Name:____Per:__________

AP Physics C

Semester 1 - Mechanics

Unit 7
Gravitation
Workbook

Unit 7 - Gravitation

Supplements to Text Readings from
Fundamentals of Physics by Halliday, Resnick, & Walker
Chapter 14

TOPIC — Pg. #

Useful numbers to know for this unit

$G = 6.67\text{x}10^{-11}$ N•m^2/kg^2

Moon mass = $7.36\text{x}10^{22}$ kg — Earth mass = $5.98\text{x}10^{24}$ kg — Sun mass = $2\text{x}10^{30}$ kg

Moon radius = $1.74\text{x}10^{6}$ m — Earth radius = $6.37\text{x}10^{6}$ m — Sun radius = $6.96\text{x}10^{8}$ m

Distance from sun to earth = $1.5\text{x}10^{11}$ m — Distance from moon to earth = $3.84\text{x}10^{8}$ m

Unit 7 – Objectives and Assignments

Text: *Fundamentals of Physics* by Halliday, Resnick, & Walker **Chapter 14**

I. Gravitation

a. Students should know Newton's Law of Universal Gravitation so they can:

(1) Determine the force that one spherically symmetrical mass exerts on another.
(2) Determine the strength of the gravitational field at a specified point outside a spherically symmetrical mass.
(3) Describe the gravitational force inside and outside a uniform sphere, and calculate how the field at the surface depends on the radius and density of the sphere.

b. Students should understand the motion of a body in orbit under the influence of gravitational forces so they can:

(1) For a circular orbit:
- i) Recognize that the motion does not depend on the body's mass, describe qualitatively how the velocity, period of revolution, and centripetal acceleration depend upon the radius of the orbit, and derive expressions for the velocity and period of revolution in such an orbit.
- ii) Prove that Kepler's Third Law must hold for this special case.
- iii) Derive and apply the relations among kinetic energy, potential energy, and total energy for such an orbit.

(2) For a general orbit:
- i) State Kepler's three laws of planetary motion and use them to describe in qualitative terms the motion of the body in an elliptic orbit.
- ii) Apply conservation of angular momentum to determine the velocity and radial distance at any point in the orbit.
- iii) Apply angular momentum conservation and energy conservation to relate the speeds of a body at the two extremes of an elliptic orbit.
- iv) Apply energy conservation in analyzing the motion of a body that is projected straight up from a planet's surface or that is projected directly toward the planet from far above the surface.

Mechanics Unit 7 Homework
Chapter 14 #9, 11, 13, 14, 23, 24, 27, 29, 30, 31, 39, 42, 45, 47, 55, 57, 63, 66

Optional Reading:
Section 14-8 Satellites
Section 14-9 Einstein & Gravitation

Kepler's 3rd Law

Directions: From the Appendix C of your text, fill in the first two columns and calculate the other columns below. (Kepler did much the same with Tycho Brahe's data to form his 3rd Law of planetary motion)

Planet	Average Distance from Sun R_{ave}	Period of Revolution T (Earth Years)	$\frac{R_{ave}}{T}$	$\frac{R^2_{ave}}{T^3}$	$\frac{R^3_{ave}}{T^2}$
Mercury					
Venus					
Earth					
Mars					
Jupiter					
Saturn					
Uranus					
Neptune					
Pluto					

From your calculations, which relationship between average distance from sun and period of revolution shows a trend and what is that trend?

AP Lab Exercise – Kepler's Laws

Purpose

Plot a planetary orbit and apply Kepler's Laws.

Concept and Skill Check

The motion of the planets has intrigued astronomers since they first gazed at the stars, moon, and planets filling the evening sky. But the old ideas of eccentrics and equants (combinations of circular motions) did not provide an accurate accounting of planetary movements. Johannes Kepler adopted the Copernican theory that Earth revolves around the sun (heliocentric, or sun-centered, view) and closely examined Tycho Brahe's meticulously recorded observations on Mars' orbit. With these data, he concluded that Mars' orbit was not circular and that there was no point around which the motion was uniform. When elliptical orbits were accepted, all the discrepancies found in the old theories of planetary motion were eliminated. From his studies, Kepler derived three laws that apply to the behavior of every satellite or planet orbiting another massive body.

1. The paths of the planets are ellipses, with the center of the sun at one focus.
2. An imaginary line from the sun to a planet sweeps out equal areas in equal time intervals, as shown in Figure 1.
3. The ratio of the squares of the periods of any two planets revolving about the sun is equal to the ratio of the cubes of their respective average distances from the sun. Mathematically, this relationship can be expressed as

$$\frac{T_a^2}{T_b^2} = \frac{r_a^3}{r_b^3}.$$

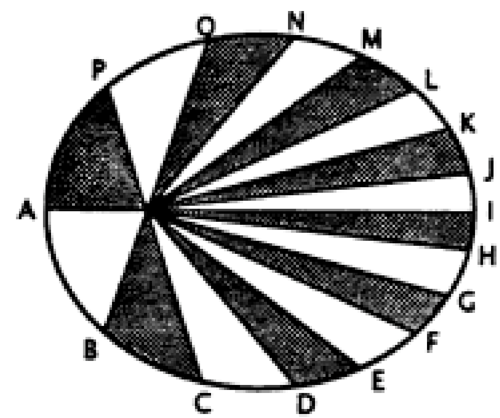


Figure 1. Kepler's law of areas.

In this experiment, you will use heliocentric data tables to plot the positions of Mercury on polar graph paper. Then you will draw Mercury's orbit. The distance from the sun, the radius vector, is compared to Earth's average distance from the sun, which is defined as 1 astronomical unit or 1 AU. The angle, or longitude, between the planet and a reference point in space is measured from the zero degree point, or vernal equinox.

Materials

polar graph paper   sharp pencil   metric ruler

Procedure

1. Orient your polar graph paper so that the zero degree point is on your right as you view the graph paper. The sun is located at the center of the paper. Label the sun without covering the center mark. Move about the center in a counter-clockwise direction as you measure and mark the longitude.
2. Select an appropriate scale to represent the values for the radius vectors of Mercury's positions. Since Mercury is closer to the sun than is Earth, the value of the radius vector will always be less than 1 AU. In this step, then, each concentric circle could represent one-tenth of an AU.

3. Table 1 provides the heliocentric positions of Mercury over a period of several months. Select the set of data for October 1 and locate the given longitude on the polar graph paper. Measure out along the longitude line an appropriate distance, in your scale, for the radius vector for this date. Make a small dot at this point to represent Mercury's distance from the sun. Write the date next to this point.
4. Repeat the procedure, plotting all given longitudes and associated radius vectors.
5. After plotting all the data, carefully connect the points of Mercury's positions and sketch the orbit of Mercury.

Observations and Data

Table 1

Some Heliocentric Positions for Mercury for 0^h Dynamical Time*

Date	Radius vector (AU)	Longitude (degrees)	Date	Radius vector (AU)	Longitude (degrees)
Oct. 1, 1990	0.319	114	Nov. 16	0.458	280
3	0.327	126	18	0.452	285
5	0.336	137	20	0.447	291
7	0.347	147	22	0.440	297
9	0.358	157	24	0.432	304
11	0.369	166	26	0.423	310
13	0.381	175	28	0.413	317
15	0.392	183	30	0.403	325
17	0.403	191	Dec. 2	0.392	332
19	0.413	198	4	0.380	340
21	0.423	205	6	0.369	349
23	0.432	211	8	0.357	358
25	0.440	217	10	0.346	8
27	0.447	223	12	0.335	18
29	0.453	229	14	0.326	29
31	0.458	235	16	0.318	41
Nov. 2	0.462	241	18	0.312	53
4	0.465	246	20	0.309	65
6	0.466	251	22	0.307	78
8	0.467	257	24	0.309	90
10	0.466	262	26	0.312	102
12	0.464	268	28	0.319	114
14	0.462	273	30	0.327	126

*Adapted from *The Astronomical Almanac for the Year 1990*, U.S. Government Printing Office, Washington, D.C., 20402, p. E9.

Analysis

1. Does your graph of Mercury's orbit support Kepler's law of orbits?

2. Draw a line from the sun to Mercury's position on December 20. Draw a second line from the sun to Mercury's position on December 30. The two lines and Mercury's orbit describe an area swept by an imaginary line between Mercury and the sun during the ten-day interval of time. Lightly shade this area. Over a small portion of an ellipse, the area can be approximated by assuming the ellipse is similar to a circle. The equation that describes this value is

$$\text{area} = (\theta/360°)\pi r^2,$$

   where r is the average radius for the orbit.

   Determine θ by finding the difference in degrees between December 20 and December 30. Measure the radius at a point midway in the orbit between the two dates. Calculate the area in AUs for this ten-day period of time.

3. Select two additional ten-day periods of time at points distant from the interval in Question 2 and shade these areas. Calculate the area in AUs for each of these ten-day periods.

4. Find the average area for the three periods of time from Questions 2 and 3. Calculate the relative error between each area and the average. Does Kepler's law of areas apply to your graph?

5. Calculate the average radius for Mercury's orbit. This can be done by averaging all the radius vectors or, more simply, by averaging the longest and shortest radii that occur along the major axis. The major axis is shown in Figure 2. Recall that the sun is at one focus; the other focus is located at a point that is the same distance from the center of the ellipse as the sun, but in the opposite direction.

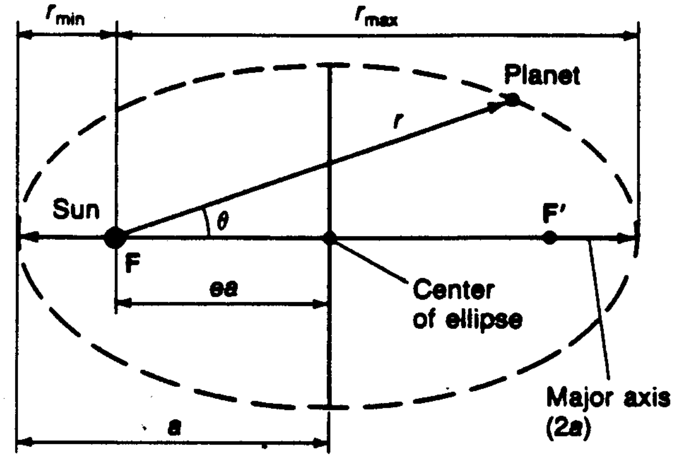


Figure 2. The major axis passes through the two foci (F and F′) and the center of the ellipse. The value *ea* determines the location of the foci; *e* is the eccentricity of the orbit. If *e* = 0, the orbit is a circle, and the foci merge at one, central point.

From Table 1, find the longest radius vector. Then, align a metric ruler so that it describes a straight line passing through the point on the orbit that represents the longest radius vector and through the center of the sun to a point opposite on the orbit. Find the shortest radius vector by reading the longitude at this opposite point and consulting Table 1 for the corresponding radius vector. Average these two radius vector values. Using the values for Earth's average radius (1.0 AU), Earth's period (365.25 days), and your calculated average radius of Mercury's orbit, apply Kepler's third law to find the period of Mercury. Show your calculations.

6. Refer again to the graph of Mercury's orbit that you plotted. Count the number of days required for Mercury to complete one orbit of the sun; recall that this orbital time is the period of Mercury. Is there a difference in the two values (from Questions 5 and 6) for the period of Mercury? Calculate the relative difference in these two values. Are the results from your graph consistent with Kepler's law of periods?

Application

There has been some discussion about a hypothetical planet **X** that is on the opposite side of the sun from Earth and that has an average radius of 1.0 AU. If this planet exists, what is its period? Show your calculations.

Extension

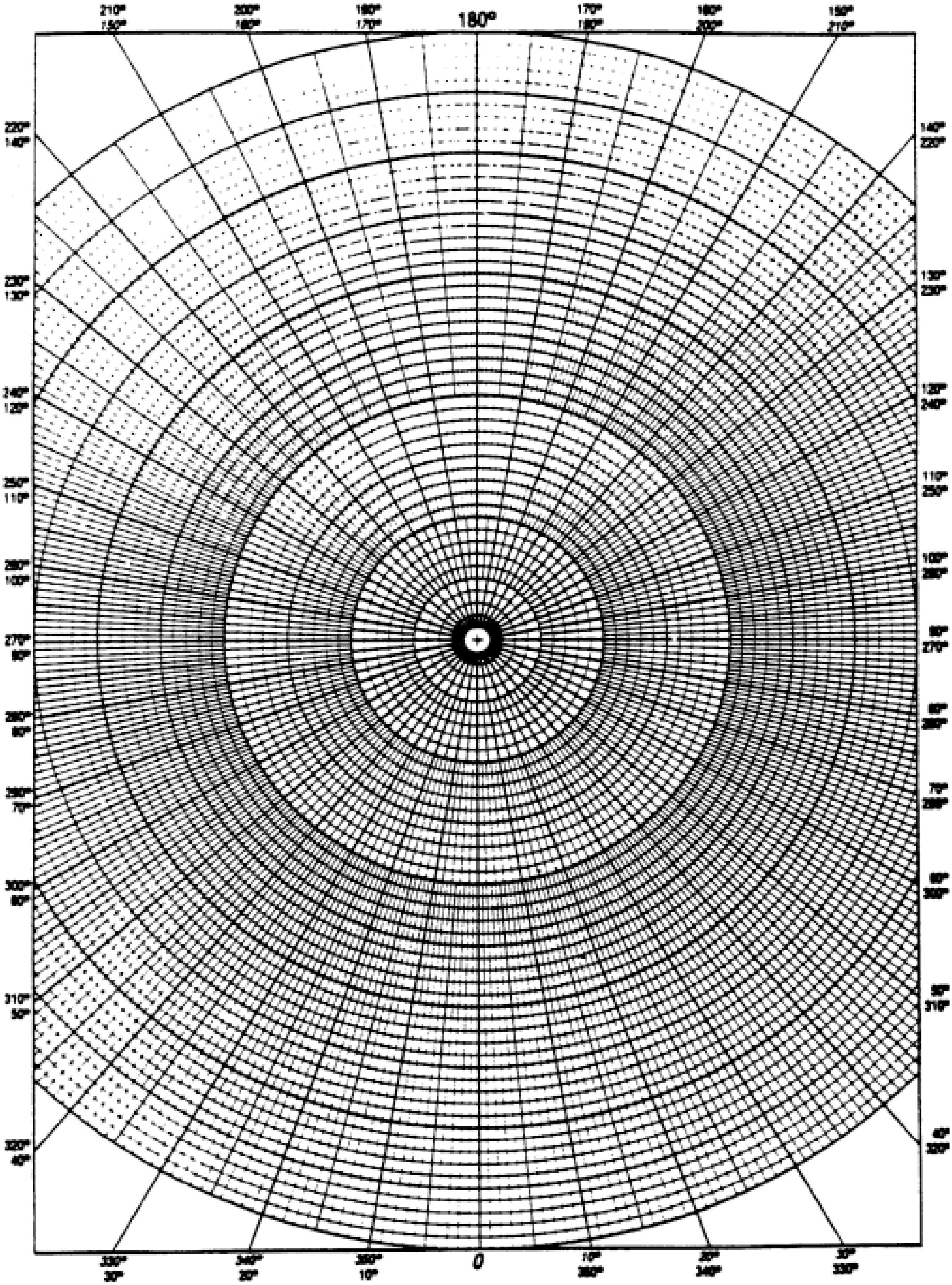
Using the data in Table 2, plot the radius vectors and corresponding longitudes for Mars. Does the orbit you drew support Kepler's law of ellipses? Select three different areas and find the area per day for each of these. Does Kepler's law of areas apply to your model of Mars?

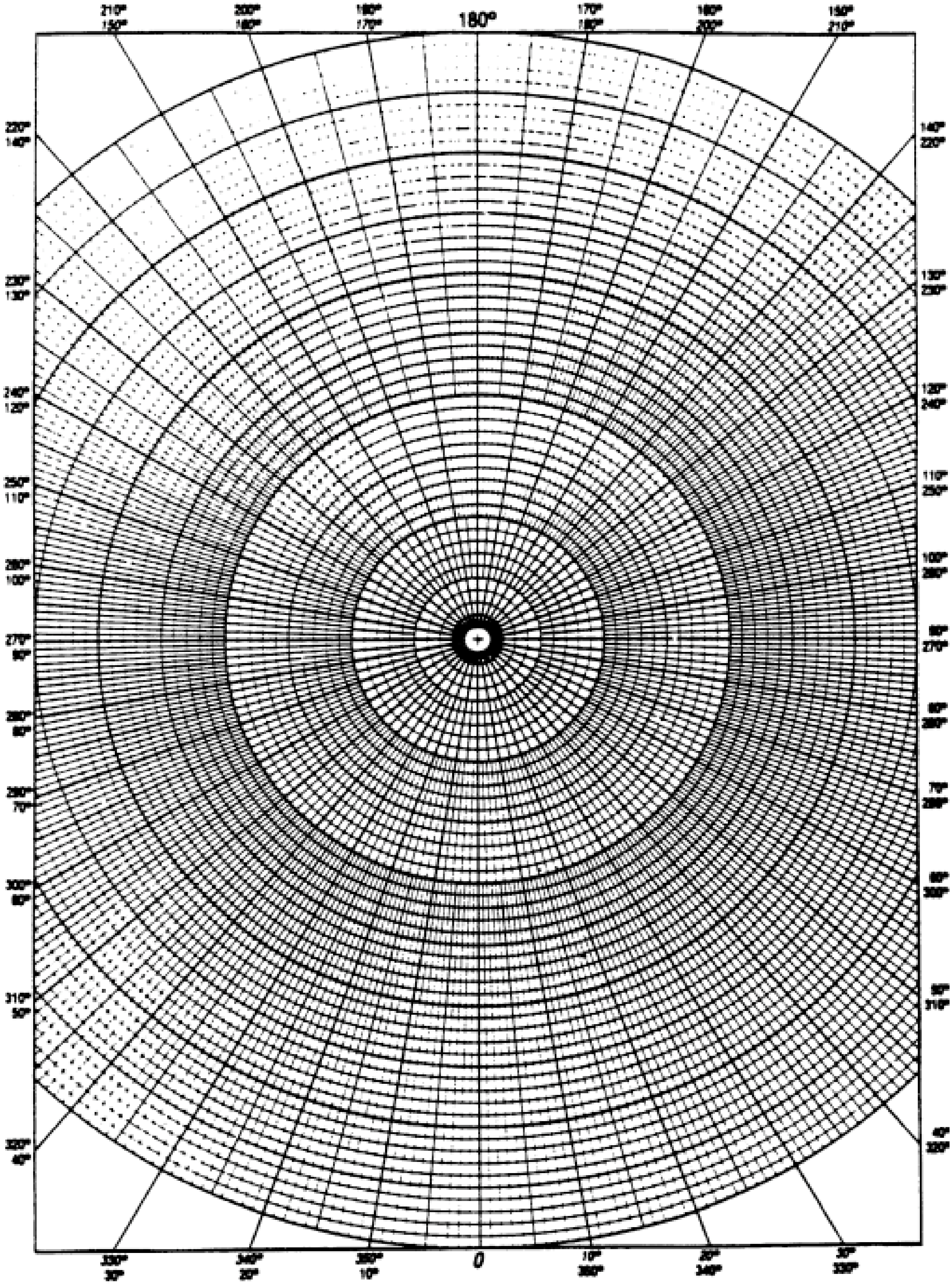
Table 2

Some Heliocentric Positions for Mars for 0^h Dynamical Time*

Date	Radius vector (AU)	Longitude (degrees)	Date	Radius vector (AU)	Longitude (degrees)
Jan. 1, 1990	1.548	231	July 12	1.382	343
17	1.527	239	28	1.387	353
Feb. 2	1.507	247	Aug. 13	1.395	3
18	1.486	256	29	1.406	13
Mar. 6	1.466	265	Sept 14	1.420	23
22	1.446	274	30	1.436	32
Apr. 7	1.429	283	Oct. 16	1.455	42
23	1.413	293	Nov. 1	1.474	51
May 9	1.401	303	17	1.495	60
25	1.391	313	Dec. 3	1.516	68
June 10	1.384	323	19	1.537	76
26	1.381	333	Jan. 4, 1991	1.557	84

*Adapted from *The Astronomical Almanac for the Year 1990*, U.S. Government Printing Office, Washington, D.C., 20402, p. E12.




180°
0

Kepler formulated his 3rd Law for planetary motion but he didn't know why planets followed this law. Not until Newton formulated his Law of Gravitation was the mystery explained. **Derive Kepler's 3rd Law for planetary motion starting from Newton's Law of Gravitation.**

1. Using Kepler's 3rd Law

a) Given that the Earth's period of revolution around the Sun is 1 year and it's average orbital radius is $R_E = 1.5\text{x}10^8$ km, find the mass[1] of the Sun.

b) Pluto is about 40 times the distance the Earth is from the Sun. How many Earth years is Pluto's period[2] of revolution around the Sun.

[1] $2\text{x}10^{30}$ kg
[2] 253 years

Using Newton's Law of Gravitation

$$\boxed{\vec{F}_g = -G\frac{m_1 m_2}{r^2}\hat{r}}$$

What is the significance of the negative sign?

1. Astronaut in Space

On the way to the Moon, the Apollo astronauts reached a point in space where the Moon's gravitational pull is stronger than the Earth's gravitational pull.

a) Determine the distance[3] of this point from the center of the Earth.
b) What is the acceleration[4] due to the Earth's gravity at this point?
c) What is weight of the astronaut due to the Earth’s pull at half of this point of no gravitational force?

2. How far does the moon fall?

The Moon is 384,400 km distant from the Earth's center and completes an orbit every 27.3 days. How far does the moon “fall” toward the Earth in one second? In other words, how far does the moon drop from a straight-line path in one second?

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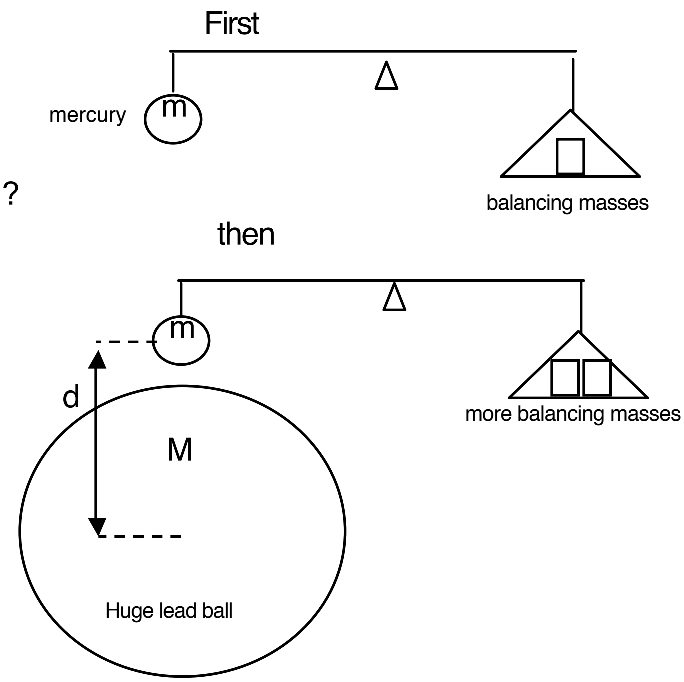
[3] 3.46×10^8 m
[4] 0.0033 m/s^2

3. Determining G with a sensitive balance

Henry Cavendish's method of determining G is the most famous; however, a different scientist used a different method of determining G. A man named Phillip von Jolly suspended a spherical container of very dense mercury of a known mass on one arm of a very sensitive balance that he put in equilibrium as shown below. He then rolled a very massive lead ball (over 3000 kg) under the mercury.

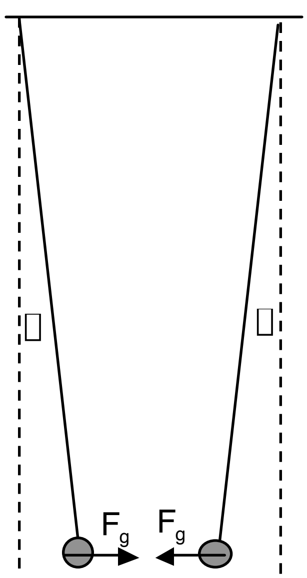
Why did he have to adjust the balance?

How did this enable him to calculate the value of G?



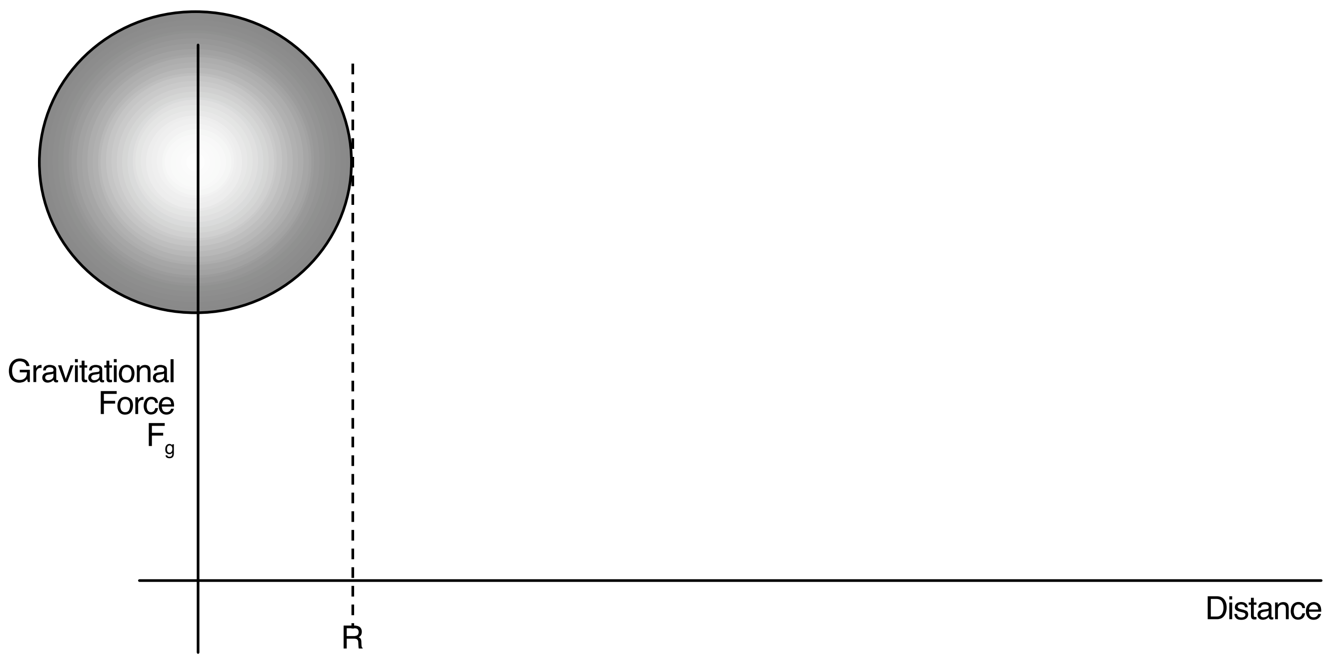
4. Determining G with two suspended metal spheres

Given two metal spheres of 100 kg suspended on 50 meter long ropes that are 1 meter apart when attached to the ceiling of a cathedral tower, show how you could determine G.



5. Graphing Gravitational Force

Make sure you read about falling into a tunnel through the Earth on page 329. Given a planet of uniform density, what does the graph of the gravitational **force** look like both inside and outside a planet of mass M and radius R?



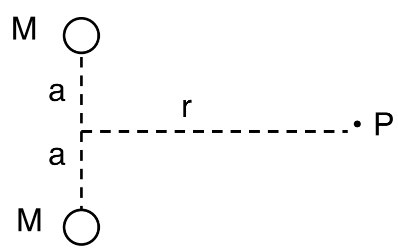
Gravitational Fields

1. Deriving Expression for General Gravitational Field

Derive an expression for the gravitational field outside any mass (considered a point mass).

2. Gravitational Field of Two Masses

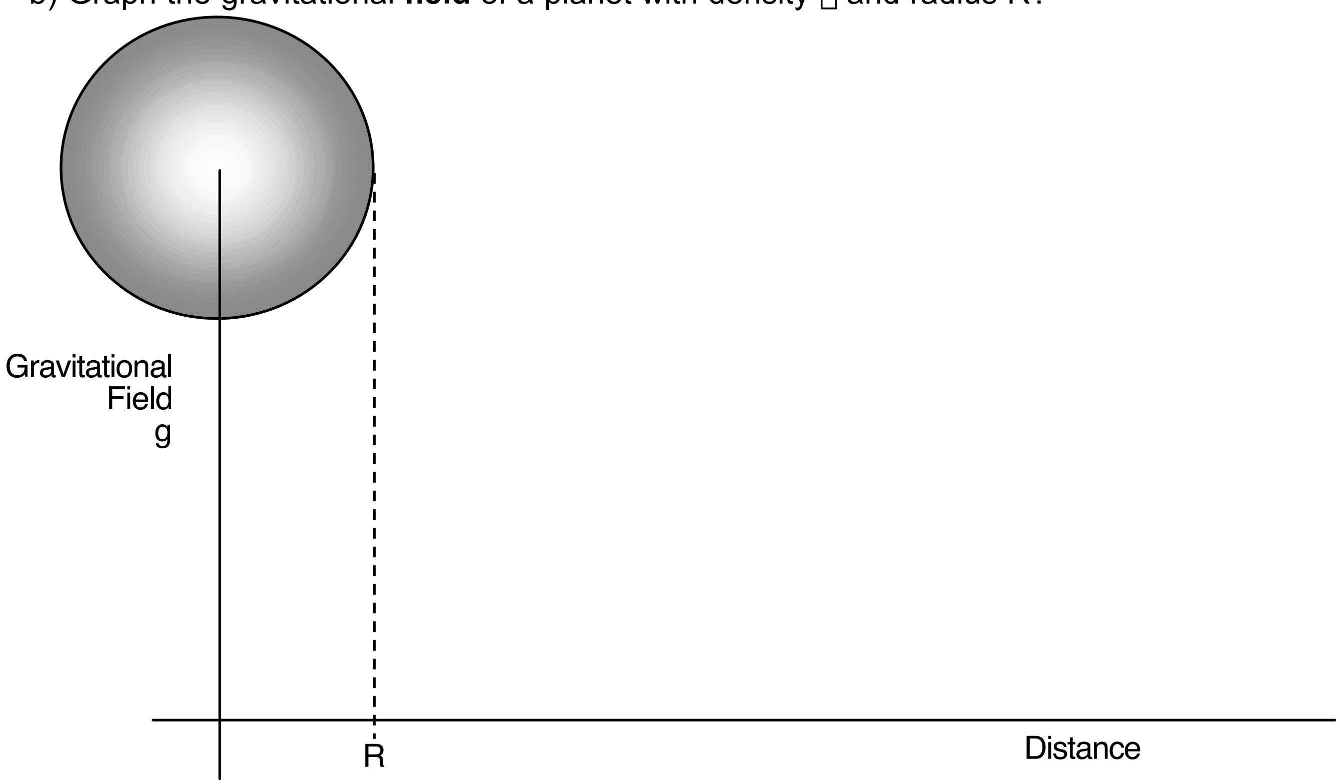
Given two masses shown below, derive the magnitude and direction for the gravitational field at point P in terms of the M, r, and a.



3. Planetary Gravity Inside and Outside

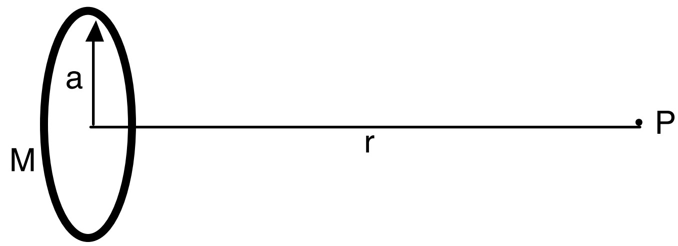
a) How does the gravitational field of a planet depend on the planet's density ρ and its radius R? In other words, what's the gravitational field like INSIDE a planet?

b) Graph the gravitational **field** of a planet with density ρ and radius R?



4. Gravitational Field of a Thin Ring

Find the magnitude and direction of the gravitational field at a point P that is a distance r along the axis of a thin ring of mass M and radius a. (Use calculus. This problem is more reality based when we get to E&M where we will do the same thing with an Electric Field from a charged ring. So an exposure here and another exposure later will solidify the method.)



5. Newton's Shell Theorem

Newton's law of gravity assumes that "the gravitational force exerted by a finite-size, spherically symmetrical mass distribution on a particle outside the sphere is the same as if the entire mass of the sphere were concentrated at its center." Newton proved this with calculus and you can try to prove it yourself mathematically. We're going to do it conceptually using what we know about the gravitational force of a thin ring (the previous example.)

Gravitational Potential Energy

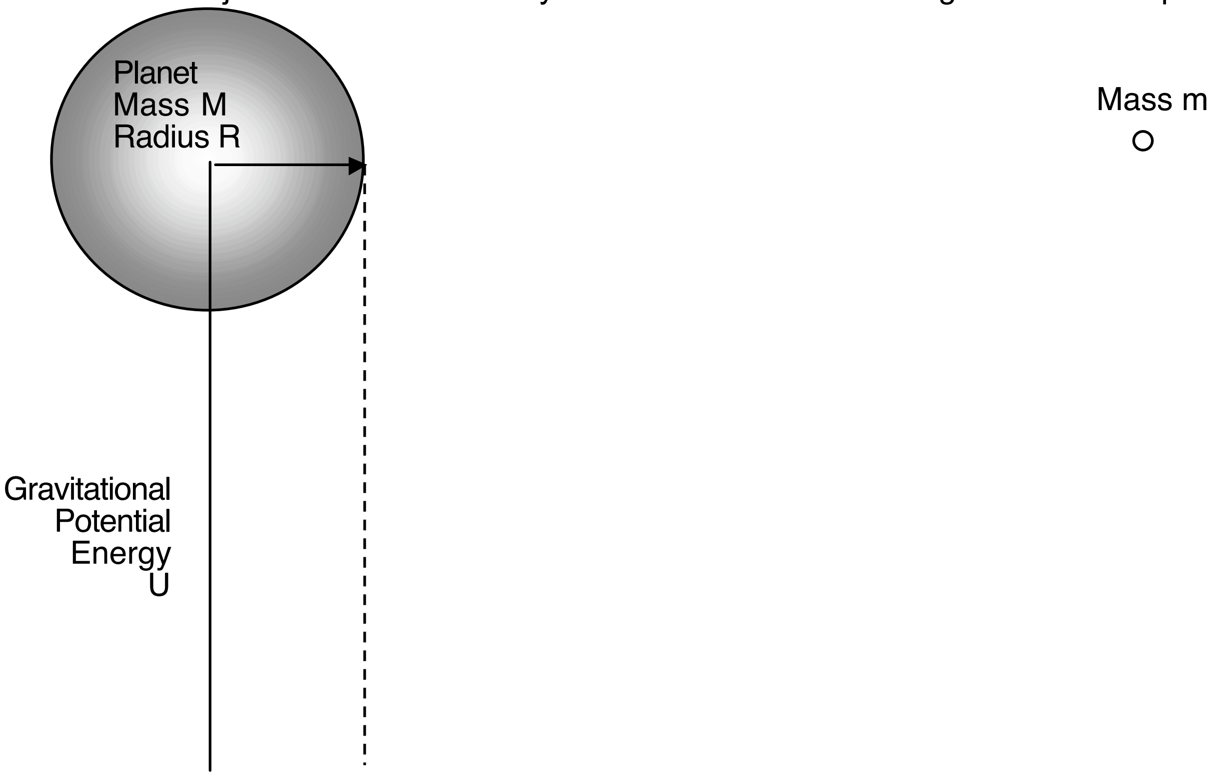
1. Deriving Gravitational Potential Energy

$$U_g(r) = -G\frac{m_1 m_2}{r}$$

a) What is the gravitational potential energy between two masses separated a distance r? This is similar to determining the $PE_{gravity} = mgh$ for an object above a particular reference point – usually the ground. Let's determine PE_g or U_g for any two objects anywhere, not just on the surface of the Earth. We'll do this by thinking of bringing a baseball from very far away or basically infinity toward the Earth. Recall that for gravitational potential energy, a reference point is required and that that reference point is arbitrary. So for us, we will choose a zero reference point at infinity. In other words, the gravitational potential energy between any two objects is zero when they are very far (infinitely) apart. (You will see this method of calculation again when we get to E&M and we derive an expression for Electric Potential Energy U_e for charges.)

There are 4 significances of Gravitational Potential Energy.

b) Graph the gravitational potential energy between a planet with mass M and radius R and another object with mass m. Do you need to consider the region inside the planet?



2. Showing $\Delta U_g = mg\Delta h$

A baseball of mass m falls a vertical distance of Δh near the Earth's surface. Show that the general expression for the change of Gravitational Potential Energy reduces to the familiar $PE_g = mg\Delta h$.

3. Meteor Crashing into Moon

a) How much work[5] is done by the Moon's gravitational field as a 1000 kg meteor comes in from outer space and crashes into the Moon? Do you expect the work done to be positive or negative?

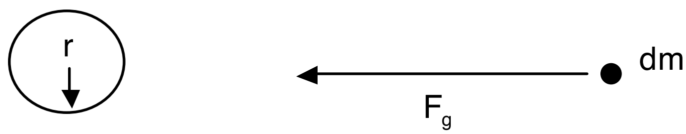
b) Given your answer in part a) and the meteor was at rest at infinity and ignoring gravitational effects of other celestial bodies, what was the meteor's velocity[6] upon impact with the Moon?

[5] $+2.8\times10^9$ J
[6] 2376 m/s

4. Formation of Uniformly Dense Planet

This is an oversimplified look at how planets are made due to gravitational forces. Planets are basically collections of space dust that were gravitationally attracted to each other. How much work is done by the force of gravity in assembling a uniform sphere of radius R and density ρ?

Solution: Let's bring in an elemental particle of mass dm into an already assembled sphere of radius r such as:



From conservation of energy, we know that the work done by gravity is the (+ , –) change of ______________________________________ of the sphere and elemental mass dm system.

The small amount of work to bring in the 1st of many elemental masses dm brings about a change of ______________________________________ of the sphere and elemental mass dm system. In short, we have

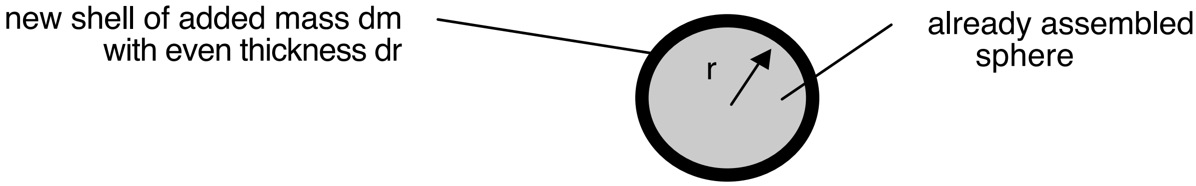
dW = _______________

Bringing in what we know, the above expression can be changed in terms of r, ρ, dm and other appropriate constants.

If you did things correctly, you should have dW as a function of dm. However, you cannot simply integrate dW with respect to dm because ____. So in order to integrate, you have to change dm to a function of ______. How do you do this?

Changing dm to a function of dr:

First assume that the elemental mass dm brought in is smoothed over the entire surface of the sphere much like icing is spread over a cake.



The new shell layer will have the same density as the already assembled sphere. So from the density ρ = dm/Vol., the mass dm becomes

dm = ρ x Vol. = ρ (________________________ of shell) x (__________________of shell)

We can then change dm as a function of dr which becomes:

dm = ____________________________________

Now our previous expression for dW = ________________________________

becomes dW = ____

All we have to do now is integrate the above expression with respect to _______ and with limits of integration from ___________ to ____________.

Work by Gravity to assemble a uniformly dense planet with mass M and radius R

Work by Gravity = ____________________

5. Unlocking Jupiter's Secrets to the Universe

a) Jupiter is a (gaseous, solid, liquid) planet. How much work was done by gravity over millions of years in collecting its current 1.9x1027 kg of material in a sphere of diameter 143,000 km?

b) If you compare Earth's atmosphere with Jupiter's and other celestial body, there seems to be a contradiction.

Fact: Earth's atmosphere has only a few parts per million of H and He.

Fact: It is known that planets and solar systems formed from interstellar gas/dust of mostly H and He with 1% or 2 % other elements.

CONTRADICTION! Can you see the problem?

From you previous chemistry class and studying thermodynamics, evidence shows that the Average Kinetic Energy for all gaseous molecules no matter the mass is ________________________ or

$$KE_{light\ molecules}\ (<, =, >)\ KE_{heavy\ molecules}$$

Since $KE = mv^2/2$, this suggests then that the lighter molecules must have (greater, same, lesser) velocity than heavy molecules. Since H and He are (light, heavy) molecules, we can conclude that the lack of H and He gas in our atmosphere means that their velocity is large enough to escape the effect of _____________________________. Similarly why does the moon have NO atmosphere?____.

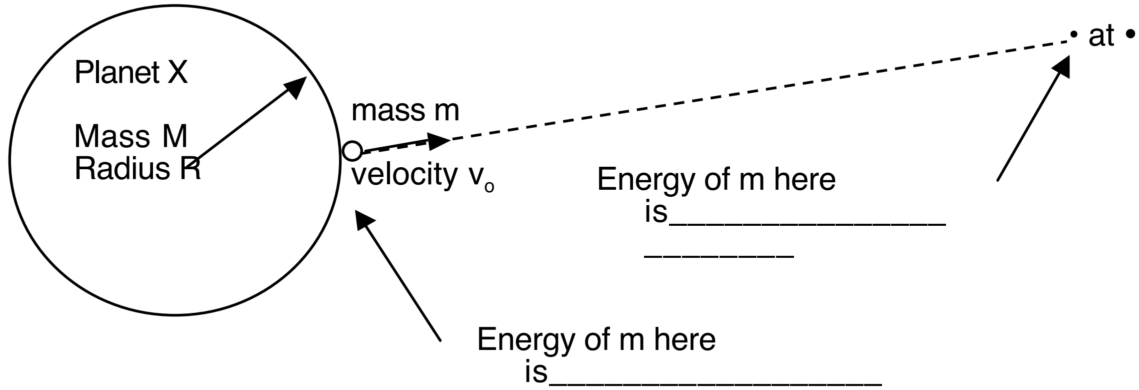
So comparing Jupiter and Earth.

Earth has lost most of its H and He gas because of its ________________________________. Jupiter, however, kept most of its H and He because of its ____________________________. In short, the random KE of gases on Jupiter do not have the velocity great enough to ___________ Jupiter's gravity. So what does that imply about the age of the gases in Jupiter's atmosphere?

What does Jupiter's atmosphere suggest about the formation of our solar system?

6. Escape Velocity from a Planet

In our previous example, we discussed how gases could escape a planet's gravitational pull. So what is the minimum velocity for a particle to overcome a planet's gravity and never return due to gravitational forces?



Conservation of Energy requires that ____.
From this, show that the Escape Velocity from a planet with mass M and radius R is given by

$$\boxed{v_{escape} = \sqrt{\frac{2GM}{R}} \quad \text{Escape Velocity}}$$

What are we ignoring in this case?

7. Escape Velocity of Earth and Gas Velocities

a) Calculate the escape velocity for Earth.

b) Calculate the escape velocity for Jupiter.

c) Compare the random kinetic energies of a hydrogen atom at escape velocities for Earth and Jupiter.

8. Earth-Sun System

The Earth-Sun distance is 1.521×10^{11} m at the aphelion and 1.471×10^{11} m at the perihelion. The Earth's orbital speed at the perihelion is 3.027×10^{4} m/s. Neglect the gravitational effects of other celestial bodies.

a) Determine the Earth's orbital speed[7] at the aphelion. Use conservation of angular momentum.

b) Determine the Earth's kinetic and potential energy at the perihelion[8].

c) Determine the Earth's kinetic and potential energy at the aphelion[9].

d) Is the total energy constant? Show why or why not.

[7] 2.93×10^{4} m/s
[8] $KE_{perihelion} = 2.74 \times 10^{33}$ J; $PE_{perihelion} = -5.4 \times 10^{33}$ J
[9] $KE_{aphelion} = 2.57 \times 10^{33}$ J; $PE_{aphelion} = -5.22 \times 10^{33}$ J

Calculating Force from Potential

Previously, we know that if given Gravitational Force F_g, then we can derive an expression for the change in Gravitational Potential Energy ΔU_g by

$$\Delta U_g = -W_g = -\int^{r} \vec{F}_g \bullet d\vec{x}$$

And our results from $\mathbf{F}_g = -G\frac{m_1 m_2}{r^2}\hat{r}$, we got U_g = ______________________.

1. Deriving Force from Potential Energy

What about going the opposite way? How do you find the gravitational force F_g if given the gravitational potential energy U_g?

We know that work and potential energy are related by Work = ___________

A general expression for work in terms of force and displacement is also __________________

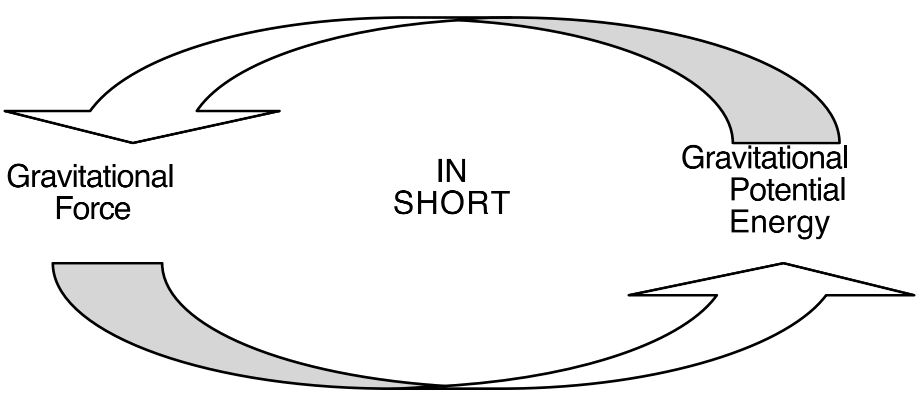
So our above expression becomes

Since gravity is a (conservative, non-conservative) force, the direction of gravitational force and dr are always (radial, perpendicular, parallel) which means that the cosø in the dot product essentially becomes __________.

Solving for F, we get

Gravitational Force as a function of Gravitational Potential Energy
F = _________________

What does the above suggest about graphs of Gravitational Potential Energy vs. radial distance r?



2. Using $F = -dU_g/dr$

Given that $U(r) = -G\dfrac{m_1 m_2}{r}$ use the above expression to derive Newton's Law of Gravitation.

Gravitation Review Problem

A satellite of mass m is in an orbit of radius R around a planet of mass M in the equatorial plane of the planet. The satellite remains above the same point on the planet at all times. The free-fall acceleration on the surface of the planet is g.

a) Find the speed[10] of the satellite.

b) Find the period [11]of the satellite.

c) Find the kinetic energy[12] of the satellite.

---

[10] $\sqrt{GM/R}$

[11] $2\pi R^{\frac{3}{2}}\sqrt{GM}$

[12] $\frac{GmM}{2R}$

d) Find the potential energy[13] of the satellite.

e) Find the radius[14] of the planet.

f) Find the minimum possible period[15] of the satellite.

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[13] $-\frac{GmM}{R}$

[14] $\sqrt{GM/g}$

[15] $2\pi(GM)^{\frac{1}{4}}g^{-\frac{3}{4}}$

g) Find the maximum kinetic energy[16] of the satellite.

h) Find the minimum potential energy[17] of the satellite.

i) Find the escape speed[18] of the satellite from this orbit.

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16 $\frac{1}{2}m(gGM)^{\frac{1}{2}}$

17 $\square m(gGM)^{\frac{1}{2}}$

18 $\sqrt{2GM/R}$