Unit Three: Propositional Logic

Welcome to Unit Three! In Unit One, we learned how to determine the validity of arguments by analyzing the relationships between the TERMS in the argument (the subject and predicate terms within the statements; e.g., major term, minor term, middle term). In this unit, we will learn how to determine the validity of arguments by analyzing the relationships between the STATEMENTS or PROPOSITIONS in the argument. This sort of logic is called "propositional logic". Let's get started.

6.1 Symbols and Translation

In unit 1, we learned what a "statement" is. Recall that a statement is just a proposition that asserts something that is either true or false. For instance, these are propositions:

"All kittens are cute." ; "I like pizza." ; "The sky is blue." ; "Triangles have three sides."

These are "simple statements". But, statements can be a lot more complicated than this. Define a "**compound statement**" as any proposition which contains either: (a) two or more simple statements, or (b) at least one negated simple statement. For instance:

"Peggy is taking Logic **and** Sue is taking Ethics."
"Chad saw a squirrel today **or** he saw a deer."
"**If** it snows today, **then** we will build a snowman."
"The cookies will turn out right **if and only if** we bake them at 350 degrees."
"**It is not the case that** the sun is shining today."

Notice that (with the exception of the last sentence about the sun shining) all of the propositions above include TWO statements. The two statements are connected by what is called an "**operator**" (in order, they are: "**and**", "**or**", "**if … then**", and "**if and only if**"). The final sentence only includes ONE statement, but it is the NEGATION of that statement. The operator in this sentence is "**not**". If we replace each of the statements above with capital letters, we will be able to see the operators more clearly:

P **and** S
S **or** D
If S **then** B.
C **if and only if** B.
Not S.

In this lesson, we will learn about each of the operators above (the words in bold). In logic, though, we actually use SYMBOLS for each of the logical operators. Like this:

P **and** S	P ● S
S **or** D	S ∨ C
If S **then** B.	S ⊃ B
C **if and only if** B.	C ≡ B
Not S.	~S

Let's look at each of these operators in more detail, beginning with "not" or "~".

<u>1. Negation:</u> Whenever we want to NEGATE a statement, this basically means that we want to say that the statement is FALSE. For instance, imagine that I told you something like:

> "I'm going to give you a million dollars... NOT."

The "NOT" here is serving the purpose of negation. Its purpose is to indicate that the entire statement just uttered is false. Only, in logic, the "not" comes first, like this:

> "NOT: I'm going to give you a million dollars."

or

> "It is not the case that I'm going to give you a million dollars."

Since, in logic, we use symbols for the operators and letters for the statements, we will replace the "NOT" with a tilde, "~", and let's use the letter "G" for "I'm going to give you a million dollars." In that case, the sentence above becomes:

> ~G

<u>Synonyms for "not":</u> All negative claims should be represented this way. So, "~G" would be the correct symbolization of all of the following:

> "**It is not the case that** I am giving you one million dollars."
> "**It is false that** I am giving you one million dollars."
> "I am **not** giving you one million dollars."

Note that the tilde comes IN FRONT of a statement. All of the other symbols we are about to look at must always come IN BETWEEN two statements.

<u>2. Conjunction:</u> Conjunction means conjoining two statements together. We typically do this with the word "and". For instance, I might say that I like vanilla AND chocolate. The claims on either side of the "and" are called the "**conjuncts**". We represent conjunction with a dot, "●".For instance:

"Peggy wants pizza, **and** Sue wants pizza."

Gets translated as:

P ● S

<u>Synonyms for "and":</u> But, "and" is not the ONLY word that indicates conjunction. Basically, ANY time that we add two claims together, we are conjoining them. For instance:

"Peggy likes pepperoni pizza, **but** Sue likes Hawaiian."
"Dominoes has a special on pepperoni; **however**, Papa John's has a special on Hawaiian."
"Peggy **and** Sue decide to get burgers instead."

ALL of the above statements are really just the conjunction of two separate claims. To see this, just replace the bold words with the word "and" and you'll see that it doesn't really change the claims being made. For instance, if we change:

"Peggy likes pepperoni pizza, **but** Sue likes Hawaiian."

To:

"Peggy likes pepperoni pizza, **and** Sue likes Hawaiian."

These two statements indicate the same thing; namely, that "Peggy likes pepperoni pizza" and also "Sue likes Hawaiian pizza".

<u>Swapping the order:</u> Notice that "Peggy likes pepperoni and Sue likes Hawaiian" means exactly the same thing as "Sue likes Hawaiian and Peggy likes pepperoni." So, for conjunction, the order of the conjuncts can be swapped, and the swapped sentence still means the same thing. So,

"P ● S" means the same thing as "S ● P"

3. Disjunction: Disjunction means presenting two statements as ALTERNATIVES. We typically do this with the word "or". For instance, I might say that I will eat strawberry OR chocolate. The claims on either side of the "or" are called the "**disjuncts**". We represent disjunction with a wedge, "v", which basically just looks like the letter "v". For instance:

"I will eat strawberry **or** I will eat chocolate."

Gets translated as:

$S \vee C$

"Or" is **inclusive:** Note that we should not interpret "or" as being "exclusive". That is, when I say that "I will eat strawberry OR chocolate" and you hand me strawberry, this does not automatically mean that I would refuse some chocolate if you offered it to me. For all you know, maybe I'd like some strawberry AND chocolate if you offered both.

What "or" means, then, is that AT LEAST one of the two disjuncts must be true. But, maybe they could BOTH be true as well. So, we say that or is "**inclusive**".

But, sometimes, we DO use the word "or" to be "**exclusive**". For instance, I might say that, "He is either five **or** six years old." Surely I do not mean to imply that he might also be five AND six years old. That is impossible. I mean "or" to be EXCLUSIVE here. That is, if the boy is five, then this rules out the possibility of him being six, and vice versa. Now, there IS a way to account for an EXCLUSIVE "or". We'd do it like this:

$(F \vee S) \bullet {\sim}(F \bullet S)$

What this basically says is that "He is either five or six, and he is NOT both five AND six." This is how we capture the idea behind the word "or" when it is meant to be exclusive. We will discuss these more complicated sorts of statements at the end of this lesson.

Synonyms for "or": But, "or" is not the ONLY word that expresses disjunction. "Unless" is another common word that means the same thing as "or". So, all of the following are disjunctions:

"I will have strawberry **or** chocolate."	Translation: $S \vee C$
"**Either** I will have strawberry, **or** I will have chocolate."	Translation: $S \vee C$
"I will choose pepperoni, **unless** you choose Hawaiian."	Translation: $P \vee H$
"**Unless** you choose Hawaiian, I will choose pepperoni."	Translation: $H \vee P$

"Unless" is a synonym for "or". Notice that "I choose pepperoni **unless** you choose Hawaiian" means the same thing as "I choose pepperoni **or** you choose Hawaiian."

Swapping the order: Notice that "I will have strawberry or chocolate" means exactly the same thing as "I will have chocolate or strawberry." So, for disjunction, the order of the disjuncts can be swapped, and the swapped sentence still means the same thing. So,

"S ∨ C" means the same thing as "C ∨ S"

4. Conditionals: Conditional statements express an "if … then" claim. For instance, I might say that "IF there is any pepperoni left, THEN I will eat pepperoni." The FIRST part of the sentence (after the "if") is called the "**antecedent**", while the SECOND part of the sentence (after the "then") is called the "**consequent**". The antecedent is the CONDITION upon which the consequent follows. The conditional relation is expressed by the horseshoe symbol, "Ↄ". For instance,

"If there is any pepperoni left, then I will eat pepperoni."

Gets translated as:

P Ↄ E

*Note: Which letters we choose are not really that important. I could have chosen "P" to represent the consequent, "I will eat **P**epperoni" but that would be confusing, since I already chose "P" to represent the antecedent, "If there is any **P**epperoni left." So, instead, I chose "E" to represent the consequent, "I will **E**at pepperoni." In general, try to pick letters that make as much sense as possible, and don't use the same letter twice for different statements.*

But, sometimes, conditionals are stated in a reversed order. For instance:

"I will eat pepperoni, if there is any pepperoni left."

The important thing here is to look for the word "if". Everything after the "if" is STILL the antecedent EVEN IF the statement after the word "if" COMES SECOND. So, this statement ALSO gets translated as:

P Ↄ E

"Only If": However—and this can be confusing so pay close attention—when the term "**ONLY if**" is used, everything after the "if" is the CONSEQUENT, and NOT the antecedent. For instance, "Athena is a cat **only if** she is a mammal."

Gets translated as:

$A \supset M$

Note that "Athena is a cat **only if** she is a mammal" does NOT mean the same thing as "Athena is a cat **if** she is a mammal" since lots of mammals are not cats (for instance, Athena might be a dog). On the other hand, all cats ARE mammals. So "if" and "only if" do not get translated the same way.

<u>Synonyms for "if … then":</u> The phrase "if … then" is not the ONLY way to express a conditional claim, however. Some synonyms for "if … then" are "**provided that**", "**on the condition that**", and "**implies that**". So, here are some sentences and their correct translations:

"**If** Peggy goes to the movies, **then** Sue will go too."	Translation: $P \supset S$
"Sue will go to the movies, **if** Peggy goes."	Translation: $P \supset S$
"Sue will go to the movies, **provided that** Peggy goes."	Translation: $P \supset S$
"Sue will go to the movies **on the condition that** Peggy goes."	Translation: $P \supset S$
"Sue's going to the movies **implies that** Peggy goes too."	Translation: $P \supset S$
"Sue will go to the movies **only if** Peggy does."	**Translation: $S \supset P$**

<u>NOT swapping the order:</u> Note that changing the order of the antecedent and the consequent does NOT result in a sentence with the same meaning. For instance, consider the difference between:

"If it is raining, then the ground is wet." Translation: $R \supset W$

And:

"If the ground is wet, then it is raining." Translation: $W \supset R$

These do NOT mean the same thing. If it is raining, then the ground will definitely be wet. On the other hand, if the ground is wet, then it is NOT definitely raining. (for instance, I might be spraying the ground with a hose, or maybe some snow is melting and getting the ground wet). So, unlike "and" and "or" where the letters can be swapped without changing the meaning, for "if … then" the order can NOT be swapped without changing the meaning.

Necessary and Sufficient Conditions: The point just made about swapping the order of the antecedent with the consequent in an "if … then" statement becomes clearer in light of a discussion about "**necessary conditions**" and "**sufficient conditions**".

Recall that we discussed necessary and sufficient conditions briefly in unit one. There, we defined these terms as follows:

> **Sufficient Condition:** A is a sufficient condition of B whenever A is all that is needed in order for B to occur, or be true, etc.
>
> **Necessary Condition:** A is a necessary condition for B whenever B CANNOT occur, or be true, etc., without A also occurring, or being true, etc.

To illustrate, consider this statement:

- "If Fido is a dog, then he is a mammal." Translation: D Ɔ M

Here, "being a dog" is a **sufficient condition** for "being a mammal": Being a dog GUARANTEES being a mammal. But, being a dog is not NECESSARY in order to be a mammal. For instance, something might be a cat, or a squirrel, or a monkey, and also be a mammal.

On the other hand, "being a mammal" is a **necessary condition** for "being a dog": Something MUST be a mammal in order to be a dog. There is no way that Fido can be a dog if he is not a mammal.

So, the sufficient condition is always on the left side of the horseshoe, and the necessary condition is always on the right side of the horseshoe. A good way to remember this is the word "SUN", because it looks like "S Ɔ N" (or, "Sufficient condition Ɔ Necessary condition"). So, here is how we should translate claims about necessary and sufficient conditions:

> "Being a **D**og is a **sufficient condition** for being a **M**ammal." D Ɔ M
>
> "Having **S**ugar is a **necessary condition** for baking **C**ookies." C Ɔ S

You can think of these statements as being translated like this:

> "If something is a dog, then it is a mammal."
>
> "If you are baking cookies, then you must have sugar."

5. Bi-Conditionals: Bi-Conditional statements are DOUBLE conditionals; that is, the "if … then" claim goes in BOTH directions, rather than just one direction. This is typically expressed by the phrase "**if and only if**", and it is symbolized by the triple-bar, "≡". Literally, bi-conditional statements such as this:

"Peggy will go to Florida **if and only if** Sue does." Translation: P ≡ S

Means this:

"**If** Peggy goes to Florida, **then** Sue will also go to Florida." Translation: P Ↄ S
AND
"**If** Sue goes to Florida, **then** Peggy will also go to Florida." Translation: S Ↄ P

So, the following two statements are equivalent:

P ≡ S

(P Ↄ S) ● (S Ↄ P)

Remember that the left side (antecedent) of the conditional "Ↄ" symbol is the **sufficient condition** and the right side (consequent) of the conditional "Ↄ" symbol is the **necessary condition**. But, in the statement, "(P Ↄ S) ● (S Ↄ P)", P is the ANTECEDENT of the FIRST conditional, and the CONSEQUENT of the SECOND conditional. The same thing is true of S. So, S is a **necessary** AND **sufficient** condition for P. Likewise, P is a **necessary** AND **sufficient** condition for S. So, another way to say "**if and only if**" is to say "**is a necessary and sufficient condition for**". For instance:

"Being H_2O **is a necessary and sufficient condition for** being water."

Gets translated as:

H ≡ W

Summary so far: The following chart summarizes everything so far:

Name	Symbol	Example
NOT	~	~P
AND	•	P • Q
OR	v	P v Q
IF … THEN	Ↄ	P Ↄ Q
IF AND ONLY IF	≡	P ≡ Q

<u>6. Using Multiple Operators:</u> So far, we have only looked at statements where there is ONE operator (for instance, "~A", "A ● B", "A v B", "A Ↄ B", and "A ≡ B"). Each of these has only ONE operator. But, often, statements can be much more complicated than this, and require TWO or more operators. For instance, we already saw one such statement:

(F v S) ● ~(F ● S)

Recall that this meant, "He is either five **or** six, **and** he is **not** both five **and** six." Notice the words in bold ("or", "and", "not", and "and"). Each of these four words are operators, and so the symbolic translation requires FOUR operator symbols.

But, we can't write the statement above just ANY old way. The statement above is a "**well-formed formula**". A well-formed formula is a formula which does not violate any of the rules for symbolic formulas. Here are several rules of thumb for formulas:

1) **Three or more letters should never appear in a row without parentheses or brackets in between them.**

Brackets are sort of like commas in English sentences. For instance, the following sentence is ambiguous:

"Buddy likes cheese and Peggy likes pepperoni or Sue likes Hawaiian."

This would be symbolized as the following:

B ● P v S

But, which of the following is the speaker saying?

"Either Buddy likes cheese and Peggy likes pepperoni … OR Sue likes Hawaiian."
OR
"Buddy likes cheese … AND Either Peggy likes pepperoni or Sue likes Hawaiian."

There is an important difference. Commas would help to clarify things. If we add commas to these two interpretations, we get the following:

"Buddy likes cheese and Peggy likes pepperoni, or Sue likes Hawaiian."

"Buddy likes cheese, and Peggy likes pepperoni or Sue likes Hawaiian."

The commas represent separators, just like parentheses do. Here are the formulas:

(B ● P) ∨ S

B ● (P ∨ S)

In the first sentence, Buddy and Peggy occur together, while Sue is separated by a comma. This indicates that the claim about Buddy and Peggy come together as a single unit. We indicate this by putting parentheses around them. Meanwhile, in the second sentence, Buddy occurs alone, and Peggy and Sue occur together after the comma. So, we put parentheses around Peggy and Sue in the second formula.

But, there will not always be a comma. The placement of the word "**either**" can be helpful in these cases. For instance, notice the difference between the following two sentences:

"**Either** Harry orders juice and Mark orders beer **or** John orders soda."

"Harry orders juice and **either** Mark orders beer **or** John orders soda."

These get symbolized as the following:

(H ● M) ∨ J

H ● (M ∨ J)

Here are a few more sentences and their translations:

"Harry likes juice or both Mark and John like soda."
H ∨ (M ● J)

"Harry likes juice and Mark or John like soda."
H ● (M ∨ J)

"If Harry orders juice, then if Mark orders beer, then John will order soda"
H Ↄ (M Ↄ J)

"If Harry will order juice provided that Mark orders beer, then John will order soda."
(M Ↄ H) Ↄ J

"If Harry and Mark order juice or John orders beer, then Larry will order water."
[(H ● M) ∨ J] Ↄ L

2) **Two letters can never appear side by side; they must always be separated by an operator (and they cannot be separated ONLY by a "~").**

The following are NOT well-formed formulas:

AB
~AB
A ∨ BC
A~B

To see that these sentences are counter-intuitive, consider how poorly they would look if we translated them into English:

Albert likes cheddar Brett likes swiss.
It is not the case that Albert likes cheddar Brett likes swiss.
Either Albert likes cheddar or Brett likes swiss Charlie likes mozzarella.
Albert likes cheddar not Brett likes swiss.

3) **All symbols except "~" must have something on either side of them. "~" must have something on its right side.**

The following are NOT well-formed formulas:

● A ∨ B	("●" needs something on both sides of it)
A~	("~" must go on the left)
A Ↄ	("Ↄ" needs something on both sides of it)

4) **Two operators can never appear side by side unless the second is a tilde.**

The following are NOT well-formed formulas:

A Ↄ ∨ B	
A ≡ ● B	
~●A	
●~A	(remember that the "●" needs something on both sides of it)

But note that the following IS a well-formed formula: A ● ~ B

For more examples of formula that are well-formed, and others that are not, please see your textbook.

Note: Do homework for section 6.1 at this time.