UNIT - III

TIME RESPONSE AND STEADY STATE ERRORS

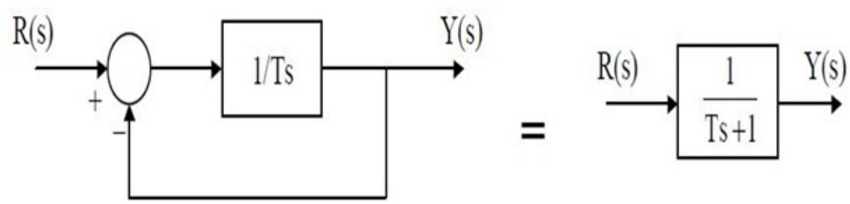
Introduction

There are two methods to analyze functioning of a control system that are time domain analysis and control domain analysis. In time domain analysis the response of a system is a function of time. It analyzes the working of a dynamic control system.

This analysis can only be applied when nature of input plus mathematical model of the control system is known. It is not easy to express the actual input signals by simple equations as the input signals of the control systems are not fully known. There are two components of any system's time response, transient response and steady response.

Typical and standard test signals are used to judge the behaviour of typical test signals. The characteristics of an input signal are constant acceleration, constant velocity, a sudden change or a sudden shock. We discussed four types of test signals that are Impulse Step, Ramp, Parabolic and another important signal is sinusoidal signal. In this article we will be discussing first order systems.

First order system



The system whose input-output equation is a first order differential equation is called first order system. The order of the differential equation is the highest degree of derivative present in an equation. First order system contains only one energy storing element. Usually a capacitor or combination of two capacitors is used for this purpose. These cannot be connected to any external energy storage element. Most of the practical models are first order systems. If a system with higher order has a dominant first order mode it can be considered as a first order system.

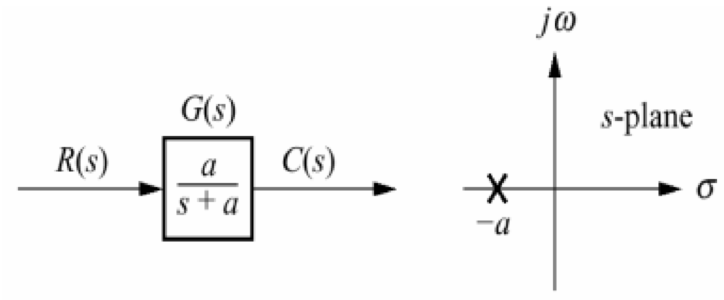
We now discuss first-order systems without zeros to define a performance specification for such a system. A first-order without zeros can be described by the transfer funtion given in the figure(a). If the input is a unit step, where R(s) = 1/s, the Laplace transform of the step response is C(s), where

$$C(s) = R(s)G(s) = \frac{a}{s(s+a)}$$

. Taking the inverse Laplace transform, the step response is given by

$$c(t) = c_f(t) + c_n(t) = 1 - e^{-at}$$

where the input pole at the origin generated the forced response cf(t)=1, and the system pole at –a, as shown in Figure.



$$c(t) = c_f(t) + c_n(t) = 1 - e^{-at}$$

Let us examine the significance of parameter a, the only parameter needed to describe the transient response.

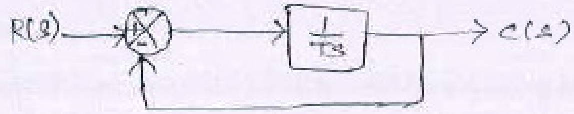
We now use these equations to define three transient response specifications.

Time Constant : We call 1/a the time constant of the response. From equationthe time constant can be described as the time for e^{-at} to decay to 37% of its initial time. Alternately, from equation (**), the time constant is the time it takes for the step response to rise to 63% of its final value. Thus, we can call the parameter a as exponential frequency.

Rise Time, Tr : Rise time is defined as the time for the waveform to go from 0.1 to 0.9 of its final value.

Settling Time, T s : Settling time is defined as the time for the response to reach and stay within 2% of its final value.

Response of first order system for unit step input



T.F $\dfrac{C(s)}{R(s)} = \dfrac{G(s)}{1+G(s)H(s)} = \dfrac{1/sT}{1+\left(\frac{1}{sT}\times 1\right)}$

$= \dfrac{\frac{1}{sT}}{\frac{sT+1}{\cancel{sT}}\times 1} = \dfrac{1}{sT+1}$

$\dfrac{C(s)}{R(s)} = \dfrac{1}{sT+1}$ $\quad C(s) = R(s) \times \dfrac{1}{Ts+1}$

If the i/p is unit step then $r(t) = 1$ and $R(s) = \frac{1}{s}$

The response in s domain, $C(s) = R(s)\dfrac{1}{1+Ts}$

$$C(s) = \frac{1}{s} \times \frac{1}{sT+1} = \frac{1}{s(sT+1)}$$

Take Partial Fraction

$$= \frac{K_1}{s} + \frac{K_2}{sT+1}$$

$K_1 = \left(\dfrac{1}{sT+1}\right)_{s=0} = 1$

$K_2 = \left(\dfrac{1}{s}\right)_{s=-1/T} = -T$ $\quad sT+1=0$, $sT=-1$, $s=-1/T$

$\therefore\ C(s) = \dfrac{1}{s} - \dfrac{T}{sT+1}$

$= \dfrac{1}{s} - \dfrac{\cancel{T}}{\cancel{T}(s+1/T)}$

$= \dfrac{1}{s} - \dfrac{1}{s+1/T}$

The response in time domain is given by

$$c(t) = \mathcal{L}^{-1}[C(s)] = \mathcal{L}^{-1}\left[\frac{1}{s} - \frac{1}{s+1/T}\right]$$

$$= 1 - e^{-t/T}$$

when $t=0$, $c(t) = 1-e^{0} = 0$

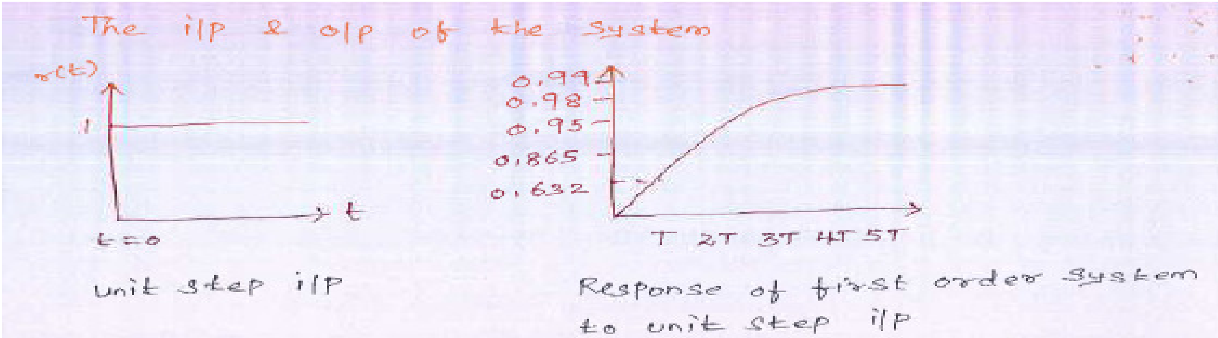
$t = 1T$, $c(t) = 1-e^{-1} = 0.632$

$t = 2T$, $c(t) = 1-e^{-2} = 0.865$

$t = 3T$, $c(t) = 1-e^{-3} = 0.95$

$t = 4T$, $c(t) = 1-e^{-4} = 0.9817$

$t = 5T$, $c(t) = 1-e^{-5} = 0.993$



Step Response of Second Order Systems :-

Consider a second order system of the form

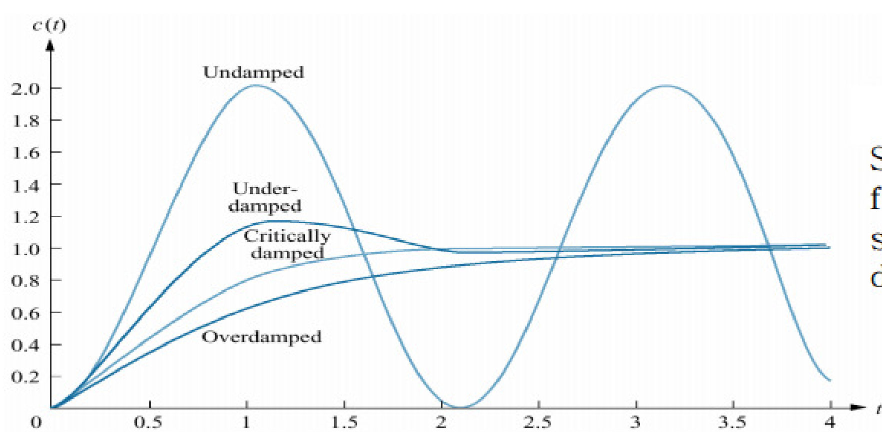
$$H(s) = \frac{b_1 s + b_0}{s^2 + a_1 s + a_0} \ .$$

In order to obtain intuition about these systems, we will be focusing on a particular form of second order system:

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}, \quad \omega_n > 0 \ .$$

The poles of this transfer function are obtained from the quadratic formula as
$s = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}.$
The location of these poles in the complex plane will vary based on the value of ζ.

We analyze four different cases:
- ζ =0: The system has two real poles in the CLHP The system is said to be undamped.
- 0 ≤ ζ < 1: The system has two complex poles in the CLHP (they will be in the OLHP if ζ > 0). The system is said to be underdamped.
- ζ = 1: The system has two repeated poles at s = −ωn. The system is said to be critically damped.
- ζ > 1: The system has two poles on the negative real axis. The system is said to be overdamped.



Step responses
for second-order
system
damping cases

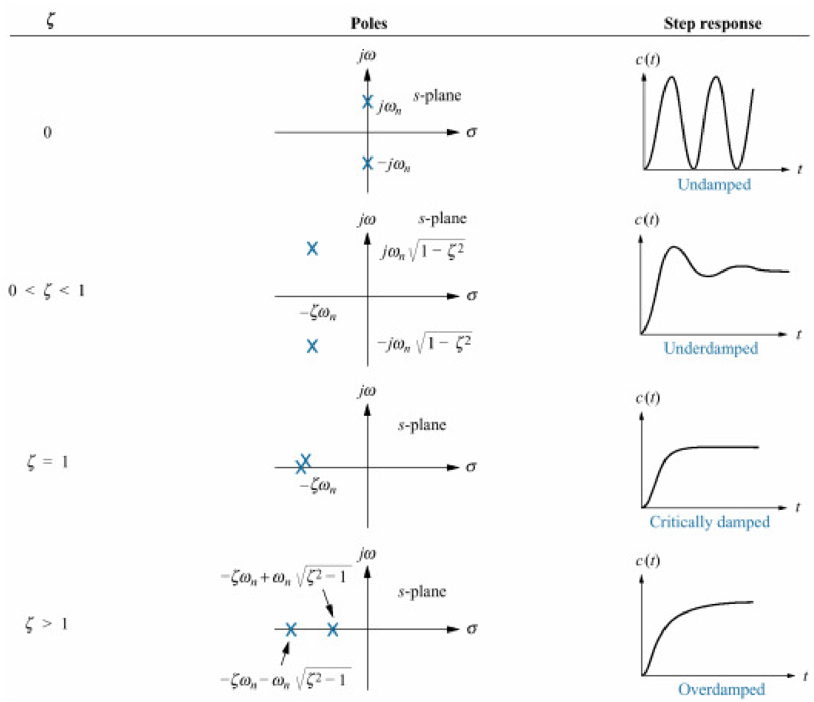
Two physically meaningful specificaitons for second-order system.

Natural Frequency, ω n : The natural frequency of a second order system is the frequency of oscillation of the system without damping. For example, the frequency of oscillation of a series RLC circuit with the resistance shorted would be natural frequency.

Damping Ratio, ζ : We define the damping ratio, ζ, to be

$$\zeta = \frac{\text{Exponential decay frequency}}{\text{Natural frequency (rad/sec)}} = \frac{1}{2\pi}\,\frac{\text{Natural period (second)}}{\text{Exponential time constant}}$$

Second-order response as a function of damping ratio



Transient Response

In analyzing and designing control systems, we must have a basis of comparison of performance of various control systems. This basis may be set up by specifying particular test input signals and by comparing the response of various systems to these input signals. Typical test signals: Step function, ramp function, impulse function, sinusoid function. The time response of a control system consists of two parts: the transient and the steady-state response.

Transient response corresponds to the behaviour of the system from the initial state to the final state.

By steady state, we mean the manner in which the system output behaves as time approaches infinity. For a step input, the transient response can be characterized by:

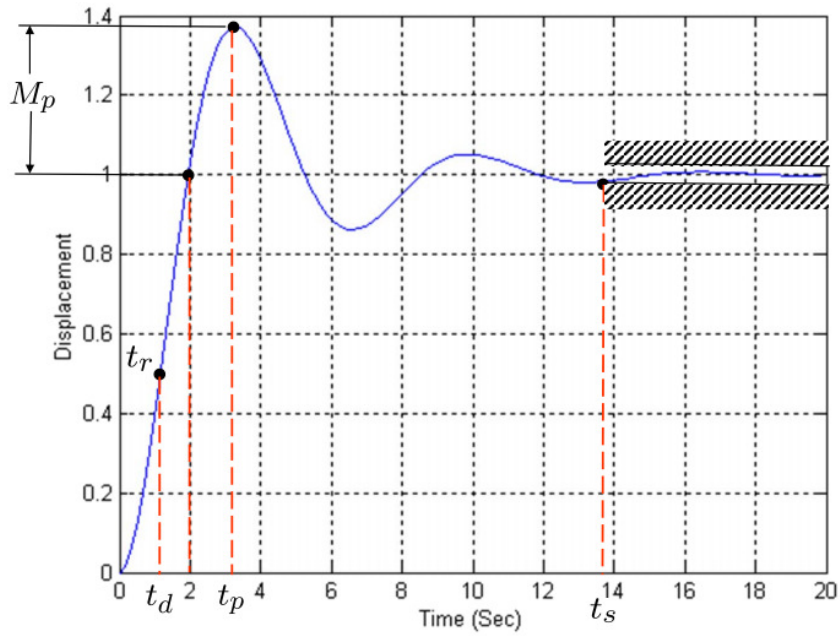
Delay time td: time to reach half the final value for the first time.

Rise time tr: time required for the response to rise from 10% to 90% for overdamped systems, and from 0% to 100% for underdamped systems

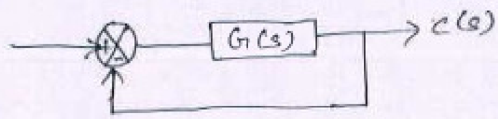
Peak time tp: time required to reach the first peak of the overshoot Percent Overshoot Mp.

Settling time ts: time required for the response curve to reach and stay within 2% or 5% of the final value. Is a function of the largest time constant of the control system.

Transient Response



1. Obtain the response of unity feedback system whose open loop T.F is $G(s) = \frac{4}{s(s+5)}$ and when the i/p is unit step.



The closed loop T.F

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s) \cdot 1}$$

$$\frac{C(s)}{R(s)} = \frac{4 / s(s+5)}{1 + \frac{4}{s(s+5)}} = \frac{4 / \cancel{s(s+5)}}{\frac{s(s+5)+4}{\cancel{s(s+5)}}} = \frac{4}{s(s+5)+4}$$

$$= \frac{4}{s^2+5s+4} = \frac{4}{(s+4)(s+1)}$$

The response in s. domain

$$C(s) = R(s) \frac{4}{(s+4)(s+1)}$$

By Partial Fraction

$$C(s) = \frac{4}{s(s+4)(s+1)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+4}$$

$$A = C(s)\, s \Big|_{s=0} = \frac{4}{(s+4)(s+1)}\Big|_{s=0} = \frac{4}{4} = 1 \quad \boxed{A=1}$$

$$B = C(s)(s+1)\Big|_{s=-1} = \frac{4}{s(s+4)}\Big|_{s=-1} = \frac{4}{-1(-1+4)} = -4/3$$

$$\boxed{B = -4/3}$$

$$C = C(s)(s+4)\Big|_{s=-4} = \frac{4}{s(s+1)} = \frac{4}{-4(-4+1)} = \frac{4}{12} = \frac{1}{3}$$

$$\boxed{C = 1/3}$$

$$C(s) = \frac{1}{s} - \frac{4}{3}\frac{1}{s+1} + \frac{1}{3}\frac{1}{s+4}$$

The response in time domain

$$c(t) = L^{-1}[C(s)]$$

$$= L^{-1}\left[\frac{1}{s} - \frac{4}{3}\frac{1}{s+1} + \frac{1}{3}\frac{1}{s+4}\right]$$

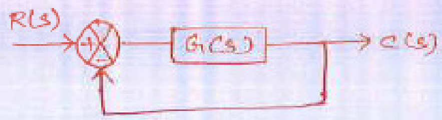
$$= 1 - \frac{4}{3}e^{-t} + \frac{1}{3}e^{-4t}$$

$$c(t) = 1 - \frac{1}{3}\left[4e^{-t} - e^{-4t}\right]$$

2. The unity feed back system is characterized by an open loop T.F $G(s) = \frac{K}{s(s+10)}$. Determine the gain K, so that the system will have a damping ratio of 0.5 for this value of K. Determine settling time, peak overshoot and time to peak overshoot for a unit step i/p.

(9)

The closed loop transfer function



$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$$

$$= \frac{\frac{K}{s(s+10)}}{1+\frac{K}{s(s+10)}} = \frac{K/\cancel{s(s+10)}}{\frac{s(s+10)+K}{\cancel{s(s+10)}}} = \frac{K}{s(s+10)+K}$$

$$= \frac{K}{s^2+10s+K}$$

The value of K can be evaluated by comparing the system T.F with standard form of second order T.F

$$\therefore \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2} = \frac{K}{s^2+10s+K}$$

on comparing we get

$$\omega_n^2 = K \qquad 2\xi\omega_n = 10 \qquad \omega_n^2 = K$$

$$\omega_n = \sqrt{K} \qquad 2\times0.5\,\omega_n = 10 \qquad \omega_n^2 = 100$$

$$2\times0.5\times\sqrt{K} = 10 \qquad \omega_n = 10 \text{ rad/sec}$$

$$\sqrt{K} = 10$$

$$K = 100$$

% Peak overshoot (%Mp) $= e^{-\xi\pi/\sqrt{1-\xi^2}} \times 100$

$$= e^{-0.5\pi/\sqrt{1-0.5^2}} \times 100$$

$$= 0.163 \times 100 = 16.3\%$$

Peak time $(t_p) = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\xi^2}} = \frac{\pi}{10\sqrt{1-(0.5)^2}} = 0.363$ sec

The value of gain (K) = 100

% Peak overshoot (%Mp) = 16.3%

Peak time (t_p) = 0.363 sec

3. The response of a feedback system to a step i/p is $c(t) = 1 + 0.2\,e^{-60t} - 1.2\,e^{-10t}$

(a) obtain the expression for the closed loop transfer function

(b) Determine the undamped natural frequency and damping ratio of the system

$R(s) = \frac{1}{s}$ $\quad c(t) = 1 + 0.2e^{-60t} - 1.2e^{-10t}$

Taking L.T for $c(t)$

$$C(s) = \frac{1}{s} + \frac{0.2}{s+60} - \frac{1.2}{s+10}$$

$$= \frac{1}{s} + \frac{0.2(s+10) - 1.2(s+60)}{(s+60)(s+10)}$$

$$= \frac{1}{s} + \frac{0.2s + 2 - 1.2s - 72}{(s+60)(s+10)}$$

$$= \frac{1}{s} + \frac{(-s-70)}{s^2+70s+600}$$

$$= \frac{\cancel{s^2} + \cancel{70s} + 600 - \cancel{s^2} - \cancel{70s}}{s(s^2+70s+600)}$$

$$= \frac{600}{s(s^2+70s+600)}$$

Closed loop transfer function

$$\frac{C(s)}{R(s)} = \frac{600}{s^2+70s+600}$$

This is a second order system. For this type standard form of $\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$

$\omega_n^2 = 600$

$\omega_n = \sqrt{600}$ $\quad \boxed{\omega_n = 24.5 \text{ rad/sec}}$

$2\xi\omega_n = 70$

$$\therefore \xi = \frac{70}{2\omega_n} = \frac{70}{2 \times 24.5} = 1.43$$

Note as $\xi > 1$, it is overdamped system.

4. A unity feed back control system has an open loop t/f function, $G(s) = \frac{10}{s(s+2)}$. Find the rise time, Percentage overshoot, Peak time and settling time for a step i/p of 12 units.

T.F ; $\dfrac{C(s)}{R(s)} = \dfrac{G(s)}{1+G(s)}$

$G(s) = \dfrac{10}{s(s+2)}$

$\therefore \dfrac{C(s)}{R(s)} = \dfrac{\dfrac{10}{s(s+2)}}{1+\dfrac{10}{s(s+2)}} = \dfrac{\dfrac{10}{s(s+2)}}{\dfrac{s(s+2)+10}{s(s+2)}} = \dfrac{10}{s^2+2s+10}$

$\therefore \dfrac{C(s)}{R(s)} = \dfrac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2} = \dfrac{10}{s^2+2s+10}$

on comparing

$\omega_n^2 = 10$

$\omega_n = \sqrt{10}$

$\omega_n = 3.162$ rad/sec

$2\xi\omega_n = 2$

$\xi = \dfrac{2}{2\omega_n} = \dfrac{1}{\omega_n} = \dfrac{1}{3.162} = 0.316$

(i) Rise time $t_r = \dfrac{\pi-\theta}{\omega_d}$

$\theta = \tan^{-1}\dfrac{\sqrt{1-\xi^2}}{\xi} = \tan^{-1}\dfrac{\sqrt{1-(0.316)^2}}{0.316} = 1.249$ rad

$\omega_d = \omega_n\sqrt{1-\xi^2} = 3.162\sqrt{1-(0.316)^2} = 3$ rad/sec

$t_r = \dfrac{\pi-\theta}{\omega_d} = \dfrac{\pi-1.249}{3} = 0.63$ sec

(ii) Percentage overshoot (% MP) $= e^{\frac{-\xi\pi}{\sqrt{1-\xi^2}}} \times 100$

$= e^{-\frac{0.316\pi}{\sqrt{1-(0.316)^2}}} \times 100$

$= 35.12\,\%$

(iii) Peak overshoot $= \dfrac{35.12}{100} \times 12 = 4.2144$ units

(iv) Peak time $t_p = \dfrac{\pi}{\omega_d} = \dfrac{\pi}{\omega_n\sqrt{1-\xi^2}} = \dfrac{\pi}{3} = 1.047$ sec

(v) Time constant $T = \dfrac{1}{\xi\omega_n} = \dfrac{1}{0.316 \times 3.162} = 1$ sec

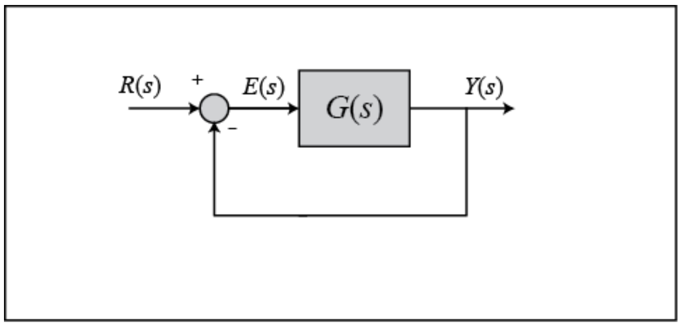
For 5% error settling time $t_s = 3T = 3$ sec

For 2% error settling time $t_s = 4T = 4$ sec

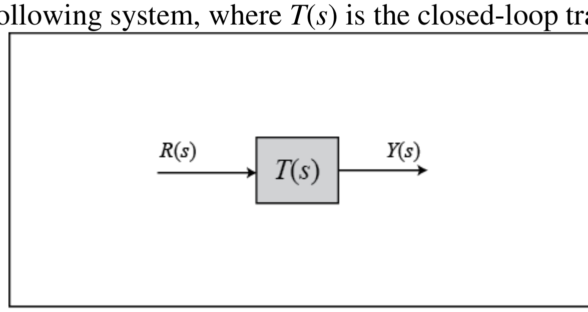
Steady-state error is defined as the difference between the input (command) and the output of a system in the limit as time goes to infinity (i.e. when the response has reached **steady state**).
The **steady-state error** will depend on the type of input (step, ramp, etc.) as well as the system type (0, I, or II).

Calculating steady-state errors

Before talking about the relationships between steady-state error and system type, we will show how to calculate error regardless of system type or input. Then, we will start deriving formulas we can apply when the system has a specific structure and the input is one of our standard functions. Steady-state error can be calculated from the open- or closed-loop transfer function for unity feedback systems. For example, let's say that we have the system given below.



This is equivalent to the following system, where $T(s)$ is the closed-loop transfer function.



We can calculate the steady-state error for this system from either the open- or closed-loop transfer function using the Final Value Theorem. **Recall that this theorem can only be applied if the subject of the limit ($sE(s)$ in this case) has poles with negative real part**.

(1) $$e(\infty) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{sR(s)}{1+G(s)}$$

(2) $$e(\infty) = \lim_{s\to 0} sE(s) = \lim_{y\to 0} sR(s)[1-T(s)]$$

Now, let's plug in the Laplace transforms for some standard inputs and determine equations to calculate steady-state error from the open-loop transfer function in each case.

- Step Input ($R(s) = 1 / s$):

  (3) $$e(\infty) = \frac{1}{1+\lim_{s\to 0} G(s)} = \frac{1}{1+K_p} \Rightarrow K_p = \lim_{y\to 0} G(s)$$

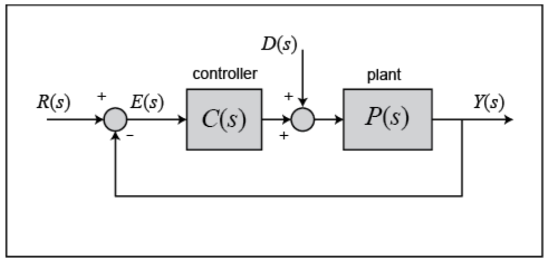
- Ramp Input ($R(s) = 1 / s$^2):

  (4) $$e(\infty) = \frac{1}{\lim_{s\to 0} sG(s)} = \frac{1}{K_v} \Rightarrow K_v = \lim_{s\to 0} sG(s)$$

- Parabolic Input ($R(s) = 1 / s$^3):

  (5) $$e(\infty) = \frac{1}{\lim_{s\to 0} s^2G(s)} = \frac{1}{K_a} \Rightarrow K_a = \lim_{s\to 0} s^2G(s)$$

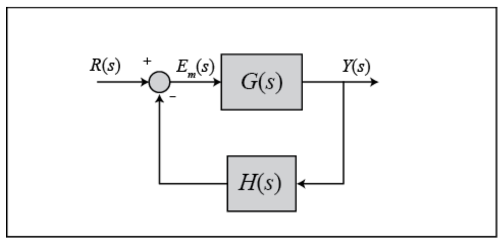
When we design a controller, we usually also want to compensate for disturbances to a system. Let's say that we have a system with a disturbance that enters in the manner shown below.



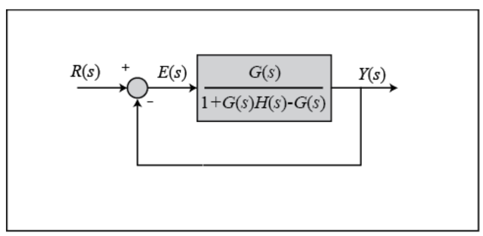
We can find the steady-state error due to a step disturbance input again employing the Final Value Theorem (treat $R(s) = 0$).

$$e(\infty) = \lim_{s\to 0} sE(s) = \frac{1}{\lim_{s\to 0} \frac{1}{P(s)} + \lim_{s\to 0} C(s)} \tag{6}$$

When we have a non-unity feedback system we need to be careful since the signal entering $G(s)$ is no longer the actual error $E(s)$. Error is the difference between the commanded reference and the actual output, $E(s) = R(s) - Y(s)$. When there is a transfer function $H(s)$ in the feedback path, the signal being substracted from $R(s)$ is no longer the true output $Y(s)$, it has been distorted by $H(s)$. This situation is depicted below.



Manipulating the blocks, we can transform the system into an equivalent unity-feedback structure as shown below.

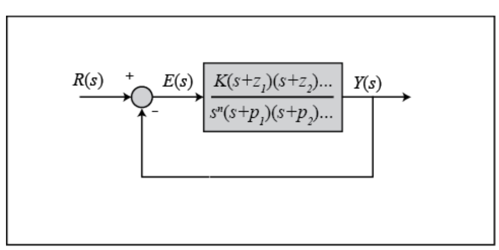


Then we can apply the equations we derived above.

System type and steady-state error

If you refer back to the equations for calculating steady-state errors for unity feedback systems, you will find that we have defined certain constants (known as the static error constants). These constants are the position constant (*Kp*), the velocity constant (*Kv*), and the acceleration constant (*Ka*). Knowing the value of these constants, as well as the system type, we can predict if our system is going to have a finite steady-state error.

First, let's talk about system type. The system type is defined as the number of pure integrators in the forward path of a unity-feedback system. That is, the system type is equal to the value of n when the system is represented as in the following figure. It does not matter if the integrators are part of the controller or the plant.



Therefore, a system can be type 0, type 1, etc. The following tables summarize how steady-state error varies with system type.

Type 0 system	***Step Input***	***Ramp Input***	***Parabolic Input***
Steady-State Error Formula	1/(1+Kp)	1/Kv	1/Ka
Static Error Constant	Kp = constant	Kv = 0	Ka = 0
Error	1/(1+Kp)	Infinity	infinity

Type 1 system	***Step Input***	***Ramp Input***	***Parabolic Input***
Steady-State Error Formula	1/(1+Kp)	1/Kv	1/Ka
Static Error Constant	Kp = infinity	Kv = constant	Ka = 0
Error	0	1/Kv	infinity

Type 2 system	***Step Input***	***Ramp Input***	***Parabolic Input***
Steady-State Error Formula	1/(1+Kp)	1/Kv	1/Ka
Static Error Constant	Kp = infinity	Kv = infinity	Ka = constant
Error	0	0	1/Ka

1. A unity Feedback system has $G(s) = \frac{40(s+2)}{s(s+1)(s+4)}$

Find the (a) Type of the system

(b) All error constants

(c) calculate ess for ramp i/p with magnitude

(a) Type of the system = 1

(b) error constants

$$K_p = \lim_{s\to 0} G(s)H(s) = \lim_{s\to 0} \frac{40(s+2)}{s(s+1)(s+4)} = \frac{40\times 2}{0} = \infty$$

$$K_v = \lim_{s\to 0} s\,G(s)H(s) = \lim_{s\to 0} \frac{\cancel{s}\cdot 40(s+2)}{\cancel{s}(s+1)(s+4)} = \frac{40\times 2}{4} = 20$$

$$K_a = \lim_{s\to 0} s^2 G(s)H(s) = \lim_{s\to 0} \frac{s^2\, 40(s+2)}{s(s+1)(s+4)} = 0$$

(c) To find ess for ramp i/p with magnitude

$$e_{ss} = e_{ss1} + e_{ss2} + e_{ss3}$$

$$= \frac{A_1}{1+K_p} + \frac{A_2}{K_v} + \frac{A_3}{K_a}$$

$$e_{ss} = \frac{4}{20} = 0.2$$

$e_{ss} = 0.2 \Rightarrow$ Final steady state error.

α. Determine the static error constants and the steady state error for i/p $r(t) = 2t^2+5t+10$. For a open loop T.F of feed back system.

$$G(s)H(s) = \frac{100}{s^2(s+4)(s+12)}$$

Error constants

$$K_p = \lim_{s\to 0} G(s)H(s) = \frac{100}{s^2(s+4)(s+12)} = \frac{100}{0} = \infty$$

$$K_v = \lim_{s\to 0} sG(s)H(s) = \frac{\cancel{s}\times 100}{s^{\cancel{2}}(s+4)(s+12)} = \infty$$

$$K_a = \lim_{s\to 0} s^2 G(s)H(s) = \frac{\cancel{s^2}\times 100}{\cancel{s^2}(s+4)(s+12)} = \frac{100}{4\times 12} = \frac{100}{48} = 2.08$$

$$r(t) = A_1 + A_2 t + A_3 \frac{t^2}{2}$$

general Form $= A_3\frac{t^2}{2} + A_2 t + A_1$ $\qquad r(t) = 2t^2+5t+10$

$$= 4\frac{t^2}{2} + 5t + 10$$

$A_1 = 10$

$A_2 = 5$

$A_3 = 4$

e_{ss} for $K_p = \frac{A_1}{1+K_p} = \infty$

e_{ss} for $K_v = \frac{A_2}{K_V} = \infty$

e_{ss} for $K_a = \frac{A_3}{K_a} = \frac{4}{2.08} = 1.923$

Alternate method for Generalised error series or Dynamic error coefficients

The error signal is s-domain

$$E(s) = \frac{R(s)}{1+G(s)H(s)} \qquad \therefore \frac{E(s)}{R(s)} = \frac{1}{1+G(s)H(s)}$$

The above eqn can be expressed as a power series of s

$$F(s) = \frac{E(s)}{R(s)} = \frac{1}{1+G(s)H(s)} = C_0 + C_1 s + \frac{C_2}{2!}s^2 + \frac{C_3}{3!}s^3 + \cdots$$

$$\therefore E(s) = C_0 R(s) + C_1 sR(s) + \frac{C_2}{2!}s^2R(s) + \frac{C_3}{3!}s^3R(s) + \cdots$$

On taking inverse laplace transform

(12)

$$e(t) = C_0 r(t) + C_1\dot{r}(t) + \frac{C_2}{2!}\ddot{r}(t) + \frac{C_3}{3!}\dddot{r}(t) + \cdots$$

where $C_0, C_1, C_2 \ldots$ are generalized error coefficient

$$\therefore e_{ss} = \lim_{t\to\infty} e(t) \qquad \dot{r}(t) = \frac{d}{dt}r(t)$$

$$C_0 = \lim_{s\to 0} F(s) \qquad \ddot{r}(t) = \frac{d^2}{dt^2}r(t)$$

$$C_1 = \underset{s \to 0}{Lt} \frac{d}{ds} F(s)$$

$$\dddot{r}(t) = \frac{d^3}{dt^3} r(t)$$

$$C_2 = \underset{s \to 0}{Lt} \frac{d^2}{ds^2} F(s)$$

(1) For a unity feed back control system the open loop T.F

$G(s) = \dfrac{10(s+2)}{s^2(s+1)}$. Find

(a) the position, velocity and acceleration error constants

(b) the steady state error when the i/p is

$$R(s) = \frac{3}{s} - \frac{2}{s^2} + \frac{1}{3s^3}$$

(a) To find static error constants

$$H(s) = 1$$

Position error constant $K_p = \underset{s \to 0}{Lt}\ G(s)H(s)$

$$= \underset{s \to 0}{Lt} \frac{10(s+2)}{s^2(s+1)} = \infty$$

Velocity error constant $K_v = \underset{s \to 0}{Lt}\ s\,G(s)H(s)$

$$= \underset{s \to 0}{Lt} \frac{s \times 10(s+2)}{s^2(s+1)} = \infty$$

Acceleration error constant $K_a = \underset{s \to 0}{Lt}\ s^2 G(s) H(s)$

$$= \underset{s \to 0}{Lt} \frac{\cancel{s^2} \times 10(s+2)}{\cancel{s^2}(s+1)} = \frac{20}{1} = 20$$

P, PI and PID Controllers

- The controller (an analogue/digital circuit, and software), is trying to keep the controlled variable such as temperature, liquid level, motor velocity, robot joint angle, at a certain value called the **set point (SP)**.
- A feedback control system does this by looking at the **error (E)** signal, which is the difference between where the controlled variable (called the **process variable (PV)**) is, and where it should be.
- Based upon the error signal, the controller decides the magnitude and the direction of the signal to the actuator.

The proportional (P), the integral (I), and the derivative (D), are all basic controllers.

Types of controllers: P, I, D, PI, PD, and PID controllers

- **Proportional Control**

With proportional control, the actuator *applies a corrective force that is proportional to the amount of error*:

$$Output_p = K_p \times E$$

$Output_p$ = system output due to proportional control

K_p = proportional constant for the system called **gain**

E = error, the difference between where the controlled variable should be and where it is. $E = SP - PV$.

One way to decrease the steady-state error is to increase the system gain (K_p), but high K_p can lead to instability problems.

Increasing K_p independently without limit is **not** a sound control strategy.

- **Integral Control**

The introduction of integral control in a control system can reduce the steady-state error to zero. Integral control *applies a restoring force that is proportional to the sum of all past errors, multiplied by time*.

$$Output_I = K_I \times \sum(E \times \Delta t)$$

$Output_I$ = controller output due to integral control

K_I = integral gain constant (sometimes expressed as $1/T_I$)

$\sum(E \times \Delta t)$ = sum of all past errors (multiplied by time)

For a constant value of error $\sum(E \times \Delta t)$ will increase with time, causing the restoring force to get larger and larger.

Eventually, the restoring force will get large enough to overcome friction and move the controlled variable in a direction to eliminate the error.

- **Derivative Control**

One solution to the overshoot problem is to include derivative control. Derivative control 'applies the brakes,' slowing the controlled variable just before it reaches its destination.

$$Output_D = K_D \times \left(\frac{\Delta E}{\Delta t}\right)$$

$Output_D$ = controller output due to derivative control

K_D = derivative gain constant

$\left(\frac{\Delta E}{\Delta t}\right)$ = error rate of change (slope of error curve)

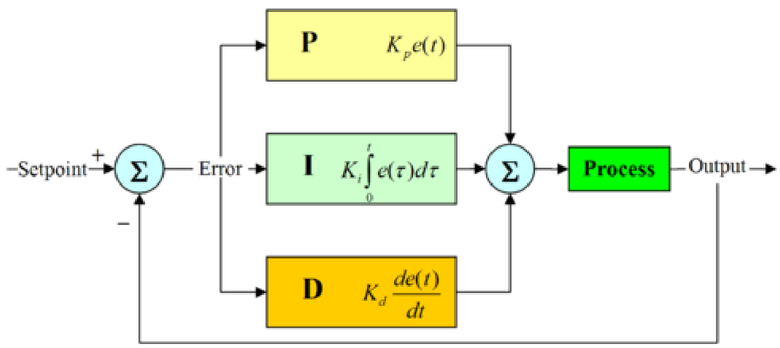
- **Combining P, I and D controllers**

As proportional, integral and derivative controllers have their individual strengths and weaknesses, they are often combined so that their strengths are maximised, whilst minimising their weaknesses. Many industrial controllers are a combination of P + I, or P + D, and are referred to as PI and PD controllers respectively.

- **PID control**

A proportional–integral–derivative controller (PID controller) is a generic control loop feedback mechanism (controller) widely used in industrial control systems.

A PID controller attempts to correct the error between a measured process variable and a desired setpoint by calculating and then outputting a corrective action that can adjust the process accordingly.



The foundation of the system is proportional control. Adding integral control provides a means to eliminate steady-state error, but increases overshoot. Derivative control increases stability by reducing the tendency to overshoot.

Simply adding together the three required control components generates the response of the PID system.

$$Output_{PID} = K_P \times E - K_I \times \sum(E \times \Delta t) - K_D \times \left(\frac{\Delta E}{\Delta t}\right)$$

$Output_{PID}$ = output from PID controller
K_P = proportional control gain

K_I = integral control gain

K_D = derivative control gain

E = error (deviation from set point)

$\sum(E \times \Delta t)$ = sum of all past errors (area under the error/time curve)

$\left(\frac{\Delta E}{\Delta t}\right)$ = rate of change of error (slope of the error curve)

Equation is:

$$Output_{PID} = K_P E - \frac{1}{T_I}\int E dt - T_D \frac{dE}{dt}$$

When you are designing a PID controller for a given system, follow the steps shown below to obtain a desired response.

1. Obtain an open-loop response and determine what needs to be improved
2. Add a proportional control to improve the rise time
3. Add a derivative control to improve the overshoot
4. Add an integral control to eliminate the steady-state error
5. Adjust each of K_p, K_i, and K_d until you obtain a desired overall response.

The characteristics of P, I, and D controllers

- A proportional controller (K_p) will have the effect of reducing the rise time and will reduce ,but never eliminate, the steady-state error.
- An integral control (K_i) will have the effect of eliminating the steady-state error, but it may make the transient response worse.
- A derivative control (K_d) will have the effect of increasing the stability of the system, reducing the overshoot, and improving the transient response.