Unit #7 - Optimization, Optimal Marginal Rates

Some problems and solutions selected or adapted from Hughes-Hallett Calculus.

Optimization Introduction

1. Let $f(x) = x^2 - 10x + 13$, and consider the interval $[0, 10]$.
   (a) Find the critical point c of $f(x)$ and compute $f(c)$.
   (b) Compute the value of $f(x)$ at the endpoints of the interval $[0, 10]$.
   (c) Determine the global min and max of $f(x)$ on $[0, 10]$.
   (d) Find the global min and max of $f(x)$ on $[0, 1]$. (Note: not the same interval as before)

(a) The critical point of $f(x)$ is the solution to $f'(x) = 0$. The derivative is$f'(x) = 2x - 10$. Setting this equal to zero and solving for x gives $x = 5$. Evaluating $f(5)$ yields -12.

(b) Evaluating $f(0)$ and $f(10)$, we find that each is equal to 13.

(c) The global min and max values must occur at critical points or at the endpoints of the interval. Since the value at the critical point is smaller than the value at the endpoints, the value of $f(5)$ is a minimum, and the value of $f(0)$ (or $f(10)$ since they are equal) is a maximum.

(d) Since there are no critical points in the interval $[0, 1]$, so the global min and max values lie at the endpoints of the interval. Computation yields $f(0) = 13$,$f(1) = 4$, so the minimum is 4 and the maximum is 13.

2. Find the maximum and minimum values of the function $f(x) = \dfrac{\ln(x)}{x}$ on the interval [1,3].

First we check for critical points. The critical point of $f(x)$ is the solution to $f'(x) = 0$.
The derivative is$f'(x) = x^{-2} - \dfrac{\ln(x)}{x^2}$.
Setting this equal to zero and solving for x gives $x = e^1 = e$.
Evaluating $f(e)$ yields the value $\frac{1}{e} = 0.3679$.
The values of the function at the endpoints of the interval are $f(1) = 0$, $f(3) = 0.3662$, so the minimum value is 0, and the maximum value is $\frac{1}{e} = 0.3679$.

3. Find the minimum and maximum values of $y = \sqrt{10}\theta - \sqrt{5}\sec\theta$ on the interval $[0, \frac{\pi}{3}]$.

Let $f(\theta) = \sqrt{10}\theta - \sqrt{5}\sec\theta$, and taking a derivative gives $f'(\theta) = \sqrt{10} - \sqrt{5}\sec\theta\tan\theta$.

At critical points,

$$\sqrt{10} - \sqrt{5}\sec\theta\tan\theta = 0$$

This is a challenging equation to solve, but there are several ways to go about solving it; what is shown below is just one. We will convert all the trig functions to sines and see what we get.

$$\sqrt{10} - \sqrt{5}\sec\theta\tan\theta = 0$$

$$\sqrt{10} = \sqrt{5}\sec\theta\tan\theta$$

Just sine/cosine: $$\sqrt{2} = \frac{1}{\cos\theta}\frac{\sin\theta}{\cos\theta}$$

Just sines: $$\sqrt{2} = \frac{\sin\theta}{1 - \sin^2\theta}$$

Multiplying up: $$\sqrt{2} - \sqrt{2}\sin^2\theta = \sin\theta$$

$$\sqrt{2}\sin^2\theta + \sin\theta - \sqrt{2} = 0$$

Now we can use the quadratic formula to solve for $\sin\theta$:

$$\begin{aligned}
\sin\theta &= \frac{-1 \pm \sqrt{1^2 - 4\sqrt{2}(-\sqrt{2})}}{2\sqrt{2}} \\
&= \frac{-1 \pm \sqrt{1+8}}{2\sqrt{2}} \\
&= \frac{-1 \pm 3}{2\sqrt{2}} \\
&= \frac{2}{2\sqrt{2}}, \text{ or } \frac{-4}{2\sqrt{2}} \\
\text{so } \sin\theta &= \frac{1}{\sqrt{2}}, \text{ or } \frac{-2}{\sqrt{2}}
\end{aligned}$$

The second value, $\frac{-2}{\sqrt{2}} \approx -1.414$ is impossible for $\sin\theta$ to achieve because $\sin\theta$ cannot go beyond the interval $[-1, 1]$.

The first value is possible: $\sin\theta = \frac{1}{\sqrt{2}}$ is in the standard 45/45 or $\frac{pi}{4}$, $\frac{\pi}{4}$ triangle, so $\theta = \frac{\pi}{4}$ is a possible solution, and it lies on the interval from the question, $[0, \frac{\pi}{3}]$.

Since were are on a closed and bounded interval, the global max and min values of y will occur at either the one critical point ($x = \frac{\pi}{4}$) or one of the end points ($x = 0$ or $x = \frac{\pi}{3}$).

Testing the value of f at each of those points, we find that the minimum value of f on this interval is at the endpoint $\theta = 0$, where $f(0) = -2.2361$,
whereas the maximum value over this interval is $f(\frac{\pi}{4}) = \sqrt{10}(\frac{\pi}{4} - 1) = -0.6786$.
At the second endpoint $\theta = \frac{\pi}{3}$,
$f(\frac{\pi}{3}) = \sqrt{10}\frac{\pi}{3} - 2\sqrt{5} = -1.1606$.

4. Find the maximum and minimum values of the function $f(x) = x - \dfrac{125x}{x+5}$ on the interval [0,21].

First we check for critical points. The critical point of $f(x)$ is the solution to $f'(x) = 0$.
The derivative is$f'(x) = 1 - \frac{625}{(x+5)^2}$.
Setting this equal to zero and solving for x gives $x = -5 \pm 25$, and of these two critical points only -5+25=20 lies in our interval.
Evaluating $f(-5+25) = f(20)$ yields the value -80.
The values of the function at the endpoints of the interval are $f(0) = 0$, $f(21) = -79.9615$, so the minimum value is -80, and the maximum value is 0.

5. The function $f(x) = -2x^3 + 21x^2 - 36x + 10$ has one local minimum and one local maximum. Find their (x, y) locations.

To identify any local extrema, we start by identifying critical points. We note that $f(x)$ is a polynomial, so its derivative is defined everywhere, so only points where $f'(x) = 0$ will be critical points.

$$\begin{aligned} & f'(x) = -6x^2 + 42x - 36 \\ \text{Setting } f'(x) = 0, \quad & 0 = -6x^2 + 42x - 36 \\ \text{Factoring,} \quad & 0 = -6(x^2 - 7x + 6) \\ & 0 = -6(x-1)(x-6) \end{aligned}$$

The two critical points are at $x = 1$ and $x = 6$. Subbing those x values back into the original function $f(x)$ gives us the points (1, -7) and $(6, 118)$.

Using test points and the first derivative test, or taking another derivative and using the second derivative test, you can find that:
there is a local minimum at (1,-7), and
there is a local maximum at (6, 118).

6. A Queen's University student decided to depart from Earth after his graduation to find work on Mars. Before building a shuttle, he conducted careful calculations. A model for the velocity of the shuttle, from liftoff at t = 0 s until the solid rocket boosters were jettisoned at t = 80 s, is given by

$$v(t) = 0.001094333t^3 - 0.08215t^2 + 28.6t - 4.3$$

(in feet per second). Using this model, estimate the global maximum value and global minimum value of the **acceleration** of the shuttle between liftoff and the jettisoning of the boosters.

For simplicity of presentation, let $c_3 = 0.001094333$ and $c_2 = 0.08215$, so

$$v(t) = c_3t^3 - c_2t^2 + 28.6t - 4.3$$

Differentiating once gives the acceleration

$$a(t) = 3c_3t^2 - 2c_2t + 28.6$$

To find the critical points of the acceleration, we need to know when its rate of change is zero:

$$a'(t) = 6c_3t - 2c_2$$

This will have a zero value when

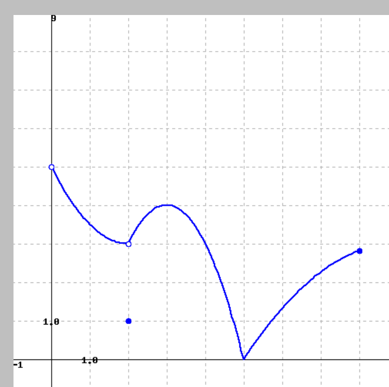
$$\begin{aligned} 0 &= 6c_3t - 2c_2 \\ t &= \frac{2c_2}{6c_3} = \frac{1}{3}\frac{0.08215}{0.001094333} \approx 25.022 \end{aligned}$$

Thus $t \approx 25.022$ is the only critical point.

We compute the acceleration at the end points of the interval ($t = 0$ and 80), and at the critical point:
$a(0) = 28.6$ ft/s^2
$a(25.022) = 26.5444$ ft/s^2
$a(80) = 36.4672$ ft/s^2

The global maximum acceleration is 36.4672 ft/s^2 and occurs at $t = 80$, while the global minimum acceleration is 26.5444 ft/s^2, and it occurs at $t = 25.022$ s.

7. Use the given graph of the function on the interval $(0, 8]$ to answer the following questions.



(a) Where does the function f have a local maximum?

(b) Where does the function f have a local minimum?

(c) What is the global maximum of f?

(d) What is the global minimum of f?

(a) x =3. ($x = 8$ is an end-point, and so is **not** considered a local max or min using our definitions.)

(b) x =2, 5

(c) none: at the left end, the interval is open, so the maximum is never reached.

(d) The global minimum of the function occurs at $x = 5$, and the value there is $f(5) = 0$.

8. Find the global maximum and minimum values of the following function on the given interval.
$f(t) = 4t\sqrt{4-t^2},\ [-1, 2]$

We are on a closed and bounded interval, and the function is continuous on that interval, so the global max and global min will occur at either:

- a critical point in the interior of the interval, or
- one of the end points ($x = -1$ or $x = 2$).

Looking for critical points, we need the derivative:

$$f'(t) = 4(4-t^2)^{1/2} + 4t\left(\frac{1}{2}\frac{1}{(4-t^2)^{1/2}}(-2t)\right)$$

Setting that equal to zero gives:

$$0 = 4\sqrt{4-t^2} + 4t\left(\frac{1}{2}\frac{1}{\sqrt{4-t^2}}(-2t)\right)$$

We note that $t = 2$ is a critical point (undefined derivative due to the $\frac{1}{\sqrt{4-2^2}}$ term), but since $t = 2$ is also an endpoint, we will look at it as a possible min/max later anyway; we are concerned for now with critical points in the *interior* of the the $t \in [-1, 2]$ interval.

Back to the $f'(t) = 0$ condition, multiplying both sides by $\sqrt{4-t^2}$,

$$0 = 4(\sqrt{4-t^2})^2 - 4t^2$$
$$\text{or } 0 = 4(4-t^2) - 4t^2$$

Simplifying (dividing all terms by 4) and solving for t,

$$0 = (4-t^2) - t^2$$
$$0 = 4 - 2t^2$$
$$2t^2 = 4$$
$$t^2 = 2$$
$$t = \pm\sqrt{2} \approx \pm 1.4142$$

Since only $t = +\sqrt{2}$ is in the domain $t \in [-1, 2]$, we can ignore the other critical point $t = -\sqrt{2}$.

To determine which points are the global max and global min, we can simply compare the values of $f(t)$ at the critical points and the end points:

$t = -1$	$f(-1) = 4(-1)\sqrt{4-(-1)^2} \approx -6.9282$
$t = \sqrt{2}$	$f(\sqrt{2}) = 4\sqrt{2}\sqrt{4-2} = 8$
$t = 2$	$f(2) = 4(2)\sqrt{4-4} = 0$

By comparing the $f(t)$ values, we find:

- The global maximum occurs at $x = 1.4142$ and $y = 8$.
- The global minimum occurs at $x = -1$, and $y = -6.9282$

9. An object with weight W is dragged along a horizontal plane by a force acting along a rope attached to the object. If the rope makes an angle θ with the plane, then the magnitude of the force is

$$F = \frac{\mu W}{\mu\sin(\theta) + \cos(\theta)}$$

where μ is a positive constant called the coefficient of friction and where $0 \le \theta \le \pi/2$. Find the value for $\tan\theta$ which minimizes the force. Your answer may depend on W and μ.

To minimize F, we differentiate with respect to θ:

$$F(\theta) = \mu W(\mu\sin(\theta) + \cos(\theta))^{-1}$$
$$\text{so}\quad F'(\theta) = -\frac{\mu W}{(\mu\sin(\theta) + \cos(\theta))^2}(\mu\cos(\theta) - \sin(\theta))$$

Setting the derivative equal to zero to identify critical points,

$$0 = -\frac{\mu W}{(\mu\sin(\theta) + \cos(\theta))^2}(\mu\cos(\theta) - \sin(\theta))$$
$$\text{requires } 0 = (\mu\cos(\theta) - \sin(\theta))$$
$$\sin(\theta) = \mu\cos(\theta)$$
$$\frac{\sin(\theta)}{\cos(\theta)} = \mu$$
$$\tan(\theta) = \mu$$

The question asked for the value of $\tan(\theta)$, so we have that now as μ.

The greater the coefficient of friction, μ, the more of our force should be directed upwards rather than forwards, to help minimize the friction effect.

10. Find the global maximum and minimum values of the following function on the given interval.

$$f(x) = 7e^{7x^3-7x},\ -1 \le x \le 0$$

The global maximum occurs at $x = -\sqrt{\frac{1}{3}}$, $y = 7e^{-7\sqrt{1/27}+7\sqrt{1/3}}$
The global minima occur at two points: $x = -1$ and $x = 0$, both with $y = 7$.

11. Find the global maximum and minimum values of the following function on the given interval.

$$f(x) = 7x - 21\ln(x),\ [1, 4]$$

The global maximum occurs at $x = 1$, $y = 7$.
The global minimum occurs at $x = 3$, $y = -2.071$

12. Find the global maximum and minimum values of the following function on the given interval.

$$f(x) = 4e^{-x} - 4e^{-2x},\ [0,1]$$

The global maximum occurs at $x = \ln(2)$, $y = 1$.
The global minimum occurs at $x = 0$, $y = 0$.

13. Find the global maximum and minimum values of the following function on the given interval.

$$f(x) = 7x - 14\cos(x),\ [-\pi, \pi]$$

Looking for critical point in the interior of the interval $[-\pi, \pi]$:

$$f'(x) = 7 - (14)(-\sin(x))$$

Setting $f'(x) = 0$,

$$0 = 7 + 14\sin(x)$$
$$\sin(x) = \frac{-1}{2}$$
$$x = \frac{-\pi}{6}$$

That's the easy angle to find, but if you check against the sine graph, or against a unit circle, you will also find $x = \frac{-5\pi}{6}$ is another solution still in the acceptable interval for this question.

Using either the first or second derivative test will show:

- $x = \frac{-\pi}{6}$ is a local **minimum**, and
- $x = \frac{-5\pi}{6}$ is a local **maximum**, and

To identify the **global** max and min, we need to look at the actual $f(x)$ values at:

- the critical points $x = \frac{-\pi}{6}$ and $x = \frac{-5\pi}{6}$, **and**
- the end points $x = -\pi$ and π.

x	$f(x)$	Comment
$-\pi$	$-7\pi - 14(-1) \approx -7.99$	
$\frac{-5\pi}{6}$	$7\frac{-5\pi}{6} - 14(-\sqrt{3}/2) \approx -6.20$	
$\frac{-\pi}{6}$	$7\frac{-\pi}{6} - 14(\sqrt{3}/2) \approx -15.79$	Smallest
π	$7\pi - 14(-1) \approx 35.99$	Largest

The global maximum occurs at $x = \pi$, $y = 7(2+\pi) \approx 35.99$.
The global minimum occurs at $x = -\pi/6$, $y = -7\pi/6 - 14\frac{\sqrt{3}}{2}) \approx -15.79$

14. Find the global and local maximum and minimum values of $f(x) = 6x^2, 0 < x \leq 6$.

There is a single global maximum at $(6, 216)$. This is an endpoint; there are no critical points on the interval $(0, 6]$
Due to the open interval at the left end, $x > 0$, there is no single point where there is a global minimum. $f(x)$ will keep decreasing as x approaches zero, but will never reach a final

15. Find the global maximum and minimum values of the following function over the given interval.

$$f(x) = \frac{3\cos x}{20 + 10\sin x},\ 0 \leq x \leq 2\pi$$

The derivative is

$$f'(x) = \frac{-3\sin(x)(20 + 10\sin(x)) - (3\cos(x))(10\cos(x))}{(20 + 10\sin(x))^2}$$

Expanding and simplifying,

$$f'(x) = \frac{-60\sin(x) - 30\sin^2(x) - 30\cos^2(x)}{(20 + 10\sin(x))^2}$$
$$= \frac{-60\sin(x) - 30(\sin^2(x) + \cos^2(x))}{(20 + 10\sin(x))^2}$$
$$= \frac{-60\sin(x) - 30}{(20 + 10\sin(x))^2}$$

Setting the derivative equal to zero to look for critical points,

$$0 = \frac{-60\sin(x) - 30}{(20 + 10\sin(x))^2}$$
$$0 = -60\sin(x) - 30$$
$$\sin(x) = \frac{-1}{2}$$
$$x = \frac{7\pi}{6}, \frac{11\pi}{6} \text{ on the given domain } [0, 2\pi]$$

We now compare the value of $f(x)$ at the critical points, and the end points.

x	$f(x)$	
0	$\frac{3}{20 + 10(0)} = \frac{3}{20} = 0.15$	
$\frac{7\pi}{6}$	$\frac{3(\frac{-\sqrt{3}}{2})}{20 + 10(\frac{-1}{2})} \approx$ **-0.1732**	Lowest
$\frac{11\pi}{6}$	$\frac{3(\frac{+\sqrt{3}}{2})}{20 + 10(\frac{-1}{2})} \approx$ **0.1732**	Highest
2π	$\frac{3}{20 + 10(0)} = \frac{3}{20} = 0.15$	

The global maximum occurs at $x = \frac{11\pi}{6}$, $y \approx 0.1732$.
The global minimum occurs at $x = \frac{7\pi}{6}$, $y \approx -0.1732$.

16. Find the global and local maximum and minimum values of $f(t) = 10/t + 4, 0 < t \leq 1$.

Due to the open interval at $t > 0$, and that $f(t)$ is a decreasing function, there is no global max on the interval $(0, 1]$. There is a global minimum at the end point $t = 1$, $f(1) = 14$.

This function has no critical points on the domain we are looking at, so it has no local min or max points.

17. Find the global and local maximum and minimum values of $f(\theta) = 3\tan\theta, -\pi/4 \leq \theta < \pi/2$.

The function is increasing for all x on the given interval (based on knowledge of the graph of tan, or the derivative $3\sec^2(\theta)$, which will always be positive).

The highest point will be at the end of the interval, but since that end is open ($x < \pi/2$, **not** $x \leq \pi/2$), there is no single point that will have the highest value. Therefore the function has no global maximum.

The left-end point of the function will be the global minimum (that endpoint **is** included in the interval). $\theta = -\pi/4$, $f(\pi/4) = -3$ will be the global minimum.

18. Find the exact global maximum and minimum values of the function $f(t) = \frac{3t}{8+t^2}$ if its domain is all real numbers.

Differentiating using the quotient rule gives

$$f'(t) = \frac{3(8+t^2) - 3t(28t)}{(8+t^2)^2} = \frac{3(8-t^2)}{(8+t^2)^2}.$$

The critical points are the solutions to $\frac{3(8-t^2)}{(8+t^2)^2} = 0$, which are $t = \pm\sqrt{8}$.

Since $f'(t) > 0$ for $-\sqrt{8} < t < \sqrt{8}$ and $f'(t) < 0$ otherwise, there is a local minimum at $t = -\sqrt{8}$ and a local maximum at $t = \sqrt{8}$.

As $t \to \pm\infty$, we have $f(t) \to 0$. Thus, the local maximum at $t = \sqrt{8}$ is a global maximum of $f(\sqrt{8}) = \frac{3\sqrt{8}}{8+8}$, and the local minimum at $t = -\sqrt{8}$ is a global minimum of $f(-\sqrt{8}) = \frac{-3\sqrt{8}}{2(8)}$.

19. A ball is thrown up on the surface of a moon. Its height above the lunar surface (in feet) after t seconds is given by the formula

$$h = 217t - \frac{7}{4}t^2.$$

(a) Find the time that the ball reaches its maximum height.

(b) Find the maximal height attained by the ball.

(a) When the ball reaches its maximum, the velocity will be zero, so we can solve for when velocity = $h'(t) = 0$.

$$h'(t) = 217 - \frac{7}{2}t$$

setting h'=0, $$0 = 217 - \frac{7}{2}t$$

$$t = \frac{2}{7}217 = 62 \text{ s}$$

(b) At the time of zero velocity, the height will be $h(62) = 6727$ feet.

20. In a certain chemical reaction, substance A combines with substance B to form substance Y. At the start of the reaction, the quantity of A present is a grams, and the quantity of B present is b grams. Assume $a < b$ and $y \leq a$. At time t seconds after the start of the reaction, the quantity of Y present is y grams. For certain types of reactions, the rate of the reaction, in grams/sec, is given by

$$\text{Rate} = k(a-y)(b-y),$$

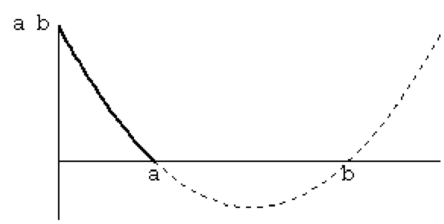
where k is a positive constant.

(a) Sketch a graph of the rate against y.

(b) For what values of y is the rate non-negative?

(c) Use your graph to find the value of y at which the rate of the reaction is fastest.

(a)



(b) If we expect the rate to be non-negative, we must have $0 \leq y \leq a$ or $b \leq y$. Since we assume $a < b$, we restrict y to $0 \leq y \leq a$.
In fact, the expression for the rate is non-negative for y greater than a but these values of y are not meaningful for the reaction. See the figure above (which shows the rate with $k = 1$).

(c) From the graph, we see that the maximum rate occurs when $y = 0$; that is, at the start of the reaction.

21. At what value(s) of x on the curve $y = 1 + 250x^3 - 3x^5$ does the tangent line have the largest slope?

The slope of the tangent line is given by $y' = 750x^2 - 15x^4$.

Consider this to be a new function, $g(x)$, that we want to maximize (to get the *largest* slope). To maximize

$g(x)$, we differentiate to find critical points:

$$\begin{aligned} g(x) &= 750x^2 - 15x^4 \\ \text{so } g'(x) &= 1500x - 60x^3 \\ \text{set } g' = 0\text{:} \quad 0 &= 1500x - 60x^3 \\ 0 &= 60x(25 - x^2) \\ 0 &= 60x(5 - x)(5 + x) \\ x &= 0, 5, -5 \end{aligned}$$

These are the critical points of the slope function. To determine which is a max, and which is a min, we can use either the first or second derivative tests. Let's use the 2nd here because differentiation of g' will be easy:
$g''(x) = 1500 - 180x^2$
$g''(-5) = -3000 < 0$: concave down; $x = -5$ is a local max.
$g''(0) = 1500 > 0$: concave up; $x = 0$ is a local min.
$g''(5) = -3000 < 0$: concave down; $x = 5$ is a local max.

The values of $g(-5) = 9375$ and $g(5) = 9375$ are the slopes of the original function at $x = -5$ and $x = 5$. They are equal, so they are both the common global maximum slope of 9375.

Optimization Word Problems

22. Some airlines have restrictions on the size of items of luggage that passengers are allowed to take with them. Suppose that one has a rule that the sum of the length, width and height of any piece of luggage must be less than or equal to 192 cm. A passenger wants to take a box of the maximum allowable volume.
    (a) If the length and width are to be equal, what should the dimensions be?
    (b) In this case, what is the volume?
    (c) If the length is be twice the width, what should the dimensions be?
    (d) In this case, what is the volume?

    Include units in all your answers.

Let the length, width and height of the box be L, w and h, respectively. Then the volume of the box is $V = Lwh$. The sum $L + w + h = 192$, and, for the first part, we know that $L = w$. Thus $2w + h = 192$, so $h = 192 - 2w$, and the volume equation becomes $V = Lwh = w \cdot w \cdot (192 - 2w) = 192 \cdot w^2 - 2w^3$. Since we need $h \geq 0$ and $h = 192 - 2w$, the domain for w is $0 \leq w \leq 96$.

Critical points are where $\frac{dV}{dw} = 2 \cdot 192 \cdot w - 6 \cdot w^2 = 0$, so $w = 0$ or $w = 64$. The global maximum must occur either at this point or at the end points. $V(64) > 0$ while $V(0) = V(96) = 0$, so the global maximum is at $L = w = 64$, in which case $h = 64$ as well. The volume is then $V = 64^3 = 262144\text{cm}^3$.

If $L = 2w$, $2w + w + h = 192$, so $V = (2w)(w)(192 - 3w) = 384w^2 - 6w^3$. Proceeding as before, we find $w = \frac{128}{3}$, $L = \frac{256}{3}$ and $h = 64$, so that $V = \frac{2097152}{9}$.

23. A wire 3 meters long is cut into two pieces. One piece is bent into a square for a frame for a stained glass ornament, while the other piece is bent into a circle for a TV antenna.
    (a) To reduce storage space, where should the wire be cut to **minimize** the total area of both figures?
    (b) Where should the wire be cut to **maximize** the total area?

(a) Note that we are interested in *the total area enclosed by the two figures* . Our first task is therefore to find an equation for this area, which will be the sum of the areas of the two figures.
Suppose we cut x meters of wire to make the circular antenna. Then there are $3-x$ meters left for the square. To find the area of the circle we need its radius. The circumference of a circle of radius r is $C = 2\pi r$, so the radius of the circle is given by $2\pi r = x$, and so $r = \frac{x}{2\pi}$. The area of the circular antenna is therefore

$$A_c = \pi r^2 = \pi \left(\frac{x}{2\pi}\right)^2 = \pi \frac{x^2}{4\pi^2} = \frac{1}{4\pi}x^2$$

Then the perimeter of a square with side length s is $P = 4s = 3-x$, so the side length is $s = \frac{1}{4}(3-x)$. Then the area of a square is $A_s = s^2$, so the area of the square is $A_s = (\frac{1}{16})(3-x)^2$.
The total area is therefore

$$A = \frac{1}{4\pi}x^2 + (\frac{1}{16})(3-x)^2 = \frac{1}{4\pi}x^2 + \frac{1}{(16)}(9-6x+x^2).$$

The domain for x is $0 \leq x \leq 3$.
The maximum and minimum of A will occur at critical or end points. Critical points are where $dA/dx = 0$, or, where

$$\frac{1}{2\pi}x + \frac{1}{(16)}(2x - 6) = 0.$$

Collecting all terms in x we have

$$\left(\frac{1}{2\pi} + \frac{2}{(16)}\right) x = \frac{6}{16},$$

so, after simplifying,

$$x = \frac{3\pi}{4+\pi}.$$

To determine if this is a local maximum or minimum, we use the second derivative test.

$$A'' = \left(\frac{1}{2\pi} + \frac{2}{(16)}\right) > 0,$$

so the function is concave up everywhere and this is a local minimum. Also, because this is the *only* critical point, this is also a *global* minimum for the area.

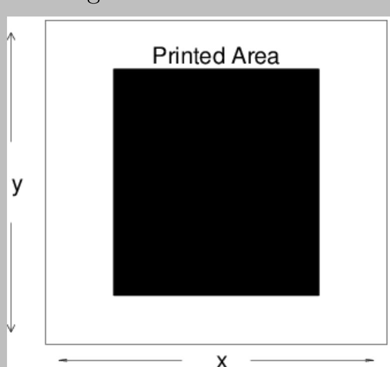
Thus to minimize area we use $\frac{3\pi}{4+\pi}$ meters of wire for the circle and $3 - \frac{3\pi}{4+\pi}$ meters for the square.

(b) To *maximize* the area, we can't use our critical point, which was a minimum; instead we must use the endpoints. The areas at the endpoints are

$$A(0) = \frac{9}{16} \approx 0.56 \qquad \text{and} \qquad A(3) = \frac{9}{4\pi} approxapprox 0.72,$$

the larger of which is $A(3)$, so the maximum area occurs when all of the wire is used for the circle and none for the square.

24. A printed poster is to have a total area of 799 square inches with top and bottom margins of 6 inches and side margins of 4 inches. What should be the dimensions of the poster so that the printed area be as large as possible? Let x denote the width of the poster and let y denote the length.



(a) Write the function of x and y that you need to maximize.

(b) Express that function in terms of x alone.

(c) Find the critical points of the function.

(d) Use the second derivative test to verify that $f(x)$ has a maximum at this critical point

(e) Find the optimal dimensions of the poster, and the resulting area. Include units.

(a) Area $= A = (x - 2\cdot 4)(y - 2\cdot 6) = (x-8)(y-12)$

(b) By using the requirement that $799 = xy$, we get $A = (x-8)\left(\frac{799}{x} - 12\right)$

(c) To find the critical points, we differentiate and set the derivative equal to zero. For this function, it is easier to expand it first before differentiating, to avoid use of the product rule.

$$A = (x-8)\left(\frac{799}{x} - 12\right)$$
$$A = 799 - \frac{6392}{x} - 12x + 96$$
$$A = 895 - \frac{6392}{x} - 12x$$

Differentiating,

$$\frac{dA}{dx} = 0 + \frac{6392}{x^2} - 12$$

Setting the derivative equal to zero,

$$0 = \frac{6392}{x^2} - 12$$
$$12 = \frac{6392}{x^2}$$
$$x^2 = \frac{6392}{12} = 532.6667$$
$$x = \pm 23.08$$

The critical points are at $x = -23.08$ and $+23.08$ inches, but only the positive length is in the domain for this problem: $x = 23.08$.

(d) Since $\frac{dA}{dx} = \frac{6392}{x^2} - 12$, the second derivative of A is given by

$$\frac{d^2A}{dx^2} = -\frac{2\cdot 6392}{x^3}$$

The second derivative of A will be negative at $x = 23.08$, so A is concave down there, indicating $x = 23.08$ is a local maximum for the printed area.

(e) The dimensions of the poster with the largest printed area will be 23.08×34.62, with a net printed area of 341.09 in^2.

25. A box with an open top has vertical sides, a square bottom, and a volume of 32 cubic meters. If the box has the least possible surface area, find its dimensions.

This is the same question studied in the course videos. Let the dimensions of the box be w and h; the bottom is square so w can represent the length of both sides of the bottom. These combine to produce

$$A = w^2 + 2(wh) + 2(wh), \qquad V = w^2h = 32$$

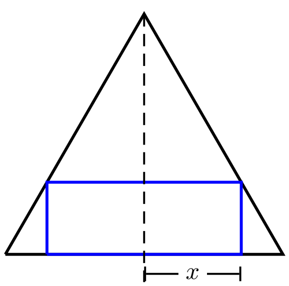
Solving for h in the V equation, $h = 32/w^2$, we can write A as just a function of w:

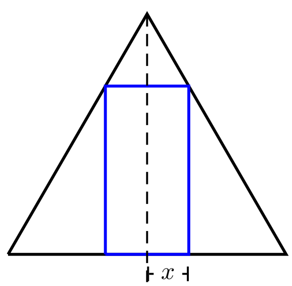
$$A = w^2 + 4w(32/w^2)$$
$$A = w^2 + 128/w$$

Differentiating and setting $A' = 0$, you will find $w = 4$, and consequently $h = 32/w^2 = 2$.

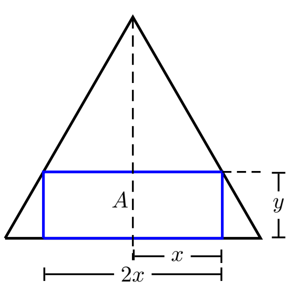
26. Find the dimensions of the rectangle of largest area that can be inscribed in an equilateral triangle with sides of length 2 if one side of the rectangle lies on the base of the triangle.

There are several ways to define the dimensions of the rectangle. We will do so using x as the **half-**width of the rectangle, to make some of the later calculations a bit simpler. Here are two examples of possible rectangles:



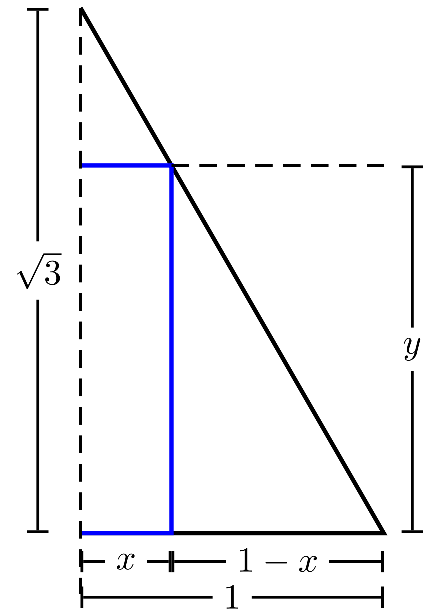


We can choose x, and want to find a formula for the resulting area A of the rectangle.



$$A = \text{width} \times \text{height} = (2x)(y)$$

To make this a function of just 1 variable, x, we need to find a formula relating y and x. We can do this by looking at the similar triangles on the right hand side of the diagram.



The large triangle has a ratio of height-to-base of $\frac{\sqrt{3}}{1}$, and the smaller triangle in the right corner has a ratio of $\frac{y}{(1-x)}$. Equating these ratios, because the triangles are similar,

$$\frac{\sqrt{3}}{1} = \frac{y}{1-x}$$
$$y = \sqrt{3}(1-x)$$

Subbing this into the Area formula,

$$A = 2xy = 2x(\sqrt{3}(1-x)) = 2\sqrt{3}(x - x^2)$$

We can now look for critical points of A by looking for where $\frac{dA}{dx} = 0$:

$$\frac{dA}{dx} = 2\sqrt{3}(1-2x)$$

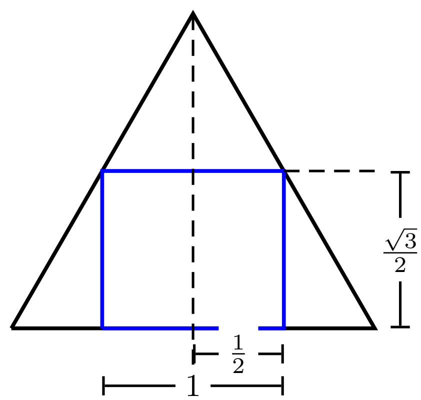
Setting the derivative equal to zero gives

$$0 = 2\sqrt{3}(1-2x)$$
$$0 = (1-2x)$$
$$2x = 1$$
$$x = \frac{1}{2}$$

Going back to the formulas for the final width, y and A, we then find

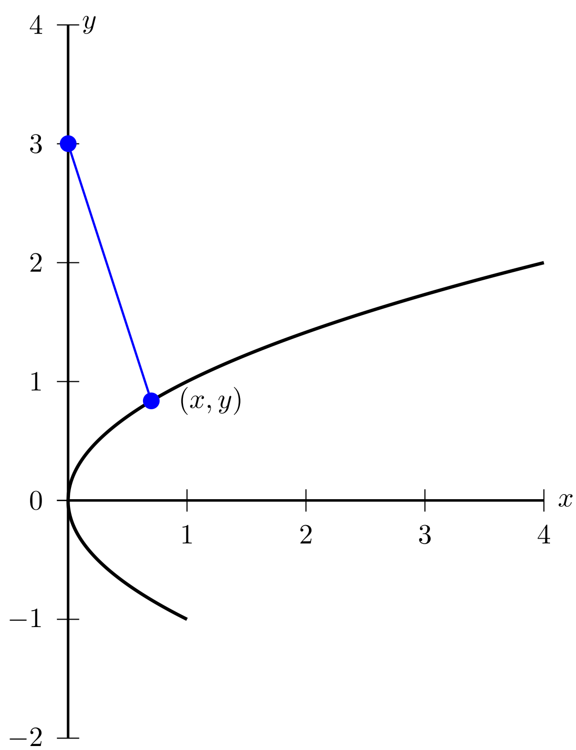
$$\text{width} = 2x = 2(1/2) = 1$$
$$y = \sqrt{3}(1 - 1/2) = \frac{\sqrt{3}}{2}$$
$$A = 2xy = 2\left(\frac{1}{2}\right)\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2} \text{ square units}$$

The optimal rectangle will be $2x = 1$ unit length on the base, have height $\frac{\sqrt{3}}{2}$ units, and an overall area of $\frac{\sqrt{3}}{2}$ units squared.



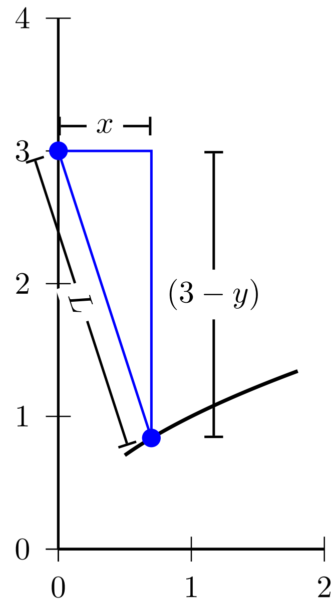
> 27. Find the minimum distance from the parabola
>
> $$x - y^2 = 0$$
>
> to the point (0,3).

The first step to solving this problem is to sketch the scenario. The parabola defined by $x - y^2 = 0$ is the sideways parabola opening right ($x = y^2$).



We want to minimize the length of the blue line, while we get to select the point (x, y) on the parabola.

We note first that the length of the line is the hypotenuse of a triangle with side lengths x and $(3 - y)$:



Also, from the equation of the parabola, we have $x = y^2$, so we can compute the length of the hypotenuse (L) which is the distance from the point on the parabola to the point (0, 3) as

$$\begin{aligned} L &= \sqrt{x^2 + (3-y)^2} \\ &= \sqrt{(y^2)^2 + (3-y)^2} \\ &= \sqrt{y^4 + (3-y)^2} \end{aligned}$$

Take the derivative with respect to y,

$$\frac{dL}{dy} = \frac{1}{2} \frac{1}{\sqrt{y^4 + (3-y)^2}} (4y^3 + 2(3-y)(-1))$$

When we set that equal to zero to look for critical points,

$$0 = \frac{1}{2} \frac{1}{\sqrt{y^4 + (3-y)^2}} (4y^3 + 2(3-y)(-1))$$

Cross-multiplying the denominator,

$$\begin{aligned} 0 &= 4y^3 + 2(3-y)(-1) \\ 0 &= 4y^3 - 6 + 2y \\ \text{or dividing by 2 } 0 &= 2y^3 + y - 3 \end{aligned}$$

Looking for factors, we try some easy integers and see that $y = 1$ satisfies the equation, so $(y - 1)$ is a factor. Doing long division or any other factoring technique gives the second factor:

$$0 = (y-1) \underbrace{(2y^2 + 2y + 3)}_{\text{has no real roots}}$$

This gives us the only critical point at $y = 1$.

Either the first or second derivative test (not shown here) will show that this only critical point is a local **minimum** for the distance $L(y)$.

This means that The point of closest approach will occur at $y = 1$ (and $x = y^2 = 1^2 = 1$), and that will give a distance of $\sqrt{1^2 + (3-1)^2} = \sqrt{5}$ to the point $(0, 3)$.

28. I have enough pure silver to coat 2 square meters of surface area. I plan to coat a sphere and a cube.

    (a) Allowing for the possibility of all the silver going onto one of the solids, what dimensions should they be if the total volume of the silvered solids is to be a maximum?

    (b) Now allowing for the possibility of all the silver going onto one of the solids, what dimensions should they be if the total volume of the silvered solids is to be a minimum?

Note that we can answer both of these parts at the same time, doing our usual critical-point-and-end-point analysis for optimization on a closed and bounded domain.

To create our variables, we define r as the radius of the sphere, s as the length of each side of the cube, A the total area (both sphere and cube together), and V the total volume (both sphere and cube together). Then the area of silver and volumes of the resulting solids are given by:

$$A = 4\pi r^2 + 6s^2$$

and

$$V = \frac{4}{3}\pi r^3 + s^3.$$

Note that $A = 2$ in the question, but we will try as much as possible to write it as A throughout to keep the solution as general as possible until we need to compute any final values.

We will eliminate s from the equations, so we can use r as our only input variable. If we do this, note that we can then use the A equation to bound the domain of r:

- $r = 0$ is clearly the smallest radius we can use (all cube, no sphere), and
- $s = 0$ (no cube, all sphere) leads to $A = 4\pi r^2$ or $r = \sqrt{\frac{A}{4\pi}} \approx 0.3989$

The area of silver available, A, is fixed or constant, so we can use that equation to solve for the relationship between r and s:

Solving the area equation for s gives

$$s = \sqrt{\frac{A - 4\pi r^2}{6}}.$$

Substituting this value in the volume equation gives

$$V = \frac{4}{3}\pi r^3 + \left(\frac{A - 4\pi r^2}{6}\right)^{\frac{3}{2}}.$$

Differentiating with respect to r and setting to zero gives:

$$V' = 4\pi r^2 - \frac{3}{2} \times \frac{8\pi r}{6}\left(\frac{A - 4\pi r^2}{6}\right)^{\frac{1}{2}} = 0$$
$$4\pi r^2 - 2\pi r\left(\frac{A - 4\pi r^2}{6}\right)^{\frac{1}{2}} = 0$$

Factoring out a common $2\pi r$,

$$2\pi r\left(2r - \left(\frac{A - 4\pi r^2}{6}\right)^{\frac{1}{2}}\right) = 0$$

Since we know $r = 0$ is an endpoint we will look at later, just need to focus on the term in brackets. If it equals zero, then

$$2r - \left(\frac{A - 4\pi r^2}{6}\right)^{\frac{1}{2}} = 0$$

$$\text{so } 2r = \sqrt{\frac{A - 4\pi r^2}{6}}.$$

Squaring gives

$$4r^2 = \frac{A - 4\pi r^2}{6}$$

which gives

$$r = \sqrt{\frac{A}{24 + 4\pi}} \approx 0.2339 \text{ meters.}$$

This is the only critical point for our function $V(r)$. Since $V(r)$ is continuous, and bounded by $r = 0$ and $r = \sqrt{\frac{A}{4\pi}} \approx 0.3989$, we can find the global max and global min of $V(r)$ by comparing the volume at:

- the endpoints of the domain, $r = 0$ and $r \approx 0.3989$, and
- the single critical point in the interior of the domain, $r \approx 0.2339$.

Using the earlier formula

$$V = \frac{4}{3}\pi r^3 + \left(\frac{A - 4\pi r^2}{6}\right)^{\frac{3}{2}},$$

here are those volumes in table form:

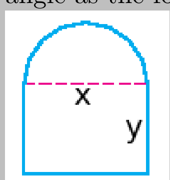
r	$V(r)$	Comment
0	0.1925	
0.2339	0.1559	Lowest V: global min
0.3989	0.2660	Highest V: global max

So the final answer to the original question is:

(a) The global **maximum** for the volume is achieved by using $r = 0.3989$, which also meant $s = 0$, or using all the silver to coat a sphere, with no cube ($s = 0$). The resulting enclosed volume was 0.226 cubic meters.

(b) The global **minimum** for the volume is achieved by splitting the silver between the sphere and the cube, using a radius of $r = 0.2339$ m, and a cube side length of $s = \sqrt{\dfrac{A - 4\pi(0.2339)^2}{6}} \approx 0.4677$ m.

29. Suppose that 241 ft of fencing are used to enclose a corral in the shape of a rectangle with a semicircle whose diameter is a side of the rectangle as the following figure:



Find the dimensions of the corral with maximum area.

From the picture, we see that x is the width of the corral, and therefore the diameter of the semicircle, and that y is the height of the rectangular section. Thus the perimeter of the corral can be expressed by the equation $2y + x + \frac{\pi}{2}x = 2y + (1 + \frac{\pi}{2})x = 241$ ft or equivalently, $y = \frac{1}{2}(241 - (1 + \frac{\pi}{2})x)$. Since x and y must both be non-negative, it follows that x must be restricted to the interval $[0, \frac{241}{1+\pi/2}]$. The area of the corral is the sum of the area of the rectangle and semicircle, $A = xy + \frac{\pi}{8}x^2$. Making the substitution for y from the constraint equation,

$A(x) = \frac{1}{2}x(241 - (1 + \frac{\pi}{2})x) + \frac{\pi}{8}x^2 = 120.5x - \frac{1}{2}(1 + \frac{\pi}{2})x^2 + \frac{\pi}{8}x^2$.
Now, $A'(x) = 120.5 - (1 + \frac{\pi}{2})x + \frac{\pi}{4}x = 0$ implies $x = \frac{120.5}{(1+\frac{\pi}{4})} \approx 67.4919$.
With $A(0) = 0$,

$$A(\tfrac{120.5}{(1+\frac{\pi}{4})}) \approx 4066.39 \quad \text{and} \quad A(\tfrac{241}{1+\frac{\pi}{2}}) \approx 3451.11,$$

it follows that the corral of maximum area has dimensions

$$x = \tfrac{120.5}{1+\frac{\pi}{4}} \quad \text{and} \quad y = \tfrac{1}{2}(241 - (1 + \tfrac{\pi}{2})\tfrac{120.5}{1+\frac{\pi}{4}}) \approx 33.746.$$

30. Find the maximum area of a triangle formed in the first quadrant by the x-axis, y-axis and a tangent line to the graph of $f = (x + 2)^{-2}$.

Let $P\left(t, \frac{1}{(t+2)^2}\right)$ be a point on the graph of the curve $y = \frac{1}{(x+2)^2}$ in the first quadrant. The tangent line to the curve at P is

$$L(x) = \frac{1}{(t+2)^2} - \frac{2(x-t)}{(t+2)^3},$$

which has x-intercept $a = \frac{3t+2}{2}$ and y-intercept $b = \frac{3t+2}{(t+2)^3}$. The area of the triangle in question is

$$A(t) = \frac{1}{2}ab = \frac{(3t+2)^2}{4(t+2)^3}.$$

Solve

$$A'(t) = \frac{(3t+2)(3 \cdot 2 - 3t)}{4(t+2)^4} = 0$$

for $0 \leq t$ to obtain $t = 2$. Because $A(0) = \frac{1}{4 \cdot 2}$, $A(2) = \frac{1}{2 \cdot 2}$ and $A(t) \to 0$ as $t \to \infty$, it follows that the maximum area is $A(2) = 0.25$.

31. A box is constructed out of two different types of metal. The metal for the top and bottom, which are both square, costs \$4 per square foot and the metal for the sides costs \$6 per square foot. Find the dimensions that minimize cost if the box has a volume of 35 cubic feet.

Let $x > 0$ be the length of a side of the square base and $z > 0$ the height of the box. With volume $x^2z = 35$, we have $z = 35/x^2$ and cost

$$C(x) = 4 \cdot 2 \cdot x^2 + 6 \cdot 4 \cdot xz = 8x^2 + 840\frac{1}{x}.$$

Solve $C'(x) = 8 \cdot 2x - 840x^{-2} = 0$ to obtain $x = \left(\frac{35 \cdot 6}{4}\right)^{1/3}$. Since $C(x) \to \infty$ as $x \to 0+$ and as $x \to \infty$, the minimum cost is $C\left((\frac{35 \cdot 6}{4})^{1/3}\right) \approx \$336.499$ when $x \approx 3.74444$ ft and $z \approx 2.49629$ ft.

32. A rectangle is inscribed with its base on the x axis and its upper corners on the parabola $y = 12 - x^2$. What are the dimensions of such a rectangle with the greatest possible area?

To solve this problem, we need to find an expression for the area of the rectangle in terms of one of its dimensions, and then use derivatives to maximize this area. First, however, we can simply things quite a bit by noting that the parabola given by $y = 12 - x^2$ is symmetric about the y-axis. Therefore, the inscribed rectangle will also be symmetric about the y-axis. So it is enough to find the dimensions of half of this rectangle and double the x value.

Our rectangle will therefore have the bottom left corner $(0, 0)$ and the top right corner $(x, 12 - x^2)$ where x is the width of the rectangle, and $12 - x^2$ is its height. Thus, the area of the rectangle is given by $f(x) = x(12 - x^2) = 12x - x^3$ where x is the width of the rectangle. We now find the derivative of this and solve for zero to find critical points.

The derivative is $f'(x) = 12 - 3x^2$. Setting this equal to 0 and recalling that we are talking about widths, so that all x values should be positive, we get:

$$
\begin{aligned}
f'(x) &= 0 \\
12 - 3x^2 &= 0 \\
3x^2 &= 12 \\
x^2 &= \frac{12}{3} = 4 \\
x &= \sqrt{4} = 2
\end{aligned}
$$

It is easy to check that the second derivative of $f(x)$ is negative everywhere, so this is a maximum of $f(x)$. Therefore, this is the width of the rectangle with the maximum area. Actually, it is the width of half of that rectangle, since we were ignoring the half on the left side of the y-axis. So the width of the whole rectangle is $2 \cdot 2 = 4$. The height is given by plugging $x = 2$ into the formula for the parabola, giving $12 - (2)^2 = 12 - 4 = 8$.

33. Centerville is the headquarters of Greedy Cablevision Inc. The cable company is about to expand service to two nearby towns, Springfield and Shelbyville. There needs to be cable connecting Centerville to both towns. The idea is to save on the cost of cable by arranging the cable in a Y-shaped configuration. Centerville is located at $(8, 0)$ in the xy-plane, Springfield is at $(0, 5)$, and Shelbyville is at $(0, -5)$. The cable runs from Centerville to some point $(x, 0)$ on the x-axis where it splits into two branches going to Springfield and Shelbyville. Find the location $(x, 0)$ that will minimize the amount of cable between the 3 towns and compute the amount of cable needed. Justify your answer.

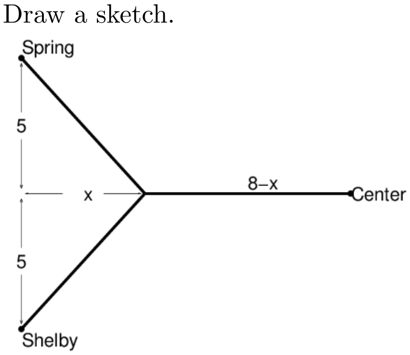
    (a) What function of x needs to be minimized to solve this problem?

    (b) Find the critical points of $f(x)$.

    (c) Use the second derivative test to verify that $f(x)$ has a minimum at this critical point.

    (d) Compute the minimum amount of wire needed.

(a) Draw a sketch.



With x being the horizontal component of the diagonal lines, the total length of the cable will be $L(x) = 2\sqrt{x^2 + 5^2} + (8 - x)$.

(b) Taking the derivative and finding critical points of $L(x)$ yields $x = 2.89$.

(c) The second derivative of $L(x)$ will be positive at $x = 2.89$, indicating that the critical point is a local minimum for the length of cable.

(d) $L(2.89) = 2\sqrt{(2.89)^2 + 25} + (8 - 2.89) = 16.66$ units of cable.

34. A cylinder is inscribed in a right circular cone of height 4 and radius (at the base) equal to 3.5. What are the dimensions of such a cylinder which has maximum volume?

As we are attempting to maximize the volume of the inscribed cylinder, we must first come up with a formula for the volume of this cylinder. Let x be the radius of the cylinder, $v(x)$ the volume. We know from basic geometry that the formula for volume is give by $\pi x^2 h$ where x is the radius and h is the height of the cylinder. So in order to come up with a formula for volume in terms of x only, we need to relate x to h.

This is where the information about the cone comes in handy. The cone is a right circular cone. Thus, inscribing the cylinder will fill up some of the base of the cone, and just touch the slanted side, leaving a similar right circular cone at the top. This new cone will have a radius of x and a height of $4 - h$ where x and h are as in the formula for the volume of our cylinder. As this cone is similar to the original, we can use ratios to get:

$$
\frac{x}{3.5} = \frac{4 - h}{4}
$$

Simplifying this, we get $h = 4 - \frac{4}{3.5}x$. Therefore, our formula for volume in terms of x becomes $v(x) = \pi x^2(4 - \frac{4}{3.5}x) = (4\pi)x^2 - (\frac{4}{3.5}\pi)x^3$

Now, we want to maximize this. So we will first take the derivative. Using the rules for differentiation of polynomials, the derivative is $v'(x) = 2(4\pi)x - 3(\frac{4}{3.5}\pi)x^2$. Solving for zero, we get, as we don't want $x = 0$, the following.

$$
\begin{aligned}
v'(x) &= 0 \\
2(4\pi)x - 3(\frac{4}{3.5}\pi)x^2 &= 0 \\
\pi x(2(4) - 3\frac{4}{3.5}x) &= 0 \\
2(4) - 3\frac{4}{3.5}x &= 0 \\
3\frac{4}{3.5}x &= 2(4) \\
x &= \frac{2}{3}(3.5) = 2.333
\end{aligned}
$$

Then, using the formula for height we came up with before, the height can be determined by:

$$
h = 4 - \frac{4}{3.5}(2.333) = 1.333
$$

35. A small island is 3 miles from the nearest point P on the straight shoreline of a large lake. If a woman on the island can row a boat 2 miles per hour and can walk 3 miles per hour, where should the boat be landed in order to arrive at a town 8 miles down the shore from P in the least time? Let x be the distance (in miles) between point P and where the boat lands on the lakeshore.

    (a) Enter a function $T(x)$ that describes the total amount of time the trip takes as a function of the distance x.

    (b) What is the distance $x = c$ that minimizes the travel time?

    (c) What is the least travel time?

See the similar example in the course notes.

(a)

$$\begin{aligned} T(x) &= \frac{\text{water dist}}{\text{water speed}} + \frac{\text{land dist}}{\text{land speed}} \\ &= \frac{\sqrt{9+x^2}}{2} + \frac{(8-x)}{3} \end{aligned}$$

(b) The minimum of $T(x)$ will occur when $x = 2.68$ miles.

(c) For that landing point, the travel time will be $T(2.68) = 3.78$ hours or about 3 hours and 45 minutes.

36. The illumination at a point is inversely proportional to the square of the distance of the point from the light source and directly proportional to the intensity of the light source. Suppose two light sources are s feet apart and their intensities are I and J, respectively. Let P be the point between them where the sum of their illuminations is a minimum: find the distance from P to I.

    (Your answer will depend on I, J, and s.)

From the proportionality statement, the illumination on a point is be given by illum $= \dfrac{\text{brightness}}{(\text{Dist. to source})^2}$

We want the total illumination at our point from the two sources, which will be given by the illumination from each source, added together to get a total:

$$Q = \frac{I}{x^2} + \frac{J}{(s-x)^2}.$$

To find the x value that minimizes Q, we look for critical points of $Q(x)$. Differentiating with respect to x gives

$$Q' = -\frac{2I}{x^3} + \frac{2J}{(s-x)^3}.$$

Setting the derivative to zero and multiplying with the denominators gives

$$-2I(s-x)^3 + 2Jx^3 = 0.$$

This can be rewritten as

$$\frac{I}{J} = \frac{x^3}{(s-x)^3}$$

Taking cube roots of both sides,

$$\left(\frac{I}{J}\right)^{1/3} = \frac{x}{(s-x)}$$

Solving for x: $$(s-x)\left(\frac{I}{J}\right)^{1/3} = x$$

$$s\left(\frac{I}{J}\right)^{1/3} - x\left(\frac{I}{J}\right)^{1/3} = x$$

$$s\left(\frac{I}{J}\right)^{1/3} = x\left(\frac{I}{J}\right)^{1/3} + x$$

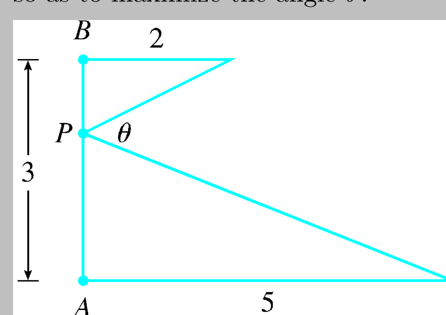
$$s\left(\frac{I}{J}\right)^{1/3} = x(1 + \left(\frac{I}{J}\right)^{1/3})$$

$$\text{or } x = \frac{s\left(\frac{I}{J}\right)^{1/3}}{(1 + \left(\frac{I}{J}\right)^{1/3})}$$

This can be tidied up a little by expanding putting everything on a common denominator of $J^{1/3}$,

$$x = \frac{sI^{1/3}}{J^{1/3} + I^{1/3}}$$

37. How far from A should the point P be chosen so as to maximize the angle θ?



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Define the distance PA to be x, so PB is $(3 - x)$.

The angle $\theta = \pi -$ (angle in PB triangle + angle in PA triangle).

$$\theta = \pi - (\arctan(2/(3-x)) + \arctan(5/x)$$

Finding crit pnts:

$$\begin{aligned}\theta' &= -\frac{1}{1+(2/(3-x))^2}\left(\frac{-2}{(3-x)^2}\right)(-1) \\ &- \frac{1}{1+(5/x)^2}\left(\frac{-5}{x^2}\right) \\ &= \frac{-2}{(3-x)^2+4} + \frac{5}{x^2+25}\end{aligned}$$

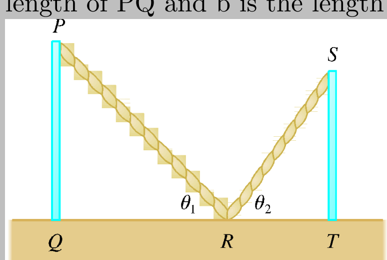
Setting $\theta' = 0$:

$$\begin{aligned}0 &= \frac{-2}{(3-x)^2+4} + \frac{5}{x^2+25} \\ \frac{2}{(3-x)^2+4} &= \frac{5}{x^2+25} \\ 2(x^2+25) &= 5((3-x)^2+4) \\ 2x^2+50 &= 45-30x+5x^2+20 \\ 0 &= 3x^2-30x+15\end{aligned}$$

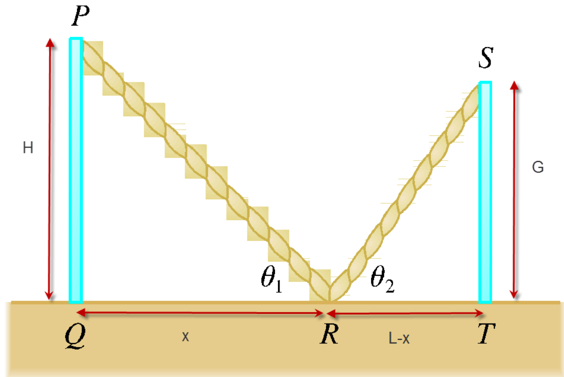
Using the quadratic formula, $x = 0.5279$ and $x = 9.4721$, but clearly only $x = 0.5279$ is in the acceptable domain.

Therefore to maximize the angle θ, we should move the point P to be 0.5279 units away from A.

38. Two vertical poles PQ and ST are secured by a rope PRS going from the top of the first pole to a point R on the ground between the poles and then to the top of the second pole as in the figure. If R is chosen to minimize the length of the rope, find the resulting relationship between θ_1 and θ_2 in terms of a and b, where a is the length of PQ and b is the length of ST.



Here is a diagram, showing the variable lengths x and $L-x$, and the constant lengths a and b.



The length of the rope is given by

$$R(x) = \sqrt{a^2+x^2} + \sqrt{(L-x)^2+b^2}$$

Taking the derivative to identify critical points,

$$\begin{aligned}R'(x) &= \left(\frac{1}{2}\right)\frac{1}{\sqrt{a^2+x^2}}(2x) \\ &+ \left(\frac{1}{2}\right)\frac{1}{\sqrt{(L-x)^2+b^2}}2(L-x)(-1)\end{aligned}$$

Setting $R'(x) = 0$:

$$\begin{aligned}0 &= \frac{x}{\sqrt{a^2+x^2}} + \frac{-(L-x)}{\sqrt{(L-x)^2+b^2}} \\ \frac{(L-x)}{\sqrt{(L-x)^2+b^2}} &= \frac{x}{\sqrt{a^2+x^2}}\end{aligned}$$

We could continue with the analysis of the optimal x value, but the question actually asks about the relationship between the angles θ_1 and θ_2. We note that in the two triangles,

$$\begin{aligned}\cos(\theta_1) &= \frac{x}{\sqrt{a^2+x^2}} \\ \text{and } \cos(\theta_2) &= \frac{L-x}{\sqrt{(L-x)^2+b^2}}\end{aligned}$$

But wait! Those are exactly the same expression we have in our current $R'(x) = 0$ equation!

$$\underbrace{\frac{(L-x)}{\sqrt{(L-x)^2+b^2}}}_{\cos(\theta_2)} = \underbrace{\frac{x}{\sqrt{a^2+x^2}}}_{\cos(\theta_1}$$

So at the optimal length of the rope, we must have

$$\cos(\theta_1) = \cos(\theta_2)$$

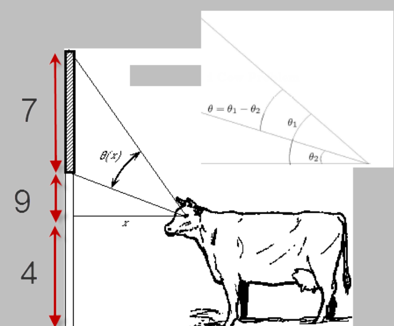
and since both angles must be in the range $[0, \pi/2]$, the only possible way to have that equality in the cosines is to have

$$\theta_1 = \theta_2$$

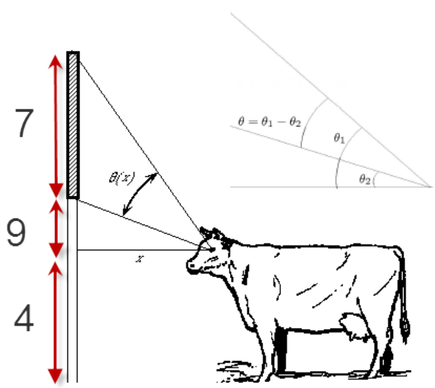
In words, the shortest rope length will be made when both of the angles in the triangles are the same.

39. The Nearsighted Cow Problem: A Calculus Classic.

A rectangular billboard 7 feet in height stands in a field so that its bottom is 13 feet above the ground. A nearsighted cow with eye level at 4 feet above the ground stands x feet from the billboard. Express θ, the vertical angle subtended by the billboard at her eye, in terms of x. Then find the distance x_0 the cow must stand from the billboard to maximize $\theta(x)$.



(a) Here is a diagram with more information added, including the definition of two related angles, θ_1 and θ_2, defined so that our desired $\theta = \theta_1 - \theta_2$.



For the diagram,

$$\begin{aligned}\theta &= \theta_2 - \theta_1\\ &= \arctan((7+9)/x) - \arctan(9/x)\\ &= \arctan(16/x) - \arctan(9/x)\end{aligned}$$

(b) To find the optimal viewing distance, we want to find x that maximizes θ. Taking the derivative with respect to x,

$$\theta' = \frac{1}{1+(16/x)^2}\left(\frac{-16}{x^2}\right) - \frac{1}{1+(9/x)^2}\left(\frac{-9}{x^2}\right)$$

Setting the derivative equal to zero to find critical points gives:

$$\frac{1}{1+(16/x)^2}\left(\frac{16}{x^2}\right) = \frac{1}{1+(9/x)^2}\left(\frac{9}{x^2}\right)$$

Cross-multiplying:

$$16x^2(1+(81/x^2)) = 9x^2(1+(256/x^2))$$

Bringing in the x^2:

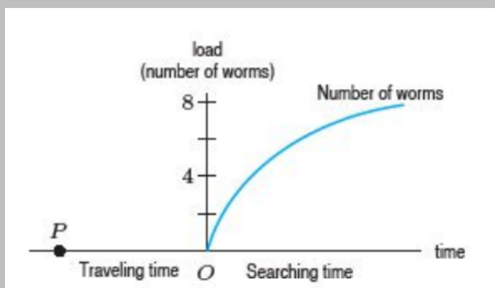
$$\begin{aligned}16(x^2+81) &= 9(x^2+256)\\ 16x^2+1296 &= 9x^2+2304\\ 7x^2 &= 1008\\ x^2 &= 144\end{aligned}$$

so the optimal viewing distance for the advertising-loving myopic bovine is

$$x = 12$$

feet from the sign.

40. While waiting for their babies to mature, a bird parent feeds the babies worms. To collect worms, the bird flies to a site where worms are to be found, picks several up, then flies back to its nest. The *loading curve* below shows how the number of worms (the load) a bird collects depends on the time it spends searching for them.



The curve is concave down because the bird can pick up worms more efficiently when its beak is empty (initial rate of collection); when its beak is partly full, the bird becomes less efficient (slower rate of collection later). The traveling time (from the nest to the site and back) is represented by the distance PO in the diagram.

The bird wants to maximize the **rate** at which it brings worms back to the nest, averaged over multiple trips, so

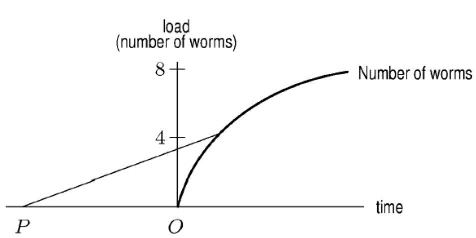
$$\text{Rate worms arrive} = \frac{\text{Load/Trip}}{\text{(Travel + Search Time)/Trip}}$$

(a) Draw a line in the figure whose slope is the desired rate.

(b) Using the graph, estimate the load which maximizes this rate.

(c) If the traveling time is increased (i.e. PO length made longer), does the optimal load increase or decrease? Why?

(a)

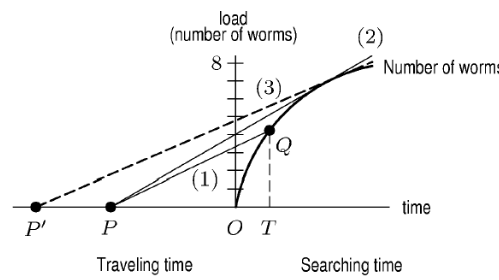


The line shown above has the **slope** equal to the rate worms arrive at the nest. To understand why, look at vertical change/rise in that line (total load collected/trip), and the horizontal length/run (total time spent on the trip).

For any point Q on the loading curve, the line PQ has slope

$$\frac{QT}{PT} = \frac{QT}{PO + OT}$$
$$= \frac{\text{Load/Trip}}{\text{(Travel + Search Time)/Trip}}$$
$$= \text{Rate worms arrive}$$

(b)



The slope of the line PQ is maximized (line made steepest) when the line is tangent to the loading curve, which happens with line (2). The load is then approximately 7 worms.

(c) If the traveling time is increased, e.g. the point P moves to the left, to point P'. If line (3) is tangent to the curve, it will be tangent to the curve further to the right than line (2), so the optimal load is larger.

This makes sense: if the bird has to fly further on each trip, you'd expect it to bring back more worms each time to make the additional travel time worthwhile in the overall average. Note that the resulting optimal slope (3) is still shallower than the slope in (2), indicating that even with the optimal strategy, birds that have to travel further to collect food will be disadvantaged because they are limited to a lower food gathering rate overall.