Unit 5: Quadratic Functions

This unit investigates quadratic functions. Students study the structure of expressions and write expressions in equivalent forms. They solve quadratic equations by inspection, by completing the square, by factoring, and by using the Quadratic Formula. Some quadratic equations will have complex solutions. Students also graph quadratic functions and analyze characteristics of those functions, including end behavior. They write functions for various situations and build functions from other functions, using operations as needed. Given bivariate data, students fit a function to the data and use it to make predictions.

KEY STANDARDS

Use complex numbers in polynomial identities and equations

MCC9-12.N.CN.7 Solve quadratic equations with real coefficients that have complex solutions.

Interpret the structure of expressions

MCC9-12.A.SSE.1 Interpret expressions that represent a quantity in terms of its context.★ (*Focus on quadratic functions; compare with linear and exponential functions studied in Coordinate Algebra*.)

MCC9-12.A.SSE.1a Interpret parts of an expression, such as terms, factors, and coefficients.★ (*Focus on quadratic functions; compare with linear and exponential functions studied in Coordinate Algebra*.)

MCC9-12.A.SSE.1b Interpret complicated expressions by viewing one or more of their parts as a single entity. For example, interpret $P(1+r)^n$ as the product of P and a factor not depending on P.★ (*Focus on quadratic functions; compare with linear and exponential functions studied in Coordinate Algebra*.)

MCC9-12.A.SSE.2 Use the structure of an expression to identify ways to rewrite it. For example, see $x^4 - y^4$ as $(x^2)^2 - (y^2)^2$, thus recognizing it as a difference of squares that can be factored as $(x^2 - y^2)(x^2 + y^2)$. (*Focus on quadratic functions; compare with linear and exponential functions studied in Coordinate Algebra*.)

Write expressions in equivalent forms to solve problems

MCC9-12.A.SSE.3 Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.★ (*Focus on quadratic functions; compare with linear and exponential functions studied in Coordinate Algebra*.)

MCC9-12.A.SSE.3a Factor a quadratic expression to reveal the zeros of the function it defines.★

MCC9-12.A.SSE.3b Complete the square in a quadratic expression to reveal the maximum or minimum value of the function it defines.★

Create equations that describe numbers or relationships

MCC9-12.A.CED.1 Create equations and inequalities in one variable and use them to solve problems. Include equations arising from ~~linear and~~ quadratic functions, ~~and simple rational and exponential functions~~.★

MCC9-12.A.CED.2 Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.★ (*Focus on quadratic functions; compare with linear and exponential functions studied in Coordinate Algebra*.)

MCC9-12.A.CED.4 Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations. *For example, rearrange Ohm's law $V = IR$ to highlight resistance R.*★ (*Focus on quadratic functions; compare with linear and exponential functions studied in Coordinate Algebra*.)

Solve equations and inequalities in one variable

MCC9-12.A.REI.4 Solve quadratic functions in one variable.

MCC9-12.A.REI.4a Use the method of completing the square to transform any quadratic equation in x into an equation of the form $(x - p)^2 = q$ that has the same solutions. Derive the quadratic formula from this form.

MCC9-12.A.REI.4b Solve quadratic equations by inspection (e.g., for $x^2 = 49$), taking square roots, completing the square, the quadratic formula and factoring, as appropriate to the initial form of the equation. Recognize when the quadratic formula gives complex solutions and write them as $a \pm bi$ for real numbers a and b.

Solve systems of equations

MCC9-12.A.REI.7 Solve a simple system consisting of a linear equation and a quadratic equation in two variables algebraically and graphically. *For example, find the points of intersection between the line $y = -3x$ and the circle $x^2 + y^2 = 3$.*

Interpret functions that arise in applications in terms of the context

MCC9-12.F.IF.4 For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; ~~and periodicity~~.★

MCC9-12.F.IF.5 Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.*★ (*Focus on quadratic functions; compare with linear and exponential functions studied in Coordinate Algebra*.)

MCC9-12.F.IF.6 Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.★ (*Focus on quadratic functions; compare with linear and exponential functions studied in Coordinate Algebra*.)

Analyze functions using different representations

MCC9-12.F.IF.7 Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.★ (*Focus on quadratic functions; compare with linear and exponential functions studied in Coordinate Algebra*.)

MCC9-12.F.IF.7a Graph ~~linear and~~ quadratic functions and show intercepts, maxima, and minima.★

MCC9-12.F.IF.8 Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function. (*Focus on quadratic functions; compare with linear and exponential functions studied in Coordinate Algebra*.)

MCC9-12.F.IF.8a Use the process of factoring and completing the square in a quadratic function to show zeros, extreme values, and symmetry of the graph, and interpret these in terms of a context.

MCC9-12.F.IF.9 Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). *For example, given a graph of one quadratic function and an algebraic expression for another, say which has the larger maximum.* (*Focus on quadratic functions; compare with linear and exponential functions studied in Coordinate Algebra*.)

Build a function that models a relationship between two quantities

MCC9-12.F.BF.1 Write a function that describes a relationship between two quantities.★ (*Focus on quadratic functions; compare with linear and exponential functions studied in Coordinate Algebra*.)

MCC9-12.F.BF.1a Determine an explicit expression, a recursive process, or steps for calculation from a context. (*Focus on quadratic functions; compare with linear and exponential functions studied in Coordinate Algebra*.)

MCC9-12.F.BF.1b Combine standard function types using arithmetic operations. *For example, build a function that models the temperature of a cooling body by adding a constant function to a decaying exponential, and relate these functions to the model.* (*Focus on quadratic functions; compare with linear and exponential functions studied in Coordinate Algebra*.)

Build new functions from existing functions

MCC9-12.F.BF.3 Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $k\,f(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology.

Include recognizing even and odd functions from their graphs and algebraic expressions for them. (*Focus on quadratic functions; compare with linear and exponential functions studied in Coordinate Algebra*.)

Construct and compare linear, quadratic, and exponential models to solve problems

MCC9-12.F.LE.3 Observe using graphs and tables that a quantity increasing exponentially eventually exceeds a quantity increasing linearly, quadratically, or (more generally) as a polynomial function.★

Summarize, represent, and interpret data on two categorical and quantitative variables

MCC9-12.S.ID.6 Represent data on two quantitative variables on a scatter plot, and describe how the variables are related.★

MCC9-12.S.ID.6a Fit a function to the data; use functions fitted to data to solve problems in the context of the data. Use given functions or choose a function suggested by the context. Emphasize ~~linear~~, quadratic, ~~and exponential~~ models.★

USE COMPLEX NUMBERS IN POLYNOMIAL IDENTITIES AND EQUATIONS

KEY IDEAS

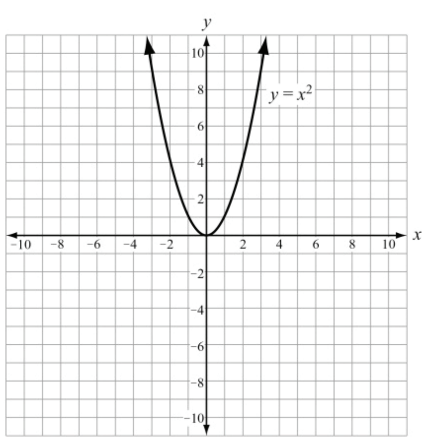
1. ***Quadratic equations*** are equations in the form $ax^2 + bx + c = 0$, where a, b, and c are real numbers and $a \neq 0$. Solutions to quadratic equations are also called the ***roots*** of the equation. Real number solutions occur at the x-intercepts of the graph of the equation.

2. There are several methods to finding the solution(s) of a quadratic equation, including graphing, factoring, completing the square, and using the quadratic formula. Using the ***quadratic formula*** will produce real and complex solutions. The quadratic formula is $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

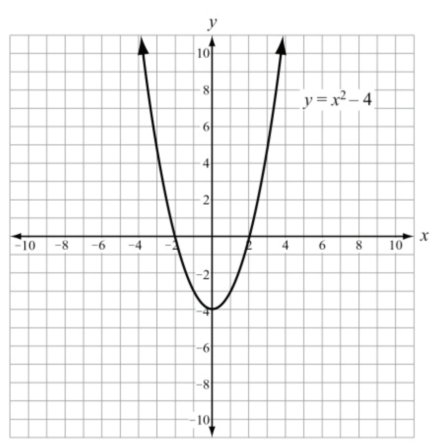
3. ***Complex solutions*** are in the form of $a + bi$, where a and b are real numbers.

Important Tip

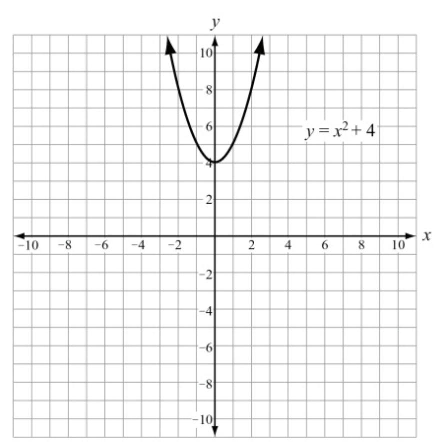
Complex solutions cannot be identified on the coordinate plane, because the graph will not have any x-intercepts. If an equation has complex solutions, they must be found algebraically. These graphs show quadratic functions with 1 real solution, 2 real solutions, and 2 complex solutions.



1 real solution



2 real solutions



2 complex solutions

INTERPRET THE STRUCTURE OF EXPRESSIONS

KEY IDEAS

1. An ***algebraic expression*** contains variables, numbers, and operation symbols.

2. A ***term*** in an algebraic expression can be a constant, a variable, or a constant multiplied by a variable or variables. Every term is separated by a plus sign or minus sign.

   Example:

   The terms in the expression $5x^2 - 3x + 8$ are $5x^2$, $-3x$, and 8.

3. A ***coefficient*** is the constant number that is multiplied by a variable in a term.

   Example:

   The coefficient in the term $7x^2$ is 7.

4. The ***degree*** of an expression in one variable is the greatest exponent in the expression.

   Example:

   The degree of the expression $n^3 - 4n^2 + 7$ is 3.

5. A ***common factor*** is a variable or number that terms can by divided by without a remainder.

   Example:

   The common factors of $30x^2$ and $6x$ are 1, 2, 3, 6, and x.

6. A ***common factor of an expression*** is a number or term that the entire expression can be divided by without a remainder.

   Example:

   The common factor for the expression $3x^3 + 6x^2 - 15x$ is $3x$
   (because $3x^3 + 6x^2 - 15x = 3x(x^2 + 2x - 5)$)

7. If parts of an expression are independent of each other, the expression can be interpreted in different ways.

Example:

In the expression $\frac{1}{2}h(b_1+b_2)$, the factors h and (b_1+b_2) are independent of each other. It can be interpreted as the product of h and a term that does not depend on h.

8. The structure of some expressions can be used to help rewrite them. For example, some fourth-degree expressions are in quadratic form.

Example:

$x^4+5x^2+4=(x^2+4)(x^2+1)$

Example:

$$\begin{aligned} x^4-y^4 &= \left(x^2\right)^2-\left(y^2\right)^2 \\ &= \left(x^2-y^2\right)\left(x^2+y^2\right) \\ &= (x+y)(x-y)\left(x^2+y^2\right) \end{aligned}$$

REVIEW EXAMPLES

1) Consider the expression $3n^2+n+2$.

   a. What is the coefficient of n?
   b. What terms are being added in the expression?

Solution:

   a. 1
   b. $3n^2$, n, and 2

WRITE EXPRESSIONS IN EQUIVALENT FORMS TO SOLVE PROBLEMS

KEY IDEAS

1. The ***zeros*** of a function are the values of the variable that make the function equal to zero. When the function is written in factored form, the zero product property can be used to find the zeros of the function. The zero product property states that if the product of two factors is zero, then one or both of the factors must be zero. So, the zeros of the function are the values that make either factor equal to zero.

Example:

$x^2 - 7x + 12 = 0$	Original equation.
$(x-3)(x-4) = 0$	Factor.

Set each factor equal to zero and solve.

$x - 3 = 0$	$x - 4 = 0$
$x = 3$	$x = 4$

The zeros of the function $y = x^2 - 7x + 12$ are $x = 3$ and $x = 4$.

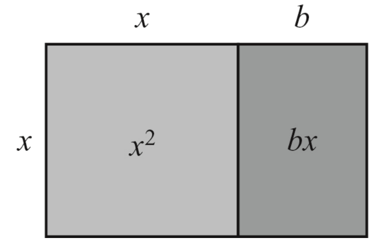
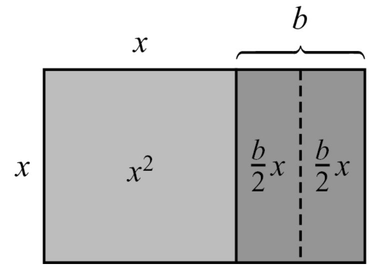
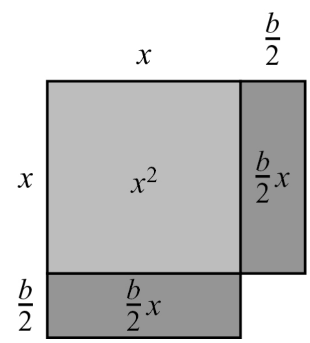
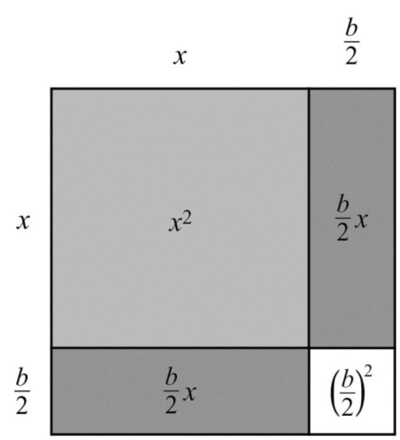
2. To ***complete the square*** of a quadratic function means to write a function as the square of a sum. The standard form for a quadratic expression is $ax^2 + bx + c$, where $a \neq 0$. When $a = 1$, completing the square of the function $x^2 + bx = d$ gives $\left(x + \frac{b}{2}\right)^2 = d + \left(\frac{b}{2}\right)^2$.
To complete the square when the value $a \neq 1$, factor the value of a from the expression.

Example:

To complete the square, take half of the coefficient of the x-term, square it, and add it to both sides of the equation.

$x^2 + bx = d$	Original expression.
$x^2 + bx + \left(\frac{b}{2}\right)^2 = d + \left(\frac{b}{2}\right)^2$	The coefficient of x is b. Half of b is $\frac{b}{2}$. Add the square of $\frac{b}{2}$ to both sides of the equation.
$\left(x + \frac{b}{2}\right)^2 = d + \left(\frac{b}{2}\right)^2$	The expression on the left side of the equation is a perfect square trinomial. Factor to write it as a binomial squared.

This figure shows how a model can represent completing the square of the expression $x^2 + bx$, where b is positive.

This model represents the expression $x^2 + bx$. To complete the square, create a model that is a square.	
Split the rectangle for bx into two rectangles that represent $\frac{b}{2}x$.	
Rearrange the two rectangles that represent $\frac{b}{2}x$.	
The missing piece of the square measures $\frac{b}{2}$ by $\frac{b}{2}$. Add and subtract $\left(\frac{b}{2}\right)^2$ to complete the model of the square. The large square has a side length of $x+\frac{b}{2}$, so this model represents $\left(x+\frac{b}{2}\right)^2$. $x^2+bx=\left(x+\frac{b}{2}\right)^2-\left(\frac{b}{2}\right)^2$.	

Important Tip

When you complete the square, make sure you are only changing the form of the expression, and not changing the value.

- When completing the square in an expression, add **and subtract** half of the coefficient of the *x* term squared.
- When completing the square in an equation, add half of the coefficient of the *x* term squared **to both sides of the equation.**

Examples:

Complete the square –

$$x^2+3x+7$$

$$\left(x^2+3x+\left(\frac{3}{2}\right)^2\right)+7-\left(\frac{3}{2}\right)^2$$

$$\left(x+\frac{3}{2}\right)^2+\frac{19}{4}$$

Complete the square –

$$x^2+3x+7=0$$

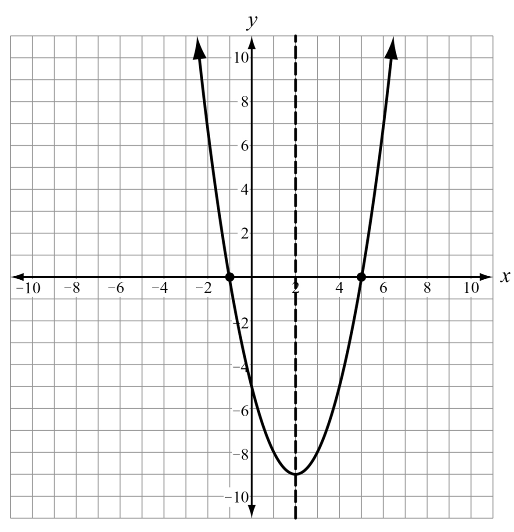
$$x^2+3x+\left(\frac{3}{2}\right)^2=-7+\left(\frac{3}{2}\right)^2$$

$$\left(x+\frac{3}{2}\right)^2=-\frac{19}{4}$$

3. Every quadratic function has a ***minimum*** or a ***maximum***. This minimum or maximum is located at the ***vertex*** (*h*, *k*). The vertex (*h, k*) also identifies the ***axis of symmetry*** and the minimum or maximum value of the function. The axis of symmetry is $x = h$.

Example:

The quadratic equation $f(x) = x^2 – 4x – 5$ is shown in this graph. The minimum of the function occurs at the vertex (2, –9). The zeros of the function are (–1, 0) and (5, 0). The axis of symmetry is $x = 2$.



4. The ***vertex form*** of a quadratic function is $f(x) = a(x - h)^2 + k$, where (h, k) is the vertex. One way to convert an equation from standard form to vertex form is to complete the square.

5. The vertex of a quadratic function can also be found by using the ***standard form*** and determining the value $\dfrac{-b}{2a}$. The vertex is $\left(\dfrac{-b}{2a}, f\left(\dfrac{-b}{2a}\right)\right)$.

REVIEW EXAMPLES

1) Write $f(x) = 2x^2 + 6x + 1$ in vertex form.

Solution:

The function is in standard form, where $a = 2$, $b = 6$, and $c = 1$.

$2x^2 + 6x + 1$	Original expression.
$2(x^2 + 3x) + 1$	Factor out 2 from the quadratic and linear terms.
$2\left(x^2 + 3x + \left(\dfrac{3}{2}\right)^2 - \left(\dfrac{3}{2}\right)^2\right) + 1$	Add and subtract the square of half of the coefficient of the linear term.
$2\left(x^2 + 3x + \left(\dfrac{3}{2}\right)^2\right) - \dfrac{9}{2} + 1$	Remove the subtracted term from the parentheses. Remember to multiply by a.
$2\left(x^2 + 3x + \left(\dfrac{3}{2}\right)^2\right) - \dfrac{7}{2}$	Combine the constant terms.
$2\left(x + \dfrac{3}{2}\right)^2 - \dfrac{7}{2}$	Write the perfect square trinomial as a binomial squared.

The vertex of the function is $\left(-\dfrac{3}{2}, -\dfrac{7}{2}\right)$.

The vertex of a quadratic function can also be found by writing the polynomial in standard form and determining the value of $\dfrac{-b}{2a}$. The vertex is $\left(\dfrac{-b}{2a}, f\left(\dfrac{-b}{2a}\right)\right)$.

For $f(x) = 2x^2 + 6x + 1$, $a = 2$, $b = 6$, and $c = 1$.

$$\frac{-b}{2a} = \frac{-6}{2(2)} = \frac{-6}{4} = \frac{-3}{2}$$

CREATE EQUATIONS THAT DESCRIBE NUMBERS OR RELATIONSHIPS

KEY IDEAS

1. Quadratic equations and inequalities can be written to model real-world situations. A quadratic equation can have 0, 1, or 2 real solutions. A quadratic inequality has a set of solutions.

   Here are some examples of real-world situations that can be modeled by quadratic functions:

   - Finding the area of a shape: Given that the length of a rectangle is 5 units more than the width, the area of the rectangle in square units can be represented by $A = x(x + 5)$.
   - Finding the product of consecutive integers: Given a number, n, the next consecutive number is $n + 1$ and the next consecutive even (or odd) number is $n + 2$. The product, P, of two consecutive numbers is $P = n(n + 1)$.
   - Finding the height of a projectile that is thrown, shot, or dropped: When heights are given in metric units, the equation used is $h(t) = -4.9t^2 + v_o t + h_o$, where v_o is the initial velocity and h_o is the initial height, in meters. The coefficient –4.9 represents half the force of gravity. When heights are given in customary units, the equation used is $h(t) = -16t^2 + v_o t + h_o$, where v_o is the initial velocity and h_o is the initial height, in feet. The coefficient –16 represents half the force of gravity. For example, the height, in feet, of a ball thrown with an initial velocity of 60 feet per second and an initial height of 4 feet can be represented by $h(t) = -16t^2 + 60t + 4$, where t is seconds.

   In each example, a quadratic inequality can be formed by using inequality symbols in place of the equal sign. For example, the product of two consecutive numbers that is less than 30 can be represented by the quadratic inequality $n(n + 1) < 30$.

2. You can use the properties of equality to solve for a variable in an equation. Use inverse operations on both sides of the equation until you have isolated the variable.

   Example:

   Solve $S = 2\pi r^2 + 2\pi rh$ for h.

Solution:

First, subtract $2\pi r^2$ from both sides. Then divide both sides by $2\pi r$.

$$S = 2\pi r^2 + 2\pi rh$$
$$S - 2\pi r^2 = 2\pi rh$$
$$\frac{S - 2\pi r^2}{2\pi r} = h$$
$$\frac{S}{2\pi r} - r = h$$

3. To graph a quadratic equation, find the vertex of the graph and the zeros of the equation. The axis of symmetry goes through the vertex, and divides the graph into two sides that are mirror images of each other. To draw the graph, you can plot points on one side of the parabola and use symmetry to find the corresponding points on the other side of the parabola.

4. The axis of symmetry is the midpoint for each corresponding pair of x-coordinates with the same y-value. If(x_1, y) is a point on the graph of a parabola and $x = h$ is the axis of symmetry, then (x_2, y) is also a point on the graph, and x_2 can be found using this equation: $\frac{x_1 + x_2}{2} = h$.

SOLVE EQUATIONS AND INEQUALITIES IN ONE VARIABLE

KEY IDEAS

1. When quadratic equations do not have a linear term, you can solve the equation by taking the ***square root*** of each side of the equation. This method provides rational and irrational values for x, as well as complex solutions. Remember, every square root has a positive value and a negative value. Earlier in the guide, we eliminated the negative answers when they represented length or distance.

Examples:

$$\begin{aligned} 3x^2 - 147 &= 0 \\ 3x^2 &= 147 \\ x^2 &= 49 \\ x &= \pm 7 \end{aligned}$$

$$\begin{aligned} 3x^2 + 147 &= 0 \\ 3x^2 &= -147 \\ x^2 &= -49 \\ x &= \pm 7i \end{aligned}$$

Check your answers:

$$\begin{aligned} 3(7)^2 - 147 &= 3(49) - 147 \\ &= 147 - 147 \\ &= 0 \end{aligned}$$

$$\begin{aligned} 3(7i)^2 + 147 &= 3(-49) + 147 \\ &= -147 + 147 \\ &= 0 \end{aligned}$$

$$\begin{aligned} 3(-7)^2 - 147 &= 3(49) - 147 \\ &= 147 - 147 \\ &= 0 \end{aligned}$$

$$\begin{aligned} 3(-7i)^2 + 147 &= 3(-49) + 147 \\ &= -147 + 147 \\ &= 0 \end{aligned}$$

2. You can ***factor*** some quadratic equations to find the solutions. To do this, rewrite the equation in standard form set equal to zero ($ax^2 + bx + c = 0$). Factor the expression, set each factor to 0 (by the Zero Product Property), and then solve for x in each resulting equation. This will provide two rational values for x.

Example:

$$\begin{aligned} x^2 - x &= 12 \\ x^2 - x - 12 &= 0 \\ (x-4)(x+3) &= 0 \end{aligned}$$

Set each factor equal to 0 and solve.

$x-4=0 \qquad x+3=0$

$x=4 \qquad x=-3$

Check your answers:

$4^2-4=16-4 \qquad (-3)^2-(-3)=9+3$

$=12 \qquad =12$

3. You can complete the square to solve a quadratic equation. First, write the expression that represents the function in standard form, $ax^2 + bx + c = 0$. Subtract the constant from both sides of the equation: $ax^2 + bx = -c$. Divide both sides of the equation by a: $x^2+\frac{b}{a}x=\frac{-c}{a}$. Add the square of half the coefficient of the x-term to both sides: $x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2=\frac{-c}{a}+\left(\frac{b}{2a}\right)^2$. Write the perfect square trinomial as a binomial squared: $\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}$. Take the square root of both sides of the equation and solve for x.

4. All quadratic equations can be solved using the quadratic formula. The ***quadratic formula*** is $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$, where $ax^2 + bx + c = 0$. The quadratic formula will yield both real and complex solutions of the equation.

Important Tip

While there may be several methods that can be used to solve a quadratic equation, some methods may be easier than others for certain equations.

REVIEW EXAMPLES

1) The standard form of a quadratic equation is $ax^2 + bx + c = 0$.

   a. After subtracting c from both sides of the equation, what would you add to both sides of the equation to complete the square?

   b. Solve for x. What formula did you derive?

SOLVE SYSTEMS OF EQUATIONS

KEY IDEAS

1. A ***system of equations*** is a collection of equations that have the same variables. A system of equations can be solved either algebraically or graphically.

2. To algebraically solve a system of equations involving a linear equation and a quadratic equation, first solve the linear equation for a variable. Then, substitute into the quadratic equation. Once you have found the solution for one variable, substitute the value into the other equation and solve for the second variable.

Example:

$$\begin{cases} y = x^2 + 2x - 9 \\ x - y = 3 \end{cases}$$

First, solve the second equation for y.

$x - y = 3$ Original equation.
$x - 3 = y$

Because both equations are solved for the same variable, substitute $x - 3$ for y in the quadratic equation and solve for x.

$$x^2 + 2x - 9 = x - 3$$
$$x^2 + x - 6 = 0$$
$$(x+3)(x-2) = 0$$
$$x + 3 = 0 \quad \text{or} \quad x - 2 = 0$$
$$x = -3 \quad \text{or} \quad x = 2$$

Substitute the x-values into one of the equations to solve for the corresponding y-values.

$x - y = 3$	$x - y = 3$
$-3 - y = 3$	$2 - y = 3$
$y = -6$	$y = -1$

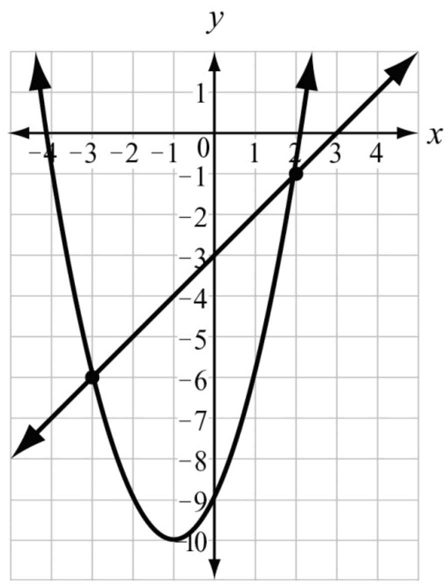
The solutions are $(-3, -6)$ and $(2, -1)$.

3. To graphically solve a system of equations involving a linear equation and a quadratic equation, graph both equations on the same coordinate plane. The point (or points) of intersection are the solutions.

Example:

For the system of equations given, graph the equations on a coordinate plane.

$$y = x^2 + 2x - 9$$
$$x - y = 3$$



The solutions appear to be (–3, –6) and (2, –1).

Important Tip

Solving a system of equations graphically will identify the approximate solutions. Solving algebraically will produce the exact solutions of the system. If you solve a system graphically, it is necessary to check your solutions algebraically by substituting them into both original equations.

INTERPRET FUNCTIONS THAT ARISE IN APPLICATIONS IN TERMS OF THE CONTEXT

KEY IDEAS

1. An ***x-intercept*** of a function is the x-coordinate of a point where the function crosses the x-axis. A function may have multiple x-intercepts. To find the x-intercepts of a quadratic function, set the function equal to 0 and solve for x. This can be done by factoring, completing the square, or using the quadratic formula.

2. The ***y-intercept*** of a function is the y-coordinate of the point where the function crosses the y-axis. A function may have zero or one y-intercepts. To find the y-intercept of a quadratic function, find the value of the function when x equals 0.

3. A function is ***increasing*** over an interval when the values of y increase as the values of x increase over that interval.

4. A function is ***decreasing*** over an interval when the values of y decrease as the values of x increase over that interval.

5. Every quadratic function has a ***minimum*** or ***maximum***, which is located at the vertex. When the function is written in standard form, the x-coordinate of the vertex is $\frac{-b}{2a}$. To find the y-coordinate of the vertex, substitute the value of $\frac{-b}{2a}$ into the function and evaluate.

6. The ***end behavior*** of a function describes how the values of the function change as the x-values approach negative infinity and positive infinity.

7. The ***domain*** of a function is the set of values for which it is possible to evaluate the function. The domain of a quadratic function is typically all real numbers, although in real-world applications it may only make sense to look at the domain values on a particular interval. For example, time must be a nonnegative number.

8. The ***average rate of change*** of a function over a specified interval is the slope of the line that connects the endpoints of the function for that interval. To calculate the average rate of change of a function over the interval from a to b, evaluate the expression $\frac{f(b)-f(a)}{b-a}$.

ANALYZE FUNCTIONS USING DIFFERENT REPRESENTATIONS

KEY IDEAS

1. Functions can be represented algebraically, graphically, numerically (in tables), or verbally (by description).

Examples:

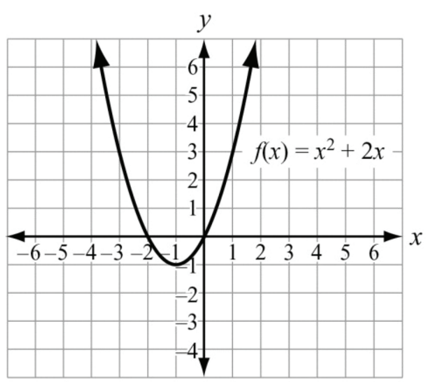
Algebraically: $f(x) = x^2 + 2x$

Verbally (by description): A function that represents the sum of the square of a number and twice the number.

Numerically (in a table):

x	***f*(*x*)**
–1	–1
0	0
1	3
2	8

Graphically:



2. You can compare key features of two functions represented in different ways.
For example, if you are given an equation of a quadratic function, and a graph of another quadratic function, you can calculate the vertex of the first function and compare it to the vertex of the graphed function.

BUILD A FUNCTION THAT MODELS A RELATIONSHIP BETWEEN TWO QUANTITIES

KEY IDEAS

1. An ***explicit expression*** contains variables, numbers, and operation symbols, and does not use an equal sign to relate the expression to another quantity.

2. A ***recursive process*** can show that a quadratic function has second differences that are equal to one another.

   Example:

   Consider the function $f(x) = x^2 + 4x - 1$.

   This table of values shows five values of the function.

x	$f(x)$
–2	–5
–1	–4
0	–1
1	4
2	11

   The first and second differences are shown.

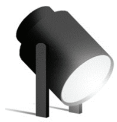
x	$f(x)$
–2	–5
–1	–4
0	–1
1	4
2	11

First differences	**Second differences**
$-4 - (-5) = 1$	$3 - 1 = 2$
$-1 - (-4) = 3$	$5 - 3 = 2$
$4 - (-1) = 5$	$7 - 5 = 2$
$11 - 4 = 7$	

3. A ***recursive function*** is one in which each function value is based on a previous value (or values) of the function.

4. When building a model function, functions can be added, subtracted, or multiplied together. The result will still be a function. This includes linear, quadratic, exponential, and constant functions.

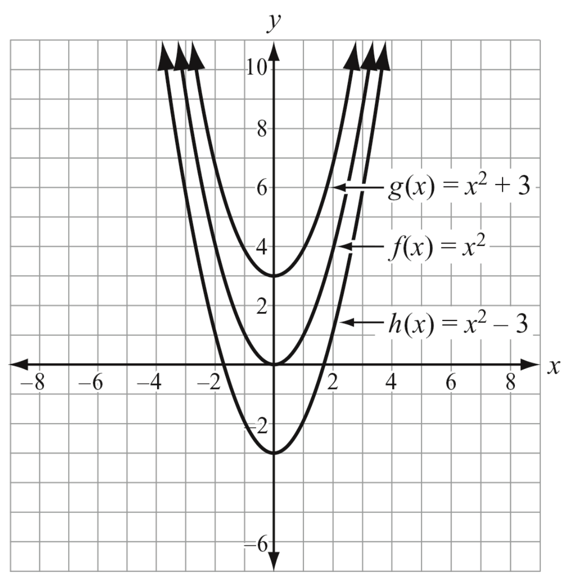
BUILD NEW FUNCTIONS FROM EXISTING FUNCTIONS



KEY IDEAS

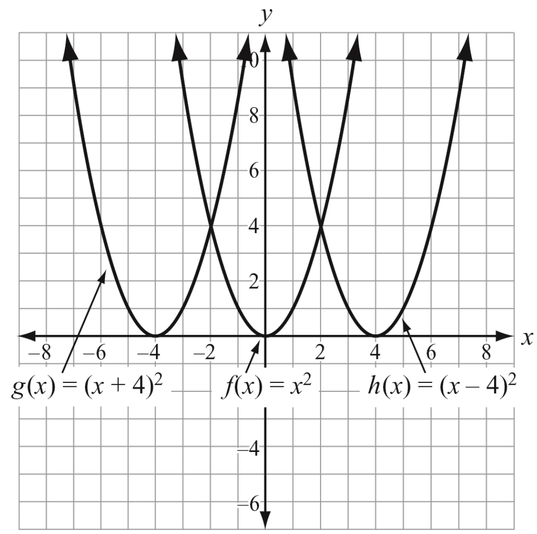
1. A ***parent function*** is the basic function from which all the other functions in a function family are modeled. For the quadratic function family, the parent function is $f(x) = x^2$.

2. For a parent function $f(x)$ and a real number k:
   - The function $f(x) + k$ will move the graph of $f(x)$ up by k units.
   - The function $f(x) - k$ will move the graph of $f(x)$ down by k units.



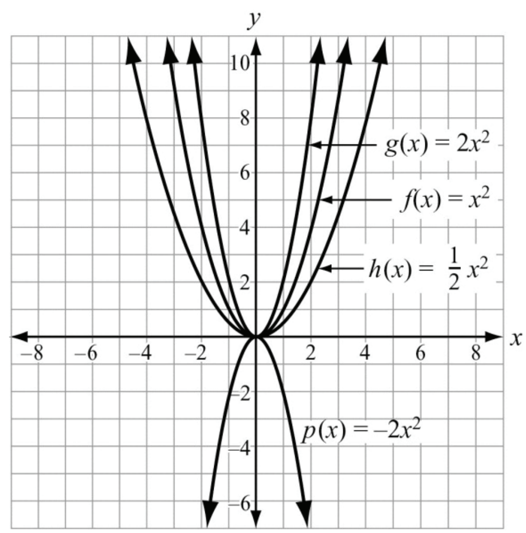
3. For a parent function $f(x)$ and a real number k:

- The function $f(x + k)$ will move the graph of $f(x)$ left by k units.
- The function $f(x - k)$ will move the graph of $f(x)$ right by k units.



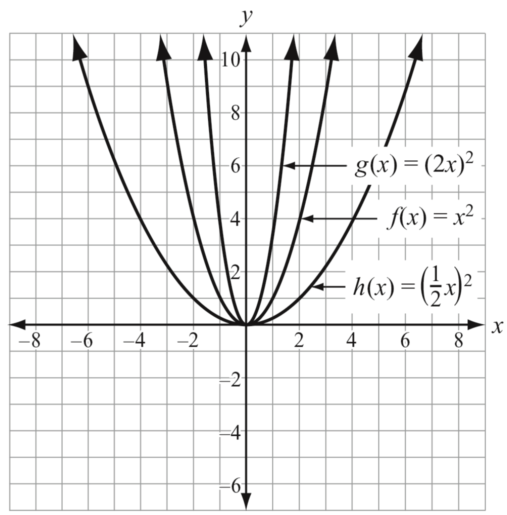
4. For a parent function $f(x)$ and a real number k:

- The function $kf(x)$ will vertically stretch the graph of $f(x)$ by a factor of k units for $|k| > 1$.
- The function $kf(x)$ will vertically shrink the graph of $f(x)$ by a factor of k units for $|k| < 1$.
- The function $kf(x)$ will reflect the graph of $f(x)$ over the x-axis for negative values of k.



5. For a parent function $f(x)$ and a real number k:

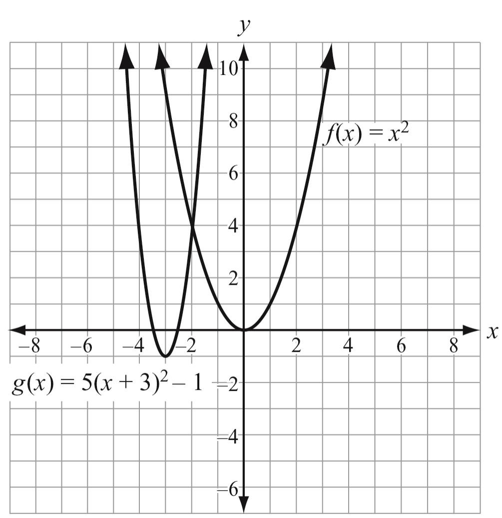
- The function $f(kx)$ will horizontally shrink the graph of $f(x)$ by a factor of $\frac{1}{k}$ units for $|k| > 1$.
- The function $f(kx)$ will horizontally stretch the graph of $f(x)$ by a factor of $\frac{1}{k}$ units for $|k| < 1$.
- The function $f(kx)$ will reflect the graph of $f(x)$ over the y-axis for negative values of k.



6. You can apply more than one of these changes at a time to a parent function.

Example:

$f(x) = 5(x + 3)^2 – 1$ will translate $f(x) = x^2$ left 3 units and down 1 unit and stretch the function vertically by a factor of 5.



7. Functions can be classified as even or odd.

- If a graph is symmetric to the y-axis, then it is an ***even function***. That is, if $f(-x) = f(x)$, then the function is even.
- If a graph is symmetric to the origin, then it is an ***odd function***. That is, if $f(-x) = -f(x)$, then the function is odd.

Important Tip

Remember that when you change $f(x)$ to $f(x+k)$, move the graph to the **left** when k is positive, and to the **right** when k is negative. This may seem different from what you would expect, so be sure to understand why this occurs in order to apply the shifts correctly.

REVIEW EXAMPLES

1) Compare the graphs of the following functions to $f(x)$.

a. $\frac{1}{2}f(x)$

b. $f(x) - 5$

c. $f(x - 2) + 1$

Solution:

a. The graph of $\frac{1}{2}f(x)$ is a vertical shrink of $f(x)$ by a factor of $\frac{1}{2}$.

b. The graph of $f(x) - 5$ is a shift of the graph of $f(x)$ down 5 units.

c. The graph of $f(x - 2) + 1$ is a shift of the graph of $f(x)$ right 2 units and up 1 unit.

CONSTRUCT AND COMPARE LINEAR, QUADRATIC, AND EXPONENTIAL MODELS TO SOLVE PROBLEMS

KEY IDEAS

1. ***Exponential functions*** have a fixed number as the base and a variable number as the exponent.

2. The value of an exponential function with a base greater than 1 will eventually exceed the value of a quadratic function. Similarly, the value of a quadratic function will eventually exceed the value of a linear function.

Example:

Exponential	
x	$y = 2^x$
1	2
2	4
3	8
4	16
5	32
6	64

Quadratic	
x	$y = x^2 + 2$
1	3
2	6
3	11
4	18
5	27
6	38

Linear	
x	$y = x + 2$
1	3
2	4
3	5
4	6
5	7
6	8

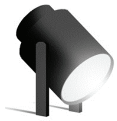
REVIEW EXAMPLES

1) This table shows that the value of $f(x) = 5x^2 + 4$ is greater than the value of $g(x) = 2^x$ over the interval [0, 8].

x	$f(x)$	$g(x)$
0	$5(0)^2 + 4 = 4$	$2^0 = 1$
2	$5(2)^2 + 4 = 24$	$2^2 = 4$
4	$5(4)^2 + 4 = 84$	$2^4 = 16$
6	$5(6)^2 + 4 = 184$	$2^6 = 64$
8	$5(8)^2 + 4 = 324$	$2^8 = 256$

As x increases, will the value of $f(x)$ always be greater than the value of $g(x)$? Explain how you know.

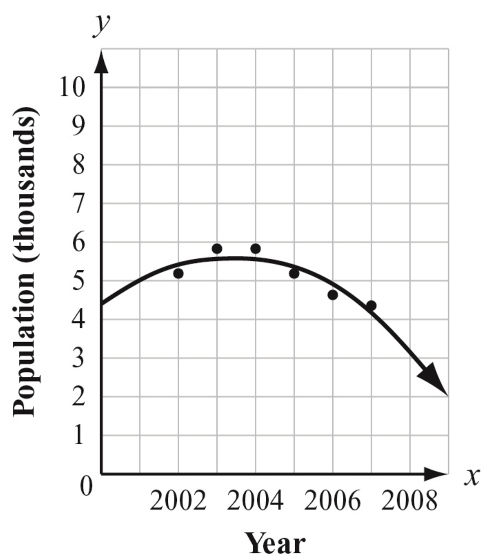
SUMMARIZE, REPRESENT, AND INTERPRET DATA ON TWO CATEGORICAL AND QUANTITATIVE VARIABLES



KEY IDEAS

1. A ***quadratic regression*** equation is a curve of best fit for data given in a scatter plot. The curve most likely will not go through all of the data points, but should come close to most of them.

Example:



2. A quadratic regression equation can be used to make predictions about data. To do this, evaluate the function for a given input value.