Pearson BTEC Levels 4 and 5 Higher Nationals in Engineering (RQF)

Unit 2: Engineering Maths (core)

Unit Workbook 1

in a series of 4 for this unit

Learning Outcome 1

Mathematical Methods

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1   Mathematical Concepts

1.1   Dimensional analysis

Very often in engineering we encounter numbers with associated physical units, for example; 3 volts (3 V), 5 kilograms per metre (5 kg/m or 5 kg.m^{-1}), 12 cubic metres (12 m^3) etc. Dimensional Analysis is a neat way to analyse those physical units, enabling us to check the validity of an equation (an equation has just one unknown and an 'equals' sign) or a formula (a formula has two or more unknowns and an 'equals' sign). We can even use dimensional analysis to generate an equation or formula.

Dimensional analysis uses the basic dimensions of **Length (L)**, **Mass (M)** and **Time (T)** to formulate the dimensions of other quantities. For example, the area of a football field can be expressed in square metres, and square metres is 'a length times a length', of course. Let's take a look at some notation for this simple analysis…

$$[Area] = [length \times length] = L^2$$

The use of the square brackets [ ] here denotes 'dimensions of'.

Another example could be 'density'. Let's look at that one…

$$[Density] = \left[\frac{mass}{volume}\right] = \frac{M}{L^3} = ML^{-3}$$

Yet another could be velocity (speed)…

$$[Velocity] = \left[\frac{metres}{seconds}\right] = \left[\frac{length}{time}\right] = \frac{L}{T} = LT^{-1}$$

Let's take a look at some rules which we need to be aware of when undertaking dimensional analysis…

Rule 1:

Constants (i.e. numbers) must be ignored. For example, if we see 3 m as a length then the dimensions are just L. The 3 is ignored.

Rule 2:

Angles must be ignored. An angle cannot be represented with our basic dimensions of length, mass or time.

Rule 3:

If physical quantities are to be added or subtracted, they MUST have the same dimensions. We cannot add apples to oranges, nor can we add voltage to current. However, we can add voltage to voltage.

At this point it is worth noting the dimensions of some common quantities encountered in engineering…

[mass]	M	M
[length]	L	L
[time]	T	T
[area]	[length × length]	L^2
[volume]	[area × length]	L^3
[density]	[mass/volume]	ML^{-3}
[velocity]	[length/time]	LT^{-1}
[acceleration]	[velocity/time]	LT^{-2}
[force]	[mass × acceleration]	MLT^{-2}
[moment of force, torque]	$MLT^{-2} \times L$	ML^2T^{-2}
[impulse]	[force × time]	MLT^{-1}
[momentum]	[mass × velocity]	MLT^{-1}
[work]	[force × distance]	ML^2T^{-2}
[kinetic energy]	[mass × (velocity)^2]	ML^2T^{-2}
[power]	[work/time]	ML^2T^{-3}
[electrical charge]	[length x (force)$^{0.5}$]	$L^{1.5}M^{0.5}T^{-1}$
[current]	[electrical charge/time]	$L^{1.5}M^{0.5}T^{-2}$
[electric potential energy]	[charge^2/distance]	L^2MT^{-2}
[voltage]	[electric potential energy/charge]	$L^{0.5}M^{0.5}T^{-1}$
[capacitance]	[charge/voltage]	L
[inductance]	[voltage x time/current]	$L^{-1}T^2$
[electrical resistance]	[voltage/current]	$L^{-1}T$

Worked Example 1

Ohm's Law relates resistance (R), voltage (V) and current (I) as;

$$I = \frac{V}{R}$$

Use the dimensions of V and R to determine the dimensions of I.

ANSWER

$$[V] = L^{0.5}M^{0.5}T^{-1}$$

$$[R] = L^{-1}T$$

$$\therefore \quad [I] = \left[\frac{V}{R}\right] = \frac{L^{0.5}M^{0.5}T^{-1}}{L^{-1}T} = L^{0.5--1}M^{0.5}T^{-1-1}$$

$$\therefore \quad \boldsymbol{[I] = L^{1.5}M^{0.5}T^{-2}}$$

Worked Example 2

A guitar string has mass (m), length (l) and tension (F, i.e. a force). It is proposed that a formula for the period of vibration (t) of the string might be;

$$t = 2\pi\sqrt{\frac{F}{ml}}$$

Use dimensional analysis to determine whether this formula might be correct.

ANSWER

The first thing to note here is that 2π is just a number, so, according to Rule 1 it should be ignored. We can now use a 'proportionality' ($\propto$) symbol instead of an equals sign to represent the proposition…

$$t \propto \sqrt{\frac{F}{ml}}$$

$$\left[\sqrt{\frac{F}{ml}}\right] = \left(\frac{MLT^{-2}}{ML}\right)^{0.5} = (T^{-2})^{0.5} = T^{-1}$$

This proposed formula has dimensions of T^{-1} which are clearly not the dimensions of time (the period of vibration of the guitar string, which has dimensions of T). Therefore, we have shown, using dimensional analysis, that the proposed answer is invalid.

Worked Example 3

Two positively charged particles, Q_1 and Q_2, are separated by a distance r. Each particle has a mass m. We are unsure whether Q, r and m are all required in an equation to represent the force (F) between the particles. Use dimensional analysis to develop a formula which represents the force between the particles.

ANSWER

We commence with an overview of the quantities which are possibly involved…

F = force $\quad$ **MLT^{-2}**

Q$_1$ = charge $\quad$ **L$^{1.5}$M$^{0.5}$T^{-1}**

Q$_2$ = charge $\quad$ **L$^{1.5}$M$^{0.5}$T^{-1}**

r = distance $\quad$ **L**

m$_1$ = mass $\quad$ **M**

m$_2$ = mass $\quad$ **M**

We may now suggest a general expression regarding the force, F…

$$F = function\{Q, r, m\}$$

Which simply means that force is possibly a function of (or governed by) charge, distance and mass.

To make progress towards a solution, we prefer to represent the problem as follows…

$$F \propto Q^A r^B m^C$$

and then we may say…

$$[F] = [Q^A r^B m^C]$$

Our job will be to find values for those unknowns A, B and C.

We now replace those quantities with their dimensions…

$$MLT^{-2} = (L^{1.5}M^{0.5}T^{-1})^A L^B M^C$$

Now let's bring together those dimensions and powers…

$$MLT^{-2} = L^{1.5A+B}M^{0.5A+C}T^{-A}$$

Now we look at each dimension in turn and equate those powers, left to right…

From L: $\quad$ 1 = 1.5A + B $\qquad\qquad\qquad$ [1]

From M: $\quad$ 1 = 0.5A + C $\qquad\qquad\qquad$ [2]

From T: $\quad$ -2 = -A $\qquad\qquad\qquad\qquad$ [3]

From [3]: **A = 2** $\qquad\qquad\qquad\qquad\qquad$ [4]

Sub [4] into [1]: $\qquad\qquad$ 1 = 1.5(2) + B $\qquad$ so **B = -2**

Sub [4] into [2]: $\qquad\qquad$ 1 = 0.5(2) + C $\qquad$ so **C = 0**

Almost there. Earlier we had…

$$F \propto Q^A r^B m^C$$

Since we know that C = 0 then the 'm' term disappears (anything to the power zero is 1). We also know values for A and B…

$$F \propto Q^A r^B m^C$$

$$F \propto Q^A r^B m^0$$

$$F \propto Q^A r^B$$

$$F \propto Q^2 r^{-2}$$

$$\therefore \quad F \propto \frac{Q^2}{r^2}$$

An important thing to note here; we suspected that the mass of the particles may have had an influence on the force between them. Our dimensional analysis determined that this was not the case – *the mass was irrelevant*. Also, the top of our solution contains a charge squared term, and we know that we had two charges, Q_1 and Q_2, so we can confidently write…

$$F \propto \frac{Q_1 Q_2}{r^2}$$

This result looks remarkably like Coulomb's Law.

1.2   Arithmetic and geometric progressions

Purpose

You will learn about the notation used for sequences, arithmetic and geometric progressions, the limit of a sequence, sigma notation, the sum of a series, arithmetic and geometric series, Pascal's triangle and the binomial theorem. Much of this material has uses in the areas of Digital Electronics, Probability and Statistics.

Terminology

Progression: A sequence of numbers, separated by commas, following a definite pattern

Series: A sequence of numbers, separated by + or - signs, following a definite pattern

Convergent: A series which has a limited sum when an infinite number of terms are added

Divergent: A series which tends to infinity when an infinite number of terms are added

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Sequence Notation and Progressions

You should be aware that many printed and electronic texts tend to blur the distinction between a sequence, progression and series. However, whenever we use the word 'sum' we are talking about a series, since there are addition signs between terms.

You can think of a progression as a sequence of terms which are formed in a regular pattern.

1, 5, 9, 13, 17,... is a progression. In fact, this is an arithmetic progression, since each term is formed by adding a common difference (4) to the previous term.

2, 6, 18, 54, 162,... is a progression. In fact, this is a geometric progression, since each term is formed by multiplying the previous term by a common factor (3).

3, 6, 7, 11, 19,... is NOT a progression, since there is no regular pattern to the numbers.

A progression may contain either an infinite or finite number of terms.

The Limit of a Sequence

For an arithmetic sequence, where the common difference is non-zero, we can clearly see that adding more and more terms will make the sum of the terms (the series) tend to get very large indeed. In fact, if we add an infinite number of terms to such a series then it will tend to $\pm$ infinity (symbol ∞) and is said to be *divergent*.

For a geometric sequence we could find the series tending to a particular value, or possibly tending to infinity. The limiting value will depend upon the common multiplying factor. Such series are said to be *convergent* or *divergent* respectively.

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Sigma Notation

Let's suppose we wish to sum all the integer terms between 1 and 10. The long way to do this would be to write…

$$S_{10} = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10$$

and we would work that out to be 55, right? A shorthand way to express this problem is to say…

$$S_{10} = 1 + 2 + 3 + \cdots + 8 + 9 + 10$$

That saved a bit of time and ink. There is an even better way to express the problem, using Sigma Notation. Here we use the Greek capital letter Sigma (∑) to denote the sum of terms…

$$\sum_{i=1}^{10} i = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10$$

We have simply used the letter i as a convenient counter, nothing more. We can then let i be any number we want between our limits, indicated below and above Sigma as 1 and 10 respectively.

If you were ever asked write down the sum of the first 1000 integers you would not need to write them all out. You would just use Sigma notation, as follows…

$$\sum_{i=1}^{1000} i$$

How neat is that?

For a *geometric* progression we can adopt a similar approach. Suppose we had a geometric progression such as…

$$2, 4, 8, 16, 32, 64, 128, 256$$

It is clear here that each new term is constructed by multiplying the previous term by 2. That multiplier is known as the 'common ratio' and is given the symbol 'r'. We always have a first term, of course, and we give this the symbol 'a'. We could introduce i as a counter again and write the Sigma notation for our series as…

$$\sum_{k=0}^{7} ar^k$$

Let's see whether this works…

$$\sum_{i=0}^{7} ar^k = 2(2)^0 + 2(2)^1 + 2(2)^2 + 2(2)^3 + 2(2)^4 + 2(2)^5 + 2(2)^6 + 2(2)^7$$

You can easily go about evaluating the answer by adding up all eight terms but that would be slow and error-prone.

Video   These videos will explain Progressions and Sigma Notation in a useful way.

2   Functions

Theory

There are six main types of Algebraic Function…

Linear Function   *of type* $\boldsymbol{y = mx + c}$

Quadratic Function   *of type* $\boldsymbol{f(x) = ax^2 + bc + c}$ where $a \neq 0$

Cubic Function   *of type* $\boldsymbol{f(x) = ax^3 + bx^2 + cx + d}$ where $a \neq 0$

Polynomial Function *of type* $\boldsymbol{f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0}$

Rational Function   *of type* $\boldsymbol{f(x) = a(x)/b(x)}$ where $a(x)$ and $b(x)$ are Polynomial Functions

Radical Function   *of type* $\boldsymbol{f(x) = \sqrt[n]{g(x)}}$

Each of these types of algebraic function are commonly used throughout all disciplines of engineering.

2.1   Exponential versus Logarithmic Functions

The logarithmic function is the inverse of the exponential function. You will remember from school work that $1000 = 10^3$ and if we take the logarithm to base 10 of BOTH sides we get…

$$\boldsymbol{log_{10}(1000) = log_{10}(10^3)}$$

You will also know that the logarithm to base 10 of 1000 really means ***raise 10 to what number to give 1000?*** You know that the answer is 3, of course. Let's review what we've just said…

$$\boldsymbol{log_{10}(1000) = log_{10}(10^3) = 3}$$

So the 'log' part has obliterated the '10' part, leaving just that 3. We can therefore confirm that the logarithmic function is the inverse of the exponential function.

Theory

2.2   Reduction of Exponential Laws to Linear Form

Assume you had an equation such as…

$$\boldsymbol{P = Ae^{kh}}$$ (where A and k are constants)

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This is clearly in exponential form (because of the e^{kh} part). To reduce this to linear form we need to take the logarithm of both sides. The natural logarithm is needed, matching the natural constant 'e', so we shall use log_e in our notation…

$$log_e(P) = log_e(Ae^{kh})$$

Employing the laws of logarithms learned at level 3…

$$log_e(P) = log_e(A) + log_e(e^{kh})$$

$$\therefore log_e(P) = kh + log_e(A)$$

This result looks remarkably like our standard linear form $y = mx + c$. Here h is the variable rather than x. Thus, we have managed to reduce the exponential form of an equation to its linear form. Such manipulations will be very useful when you are solving equations or transposing formulae.

Video   Check these videos if you need to revise logarithms.

2.3   Solution of Exponential and Logarithmic Equations

Below is a plot of the curves $f(x) = 2^x$ and $f(x) = 2^{-x}$. You will notice that they exhibit growth and decay respectively.
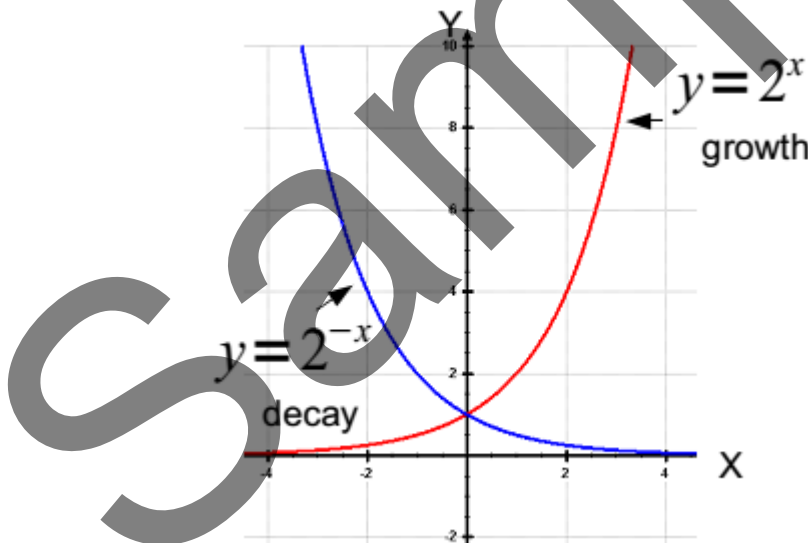
We may choose any base we like and manipulate exponents when multiplied...

$$(4^x)(4^{x+1}) = 4^{(x)+(x+1)} = 4^{2x+1}$$

Choosing another arbitrary base we may see the division rule in action...

$$\frac{6^{2x+4}}{6^{x+3}} = 6^{(2x+4)-(x+3)} = 6^{x+1}$$

Engineers very often meet the exponential function e^x and its variants. The symbol 'e' was chosen by an 18th century mathematician named Leonhard Euler, and is known as 'Euler's number'. The number 'e' is related to many situations in the natural universe, including…

- initial growth/decay of biological organisms
- population growth/decay
- radioactive decay
- a charging/discharging capacitor
- an energising/de-energising inductor
- financial profit/loss

Regarding the last point on this list, consider that you have £1 to invest with a bank. The bank generously allows you to accumulate interest as many times as you like throughout the course of your deposit. Assume that the annual interest rate quoted to you was a generous 100%. You decide that you would like to know which interest period will yield the most money. You can then use the compound interest formula to calculate the value of your investment, in each case, after a full year…

Interest period	calculation	account balance (£)
annual	$\left(1+\frac{1}{1}\right)^1$	2.000000
quarterly	$\left(1+\frac{1}{4}\right)^4$	2.441406
monthly	$\left(1+\frac{1}{12}\right)^{12}$	2.613035
weekly	$\left(1+\frac{1}{52}\right)^{52}$	2.692597
daily	$\left(1+\frac{1}{365}\right)^{365}$	2.714567
each minute	$\left(1+\frac{1}{262800}\right)^{262800}$	2.718277
each second	$\left(1+\frac{1}{15678000}\right)^{15678000}$	2.718282

As you can see, it is more profitable to have interest calculated each second. You will also notice that the amount is reaching a limit. This limit is in fact 'e', which, to 9 decimal places, is 2.718281828. Therefore, there will be no tangible increased profit from calculations every millisecond or microsecond. Now you know why banks like to pay interest annually, rather than more frequently!

So then, what Euler actually discovered was a universal constant of growth and decay. This constant can be applied to the analysis of any *natural* growth or decay scenario. Humans generally have 10 fingers (if you count thumbs as fingers). When you were at school you learned about log_{10} of numbers. The base of 10 was chosen as an obvious human reference number. As an engineer you will almost always discard the base 10 and use the universal constant 'e' as the regular base. You might see this function displayed on your calculator buttons as l_n, log_n or log_e but they are all the same thing.

Let's take a look at the function $y = log_e(x)$…
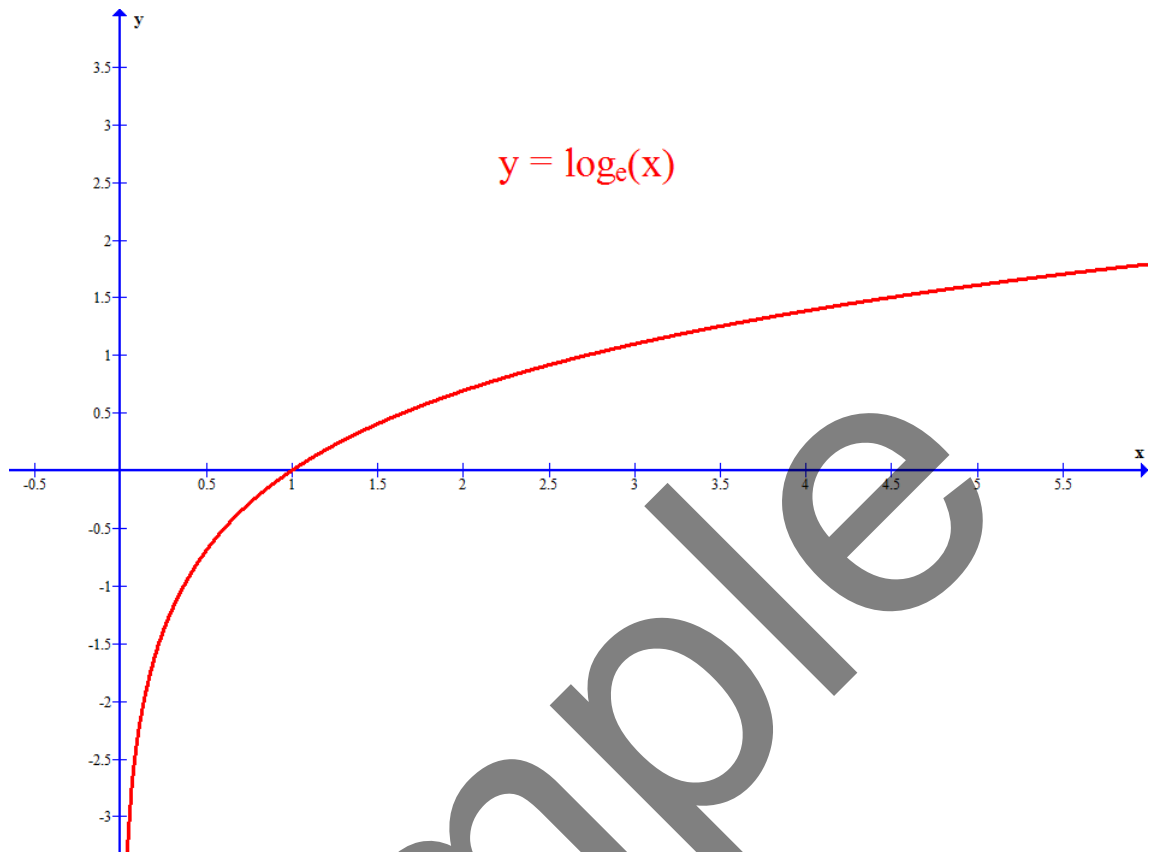
You will notice that $log_e(1) = 0$ since the graph crosses $y = 0$ when $x = 1$. Think of this as $e^0 = 1$, remembering that anything to the power 0 is 1.

A rule of logarithms learned at level 3 is…

$$log_e(x^n) = n\ log_e(x)$$

This rule will be used very often to transpose formulae which you encounter in engineering problems. Let's take a look at one such problem…
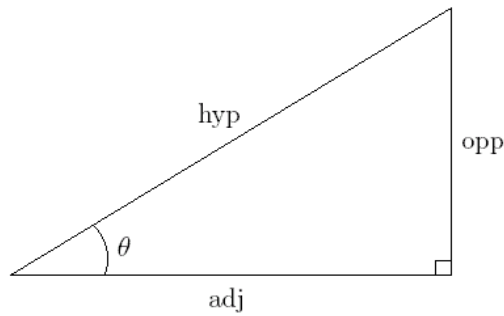
Worked Example 7

The instantaneous voltage (v_c) across a capacitor (C), having an initial peak voltage (V_p), is discharged through a resistor (R). The formula for the voltage across the capacitor is given by…

$$v_c = V_p\ e^{(-t/RC)}$$

Rearrange this formula to find a formula for C.

Our first thought in this problem is that the V_p term may be moved across diagonally…

$$\sin(\theta) = \frac{opp}{hyp}; \quad \cos(\theta) = \frac{adj}{hyp}; \quad \tan(\theta) = \frac{opp}{adj}$$
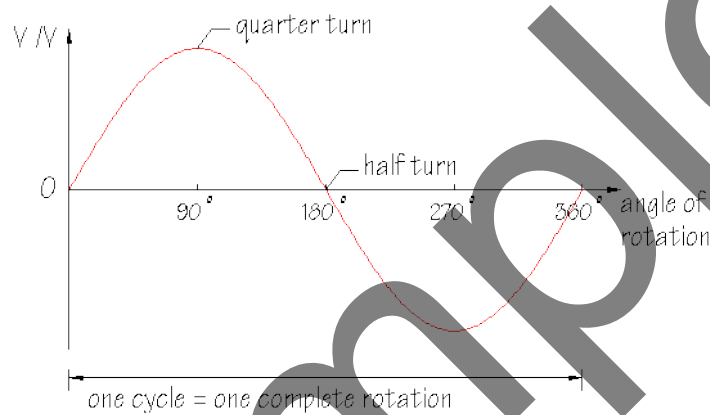
If we say $\sin(90^o) = 1$ then what this means, in relation to the sine wave above, is that the sine function reaches an amplitude of 1 at 90°. This is reading the function from the bottom up and then left. What if we were to read the function from the left first of all and then across and to the bottom? This could be expressed mathematically as…

$$\boldsymbol{sin^{-1}(1) = 90}$$

By the same token we could look at a cosine wave…
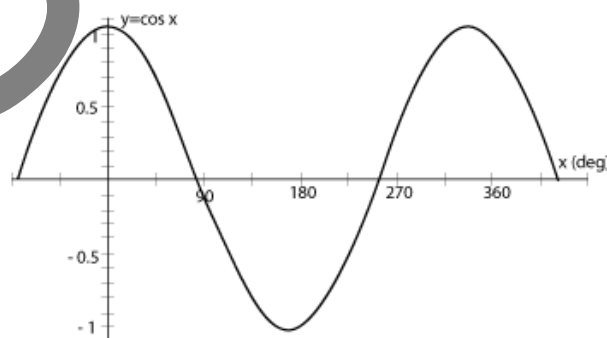
Here we can say…

$$\boldsymbol{cos(0) = 1}$$

and we can inverse this as…

$$cos^{-1}(1) = 0$$

These inverse trigonometric functions are very useful to us because they allow the release of variables which are 'trapped' inside circular functions. When sin^{-1} directly meets sin then the two disappear (handy). The same goes for cos^{-1} directly meeting cos.

> **Worked Example 9**

An instantaneous signal voltage (v_s) is described by the equation…

$$v_s = 10\ sin(2\pi ft - \pi/8)$$

Make f the subject of this formula.

We commence our solution by tidying up a bit and then observing that our target variable f is 'trapped' inside the sine function. Taking sin^{-1} of both sides should free up our f variable…

$$\frac{v_s}{10} = sin(2\pi ft - \pi/8)$$

$$\therefore\ sin^{-1}\left(\frac{v_s}{10}\right) = sin^{-1}(sin(2\pi ft - \pi/8))$$

$$\therefore\ sin^{-1}\left(\frac{v_s}{10}\right) = 2\pi ft - \pi/8$$

$$\therefore\ sin^{-1}\left(\frac{v_s}{10}\right) + \frac{\pi}{8} = 2\pi ft$$

$$\therefore\ f = \frac{sin^{-1}\left(\frac{v_s}{10}\right) + \frac{\pi}{8}}{2\pi t}$$

NOTE: Here the angle involves π. Whenever π is involved in the angle you MUST have your calculator in radians mode (RAD) otherwise you will be in degrees mode and find the wrong answer.

There are many facets of science, engineering and nature where simple trigonometry is an inadequate tool for analysis. Let's move on to look at HYPERBOLIC functions. Hyperbolae arise in many situations, such as...

- The path of a spacecraft in open orbit around a planet and exceeding the planet's escape velocity
- The trace of a sundial shadow

- ▪ The shape of a hanging cable, chain or rope
- ▪ The scattering trajectory of a subatomic particle acted upon by a repulsive force
- ▪ The current at any point along a transmission line exhibiting leakage
- ▪ The shape of an object produced by cutting a vertical section through a cone

A simple hyperbola is produced by plotting the graph of $y = 1/x$...
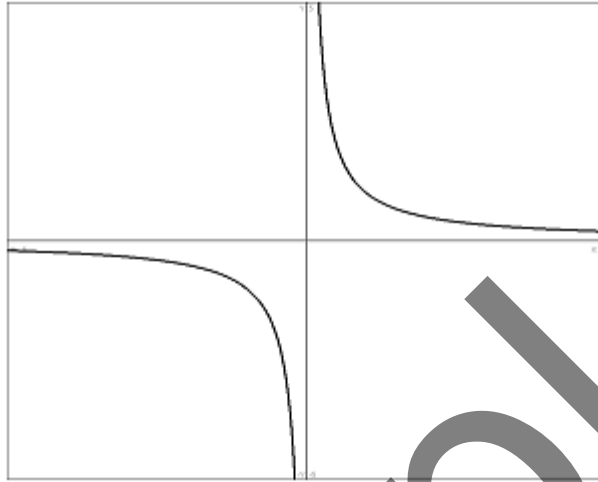
The graphic below helps us to analyse the hyperbola. From this we may define the hyperbolic sine and hyperbolic cosine functions (pronounced 'shine' and 'cosh' respectively). Notice that the apex of the hyperbola is one unit away from the vertical axis (much akin to our unit circle).
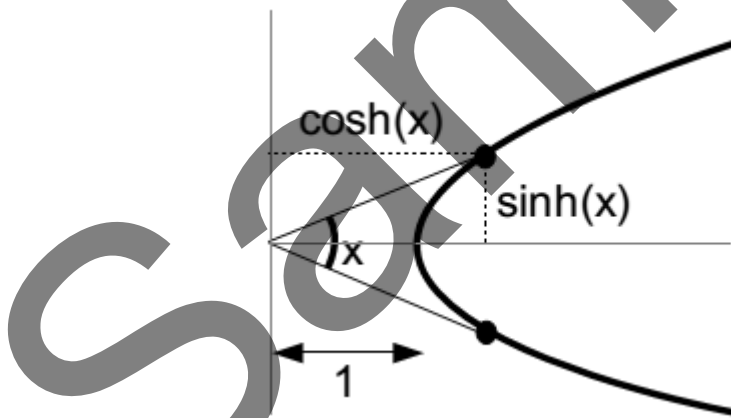
Let's take a look at the combined graphs of $\sinh(x)$ and $\cosh(x)$...
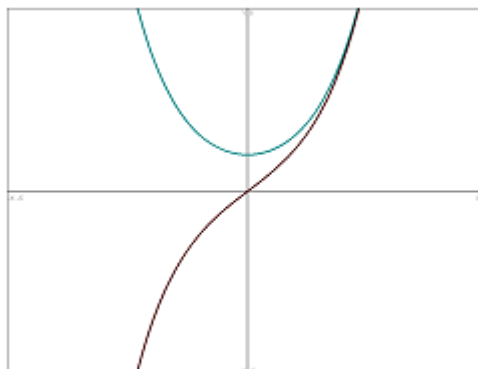
The black curve represents $\sinh(x)$ and the curve resembling a hanging cable represents $\cosh(x)$. An apparently strange thing happens if we add these two graphs together...
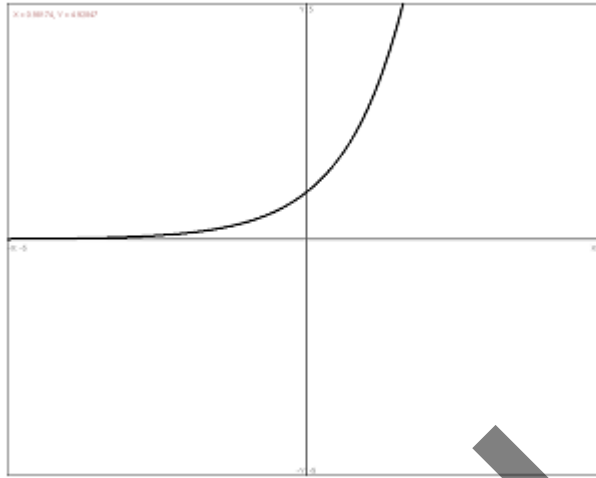
We end up with a graph of e^x, as above. Let's write down mathematically what we have just noticed graphically...

$$e^x = \boldsymbol{cosh}(x) + \boldsymbol{sinh}(x)$$

Now, rather than add the $\cosh(x)$ and the $\sinh(x)$ we can also subtract them. This will give us a graph of e^{-x}. Now we may also write...

$$e^{-x} = \boldsymbol{cosh}(x) - \boldsymbol{sinh}(x)$$

If we add these last two results we get...

$$\boldsymbol{cosh}(x) = \frac{e^x + e^{-x}}{2}$$

and subtracting the second from the first gives...

$$\boldsymbol{sinh}(x) = \frac{e^x - e^{-x}}{2}$$

We may also try $cosh^2(x) - sinh^2(x)$ and realise that this comes out to be 1...

$$\boldsymbol{cosh^2(x) - sinh^2(x) = 1}$$

We can also divide $\sinh(x)$ by $\cosh(x)$ …

$$\boldsymbol{\tanh(x) = \frac{\sinh(x)}{\cosh(x)}}$$

These last four results are highlighted in blue because they are extremely important when you need to take your maths knowledge through to level 5 and beyond. Those topics will involve Complex Numbers, covered in Unit 35: Further Analytical Methods for Engineers.

| Video |

These videos will further boost your understanding of hyperbolic functions.

$$\therefore \ (e^x)^2 - 8e^x - 1 = 0$$

We can't see a straightforward way to find factors here so we must use the quadratic formula which we loved so much back in our school days. Here's a quick reminder for you…

Given $ax^2 + bx + c = 0$ we may solve for x by using the formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

In our problem we have:

$a = 1, b = -8, c = -1$

Let's plug in our values to attempt a solution…

$$e^x = \frac{-(-8) \pm \sqrt{(-8)^2 - 4(1)(-1)}}{2(1)} = \frac{8 \pm \sqrt{68}}{2} = \frac{8 \pm 8.246}{2} = 8.123 \ \ or \ -0.123$$

Taking logs of both sides will give us…

$$x = log_e(8.123) \ \ or \ \ x = log_e(-0.123)$$

That last solution is no good because we cannot take the log of a negative number (not a real one anyway). So, the first one is what we're after…

$$x = log_e(8.123)$$

Trying that on your calculator gives $x = 2.10$, correct to 3 significant figures.

The handy inverse trigonometric functions looked at earlier also have their counterparts in the hyperbolic realm. These are known as $sinh^{-1}, cosh^{-1}$ and $tanh^{-1}$. Let's look at an example…

Worked Example 11

Find x if $sinh(x) = 0.521$

We simply apply the inverse, $sinh^{-1}$, to both sides…

$$sinh^{-1}(sinh(x)) = sinh^{-1}(0.521)$$

$$\therefore \ x = sinh^{-1}(0.521) = 0.5$$