24 hrs +3 hrs = 108 units

Flight 2 to 104 will be (9.00 am arrives jaipur and depart from jaipur 5.30 pm same day) = 8 hrs +30 minutes= 34 units

The layover time between Flight 3 to 101 will be (2.30 pm arrival and depart from jaipur 8.00 am next day) =17 hrs + 30 minutes = 70 units

Flight 3 to 102 will be (2.30 pm arrives jaipur and depart from jaipur 8.30 am next day) = 18hrs = 72 units

Flight 3 to 103 will be (2.30 pm arrives jaipur and depart from jaipur 12.00 noon next day) = 21hrs + 30 minutes = 86 units

Flight 3 to 104 will be (2.30 pm arrives jaipur and depart from jaipur 5.30 pm next day) = 24hrs+3hrs= 108 units

The layover time between Flight 4 to 101 will be (7.30 pm arrival and depart from jaipur 8.00 am next day) =12 hrs +30 minutes = 50 units

Flight 4 to 102 will be (7.30 pm arrives jaipur and depart from jaipur 8.30 am next day) = 13hrs = 52 units

Flight 4 to 103 will be (7.30 pm arrives jaipur and depart from jaipur 12.00 noon next day) = 16hrs + 30 minutes = 66 units

Flight 4 to 104 will be (7.30 pm arrives jaipur and depart from jaipur 5.30 pm next day) = 22hrs= 88 units

Table 2: layover times when crew based at jaipur

Flights	101	102	103	104
1	87	85	71	49
2	91	89	75	53
3	113	111	97	75
4	37	35	21	95

Arrival and depart when crew is based in jaipur

Flight No.	Arrive(delhi)	Flight No.	Depart (Delhi)
101	9.15 AM	1	7.00 AM
102	9.45 AM	2	8.00 AM
103	1.15 PM	3	1.30 PM
104	6.45 PM	4	6.30AM

Since, the crew must have a minimum layover of 5 hrs between flights

The layover time between flights 101 and 1 will be 21 hrs+ 45 minutes (87 units) from 9.15 AM to 7.00 AM next day by flight no 1 i.e flight 101 arrives Delhi at 9.15 am and leaves the Delhi 7.00 am next day by flight no 1 because of minimum layover is 5 hrs between flights and no other flights is there in between so flight will there next day only.

Flight 101 to 2 will be (91 units) 9.15 am arrives Delhi leaves Delhi 8.00 am next day= 22 hrs+45 minutes

Flight 101 to 3 will be (113 units) 9.15 am arrives Delhi leaves Delhi 1.30 pm next day= 28 hrs +15 minutes = 113 units

Flight 101 to 4 will be (38 units) 9.15 am arrives Delhi leaves Delhi 6.30 pm on the same day = 9 hrs +15 min = 37 mins

The layover time between Flight 102 to 1 will be (9.45 am arrival and depart from Delhi 7.00 am next day) =21 hrs+15 minutes = 85 units

Flight 102 to 2 will be (9.45 am arrives Delhi and depart from Delhi 8.00 am next day) = 22hrs + 15 minutes = 89 units

Flight 102 to 3 will be (9.45 am arrives Delhi and depart from Delhi 1.30 pm next day) = 27 hrs +45 minutes = 111 units

Flight 102 to 4 will be (9.45 am arrives Delhi and depart from Delhi 6.30 pm same day) = 8 hrs +45 minutes= 35 units

The layover time between Flight 103 to 1 will be (1.15 pm arrival and depart from Delhi 7.00 am next day) =17hrs + 45 minutes = 71 units

Flight 103 to 2 will be (1.15 pm arrives Delhi and depart from Delhi 8.00 am next day) = 18hrs + 45 minutes = 75 units

Flight 103 to 3 will be (1.15 pm arrives Delhi and depart from Delhi 1.30 pm next day) = 24 hrs +15 minutes = 97 units

Flight 103 to 4 will be (1.15 pm arrives Delhi and depart from Delhi 6.30 pm same day) = 5 hrs + 15 minutes= 21 units

The layover time between Flight 104 to 1 will be (6.45 pm arrival and depart from Delhi 7.00 am next day) =12hrs + 15 minutes = 49 units

Flight 104 to 2 will be (6.45 pm arrives Delhi and depart from Delhi 8.00 am next day) = 13hrs + 15 minutes = 53 units

Flight 104 to 3 will be (6.45 pm arrives Delhi and depart from Delhi 1.30 pm next day) =18 hrs +45 minutes = 75 units

Flight 104 to 4 will be (6.45 pm arrives Delhi and depart from Delhi 6.30 pm same day) = 23 hrs + 45 minutes= 95 units

Step 3:construct the table for minimum layover times between flights with the help of Table 1 and Table 2 layover times marked * denote that the crew is based at jaipur.

Table 3:

Flights	101	102	103	104
1	87*	85*	71*	38
2	91*	89*	75	34
3	70	72	86	75
4	37*	35*	21*	88

Using Hungarian method we solve the above table and the assignment are as shown in the table

~~0~~	~~0~~	[0]	~~0~~
12	8	8	[0]
[0]	~~0~~	28	50
4	[0]	~~0~~	100

The optimal assignments are

Flight 1-103

Flight 2-104

Flight 3-101

Flight 4-102

Unbalanced Assignment Problem

If the cost matrix of an assignment problem is not a square matrix (number of sources is not equal to the number of destinations),

The assignment problem is called an unbalanced assignment problem.

In such cases, fictitious rows and columns are added in the matrix so as to form a square matrix.

Then the usual assignment algorithm can be applied to this resulting balanced problem.

Problem 1:

A company is faced with the problem of assigning six different machines to five different jobs. The costs are estimated as follows (in hundreds of rupees):

	Jobs					
		1	2	3	4	5
Machines	1	2.5	5.0	1.0	6	1.0
	2	2.0	5.0	1.5	7	3.0
	3	3.0	6.5	2.0	9	4.5
	4	3.5	7.0	2.0	9	4.5
	5	4.0	7.0	3.0	9	6.0
	6	6.0	9.0	5.0	10	6.0

Solve the problem assuming that the objective is to minimize the total cost.

Solution:

The matrix is 6x5, and then given problem is unbalanced assignment problem. So we introduce fictitious job i.e Job 6 in the cost matrix in order to get the balanced assignment problem. The costs corresponding to such column are always taken as zero.

	Jobs						
		1	2	3	4	5	6
Machines	1	2.5	5.0	1.0	6	1.0	0
	2	2.0	5.0	1.5	7	3.0	0
	3	3.0	6.5	2.0	9	4.5	0
	4	3.5	7.0	2.0	9	4.5	0
	5	4.0	7.0	3.0	9	6.0	0
	6	6.0	9.0	5.0	10	6.0	0

Then, the problem can be solved by usual manner by Hungarian method.

Row Minimization

2.5	5.0	1.0	6	1.0	0
2.0	5.0	1.5	7	3.0	0
3.0	6.5	2.0	9	4.5	0
3.5	7.0	2.0	9	4.5	0
4.0	7.0	3.0	9	6.0	0
6.0	9.0	5.0	10	6.0	0

Row minimization remains as same as original problem

Column minimization

~~0.5~~	~~0~~	~~0~~	~~0~~	~~0~~	~~0~~
~~0~~	~~0~~	~~0.5~~	~~1~~	~~2.0~~	~~0~~
1.0	1.5	1.0	3	3.5	~~0~~
1.5	2.0	1.0	3	3.5	~~0~~
2.0	2.0	2.0	3	5.0	~~0~~
4.0	4.0	4.0	4	5.0	~~0~~

N=3, n=5x5, N≠n, so the least value in the uncrossed value is 1.0 and with this value all the uncrossed cell is subtracted and the resulting matrix

~~0.5~~	~~0~~	~~0~~	~~0~~	~~0~~	~~1~~
~~0~~	~~0~~	~~0.5~~	~~1~~	~~2.0~~	~~1~~
~~0~~	~~0.5~~	~~0~~	~~2~~	~~3.5~~	~~1~~
~~0.5~~	~~1.0~~	~~0~~	~~2~~	~~3.5~~	~~1~~
1.0	1.0	1.0	2	5.0	~~0~~
3.0	3.0	3.0	3	5.0	~~0~~

N=5, n=6x6, N≠n

The least value in the uncrossed cell is 1 and using this value it is subtracted to the entire uncrossed cell and for the intersection of horizontal and vertical lines it is added and the resulting matrix is given below.

0.5	0	0	0	0	2
0	0	0.5	1	2.0	2
0	0.5	0	2	3.5	2
0.5	1.0	0	2	3.5	2
0	0	0	1	5.0	0
2.0	2.0	2.0	2	5.0	0

N=5, n=6, N≠n, the least value is 1 and using this it is subtracted to the uncrossed cell of the above matrix and the resulting is shown below.

~~0.5~~	~~0~~	~~0~~	~~0~~	~~0~~	~~2~~
~~0~~	~~0~~	~~0.5~~	~~0~~	~~1.0~~	~~2~~
~~0~~	0.5	~~0~~	1	2.5	~~2~~
~~0.5~~	1.0	~~0~~	1	2.5	~~2~~
~~0~~	~~0~~	~~0~~	~~0~~	~~4.0~~	~~0~~
~~2.0~~	~~2.0~~	~~2.0~~	~~1~~	~~4.0~~	~~0~~

The table satisfies N=n and the optimal assignment table is obtained and using the matrix the assignment can be done.

~~0.5~~	~~0~~	~~0~~	~~0~~	[0]	2
~~0~~	[0]	~~0.5~~	~~0~~	1.0	2
[0]	0.5	~~0~~	1	2.5	2
0.5	1.0	[0]	1	2.5	2
~~0~~	~~0~~	~~0~~	[0]	4.0	~~0~~
2.0	2.0	2.0	1	4.0	[0]

Machine 1 ---------------Job5 = 1

Machine 2 ---------------Job2 = 5

Machine 3 ---------------Job1 = 3

Machine 4 ---------------Job3 = 2

Machine 5 ---------------Job4 = 9

Machine 6 ---------------Job6 = 6

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Problem 2.

Manager of Transportation Company must order 5 trucks out of a fleet to be present at 5 specific locations for loading goods that are awaiting shipment eight trucks are at different location. The costs are given in the table below. Assign 5 trucks so as to minimize the cost.

		Loading Locations				
		A	B	C	D	E
Trucks	1	300	290	280	290	210
	2	250	310	290	300	200
	3	180	190	300	190	180
	4	320	180	190	240	170
	5	270	210	190	250	160
	6	190	200	220	190	140
	7	220	300	230	180	160
	8	200	190	260	210	180

Solution;

The given problem is 8x5, so it is a unbalanced assignment problem, here we must had 3 fictitious loading locations as shown below.

		Loading Locations							
		A	B	C	D	E	F	G	H
Trucks	1	300	290	280	290	210	0	0	0
	2	250	310	290	300	200	0	0	0
	3	180	190	300	190	180	0	0	0
	4	320	180	190	240	170	0	0	0
	5	270	210	190	250	160	0	0	0
	6	190	200	220	190	140	0	0	0
	7	220	300	230	180	160	0	0	0
	8	200	190	260	210	180	0	0	0

Once the matrix is balanced and then usual procedure is used to solve the problem for assignment.

Row minimization

300	290	280	290	210	0	0	0
250	310	290	300	200	0	0	0
180	190	300	190	180	0	0	0
320	180	190	240	170	0	0	0
270	210	190	250	160	0	0	0
190	200	220	190	140	0	0	0
220	300	230	180	160	0	0	0
200	190	260	210	180	0	0	0

Since, each row has zero (0) the row minimization remains same as the original problem.

Column minimization

120	110	90	110	70	0	0	0
70	130	100	120	60	0	0	0
0	10	110	10	40	0	0	0
140	0	0	60	30	0	0	0
90	30	0	70	20	0	0	0
10	20	30	10	0	0	0	0
40	120	40	0	20	0	0	0
20	10	70	30	40	0	0	0

120	110	90	110	70	~~0~~	[0]	~~0~~
70	130	100	120	60	[0]	~~0~~	~~0~~
[0]	10	110	10	40	~~0~~	~~0~~	~~0~~
140	[0]	~~0~~	~~60~~	~~30~~	~~0~~	~~0~~	~~0~~
90	30	[0]	70	20	~~0~~	~~0~~	~~0~~
10	20	30	10	[0]	~~0~~	~~0~~	~~0~~
40	120	40	[0]	20	~~0~~	~~0~~	~~0~~
20	10	70	30	40	~~0~~	~~0~~	[0]

T1---------------------------- G = 0

T2---------------------------- F = 0

T3---------------------------- A = 180

T4---------------------------- B = 180

T5---------------------------- C = 190

T6---------------------------- E = 140

T7---------------------------- D = 180

T8---------------------------- H = 0

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The Travelling salesman (routing) problem

The travelling salesman problem is one of the problems considered as puzzles by the mathematicians.

Suppose a salesman wants to visit a certain number of cities allotted to him.

He knows the distances (or cost or time) of journey between every pair of cities, usually denoted by c_{ij} i.e. city I to city j.

His problem is to select such a route that stars from his home city. Passes through each city once and only once and returns to his home icty in the shortest possible distance (or at the least cost or in least time).

FORMULATION OF A TRAVELLING – SALESMAN PROBLEM AS ASSIGNMENT PROBLEM

Suppose c_{ij} is the distance 9or cost or time) from city I to city j and x_{ij} =1, if the salesman goes directly from city I to city j, and zero otherwise. Then minimize $\sum_i \sum_j x_{ij} c_{ij}$ with the additional restriction that the x_{ij} must be so chosen that no city is visited twice before the tour of all cities is completed. In particular, he cannot go directly from city I to I itself. This possibility may be avoided in the minimization process by adopting the convention $c_{ij} = \infty$ which ensures that x_{ij} can never be unity.

Alternatively, omit the variable x_{ij} from the problem specification. It is also important to note that onl single x_{ij} =1 for each value of I and j. the distance (or cost or time) matrix for this problem is given in table 1.

	To			
	∞	C12	---	C1n
	C21	∞	---	C2n
From	---	--	∞	--
	Cn1	Cn2	---	∞

Problem 1:

Given the matrix of set-up costs, show how to sequence the production so as to minimize the set-up cost per cycle.

	To					
From		A1	A2	A3	A4	A5
	A1	∞	2	5	7	1
	A2	6	∞	3	8	2
	A3	8	7	∞	4	7
	A4	12	4	6	∞	5
	A5	1	3	2	8	∞

Solution:

Row minimization

	A1	A2	A3	A4	A5
A1	∞	1	4	6	0
A2	4	∞	1	6	0
A3	4	3	∞	0	3
A4	8	0	2	∞	1
A5	0	2	1	7	∞

Column minimization

	A1	A2	A3	A4	A5
A1	∞	1	3	6	0
A2	4	∞	0	6	0
A3	4	3	∞	0	3
A4	8	0	1	∞	1
A5	0	2	0	7	∞

N=n, 5=5x5

∞	1	3	6	[0]
4	∞	[0]	6	~~0~~
4	3	∞	[0]	3
8	[0]	1	∞	1
[0]	2	~~0~~	7	∞

A1-------A5-----A1, A2-------A3------A4------A2

Cost = 1+3+4+4+1=13

As per the sequence from the above assignment indicates to produce the products A1, then A5 and then again A1, without producing the products A2,A3 and A4 thereby violates the additional restriction of producing each product once and only once before returning to the first product.

Step 2: next to examine the matrix for the best solutions to the assignment problem and first we try with value one (1) the cells having 1 are c12, c43, c45, using this cells we try for one sequence

Let us try with assigning with c15 to c12 and c42 to c45

∞	[1]	3	6	~~0~~
4	∞	[0]	6	~~0~~
4	3	∞	[0]	3
8	~~0~~	1	∞	[1]
[0]	2	~~0~~	7	∞

The sequence will be A1---A2---A3----A4---A5----A1

The cost will be 2+3+4+5+1 =15

Here the cost is increased by Rs.2. if you see in the matrix to get one sequence we have changed the assignment from c15 to c12 contains 1 and c42 to c45 contains due this penalty cost. The cost is increased by Rs 2.

Problem 2

A machine operator processes 5 types of items on his machine each week, and must choose a sequence for them. The set-up coast per change depends on the item presently on the machine and the set-up to be made according to the following table:

	To Item				
	A	B	C	D	E
A	∞	4	7	3	4
B	4	∞	6	3	4
C	7	6	∞	7	5
D	3	3	7	∞	7
E	4	4	5	7	∞

if the processes each type of item once and only once each week how should he sequence the items on his machine in order to minimize the total set-up cost?

Solution:

Using usual assignment problem

Row Minimization

∞	1	4	1	1
1	∞	3	0	1
2	1	∞	2	0
0	0	4	∞	4
0	0	1	3	∞

Column minimization

∞	1	3	1	1
1	∞	2	0	1
2	1	∞	2	0
0	0	3	∞	4
0	0	0	3	∞

Draw horizontal and vertical lines

∞	1	3	~~1~~	~~1~~
1	∞	2	~~0~~	~~1~~
2	1	∞	~~2~~	~~0~~
~~0~~	~~0~~	~~3~~	~~∞~~	4
~~0~~	~~0~~	~~0~~	~~3~~	~~∞~~

N≠n, 4≠5, least value is 1

∞	0	2	1	1
0	∞	1	0	1
1	0	∞	2	0
0	0	3	∞	5
0	0	0	4	∞

N=n

∞	[0]	2	1	1
~~0~~	∞	1	[0]	1
1	~~0~~	∞	2	[0]
[0]	~~0~~	3	∞	5
~~0~~	~~0~~	[0]	4	∞

A---B----D----A, C----E---C, THE COST =4+3+5+3+5=20

In the given problem we have 2 sequence and doesn't satisfies the travelling salesmen procedure

So we try with shifting of cells from c12 to c15 and c35 to c32

∞	~~0~~	2	1	[1]
~~0~~	∞	1	[0]	1
1	~~[0]~~	∞	2	~~0~~
[0]	~~0~~	3	∞	5
~~0~~	~~0~~	[0]	4	∞

A---E---B---D—A, C---B, This also not satisfied the travelling salesmen

COST =4+3+6+3+5=21

Now we try with shifting c32 to c31 and c41 to c42

∞	~~0~~	2	1	[1]
~~0~~	∞	1	[0]	1
[1]	~~0~~	∞	2	~~0~~
~~0~~	[0]	3	∞	5
~~0~~	~~0~~	[0]	4	∞

A---E---C----A, B---D---B this is also not suitable to travelling salesmen

COST= 4+3+7+3+5=22

NOW WE TRY WITH SHIFITNG OF C15 TO C14 AND C24 TO C25

∞	~~0~~	2	[1]	~~1~~
~~0~~	∞	1	~~0~~	[1]
[1]	~~0~~	∞	2	~~0~~
~~0~~	[0]	3	∞	5
~~0~~	~~0~~	[0]	4	∞

A---D—B---E---C—A COST=4+4+7+3+5=23

Problem3

Solve the given travelling salesman Problem

	1	2	3	4	5
1	∞	20	4	10	∞
2	20	∞	5	∞	10
3	4	5	∞	6	5
4	10	∞	6	∞	2
5	∞	10	5	2	∞

Problem 4.

Solve the travelling salesman problem given by the following data

$C_{12}=20$, $C_{13}=4$, $C_{14}=10$, $C_{23}=5$, $C_{34}=6$, $C_{25}=10$, $C_{35}=6$, $C_{45}=20$ When = $C_{ij}= C_{ji}$ there is no route between cities I and j if the value is not shown.

Solution :

Step1 first express the given problem in the form of an assignment problem by taking $C_{ij} = \infty$, for i=j

Apply the usual assignment problem

Row minimization:

∞	16	0	6	∞
15	∞	0	∞	5
0	1	∞	2	1
8	∞	4	∞	0
∞	8	3	0	∞

Column minimization

∞	15	0	6	∞
15	∞	0	∞	5
0	0	∞	2	1
8	∞	4	∞	0
∞	7	3	0	∞

Draw the lines horizontal and vertical lines

∞	15	~~0~~	6	∞
15	∞	~~0~~	∞	5
~~0~~	~~0~~	~~∞~~	~~2~~	~~1~~
~~8~~	~~∞~~	4	~~∞~~	~~0~~
~~∞~~	~~7~~	~~3~~	~~0~~	~~∞~~

N≠n, 4=5 the least value in the uncrossed cell is 5

∞	10	~~0~~	~~1~~	~~∞~~
10	∞	~~0~~	~~∞~~	~~0~~
~~0~~	~~0~~	~~∞~~	~~7~~	~~6~~
8	∞	~~4~~	~~∞~~	~~0~~
∞	7	~~3~~	~~0~~	~~∞~~

N ≠ n, 4≠5 the least value is 7

∞	3	~~0~~	1	~~∞~~
3	∞	~~0~~	∞	~~0~~
~~0~~	~~0~~	~~∞~~	~~14~~	~~13~~
1	∞	~~4~~	∞	~~0~~
~~∞~~	~~0~~	~~3~~	~~0~~	~~∞~~

N≠n, 4≠5 the least value is 1

∞	2	0	0	∞
2	∞	0	∞	0
0	0	∞	2	2
0	∞	4	∞	0
∞	0	4	0	∞

N=n,

Using our optimal assignment table we can assign

∞	2	~~0~~	[0]	∞
2	∞	[0]	∞	~~0~~
[0]	~~0~~	∞	2	2
~~0~~	∞	4	∞	[0]
∞	[0]	4	0	∞

1-----4-----5----2----3---1

Cost = 10+5+4+20+10 = Rs. 49/-

Using penalty method

A salesman has to visit 5 cities ABCD &E the distance (miles) between the 5 cities are as follows

	A	B	C	D	E
A	∞	7	6	8	4
B	7	∞	8	5	6
C	6	8	∞	9	7
D	8	5	9	∞	8
E	4	7	7	8	∞

Row minimization

∞	3	2	4	0
4	∞	3	0	1
0	2	∞	3	1
3	0	4	∞	3
0	3	3	1	∞

Column minimization

~~∞~~	~~3~~	~~0~~	~~4~~	~~0~~
~~4~~	~~∞~~	1	~~0~~	1
~~0~~	~~2~~	∞	~~3~~	1
~~3~~	~~0~~	2	~~∞~~	3
~~0~~	~~3~~	1	~~1~~	∞

N≠n, 4≠5, least value is 1

~~∞~~	~~3~~	~~0~~	~~4~~	~~0~~
~~4~~	~~∞~~	~~0~~	~~0~~	~~0~~
~~0~~	~~2~~	~~∞~~	~~3~~	~~0~~
~~3~~	~~0~~	~~1~~	~~∞~~	~~2~~
~~0~~	~~3~~	~~0~~	~~1~~	~~∞~~

N=n 5=5

The optimal assignment

∞	3	[0]	4	~~0~~
4	∞	~~0~~	[0]	~~0~~
0	2	∞	3	[0]
3	[0]	1	∞	2
[0]	3	~~0~~	1	∞

A----C---E----A, B---D---B COST=6+7+4+5+5=27

Using penalty method we can get one sequence

∞	~~3~~	$0^{0+0=0}$	4	$0^{0+0=0}$
4	~~∞~~	$0^{0+0=0}$	$0^{0+1=1}$	$0^{0+0=0}$
$0^{0+0=0}$	~~2~~	∞	3	$0^{0+0=0}$
~~3~~	~~$0^{2+1=3}$~~	~~1~~	~~∞~~	~~2~~
$0^{0+0=0}$	~~3~~	$0^{0+0=0}$	1	∞

D----B

	A	C	D	E
A	∞	$0^{0+0=0}$	4	$0^{0+0=0}$
B	4	$0^{0+0=0}$	----	$0^{0+0=0}$
C	$0^{0+0=0}$	∞	3	$0^{0+0=0}$
E	$0^{0+0=0}$	$0^{0+0=0}$	1	∞

since, the above table in the column d there are no zeros, so select the least value in the that row which is 1 and subtract with all other elements in that row.

	A	C	~~D~~	E
A	∞	$0^{0+0=0}$	~~3~~	$0^{0+0=0}$
B	4	$0^{0+0=0}$	~~----~~	$0^{0+0=0}$
C	$0^{0+0=0}$	∞	~~2~~	$0^{0+0=0}$
~~E~~	$\cancel{0^{0+0=0}}$	$\cancel{0^{0+0=0}}$	$\cancel{0^{2+0=2}}$	~~∞~~

E------D and the resulting matrix is

	A	C	E
A	∞	$0^{0+0=0}$	$0^{0+0=0}$
B	4	$0^{0+0=0}$	$0^{0+0=0}$
C	$0^{0+4=0}$	∞	$0^{0+0=0}$

C-----A and the resulting matrix

	C	E
A	----	$0^{0+0=0}$
B	$0^{0+0=0}$	$0^{0+0=0}$

B-------C

A-------E

The Optimal Assignment

A-----E-----D------B-----C-----A

COST=4+8+5+8+6=31

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3. **D.S.Hira**, Operation Research, S.Chand & Company Ltd., New Delhi, 2004
4.