Unit 6 Exponential and Logarithmic Functions

Lesson 1: Graphing Exponential Growth/Decay Function

Lesson Goals:

- Identify transformations of exponential functions
- Identify the domain/range and key features of exponential functions

Why do I need to Learn This?

- Many real life applications involve exponential functions.
- Visually the graph can help you understand a problem better.

Lesson 2: Exponential Applications

Lesson Goals:

- Set up an exponential model for a real-life situation
- Understand the difference between a linear growth/decay and exponential growth/decay
- Solve financial equations involving simple and compound interest

Why do I need to Learn This?

- There are many real-life examples modeled by exponential growth or decay.
- Many loans and bank interest formulas use exponential growth.

Lesson 3: Defining and Evaluating Logarithmic Functions

Lesson Goals:

- Rewrite between exponential and logarithmic forms of an expression
- Evaluate simple logarithmic expressions without a calculator
- Use a formula modeling a real-life situation that incorporates a logarithm.

Why do I need to Learn This?

- Logarithms are used in a variety of scientific applications.

Lesson 4: Properties of Logarithms

Lesson Goals:

- Expand or condense logarithmic expressions in order to evaluate or simplify.
- Use the change-of-base formula to find decimal approximations of logarithms.
- Use formulas modeling real-life situation that incorporates a logarithm.

Why do I need to Learn This?

- Logarithms are used in a variety of scientific applications.

Lesson 5: Solving Exponential and Logarithmic Equations

Lesson Goals:

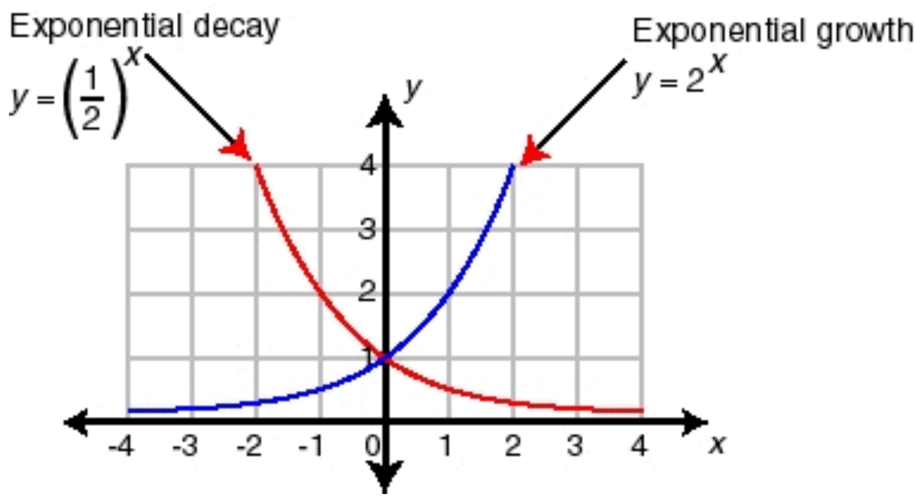
- Solve an exponential equation by rewriting in log form or using inverse operations.
- Solve a log equation by rewriting in exponential form or using inverse operations.
- Solve a real-world problem modeled by an exponential or logarithmic equation.

Why do I need to Learn This?

- Exponential Function and logarithms are used in a variety of scientific and financial applications.

Unit 6 Lesson 1
Graphing Exponential Functions

- **Exponential Function**: a function in the form $f(x) = b^x$, where *b* is a positive constant other than 1. There is no single parent exponential function since each base *b* determines a different function



$\boldsymbol{f(x) = 2^x}$

Domain:

Range:

y-intercept:

End behavior as $x \to \infty$:

End behavior as $x \to -\infty$:

$\boldsymbol{f(x) = \left(\frac{1}{2}\right)^x}$

Domain:

Range:

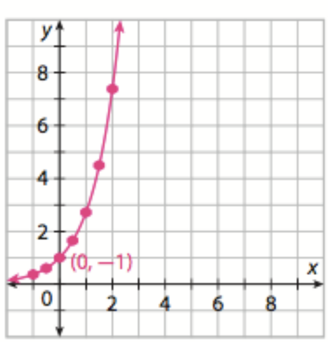
y-intercept:

End behavior as $x \to \infty$:

End behavior as $x \to -\infty$:

Base e: The function $f(x) = e^x$ is an exponential growth function.

$\boldsymbol{e} \approx$



Even though *e* is an irrational number, it can be used as the base of an exponential function. The number *e* is sometimes called the natural base of an exponential function and is used extensively in scientific and other applications involving exponential growth and decay.

I. Exponential Growth Functions

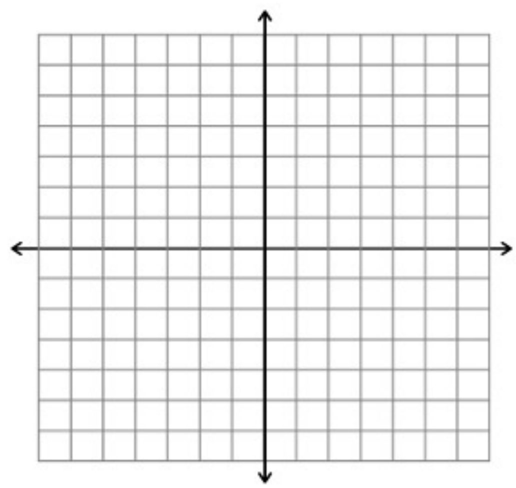
NOTE: Every parent function has different reference points!

Step 1: Identify the transformations from the parent function.
Step 2: Draw in the horizontal asymptote. (check vertical shift)
Step 3: Identify the reference points of the parent function. Always use (0,1)
Step 4: Graph 2+ reference points.

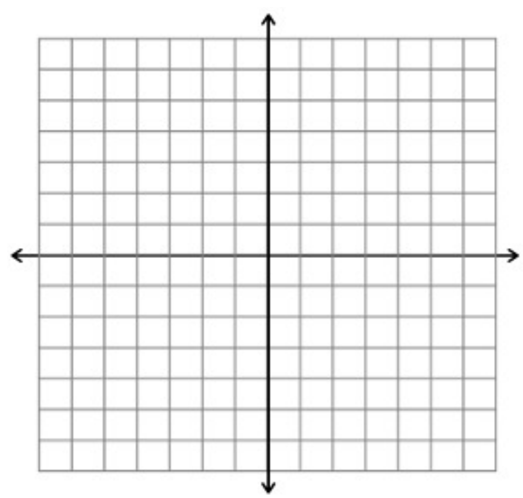
- Find (0,1) reference point FROM THE ASYMPTOTE
- Shift the point left/right (vertical shift has already occurred if counting from the asymptote)
- Vertically stretch/compress the point (from the asymptote)
- Reflect the point if necessary

OR…. You can always make an x-y table!!!

A. $g(x) = -3(2^{x-2}) + 1$



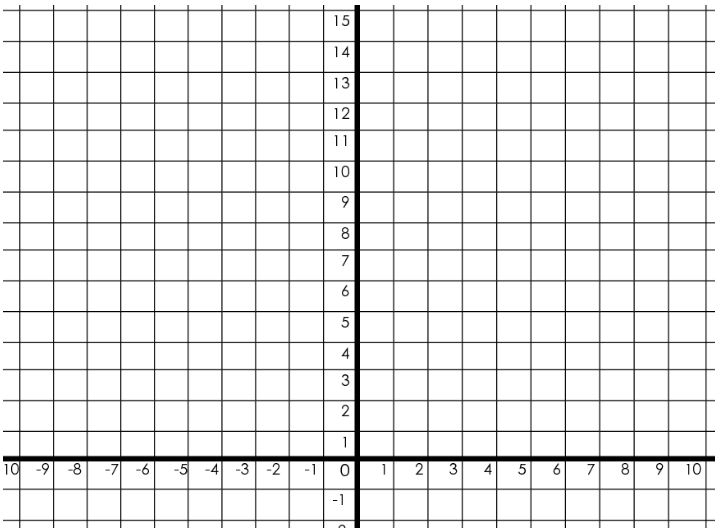
B. $g(x) = 0.5(3^{x}) + 1$



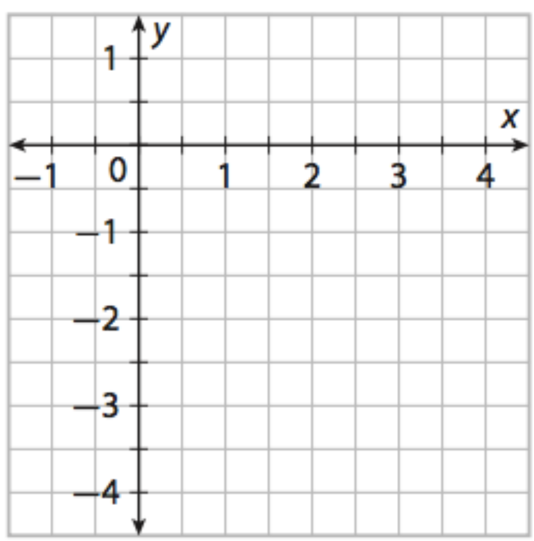
II. Exponential Growth Functions with base *e*

	$f(x) = e^x$
First reference point	$(0, 1)$
Second reference point	$(1, e)$
Asymptote	$y = 0$

(A) $g(x) = 3 \cdot e^{x+1} + 4$

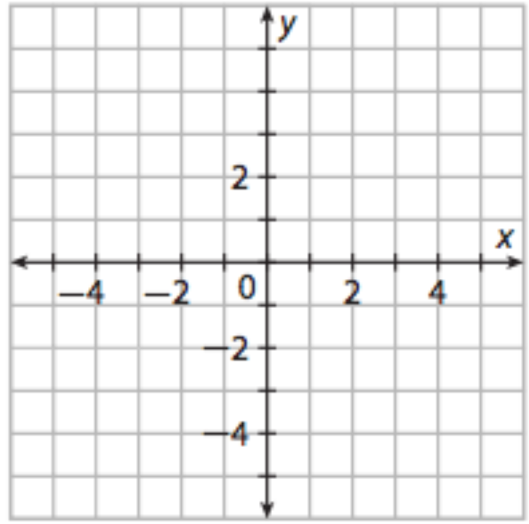


(B) $g(x) = -0.5 \cdot e^{x-2} - 1$

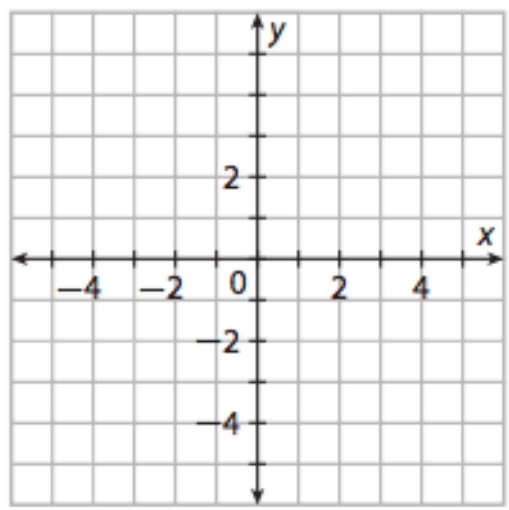


III. Exponential Decay Functions

A. $g(x) = 3\left(\frac{1}{2}\right)^{x-2} - 2$



B. $g(x) = -2\left(\frac{1}{3}\right)^{x+1} + 4$



Unit 6 Lesson 2 Exponential Applications

I. Basic growth & decay functions

An ***exponential function*** has the form $f(t) = a(1 \pm r)^t$ where $a > 0$ and *r* is a constant percent increase for growth or decrease for decay (expressed as a decimal) for each unit increase in *t* time.

A quantity is ***growing exponentially*** if it increases by the same percent in each time period.

A quantity is ***decaying exponentially*** if it decreases by the same percent in each time period.

Growth: $y = a(1+r)^x$	**Decay:** $y = a(1-r)^x$

a = initial **amount** before measuring growth/decay
r = growth/decay **rate** (often a percent)
x = number of **time** intervals that have passed

Appreciation:

Depreciation:

<u>Example 1:</u>
Tony purchased a rare guitar in 2012 for $12,000. Experts estimate that its value will increase by 14% per year. How much will the guitar be worth in 2020?

<u>Example 2:</u>
The value of a truck purchased new for $28,000 depreciates in value by 9.5% each year. Predict how much the truck will be worth after 6 years.

<u>Example 3:</u>
The Dow Jones index is a stock market index for the New York Stock Exchange. The Dow Jones index for the period 1980-2000 can be modeled by $D(t) = 878e^{0.121t}$, where t is the number of years after 1980. Determine the Dow Jones index for the year 1993.

II. Compound Interest

Simple Interest: Interest earned on the principle amount of money.

Compound Interest: Interest earned on the principle amount of money AND on the accumulated interest of previous periods

Principle: The initial amount of money invested

Common words for compounding	***n***
Annual	1
Semi-annual	2
Quarterly	4
Monthly	12
Daily	365

The more often interest is ***compounded*** (reinvested), the more money you earn!

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

A = final amount
P = initial principle amount
r = interest rate (as a decimal)
t = time (years)
n = number of times per year interest is compounded

Notice that if the interest is compounded once annually, the following equation will be used:

$$A = P(1 + r)^t$$

Example 4:
A person invests $3500 in an account that earns 3% annual interest. How much money will the person have in the account after 10 years?

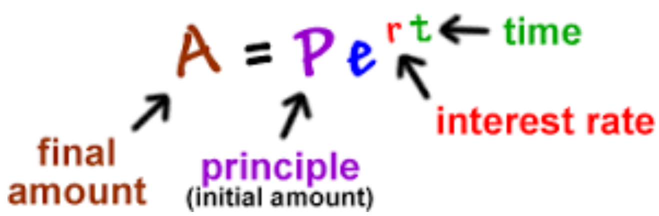
Example 5:
A person invests $8000 in an account that earns 6.5% annual interest compounded daily. How much money will the person have in the account after 3 years?

III. Interest Compounded Continuously

Recall that base e comes from $\left(1+\frac{1}{x}\right)^x$ as $x \to \infty$ and $e \approx 2.72$.

$A = P\left(1+\frac{r}{n}\right)^{nt}$ can be rewritten as $A = P\left(\mathbf{1}+\frac{\mathbf{1}}{\boldsymbol{n}}\right)^{\boldsymbol{rnt}} = P\left(\left(\mathbf{1}+\frac{\mathbf{1}}{\boldsymbol{n}}\right)^{\boldsymbol{n}}\right)^{rt}$

As the value of n increases, the bolded part of this formula becomes *e*. This can be rewritten as $A = P\boldsymbol{e}^{rt}$.



Example 6:
A person invests $5000 in an account that earns 3.5% annual interest compounded continuously. How much money is in the person's account after 4 years?

Example 7:
A person invests $1550 in an account that earns 4% annual interest compounded continuously. How much money is in the person's account after 9 years?

Unit 6 Lesson 3
Defining and Evaluating Logarithmic Functions

Exponential and Logarithmic Functions are ***inverses*** of each other.

Explain 1 Converting Between Exponential and Logarithmic Forms of Equations

In general, the exponential function $f(x) = b^x$, where $b > 0$ and $b \neq 1$, has the logarithmic function $f^{-1}(x) = \log_b x$ as its inverse. For instance, if $f(x) = 3^x$, then $f^{-1}(x) = \log_3 x$, and if $f(x) = \left(\frac{1}{4}\right)^x$, then $f^{-1}(x) = \log_{\frac{1}{4}} x$. The inverse relationship between exponential functions and logarithmic functions also means that you can write any exponential equation as a logarithmic equation and any logarithmic equation as an exponential equation.

Exponential Equation	Logarithmic Equation
$b^x = a$	$\log_b a = x$

$$b > 0, b \neq 1$$

Q: Why do we need logarithms?
A: To find non-exact exponents.

Exponential form:		Logarithmic form:	
$2^2 = 4$		$\log_2 4 = 2$	
$2^3 = 8$		$\log_2 8 = 3$	
$2^? = 6$	? between 2 and 3	$\log_2 6 = ?$	$? \approx 2.585$

Example: **Complete each table by writing each given equation in its alternate form.**

Exponential Equation	Logarithmic Equation
$4^3 = 64$	?
?	$\log_5 \frac{1}{25} = -2$
$\left(\frac{2}{3}\right)^p = q$	?
?	$\log_{\frac{1}{2}} m = n$

Exponential Equation	Logarithmic Equation
$3^5 = 243$	
	$\log_4 \frac{1}{64} = -3$
$\left(\frac{3}{4}\right)^r = s$	
	$\log_{\frac{1}{5}} v = w$

Reflect

A student wrote the logarithmic form of the exponential equation $5^0 = 1$ as $\log_5 0 = 1$. What did the student do wrong? What is the correct logarithmic equation?

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Explain 2 Evaluating Logarithmic Functions by Thinking in Terms of Exponents

The logarithmic function $f(x) = \log_b x$ accepts a power of b as an input and delivers an exponent as an output. In cases where the input of a logarithmic function is a recognizable power of b, you should be able to determine the function's output. You may find it helpful first to write a logarithmic equation by letting the output equal y and then to rewrite the equation in exponential form. Once the bases on each side of the exponential equation are equal, you can equate their exponents to find y.

Special Logarithmic Values to Memorize $b \neq 1,\ b > 0$

Logarithm of 1	$\log_b 1 = 0$ because $b^0 = 1$	example: $\log_5 1 = 0$
Logarithm of base b	$\log_b b = 1$ because $b^1 = b$	example: $\log_4 4 = 1$
Inverse Functions	$\log_b b^x = x$ and $b^{\log_b x} = x$ because exponential and logarithmic functions are inverses	example: $6^{\log_6 8} = 8$

Evaluate the expression without a calculator.

1. $\log_3 81$

2. $\log_5 0.04$

3. $\log_9 3$

4. $\log_{\frac{1}{2}} 8$

5. $\log_6 6^2$

6. $7^{\log_7 12}$

7. If $f(x) = \log_{10} x$, find $f(1000)$, $f(0.01)$, and $f(\sqrt{10})$.

Explain 3 Evaluating Logarithmic Functions Using a Scientific Calculator

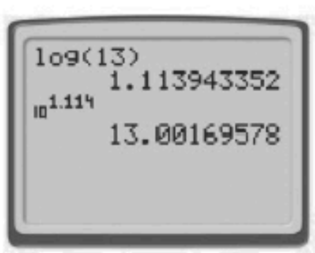
You can use a scientific calculator to find the logarithm of any positive number x when the logarithm's base is either 10 or e. When the base is 10, you are finding what is called the *common logarithm* of x, and you use the calculator's LOG key because $\log_{10} x$ is also written as $\log x$ (where the base is understood to be 10). When the base is e, you are finding what is called the *natural logarithm* of x, and you use the calculator's LN key because $\log_e x$ is also written as $\ln x$.

Common Logarithm	**Natural Logarithm**
$\log_{10} x = \log x$	$\log_e x = \ln x$

* If a base is NOT shown, we assume base 10.*
If you see *ln* (natural log), we assume base *e*.

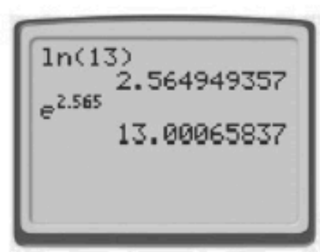
Ⓐ 13

First, find the common logarithm of 13. Round the result to the thousandths place and raise 10 to that number to confirm that the power is close to 13.



So, $\log 13 \approx 1.114$.

Next, find the natural logarithm of 13. Round the result to the thousandths place and raise e to that number to confirm that the power is close to 13.



So, $\ln 13 \approx 2.565$.

Evaluate using your calculator.

1. $\log 5$

2. $\ln 0.1$

Explain 4 Evaluating a Logarithmic Model

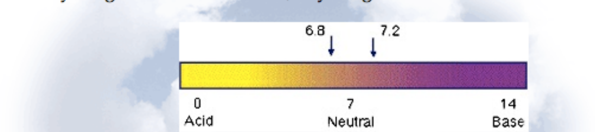
There are standard scientific formulas that involve logarithms, such as the formulas for the acidity level (pH) of a liquid and the intensity level of a sound. It's also possible to develop your own models involving logarithms by finding the inverses of exponential growth and decay models.

Example 4

(A) The acidity level, or pH, of a liquid is given by the formula $\text{pH} = \log \frac{1}{[\text{H}^+]}$ where $[\text{H}^+]$ is the concentration (in moles per liter) of hydrogen ions in the liquid. In a typical chlorinated swimming pool, the concentration of hydrogen ions ranges from 1.58×10^{-8} moles per liter to 6.31×10^{-8} moles per liter. What is the range of the pH for a typical swimming pool?

Using the pH formula, substitute the given values of $[\text{H}^+]$.

pH: What does it mean? pH is the abbreviation for potential hydrogen. The pH of any solution is the measure of its hydrogen-ion concentration. The higher the pH reading, the more alkaline and oxygen rich the fluid is. The lower the pH reading, the more acidic and oxygen deprived the fluid is. The pH range is from 0 to 14, with 7.0 being neutral. Anything above 7.0 is alkaline, anything below 7.0 is considered acidic.



(B) *Lactobacillus acidophilus* is one of the bacteria used to turn milk into yogurt. The population P of a colony of 3500 bacteria at time t (in minutes) can be modeled by the function $P(t) = 3500(2)^{\frac{t}{73}}$. How long does it take the population to reach 1,792,000?

Step 1 Solve $P = 3500(2)^{\frac{t}{73}}$ for t.

Step 2 Use the logarithmic model to find t when $P = 1{,}792{,}000$.

Unit 6 Lesson 4
Properties of Logarithms

Properties of Logarithms			
For any positive numbers $a, m, n, b\ (b \neq 1)$, and $c\ (c \neq 1)$, the following properties hold.			
Definition-Based Properties	$\log_b b^m = m$	$\log_b 1 = 0$	$\log_b b = 1$
Product Property of Logarithms	$\log_b mn = \log_b m + \log_b n$		
Quotient Property of Logarithms	$\log_b \frac{m}{n} = \log_b m - \log_b n$		
Power Property of Logarithms	$\log_b m^n = n\log_b m$		
Change of Base Property of Logarithms	$\log_c a = \frac{\log_b a}{\log_b c}$		

I. Using properties to expand a logarithmic expression.

(a) $\log_2 \frac{7x^3}{y}$ Assume *x* and *y* are positive.

(b) $\ln x^{\frac{1}{2}} y^3$

II. Using properties to condense a logarithmic expression.

Express each expression as a single logarithm. Simplify if possible.

(a) $\log 6 + 2\log 2 - \log 3$

(b) $3(\ln 3 - \ln x) + (\ln x - \ln 9)$

(c) $\log_3 27 - \log_3 81$

(d) $\log_5 \left(\frac{1}{25}\right) + \log_5 625$

III. Using the change-of-base formula to evaluate logarithms.

Change-of-Base Formula Let u, b, and c be positive numbers with $b \neq 1$ and $c \neq 1$.

$$\log_c u = \frac{\log_b u}{\log_b c}$$

To use your calculator, let $b = 10$ or $b = e$.

(a) Evaluate $\log_3 7$.

(b) Evaluate $\log_9 \frac{5}{16}$.

Explain 2 Rewriting a Logarithmic Model

There are standard formulas that involve logarithms, such as the formula for measuring the loudness of sounds. The loudness of a sound $L(I)$, in decibels, is given by the function $L(I) = 10\log\left(\frac{I}{I_0}\right)$, where I is the sound's intensity in watts per square meter and I_0 is the intensity of a barely audible sound. It's also possible to develop logarithmic models from exponential growth or decay models of the form $f(t) = a(1 + r)^t$ or $f(t) = a(1 - r)^t$ by finding the inverse.

(A) During a concert, an orchestra plays a piece of music in which its volume increases from one measure to the next, tripling the sound's intensity. Find how many decibels the loudness of the sound increases between the two measures.

Let I be the intensity in the first measure. So $3I$ is the intensity in the second measure.

(B) The population of the United States in 2012 was 313.9 million. If the population increases exponentially at an average rate of 1% each year, how long will it take for the population to double?

The exponential growth model is $P = P_0(1+ r)^t$, where P is the population in millions after t years, P_0 is the population in 2012, and r is the average growth rate.

Unit 6 Lesson 5
Solving Exponential & Logarithmic Equations

Remember that ***exponential*** and ***logarithmic*** functions are inverses

I. Solving Exponential Equations – SOLVE WITH LOGARITHMS!

1. $4^{3x} = 8^{x+1}$ Notice that $2^2 = 4$ and $2^3 = 8$

2. $2^x = 7$

3. $10^{2x-3} + 4 = 21$

4. $10 = 5e^{4x}$

5. $6^{3x-9} - 10 = -3$

Remember that ***exponential*** and ***logarithmic*** functions are inverses

II. Solving Logarithmic Equations – SOLVE WITH EXPONENTS!

Check for ***extraneous*** solutions! (extra solutions that don't work)

1. $\log_3(5x - 1) = \log_3(x + 7)$ Use the fact that BASES are the same.

2. $\log_5(3x + 1) = 2$

3. $2 = \log 5x + \log(x - 1)$

III. Solving a Real-World Problem using exponents and logarithms.

1. Use Newton's Law of Cooling to answer the following. You are cooking stew. When you take it off the stove, its temperature is $212°\mathrm{F}$. The room temperature is $70°\mathrm{F}$. The cooling rate of the stew is $r = 0.048$. How long (minutes) will it take to cool to stew to a serving temperature of $100°\mathrm{F}$?

Newton's Law of Cooling: $T = (T_0 - T_R)e^{-rt} + T_R$

- T = the ending temperature
- T_0 = the initial temperature (beginning temperature)
- T_R = the room temperature
- r = the constant cooling rate
- t = the time (in minutes) of cooling

2. The moment magnitude, *M*, of an earthquake that releases energy, *E* (in ergs), can be modeled by the equation: $M = 0.291 \ln E + 1.17$

On May 22, 1960, a powerful earthquake took place in Chile. It had a moment magnitude of 9.5. How much energy did this earthquake release?

3. Suppose that \$250 is deposited into an account that pays 4.5% compounded quarterly. The equation $A = P\left(1 + \frac{r}{4}\right)^n$ gives the amount *A* in the account after *n* quarters for an initial investment *P* that earns interest at a rate *r*. Solve for *n* to find how long it will take for the account to contain at least \$500.