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Mimetic Gravity: Exploring an Alternative Theory of Gravity

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“Take up one idea. Make that one idea your life; dream of it; think of it; live on that idea. Let the brain, the body, muscles, nerves, every part of your body be full of that idea, and just leave every other idea alone. This is the way to success, and this is the way great spiritual giants are produced.”

-Swami Vivekananda-

*“I would like to dedicate this thesis to **Swami Vivekananda**, whose philosophy and ideology provides me the eternal strength to stand up, move forward irrespective of the difficult situation, live a dynamic life, find a better tomorrow. Most of all, it teaches me to ask questions before accepting anything and, above all, to live a life.”*

0.1 Abstract (English)

The origin of the late-time accelerated expansion of the universe is still a great mystery. Numerous cosmological models have been proposed to explain this phenomenon. Modern days' technology and equipment have allowed scientists to successfully execute many observations in cosmology and astrophysics: space missions, large ground-based telescopes and gravitational-wave antennas have led to important discoveries and ruled out many models. The Lambda-Cold Dark Matter, Λ CDM model provides a coherent and satisfactory framework to accommodate all fundamental observations. Therefore it is called the "standard model of cosmology". Despite its many successes, Λ CDM requires the introduction of dark energy in the form of an unnaturally small cosmological constant and is plagued by fine-tuning problems ("why do dark energy, dark matter and baryons have comparable energy densities today?"). The elementary particle candidates which are assumed to form the cold dark matter component have never been directly detected. These facts can be taken as possible indications of a potential crisis. This has motivated the introduction of various alternative models, among which a novel class of modified gravity theories, called "mimetic gravity" or "mimetic dark matter theory", which aims at explaining both the dark energy and (at least part of) the dark matter components as consequences of a suitable modification of the gravitational theory w.r.t. Einstein General Relativity. (Chapter 1 and 2)

In this PhD thesis, we propose the 'generalized mimetic gravity theory', which arises in full generality by means of a non-invertible disformal transformation of the most general single scalar field scalar-tensor theory of gravity and implemented our idea for Horndeski and beyond-Horndeski models.

This novel class of models is a generalization of the so-called mimetic dark matter theory recently introduced by Chamseddine and Mukhanov, as discussed in Chapters 2 and 3. It can source the background evolution of the universe by mimicking any perfect fluid, including radiation, dark matter, and dark energy. In this chapter, we also show that very general single-scalar-field scalar-tensor theories of gravity are generically invariant under invertible disformal transformations.

In Chapter 4 we analyze linear scalar perturbations around a flat Friedmann-Lemaître-Robertson-Walker (FLRW) background in mimetic Horndeski gravity and show that the sound speed is zero on all backgrounds and therefore the system does not have any wave-like scalar degrees of freedom.

Further, we present mimetic vector-tensor theories. In particular, we establish that the non-invertible disformal transformation at the origin of the normalization constraint term in the Einstein-Aether theory, i.e., that the Einstein-Aether theory is also in the class of mimetic theories. We shall also show that an Einstein-Maxwell system sourced by dust can be recovered in the weak limit of a minimal Einstein-Aether theory and that vector field becomes rotation and acceleration free in such a limit (Chapter 5).

Finally, in the concluding Chapter 6, we wind up the thesis by discussing some applications and future research directions in mimetic theories of gravity.

The Chapters 3 and 4 are based on our published papers [2, 3] and Chapter 5 is based on the material which will appear in a forthcoming paper (P. Karmakar, T. Koivisto, D. Mota and S. Mukohyama.)[4].

0.2 Sommario (Italiano)

L'origine dell' accelerazione con cui attualmente l' universo si sta espandendo è ancora uno dei più grandi misteri della cosmologia. Diversi modelli cosmologici sono stati proposti per spiegare questo fenomeno. Le tecnologie e gli strumenti di misura moderni hanno permesso agli scienziati di eseguire con successo molte osservazioni in cosmologia e astrofisica: missioni spaziali, grandi telescopi terrestri e antenne per misurare le onde gravitazionali hanno portato a importanti scoperte ed escluso molti modelli. Il modello cosmologico cosiddetto ' Lambda-Cold Dark Matter ' (Λ CDM) è il modello che meglio spiega in un quadro coerente e soddisfacente tutte le osservazioni fondamentali. Per questo è chiamato il modello "standard della cosmologia". Nonostante i suoi numerosi successi, il modello Λ CDM richiede l'introduzione della cosiddetta energia oscura sotto forma di un' innaturale piccola costante cosmologica ed è afflitto da problemi di 'fine-tuning' ('perchè l' energia oscura, la materia oscura e i barioni hanno densità di energia paragonabili oggi?'). I candidati di particelle elementari che si presume possano formare la componente di materia oscura fredda non sono mai stati rilevati direttamente. Questi fatti possono essere presi come possibili indicazioni di una potenziale crisi. Ciò ha portato all'introduzione di vari modelli alternativi, tra cui una nuova classe di teorie di gravità modificata, detta 'gravità mimetica' o 'teoria della materia oscura mimetica', che mira a spiegare sia l'energia oscura e (almeno parte de) i componenti di materia oscura come conseguenza di un' opportuna modifica della teoria della gravità rispetto alla Teoria della Relatività Generale di Einstein. (Capitolo 1 e 2)

In questa tesi di dottorato, proponiamo la teoria della 'gravità mimetica generalizzata' , che emerge in piena generalità per mezzo di una trasformazione

disforme non-invertibile della teoria scalare-tensoriale della gravità a singolo campo scalare più generale possibile, implementandola poi al caso dei modelli di Horndeski e di modelli che vanno oltre Horndeski. Questa nuova classe di modelli è una generalizzazione della cosiddetta teoria della materia oscura ‘mimetica’, recentemente introdotta da Chamseddine e Mukhanov, come discusso nei capitoli 2 e 3. Essa può far da sorgente all’evoluzione di background dell’universo mimando qualsiasi fluido perfetto, tra cui un fluido di radiazione, di materia oscura e l’energia oscura. In questi capitoli mostriamo anche che teorie scalari-tensoriali della gravità molto generali a singolo campo scalare sono genericamente invarianti per trasformazioni disformi invertibili.

Nel Capitolo 4 analizziamo le perturbazioni scalari lineari intorno ad un background di Friedmann-Lemaître-Robertson-Walker (FLRW) spazialmente piatto nell’ambito della gravità mimetica di Horndeski e dimostriamo che la velocità del suono è nulla su qualsiasi background e pertanto il sistema non dispone di eventuali gradi di libertà scalari che si propagano.

Inoltre, discutiamo teorie mimetiche vettoriali-tensoriali. In particolare, si stabilisce che la condizione di non-invertibilità della trasformazione disforme è all’origine del termine di vincolo di normalizzazione nella teoria di Einstein-Aether, ovvero che la teoria di Einstein-Aether rientra anch’essa nella classe di teorie mimetiche. Si mostrerà anche che un sistema di Einstein-Maxwell con polvere può essere recuperato nel limite debole di una teoria minimale di Einstein-Aether e che il campo vettoriale di questa teoria diventa irrotazionale e senza accelerazione in tale limite (capitolo 5).

Infine, nel Capitolo conclusivo 6, finiamo la tesi discutendo alcune applicazioni e le direzioni future della ricerca in teorie di gravità mimetica.

I capitoli 3 e 4 si basano sulle nostre pubblicazioni [2, 3] e il Capitolo 5 si

basa sul materiale che apparirà in un prossimo articolo (P. Karmakar, T. Koivisto, D. Mota e S. Mukohyama.)[4].

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Chapter 1

Introduction

1.1 Content of the expanding universe

Galaxies, along with planets, stars, clouds of gas and dust etc., in fact everything that we can see, is just the tip of a cosmic iceberg, a small percentage of the mass and energy of the whole universe. Considering the standard model of cosmology, Λ CDM, the recent successful Planck mission confirms that our 13.8 billion years old universe is composed of 68.3% dark energy, 26.8% dark matter, and 4.9% ordinary matter [5, 1]. Briefly put, dark energy or cosmological constant repels, causing the accelerated expansion of the universe, and dark matter attracts, a feature which plays an important role in the structure formation, galaxy formation, and the cosmic microwave background (CMB) anisotropy [5, 1].

In fact we do not know what is the fundamental nature of those two dark components. Dark energy, dark matter (and quantum gravity) are not only three of the biggest problems with regards to Gravitation and Cosmology, but have also secured their place in the top ten of science problems in this century. We explain some of the main issues related to dark energy and dark

matter in Section 1.2.2 and 1.3.

It is well established today that our universe is expanding. A function, a , dubbed scale factor is introduced to describe the relative expansion between two points in the universe. Therefore this scale factor is a function of cosmic time, t , which explains the expansion rate of the universe in the different eras of the evolution of the universe. After the Big Bang, the early universe has expanded very fast, according to the inflationary extension of the Λ CDM model, in the first few fractions of a second, which is called inflation. The expansion rate became relatively slower after inflation. The scale factor dependence was $a \propto t^{1/2}$ during this radiation dominated era. The dark matter dominated the expansion in the subsequent period, and the expansion became $a \propto t^{2/3}$. Currently the accelerated expansion is dominated by a mysterious new form of energy source, with a negative pressure, called dark energy. The expansion is characterized by the Hubble parameter H ,

$$H = \frac{da/dt}{a}. \quad (1.1)$$

According to GR, H is sourced as,

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{\rho}{3} - \frac{K}{a^2}, \quad (1.2)$$

which is called Friedmann equation.

The spatial geometry of the universe can be closed, flat or open, which corresponds, respectively, to $K = +1$, $K = 0$ or $K = -1$ or equivalently $\Omega > 1$, $\Omega = 1$ or $\Omega < 1$ respectively (see, e.g., [6]). K sets the spatial curvature and Ω is the density parameter of the Friedmann universe. The density parameter is the ratio of the observed density, ρ to the critical density, ρ_{crit} . This density parameter is responsible for the curvature of the universe. The

relation between K and Ω is given by

$$\frac{K}{H^2 a^2} = \Omega_m(a) + \Omega_r(a) + \Omega_v - 1, \quad (1.3)$$

where Ω_m , Ω_r and Ω_v refer, respectively, to matter, radiation and vacuum density parameter. One can also define a curvature density parameter, $\Omega_K = -\frac{K}{H^2 a^2}$.

Observations suggest to us that flat spatial geometry is a good approximation, $\Omega_K = 0.000 \pm 0.005$ (95%, Planck TT+lowP+lensing+BAO)[1]. Therefore the spatial curvature, K , can be neglected.

Using the continuity equation of the barotropic perfect fluid with constant equation of state w [7], the energy density is

$$\rho = \rho_0 a^{-3(1+w)}, \quad (1.4)$$

where today's energy density is ρ_0 and scale factor is $a_0 = 1$.

Radiation-dominated era	$\rho \propto a^{-4}$	$a \propto t^{1/2}$
Matter-dominated era	$\rho \propto a^{-3}$	$a \propto t^{2/3}$
Dark energy dominated era	$\rho = const.$	$a \propto e^{Ht}$

In this thesis, we will mainly concentrate on the expansion of the universe, proposing a promising model which may explain the above-mentioned expansion history and we will analyze this model in order to confront it with observations.

In this chapter, we will first explain briefly General Relativity and afterwards some of the challenges that it faces. Then we shall move on to a possible modification to GR in order to confront it to observations. After that we will explain the standard model of cosmology, its consequences and difficulties within the standard model.

1.2 General Relativity

Einstein's description of gravity as a geometric property of spacetime is known as General Relativity (GR) [8]. In his theory, spacetime is quantified as a metric tensor, which is also called metric. This metric tensor is a symmetric tensor of rank two and represents the gravitational potential in the weak field approximation. All the geometric properties of spacetime are encoded in the metric $g_{\mu\nu}$. The distance between two points in curved spacetime is given by the line element constructed by the metric, $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$. The trajectory of a particle in curved spacetime is described by the geodesic equation which is constructed by the metric. In this thesis, we use the metric signature $(-, +, +, +)$.

In Riemannian geometry, the curvature of the manifold is given by the Riemann tensor, $R_{\mu\nu\rho\sigma}$ and the volume of a small wedge of a geodesic ball is given by the contraction of the Riemann tensor by a metric, $R_{\mu\nu} = R^\rho{}_{\mu\rho\nu}$ and the curvature invariant in curved spacetime is given by the Ricci scalar, $R = g^{\mu\nu} R_{\mu\nu}$.

The field equation of General Relativity given by Einstein in terms of the metric, the derivative of the metric up to second order, and the matter sector is

$$G_{\mu\nu} = T_{\mu\nu}, \tag{1.5}$$

where the Einstein tensor, is $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$ and the energy momentum tensor, $T_{\mu\nu}$ which includes the matter sector of the universe. We set the speed of light, c , the geometrical unit, $8\pi G$, and the Planck mass, M_{pl} to unity, i.e., $c = M_{pl} = 8\pi G = 1$. This GR field equation can also be derived from the

Einstein-Hilbert action,

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{2} R(g_{\mu\nu}) + \mathcal{L}_m(g_{\mu\nu}, \dots) \right], \quad (1.6)$$

$$= S_{EH} + S_m, \quad (1.7)$$

where \mathcal{L}_m is the matter Lagrangian and corresponding matter action is S_m .

The properties of the Einstein equation are

1. The Einstein tensor $G_{\mu\nu}$ is a tensor by definition.
2. The Einstein tensor contains up to second order derivatives in the metric.
3. $G_{\mu\nu}$ and $T_{\mu\nu}$ are symmetric tensors.
4. The Bianchi identity ensures the conservation of Einstein's tensor, $\nabla^\nu G_{\mu\nu} = 0$, which also forces the four-divergence of the energy momentum tensor to zero, $\nabla^\nu T_{\mu\nu} = 0$.
5. In the weak field limit, the time-time component of the Einstein equation must be converted to $\partial^2 g_{00} = T_{00}$. This will lead to the relation, $G_{00} = \partial^2 g_{00}$.

The matter action S_m , which is made of the Lagrangian, \mathcal{L}_m is determined by particle physics.

1.2.1 Successes of General Relativity

General Relativity successfully explains phenomena such as the perihelion precession of Mercury, the deflection of light by massive objects like the Sun, the gravitational redshift of light, and energy loss from binary pulsars by emission of gravitational waves. In modern cosmology, General Relativity is

well tested and established on small scales, the solar system scales as well as down to the micrometer ranges. Recently, the LIGO directly detected gravitational waves from the merger of a pair of black holes [9], thus confirming the predictions of General Relativity and enhancing its acceptability.

1.2.2 Challenges in General Relativity

Gravity is everywhere and has to be tested everywhere. The acceleration of the universe was confirmed by the high redshift type Ia supernovae in 1998 and upheld by a few other subsequent observations [10, 11]. The accelerating expansion and the age of the universe puts a strong challenge to the General Relativity. On the other hand, General Relativity is not renormalizable, which limits its ability to be used as a quantum theory of gravity and suggests a need to upgrade it to a UV complete theory.

Despite being the simple and elegant theory, due to the aforementioned reasons General Relativity is not the final result. One may, of course, think to propose an entirely new theory which may explain all the phenomena of the universe but, due to the huge success of General Relativity, it might be smarter to try to modify the General Relativity theory, which may help to provide explanations of large scale phenomena while preserving the behavior of General Relativity in the small scale cases. One possibility would be to introduce the modification in the gravity sector, the spacetime geometry, which is called modified gravity (“modified gravity” as in the modification of General Relativity). A mysterious form of a new fluid or field, which is known as dark energy approach, might be an alternative choice of explanation. However, such energy sources have not been directly detected yet.

By relaxing the property number (5) of the Einstein field equation mentioned previously in Section 1.2, we are also allowed to introduce a pure metric term

multiplied by a constant, $-\Lambda g_{\mu\mu}$ in the Eq. (1.5) and the equivalent term, $\int d^4x \sqrt{-g} \frac{1}{2} (2\Lambda)$ should be added in the action, Eq. (2.2). Λ is called the **cosmological constant**. We shall discuss this in more detail in the following section.

Researchers have tried to modify the gravitational sector by replacing R by a function of R , $f(R)$ [12, 13]. We are allowed to extend GR by replacing the cosmological constant by a scalar field, or a function of the scalar field. Since the proposal of the Brans-Dicke theory [14], several such modifications have been proposed. These general classes of modifications of gravity are called scalar-tensor theory.

The majority of the modifications to General Relativity inevitably introduce additional degrees of freedom (often scalar), and these generally mediate a fifth force [15]. The strength of the coupling between this new degree of freedom to the baryonic field is very tightly constrained by searches for a fifth force and violations of the weak equivalence principle. The strength of the scalar mediated interaction is required to be orders of magnitude weaker than gravity. Tuning this coupling to very small magnitudes introduces additional naturalness problems. We need to find a mechanism, or screening, which can suppress the fifth force mediated by the new degree of freedom, without destroying modification on all scales [16, 17, 18]. Plenty of other observational results also put a tight constraint on many of those [19].

Over the next decade a significant amount of research time, energy, and money will be invested in understanding dark energy: searching for the nature of dark energy lies at the heart of the European Space Agency's PLANCK mission[20] and of the planned EUCLID mission [21, 22], as well as of both SNAP[23] and JDEM[24] of NASA's Beyond Einstein program. EUCLID is of particular interest, as it will be able to constrain a large number

of dark energy and modified gravity models with high precision and accuracy.

1.3 Standard model of cosmology, Λ CDM

Numerous cosmological models have been proposed to explain the acceleration of the universe since its discovery, that was made by observing supernovae type Ia in 1998. Thanks to the modern observations, such as the WMAP and Planck missions, many of those proposed models have now been ruled out.

The so-called Lambda-Cold Dark Matter (Λ CDM) model of cosmology has been very successful at explaining all cosmological observations with a minimal set of six cosmological parameters. These six parameters have recently been measured to an unprecedented accuracy with the Planck satellite [5, 1]. Several deviations from this simple model have been constrained to be relatively small [5, 1, 25, 26, 27, 28]. This model represents what we call the standard model of cosmology.

The values of the six cosmological parameters given by the latest Planck observation are given [1],

where r_s is the sound horizon and $D_A(z_*)$ is the comoving angular diameter distance to last scattering.

The Λ CDM model has two components, Λ , which is known as the cosmological constant (introduced in the previous section 1.2.2) and CDM, which is known as cold dark matter. The presence of the later component has been confirmed by different cosmological and astrophysical observations, and possible explanations are based on ideas from particle physics and astrophysical observations.

Parameter	Symbol	Value
Physical baryon density parameter	$\Omega_b h^2$	0.02230 ± 0.00014
Physical dark matter density parameter	$\Omega_c h^2$	0.1188 ± 0.0010
100 x approximation to r_s/D_A (CosmoMC)	$100\theta_{MC}$	1.04093 ± 0.00030
Reionization optical depth	τ	0.066 ± 0.012
Log power of the primordial curvature perturbations ($k_0 = 0.05$) Mpc^{-1}	$\ln(10^{10} A_s)$	3.064 ± 0.023
Scalar spectral index	n_s	0.9667 ± 0.0040

Table 1.1: Parameter 68% confidence limits for the base Λ CDM model from Planck CMB power spectra, in combination with TT,TE,EE+lowP+lensing+ext [1].

1.3.1 Cosmological constant problem

There is a huge discrepancy between the theoretical predictions and the calculated value from the observations of the cosmological constant, which is known as the “cosmological constant problem”.

The old and the new cosmological constant problem

This problem is classified into two categories. The strong or “old” cosmological constant (CC) problem is why the vacuum energy density, ρ_{vac} (or ρ_Λ) is small and positive, and the weak, or “new” CC problem is why Λ is non-zero and exists at all [29].

As per the latest Planck satellite measurement, [1], the dark energy density parameter is $\Omega_\Lambda = 0.6911 \pm 0.0062$ and the Hubble constant today is $H_0 = 67.74 \pm 0.46 \text{ km Mpc}^{-1} \text{ s}^{-1}$. We know that the critical density (in natural units), $\rho_{crit} = 3H_0^2 \simeq 8.62 \times 10^{-27} \text{ kg/m}^3$. Therefore, the observed energy density of Λ is given by

$$\rho_{vac}^{obs} = \Omega_{\Lambda} \rho_{crit}, \quad (1.8)$$

$$\approx 5.96 \times 10^{-27} \text{kg/m}^3 \quad \text{or} \quad 8.16 \times 10^{-47} \text{GeV}^4. \quad (1.9)$$

In quantum field theory, the vacuum energy density or zero point energy depends on the frequency interval of the field modes (see for example [30]), and is given by,

$$\rho_{vac}^{QM} = \frac{E}{V} = \frac{1}{V} \sum_k \frac{1}{2} \hbar \omega_k \approx \frac{\hbar}{2\pi^2 c^3} \int_0^{\omega_{max}} \omega^3 d\omega = \frac{\hbar}{8\pi^2 c^3} \omega_{max}^4, \quad (1.10)$$

where ω_{max} is the maximum frequency of the considered frequency interval. For example, if we consider the electroweak scale, then the approximated vacuum energy density is $\rho_{vac}^{EW} \sim 10^8 \text{GeV}^4$, which is $\sim 10^{55}$ times larger than the observed value, and if we consider the Planck energy scale, then the approximated vacuum energy density is $\rho_{vac}^{Planck} \sim 10^{76} \text{GeV}^4$, which is approximately 10^{123} times larger than the observed value given in Eq. (1.9) [31, 32, 33]. Therefore, there is a huge discrepancy between the observed value of the CC and its order of magnitude theoretical estimate.

Of course, if we manage to understand the first issue, why it is so small, then we will be likely to comprehend its value partially.

Coincidence problem

Another problem related to the CC is why this CC appears to be dominating the evolution today (why “now”?) [29, 34]. The ρ_{vac} and ρ_{matter} evolve very differently with a , however they have comparable value “today”. It raises the question on the dynamics of Λ .

Classical problems with CC

While there exists the aforementioned quantum problem of CC, there are also some serious unanswered classical questions [29]. If we introduce the cosmological constant in the GR equations, then is there any geometrical meaning of it? Also, the relation of the cosmological constant, Λ , with Newton's constant, G , is also unclear.

1.3.2 Cold dark matter (CDM)

The dominant dark matter component in Λ CDM is referred to the cold dark matter (CDM). We can describe well all cosmological observations by including CDM in our standard model of cosmology, which is unexplainable without CDM. The pressure of such cold dark matter is negligibly small. This dark matter component is assumed to be cold, collisionless and does not interact with the other particles of the standard model of particle physics except gravitationally. It amounts to about one quarter of the total energy density budget of the current universe [1]. Such hypothetical non-baryonic, almost pressureless, clustered fluid is necessary to explain the observed structure formation of the universe, galaxy clustering and acoustic oscillation in the CMB, large scale structure etc.

Regardless the success of the Λ CDM model to explain observations, dark energy and dark matter are the two major unknown ingredients of this model. As a consequence of introducing dark energy, the well-known, unavoidable cosmological constant problem and the fine tuning problem emerge. On the other hand, no cold dark matter particles have been experimentally found either on earth or in space [35, 36]. These reasons motivated us to modify the laws of gravitation without introducing new energy sources, which might

be a possible alternative to the Λ CDM model.

Chapter 2

Alternative theories of gravity: the mimetic gravity scenario

2.1 Motivation of mimetic dark matter

The indications of a crisis with regards to appropriate cosmological models and the presence of unknown energy components, as explained in the previous chapter, have motivated many studies that try to explain the phenomena that they give rise to, by modifying the law of gravitation without introducing new energy sources, see e.g. [37, 38]. In this chapter, we will explain an alternative model to the Λ CDM model, called mimetic gravity, and its major ingredient, disformal transformations.

In 2013, Chamseddine and Mukhanov introduced a modification to General Relativity, called “mimetic dark matter”, reformulating it in terms of an auxiliary metric which is conformally related to the original “physical” metric, where the conformal factor is a certain function of the new metric and the first derivative of a scalar field [39] (the term “conformal” is explained in the next section, 2.2). In these new variables, the conformal degree of freedom

becomes dynamical even in the absence of matter. This mimics the phenomenon of cold dark matter. In their subsequent article [40], they showed that, with an additional potential for the new scalar field, the scalar field can mimic the gravitational behavior of any form of matter (see also [41]). Those results are valid for the general disformal transformation too [42] (the term “disformal” is explained in section 2.3).

In the next chapter, we shall explain the “generalized mimetic gravity”.

The name “mimetic” was given because this model explains and “mimics” the cold dark matter evolution of the universe in the background. In our “generalized mimetic gravity”, the “generalized” term came about because it is the generalization of the “mimetic dark matter” model, which can source the background evolution of the universe by mimicking any perfect fluid, including radiation, dark matter, and dark energy. We generalized the Einstein action by considering the most general scalar-tensor theory and extended the conformal transformation by considering general disformal transformations.

2.2 Simple model of mimetic dark matter

In a conformal transformations, the “physical metric”, $g_{\mu\nu}$, is proportional to the “auxiliary metric”, $\ell_{\mu\nu}$, by a function, $f(x)$ as given below,

$$g_{\mu\nu} = f(x)\ell_{\mu\nu}, \quad (2.1)$$

where $f(x)$ is some specified function of the spacetime coordinates.

In Ref. [39], the authors performed a particular conformal transformation with $f(x) = -w$ on the Einstein-Hilbert (EH) action,

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{2} R(g_{\mu\nu}) + \mathcal{L}_m(g_{\mu\nu}, \dots) \right], \quad (2.2)$$

$$w = \ell^{\alpha\beta} \partial_\alpha \varphi \partial_\beta \varphi, \quad (2.3)$$

where φ is a scalar field.

One may compute the alternative identity of that conformal transformation,

$$g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi = -1. \quad (2.4)$$

The above identity can be incorporated into the original EH action by the Lagrange multiplier method,

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{2} R(g_{\mu\nu}) + \mathcal{L}_m(g_{\mu\nu}, \dots) + \lambda (-g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - 1) \right]. \quad (2.5)$$

One may derive the EOM of the new metric field, $\ell_{\mu\nu}$,

$$(G^{\mu\nu} - T^{\mu\nu}) + (G - T) g^{\mu\alpha} g^{\nu\beta} \partial_\alpha \varphi \partial_\beta \varphi = 0. \quad (2.6)$$

G and T are the trace of the $G^{\mu\nu}$ and $T^{\mu\nu}$.

In these new variables, $g_{\mu\nu}$ and φ , the theory becomes invariant under the Weyl rescaling of the new metric, $\ell_{\mu\nu}$, and therefore provides traceless equations of motion.

The extra contribution from the gravitational sector as the modified energy momentum tensor, can be redefined as, $\tilde{T}^{\mu\nu}$, so that the Eq. (2.6) can be written in terms of two different energy momentum tensors,

$$G^{\mu\nu} = T^{\mu\nu} + \tilde{T}^{\mu\nu}, \quad (2.7)$$

where

$$\tilde{T}^{\mu\nu} = -(G - T) g^{\mu\alpha} g^{\nu\beta} \partial_\alpha \varphi \partial_\beta \varphi. \quad (2.8)$$

Comparing it with the expression of a pressure-less perfect fluid, the energy density, ε and normalized four velocity u^μ are,

$$\varepsilon = G - T, \quad (2.9)$$

$$u^\mu = g^{\mu\rho} \partial_\rho \varphi, \quad (2.10)$$

respectively. Using the conservation of the energy momentum tensor, one can show that,

$$\nabla_\mu [(G - T) \partial^\mu \varphi] = 0. \quad (2.11)$$

The cosmological solution of Eq. (2.6) for a flat Friedmann-Lemaître-Robertson-Walker (FLRW) background (in the synchronous gauge) is,

$$G - T \propto \frac{1}{a^3}, \quad (2.12)$$

where a is the scale factor.

Therefore, the scale factor behaves as in the dark matter dominated era in the background without “dark matter.” This switches on a new conformal degree of freedom of gravity, which behaves as an irrotational pressureless perfect fluid, i.e. it can mimic a cold dark matter component. In this model, the observed cold dark matter energy density would, in general, be the sum of two unknown amounts of energy density contributions, one coming from hypothetical dark matter particles and the other from the “mimetic” dark matter, which is only a gravitational effect. In a subsequent work [40] it was shown that by introducing a potential for the new scalar field, one can mimic the gravitational behavior of almost any form of matter (see also [41] for earlier work).

Refs. [40, 43, 44] considered the effects of higher-derivative interactions on the cosmology of the “mimetic” dark matter model. In [43], the authors showed that the energy-momentum tensor of the mimetic theory with higher-derivatives is actually of the imperfect fluid type and the theory can support vorticity. Ref. [44] argues that these higher-derivative interactions could possibly help to solve the small-scale problems of the cold dark matter model. Ref. [45] proposed an alternative conformal extension of General Relativity, using a vector field, that can also support rotational flows for the mimetic dark matter.

The stability of the “mimetic” dark matter model was analyzed in [45], where it was shown that the positiveness of the energy density of the fluid is a sufficient condition for the absence of ghost instabilities. The puzzle of why a simple reparametrization of variables can lead to new additional solutions of the equations of motion was also explained in [45]. The point is that we have additional solutions because we are doing a non-invertible conformal transformation.

2.3 Disformal transformation

The conformal relation was quite popular in many research fields including the Brans-Dicke theory and string theory. In 1993, Bekenstein noticed that the relation between physical and gravitational geometry can be generalized by adding the derivative of the scalar field.

The disformal transformation is expressed as the sum of a conformal function times the auxiliary metric and a disformal function times a term involving

the first derivative of a scalar field [46].

$$g_{\mu\nu} = A(\varphi, w)\ell_{\mu\nu} + B(\varphi, w)\partial_\mu\varphi\partial_\nu\varphi, \quad (2.13)$$

where w is defined as

$$w \equiv \ell^{\rho\sigma}\partial_\rho\varphi\partial_\sigma\varphi. \quad (2.14)$$

A and B are conformal and, respectively, disformal functions of two variables. $g_{\mu\nu}$ is the original metric and $\ell_{\mu\nu}$ is an auxiliary new metric. φ is a scalar field that defines the transformation.

2.3.1 Properties of disformal transformation and function

The transformation given in Eq. (2.13) will be called pure conformal if the disformal function, $B = 0$, and pure disformal if the conformal function is unit and the disformal function is arbitrary general, i.e., $A = 1$ and $B = B(\varphi, w)$.

The properties and conditions of the free functions were explained in detail in Ref. [47].

The transformation should preserve the Lorentzian signature. The time-time component of the Eq. (2.13)

$$g_{00} = A(\varphi, w)\ell_{00} + B(\varphi, w)\partial_0\varphi\partial_0\varphi < 0. \quad (2.15)$$

The above relation should hold for any value of fields and derivatives. So B can also take zero value. The above relation would be true if $A > 0$, which also holds in scalar-tensor theory.

Contraction of Eq. (2.15) with g^{00} (which is negative as per our signature

convention) will lead to

$$A(\varphi, w) + B(\varphi, w)w > 0. \quad (2.16)$$

Therefore, one of the functions should always have the kinetic dependency when the disformal function B is non-zero [48]. However author argued in [49] that, consistent models can avoid violating it without explicit kinetic dependence of the disformal function(s).

To have causal behavior, the line element must be less than zero [50].

The inverse disformal transformation of Eq. (2.13) is

$$g^{\mu\nu} = \frac{1}{A(\varphi, w)}\ell^{\mu\nu} + \frac{B(\varphi, w)/A(\varphi, w)}{A(\varphi, w) + 2wB(\varphi, w)}\partial^\mu\varphi\partial^\nu\varphi. \quad (2.17)$$

The transformation for the inverse metric and the volume element should be non-singular. However, this condition will not impose any new constraint besides the aforementioned constraints.

2.4 Disformal transformation behind mimetic gravity

In [42], the issue of why new extra solutions in the mimetic dark matter are introduced by a reparametrization of variables by Chamseddine and Mukhanov was revisited from a different viewpoint. They performed the full disformal transformation of the type, Eq. (2.13) on the EH action Eq. (2.2). The Jacobian of the system can be derived from the set of the equations of motion

in the new frame under such diffeomorphism,

$$\det = w^2 A \frac{\partial}{\partial w} \left(B + \frac{A}{w} \right). \quad (2.18)$$

The Einstein equation, $G_{\mu\nu} = T_{\mu\nu}$ can be recovered when the determinant is non-zero (generic case). It is established in a clear and elegant way that Einstein's theory of General Relativity is invariant under generic disformal transformations. See also for example, [51] and references therein.

However, there exists a particular subset (when the determinant of the system is zero) of the previous general case, such that the resulting equations of motion are no longer the general relativistic equations, but instead one finds the equations of motion of the so-called ‘‘mimetic’’ dark matter model (also called ‘‘mimetic gravity’’) [39]. They also showed that the transformation used in the Chamseddine and Mukhanov's article is a special type of disformal transformation with $A \equiv w$ and $B = 0$.

Therefore, the disformal transformation plays a crucial role in mimetic dark matter.

In [52] (see also [45, 40]), it was shown that the equations of motion of mimetic gravity can be derived by extremizing, with respect to $g_{\mu\nu}$, the Einstein-Hilbert action with the addition of the term $\int d^4x \sqrt{-g} \lambda (g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - 1)$, where λ is a new field playing the role of a Lagrange multiplier.

The invariance of cosmological perturbations under disformal transformations has been recently studied in, for example, [53, 54, 55, 56, 57, 58] and mimetic theories of modified gravity have been considered in [59] and references therein.

Disformal transformations became of recent interest in other related areas of cosmology.

The authors of Ref. [47] showed that the Horndeski ‘‘structure’’ would be

invariant under the disformal transformation (the action will remain in Horndeski structure after field redefinition), if the arbitrary functions in the disformal transformation are only restricted to be functions of the scalar field, i.e., $A \equiv A(\varphi)$ and $B \equiv B(\varphi)$. In other words, if we perform such a restricted disformal transformation on the full Horndeski action, we will be still able to ensure the second order equations of motion.

However, the authors of Ref. [60, 61] extended the Horndeski theory to beyond Horndeski by applying a disformal transformation with $A \equiv A(\varphi)$ and $B \equiv B(\varphi, w)$ partially on the Horndeski action such that the equations of motion will be second order, i.e., free of Ostrogradski instabilities.

In GR the scalar field may strongly couple with gravity by disformal transformation; this is called disformal coupling. In Ref. [62], the authors recovered GR in the range of solar system scales, when the scalar field is static and smooth. This is called a disformal screening mechanism.

Chapter 3

The two faces of mimetic Horndeski gravity

3.1 Introduction

In this chapter, we will generalize some results of [52, 45, 42] to a very general scalar-tensor theory of gravity. Our results will be valid for a very general theory, however, for concreteness one may think of the scalar-tensor theory as being the most general second-order theory known as the Horndeski model [63], which is free from Ostrogradski instability (see also [64] for a recent rederivation and [65] for another proof of equivalence with the original formulation of Horndeski). One may also extend our formalism to the recently proposed extensions of the Horndeski model, the so-called G^3 theories [60, 61] or even their extensions [66].

This chapter is organized as follows. First we shall explain the general Horndeski action. In the next section, we will show under which conditions the previous disformal transformation is non-invertible. Then we will show that very general scalar-tensor theories of gravity are invariant under generic dis-

formal transformations. For a particular special subset of those generic disformal transformations the invariance is broken and one finds new equations of motion which are a generalization of the so-called “mimetic” dark matter theory. We will show that the invariance is broken exactly for transformations that satisfy the non-invertibility condition. In section 3.4, we will demonstrate that the new mimetic general scalar-tensor theory equations of motion can also be derived by the use of a Lagrange multiplier as in the General Relativity case. We also briefly comment on the higher-derivative nature of the resulting equations. In section 3.5, we shall present some applications of our results. For instance we will show that the simplest mimetic scalar-tensor model is able to mimic the cosmological background of a flat FLRW model with a barotropic perfect fluid with an arbitrary equation of state. Finally, section 4.5 is devoted to the conclusions.

3.2 Galileon and Horndeski actions

A generalization of the 4D effective action of Dvali, Gabadadze, and Porrati (DGP) [67] was introduced to explain the accelerated expansion of the Universe without introducing any dark energy nor cosmological constant, which is called the galileon theory [68]. It was claimed that this infrared modification of gravity suffers from ghost instability on the “self-accelerated” de Sitter branch. This theory has an internal “Galilean” invariance, or a shift symmetry,

$$\varphi(x) \rightarrow \varphi(x) + b_\mu x^\mu + c. \quad (3.1)$$

In the effective theory, under this constant shift, c , to the scalar field, φ represents a Goldstone boson, the vectorial parameter, b_μ , corresponds to a constant shift of the gradient of the scalar field, $\partial_\mu \varphi$.

The Galilean Lagrangian term is given as [68],

$$\mathcal{L}_{n+1}^G = (T^{\mu_1 \dots \mu_n \nu_1 \dots \nu_n} \partial_{\mu_1} \varphi \partial_{\nu_1} \varphi) \partial_{\nu_2} \partial_{\mu_2} \varphi \dots \partial_{\nu_n} \partial_{\mu_n} \varphi, \quad (3.2)$$

where $T^{\mu_1 \dots \mu_n \nu_1 \dots \nu_n}$ is the antisymmetric tensor over $\mu_i \leftrightarrow \nu_i$ and $n \leq 5$ for four dimensions.

The flat space Galilean theory guarantees the second order derivatives in the equation of motion, hence free from the Ostrogradski instability. This theory is unitary and stable under quantum corrections.

Although the Galilean Lagrangian was originally introduced in Minkowski space, it can be extended to general curved spacetime by replacing the partial derivatives to covariant derivatives. Although, the simplest covariantization ($\partial \rightarrow \nabla$) led to higher than second order derivatives (third order) in the field equations of the scalar field. However, the authors in Ref. [69, 70] showed that one could eliminate these higher derivatives by introducing suitable nonminimal, curvature, couplings. Although that lead to breaking of shift symmetry. However, shift symmetry is not meaningful because we cannot define (covariantly) a constant vector in a curved spacetime. The explicit form of the covariant Galilean Lagrangian is [70, 71, 26],

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{2} R + \sum_{n=0}^3 \mathcal{L}_n \right]. \quad (3.3)$$

The four Lagrangian densities for the scalar field are:

$$\mathcal{L}_0^G = c_2 X, \quad (3.4)$$

$$\mathcal{L}_1^G = -2(c_3/\Lambda^3)X\Box\varphi, \quad (3.5)$$

$$\mathcal{L}_2^G = 2(c_4/\Lambda^6)X [(\Box\varphi)^2 - (\nabla_\mu\nabla_\nu\varphi)^2] + (c_4/\Lambda^6)X^2R, \quad (3.6)$$

$$\begin{aligned} \mathcal{L}_3^G &= -2(c_5/\Lambda^9)X [(\Box\varphi)^3 - 3\Box\varphi(\nabla_\mu\nabla_\nu\varphi)^2 + 2(\nabla_\mu\nabla_\nu\varphi)^3] \\ &\quad + 6(c_5/\Lambda^9)X^2G_{\mu\nu}\nabla^\mu\nabla^\nu\varphi. \end{aligned} \quad (3.7)$$

where $X = -\frac{1}{2}\nabla_\mu\nabla^\mu\varphi$. Λ is a constant with the dimension of mass. The coupling coefficients c_n may also includes the potential term.

One may generalize the above action, replacing the model parameters, c_i by general function of φ and its kinetic energy and simplifying it by integration by parts, which lead to an action the so called Horndeski model [64].

The most general class of 4D local scalar-tensor theories that contains second-order equations of motion and that can be derived from an action is known as the Horndeski theory [63]. Its action is

$$S_H = \int d^4x\sqrt{-g}\mathcal{L}_H = \int d^4x\sqrt{-g}\sum_{n=0}^3\mathcal{L}_n, \quad (3.8)$$

where

$$\mathcal{L}_0 = K(X, \varphi), \quad (3.9)$$

$$\mathcal{L}_1 = -G_3(X, \varphi)\Box\varphi, \quad (3.10)$$

$$\mathcal{L}_2 = G_{4,X}(X, \varphi) [(\Box\varphi)^2 - (\nabla_\mu\nabla_\nu\varphi)^2] + RG_4(X, \varphi), \quad (3.11)$$

$$\begin{aligned} \mathcal{L}_3 &= -\frac{1}{6}G_{5,X}(X, \varphi) [(\Box\varphi)^3 - 3\Box\varphi(\nabla_\mu\nabla_\nu\varphi)^2 + 2(\nabla_\mu\nabla_\nu\varphi)^3] \\ &\quad + G_{\mu\nu}\nabla^\mu\nabla^\nu\varphi G_5(X, \varphi). \end{aligned} \quad (3.12)$$

The functions $K(X, \varphi)$, $G_3(X, \varphi)$, $G_4(X, \varphi)$ and $G_5(X, \varphi)$ are free functions of their two variables and define a particular theory in the Horndeski class. The subscript $,_X$ denotes derivative with respect to X .

The above action has been considered as the most general single scalar field action which is free from Ostrogradski ghosts. However recent studies in [60, 66, 72] say that there can be a possibility to extend the Horndeski action without introducing the ghosts. The questions of which action should be the “most general” single scalar-tensor theory is still an open question up to date.

3.3 Mimetic gravity from a disformal transformation

In this section, we will consider disformal transformations of very general scalar-tensor theories of gravity. We will show that these theories are invariant under generic disformal transformations. However for a special subset of non-invertible disformal transformations the theory is modified resulting in new equations of motion which may possess novel solutions.

3.3.1 Non-invertibility condition of a disformal transformation

In this first subsection we will derive what is the condition for non-invertibility of a disformal transformation of the type

$$g_{\mu\nu} = A(\varphi, w)\ell_{\mu\nu} + B(\varphi, w)\partial_\mu\varphi\partial_\nu\varphi, \quad (3.13)$$

where w is defined as

$$w \equiv \ell^{\rho\sigma}\partial_\rho\varphi\partial_\sigma\varphi. \quad (3.14)$$

A and B are arbitrary functions (see footnote 1) of two variables. $g_{\mu\nu}$ is the original metric and $\ell_{\mu\nu}$ is an auxiliary new metric. φ is a scalar field that defines the transformation. In this section, we assume that φ is the same scalar field that is present in the action of the scalar-tensor theory. In appendix A.1 we will consider the case when the disformal transformation introduces a different new field. The issue of the non-invertibility for a conformal transformation in the context of "mimetic" gravity was first discussed in [45] and here we will generalize their arguments to disformal transformation in scalar-tensor theories. The conditions under which disformally coupled theories can be re-written in the so-called Jordan frame were studied in [51].

The inverse of $g_{\mu\nu}$ can be written as

$$g^{\mu\nu} = \frac{1}{A(\varphi, w)} \ell^{\mu\nu} + \frac{B(\varphi, w)}{B(\varphi, w)g^{\rho\sigma}\partial_\rho\varphi\partial_\sigma\varphi - 1} g^{\mu\alpha}g^{\nu\beta}\partial_\alpha\varphi\partial_\beta\varphi, \quad (3.15)$$

where $B(\varphi, w)g^{\rho\sigma}\partial_\rho\varphi\partial_\sigma\varphi - 1 \neq 0$ for obvious reasons. Using the previous equation one finds $\ell_{\mu\alpha}\ell^{\alpha\nu} = \delta_\mu^\nu$.

Eq. (3.13) is a convolved transformation law for $g_{\mu\nu}$ in terms of $\ell_{\mu\nu}$ because $\ell_{\mu\nu}$ enters in w . In other words, for a fixed φ in Eq. (3.13), one can see that in order to write $\ell_{\mu\nu}$ in terms of $g_{\mu\nu}$ one needs to solve w in terms of $g_{\mu\nu}$. Here we are assuming that φ is not a new variable (i.e. it is already present in the action). Despite this assumption, in the following subsection we will show that the condition we find here for non-invertibility of the transformation is the same as the condition found in the next subsection for the system of equations of motion to be indeterminate. This later condition is valid independently of the assumption that φ is a field already present in the action.

Using Eq. (3.15) one can show that

$$w = \frac{A(\varphi, w)g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi}{1 - B(\varphi, w)g^{\alpha\beta}\partial_\alpha\varphi\partial_\beta\varphi}. \quad (3.16)$$

The previous equation can be written as

$$G(\varphi, w) = g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi, \quad (3.17)$$

where the function $G(\varphi, w)$ is defined as

$$G(\varphi, w) \equiv \frac{w(1 - B(\varphi, w)g^{\alpha\beta}\partial_\alpha\varphi\partial_\beta\varphi)}{A(\varphi, w)}. \quad (3.18)$$

For a fixed given φ and using the inverse function theorem, if $\frac{dG(\varphi, w)}{dw}|_{w=w_*} \neq 0$ then the inverse function G^{-1} exists in a neighborhood of w_* so one can write w as a function of $g_{\mu\nu}$ only as $w = G^{-1}(g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi)$. Finally one can use Eq. (3.13) (or Eq. (3.15)) to write $\ell_{\mu\nu}$ as a function of $g_{\mu\nu}$. This completes the proof that the inverse transformation, i.e. $\ell_{\mu\nu}(g_{\alpha\beta})$, exists. Furthermore, the non-existence of G^{-1} implies that $\frac{dG(\varphi, w)}{dw}|_{w=w_*} = 0$. One can solve this as

$$G(\varphi, w) = 1/b(\varphi), \quad (3.19)$$

where in the right-hand-side we wrote $1/b(\varphi)$ to use the same conventions of notation as in the literature.

If we are in the exceptional case of the previous equation then the transformation from $g_{\mu\nu}$ to $\ell_{\mu\nu}$ cannot be inverted even implicitly and from Eq. (3.17) one can find that

$$b(\varphi) = \frac{1}{g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi}. \quad (3.20)$$

The previous equation can be used with Eq. (3.16) to find

$$B(\varphi, w) = -\frac{A(\varphi, w)}{w} + b(\varphi). \quad (3.21)$$

This condition for having a non-invertible transformation is the same as the condition that Deruelle and Rua [42] found for the system of equations of motion of mimetic dark matter to be indeterminate. We will generalize their analysis in the next subsection. Note that here we never assumed any explicit scalar-tensor theory so this result is very general. Eq. (3.20) is a kinematical constraint valid independently of the dynamics. Furthermore, the results of this subsection also explain why it is not so surprising that the transformed scalar-tensor theory, i.e. mimetic gravity, may contain new solutions with respect to the original theory. The reason is that we are performing a non-invertible change of variables.

3.3.2 Disformal transformation method

In this subsection, we will perform a disformal transformation of the type (3.13) on a very general scalar-tensor theory and compute the equations of motion that result. This is a generalization of the results in Deruelle and Rua [42]. The further generalization for the case when the transformation field is different from the scalar field in the action is discussed in Appendix A.1. This is the case in the ‘‘mimetic’’ dark matter model [39].

We start with a very general local action of the type

$$S = \int d^4x \sqrt{-g} \mathcal{L}[g_{\mu\nu}, \partial_{\lambda_1} g_{\mu\nu}, \dots, \partial_{\lambda_1} \dots \partial_{\lambda_p} g_{\mu\nu}, \varphi, \partial_{\lambda_1} \varphi, \dots, \partial_{\lambda_1} \dots \partial_{\lambda_q} \varphi] + S_m[g_{\mu\nu}, \phi_m], \quad (3.22)$$

where the integers $p, q \geq 2$, \mathcal{L} is the Lagrangian density which is a functional of the metric, the scalar field and their derivatives. S_m is the action for the matter field ϕ_m which we assume to be uncoupled with φ . For the sake of concreteness, the Lagrangian \mathcal{L} may be thought of as being the Lagrangian of Horndeski's theory [63] or one of its recently proposed healthy extensions [60, 61, 66].

The variation of the action with respect to the fundamental fields, φ , $g_{\mu\nu}$ and ϕ_m , is given by,

$$\delta S = \frac{1}{2} \int d^4x \sqrt{-g} (E^{\mu\nu} + T^{\mu\nu}) \delta g_{\mu\nu} + \int d^4x \Omega_\varphi \delta\varphi + \int d^4x \Omega_m \delta\phi_m, \quad (3.23)$$

where

$$\Omega_\varphi = \frac{\delta(\sqrt{-g}\mathcal{L})}{\delta\varphi} = \frac{\partial(\sqrt{-g}\mathcal{L})}{\partial\varphi} + \sum_{h=1}^q (-1)^h \frac{d}{dx^{\lambda_1}} \cdots \frac{d}{dx^{\lambda_h}} \frac{\partial(\sqrt{-g}\mathcal{L})}{\partial(\partial_{\lambda_1} \dots \partial_{\lambda_h} \varphi)}, \quad (3.24)$$

$$\begin{aligned} E^{\mu\nu} &= \frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L})}{\delta g_{\mu\nu}} \\ &= \frac{2}{\sqrt{-g}} \left(\frac{\partial(\sqrt{-g}\mathcal{L})}{\partial g_{\mu\nu}} + \sum_{h=1}^p (-1)^h \frac{d}{dx^{\lambda_1}} \cdots \frac{d}{dx^{\lambda_h}} \frac{\partial(\sqrt{-g}\mathcal{L})}{\partial(\partial_{\lambda_1} \dots \partial_{\lambda_h} g_{\mu\nu})} \right), \end{aligned} \quad (3.25)$$

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g_{\mu\nu}}, \quad \Omega_m = \frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta\phi_m}, \quad (3.26)$$

$$\text{where } S_m[g_{\mu\nu}] = \int d^4x \sqrt{-g} \mathcal{L}_m[g_{\mu\nu}, \phi_m].$$

\mathcal{L}_m is the matter Lagrangian density and $T^{\mu\nu}$ is the matter energy-momentum tensor. In the case of General Relativity, the tensor $E^{\mu\nu}$ is $E^{\mu\nu} = -G^{\mu\nu}$, where $G^{\mu\nu}$ is the Einstein tensor.

We consider a disformal transformation of the type (3.13) from where one

can obtain its variation as

$$\begin{aligned}
 \delta g_{\mu\nu} = & A \delta \ell_{\mu\nu} - \left(\ell_{\mu\nu} \frac{\partial A}{\partial w} + \partial_\mu \varphi \partial_\nu \varphi \frac{\partial B}{\partial w} \right) \left[(\ell^{\alpha\rho} \partial_\alpha \varphi) (\ell^{\beta\sigma} \partial_\beta \varphi) \delta \ell_{\rho\sigma} \right. \\
 & \left. - 2 \ell^{\rho\sigma} (\partial_\rho \varphi) (\partial_\sigma \delta \varphi) \right] + \left(\ell_{\mu\nu} \frac{\partial A}{\partial \varphi} + \partial_\mu \varphi \partial_\nu \varphi \frac{\partial B}{\partial \varphi} \right) \delta \varphi \\
 & + B [(\partial_\mu \varphi) (\partial_\nu \delta \varphi) + (\partial_\nu \varphi) (\partial_\mu \delta \varphi)]. \tag{3.27}
 \end{aligned}$$

Inserting Eq. (3.27) into Eq. (3.23), the generalized Einstein equations of motion, $\delta S / \delta \ell_{\mu\nu} = 0$, are

$$A(E^{\mu\nu} + T^{\mu\nu}) = \left(\alpha_1 \frac{\partial A}{\partial w} + \alpha_2 \frac{\partial B}{\partial w} \right) (\ell^{\mu\rho} \partial_\rho \varphi) (\ell^{\nu\sigma} \partial_\sigma \varphi), \tag{3.28}$$

and the generalized Klein-Gordon equation, $\delta S / \delta \varphi = 0$, is,

$$\begin{aligned}
 \frac{1}{\sqrt{-g}} \partial_\rho \left\{ \sqrt{-g} \partial_\sigma \varphi \left[B(E^{\rho\sigma} + T^{\rho\sigma}) + \left(\alpha_1 \frac{\partial A}{\partial w} + \alpha_2 \frac{\partial B}{\partial w} \right) \ell^{\rho\sigma} \right] \right\} \\
 - \frac{\Omega_\varphi}{\sqrt{-g}} = \frac{1}{2} \left(\alpha_1 \frac{\partial A}{\partial \varphi} + \alpha_2 \frac{\partial B}{\partial \varphi} \right), \tag{3.29}
 \end{aligned}$$

where we have defined two new quantities as

$$\alpha_1 \equiv (E^{\rho\sigma} + T^{\rho\sigma}) \ell_{\rho\sigma} \quad \text{and} \quad \alpha_2 \equiv (E^{\rho\sigma} + T^{\rho\sigma}) \partial_\rho \varphi \partial_\sigma \varphi. \tag{3.30}$$

In addition, the equation of motion for the matter field is $\Omega_m = 0$.

By contracting the metric equations of motion (3.28) with $\ell_{\mu\nu}$ and with $\partial_\mu \varphi \partial_\nu \varphi$, we find

$$\alpha_1 \left(A - w \frac{\partial A}{\partial w} \right) - \alpha_2 w \frac{\partial B}{\partial w} = 0, \quad \alpha_1 w^2 \frac{\partial A}{\partial w} - \alpha_2 \left(A - w^2 \frac{\partial B}{\partial w} \right) = 0. \tag{3.31}$$

These two equations form a two-dimensional linear system of algebraic equations for α_1 and α_2 . The solutions of the system are different depending on

whether its determinant is zero or non-zero. In the next two subsections we study these two cases separately.

Generic case

We may write the system of equations (3.31) in matrix form, as

$$M \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = 0, \quad \text{where} \quad M = \begin{pmatrix} A - w \frac{\partial A}{\partial w} & -w \frac{\partial B}{\partial w} \\ w^2 \frac{\partial A}{\partial w} & -A + w^2 \frac{\partial B}{\partial w} \end{pmatrix}. \quad (3.32)$$

The determinant of the system is

$$\det(M) = w^2 A \frac{\partial}{\partial w} \left(B + \frac{A}{w} \right). \quad (3.33)$$

If $\det(M) \neq 0$ then the only solution is $\alpha_1 = \alpha_2 = 0$. For this generic case the equations of motion, Eqs. (3.28) and (3.29), reduce to

$$E^{\mu\nu} + T^{\mu\nu} = 0, \quad (3.34)$$

$$\Omega_\varphi = 0. \quad (3.35)$$

When written in terms of the metric $g_{\mu\nu}$, these two equations in addition to $\Omega_m = 0$ are the same equations as in the original theory before doing any disformal transformation. In other words, by taking the variation with respect to the original metric $g_{\mu\nu}$ or with respect to $\ell_{\mu\nu}$ and φ we get, in the end, the same equations of motion. This shows that generically (i.e. $\det(M) \neq 0$) the theory (physics) is invariant under disformal transformations of the type (3.13), which only act here as field redefinitions. This generalizes the results of [42] (obtained for Einstein gravity) to a very general scalar-tensor theory

of the kind (3.22). This result is less surprising if one recalls that all one is doing is a well-behaved invertible change of variables.

Mimetic gravity

If the determinant of the system is zero then one can solve the differential equation (3.33) to find that the free function $B(\varphi, w)$ has to be of the form

$$B(\varphi, w) = -\frac{A(\varphi, w)}{w} + b(\varphi), \quad (3.36)$$

where $b(\varphi)$ is an integration constant (it does not depend on w but it may depend on φ) and we assume it is non-zero for all φ . This solution was previously found in [42] for the case when the starting action in Eq. (3.22) is simply the Einstein-Hilbert action. Here we show that solution (3.36) is still valid for a general action of the form (3.22), irrespective of whether the scalar field in the action is the same or different than the scalar field involved in the transformation, as shown in Appendix A.1. This is a consequence of the fact that the determinant of the system, Eq. (3.33), does not depend on the form of the starting action (3.22) and it is the same as the determinant found in [42]. Substituting solution (3.36) into the system (3.31) gives us $\alpha_2 = w\alpha_1$. Hence, the equations of motion (3.28) and (3.29) become

$$\begin{aligned} E^{\mu\nu} + T^{\mu\nu} &= \frac{\alpha_1}{w} (\ell^{\mu\rho} \partial_\rho \varphi) (\ell^{\nu\sigma} \partial_\sigma \varphi), \\ \frac{1}{\sqrt{-g}} \partial_\rho (\sqrt{-g} b \alpha_1 \ell^{\rho\sigma} \partial_\sigma \varphi) - \frac{\Omega_\varphi}{\sqrt{-g}} &= \frac{1}{2} \alpha_1 w \frac{db}{d\varphi}. \end{aligned} \quad (3.37)$$

Now, the disformal transformation is of the particular type

$$g_{\mu\nu} = A(\varphi, w) \ell_{\mu\nu} + \partial_\mu \varphi \partial_\nu \varphi \left(b(\varphi) - \frac{A(\varphi, w)}{w} \right). \quad (3.38)$$

The inverse metric transforms (recall we assume $b \neq 0$) as

$$g^{\mu\nu} = \frac{\ell^{\mu\nu}}{A} + \frac{A - w b}{A b w^2} (\ell^{\mu\rho} \partial_\rho \varphi) (\ell^{\nu\sigma} \partial_\sigma \varphi), \quad (3.39)$$

and these equations can be used to write (3.37) in terms of $g_{\mu\nu}$ (explicitly) only. Similarly to [42], we have $\ell^{\mu\rho} \partial_\rho \varphi = b w \partial^\mu \varphi$ and $\alpha_1 = (E+T)/(b w)$ where $\partial^\mu \varphi \equiv g^{\mu\rho} \partial_\rho \varphi$ and $E + T \equiv g_{\rho\sigma} (E^{\rho\sigma} + T^{\rho\sigma})$. By contracting $\ell^{\mu\rho} \partial_\rho \varphi = b w \partial^\mu \varphi$ with $\partial_\mu \varphi$ and using the definition of w one can also find that

$$b(\varphi) g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi = 1. \quad (3.40)$$

So the equations of motion (3.37) simplify to

$$E_{\mu\nu} + T_{\mu\nu} = (E+T) b \partial_\mu \varphi \partial_\nu \varphi, \quad \nabla_\rho [(E+T) b \partial^\rho \varphi] - \frac{\Omega_\varphi}{\sqrt{-g}} = \frac{1}{2} (E+T) \frac{1}{b} \frac{db}{d\varphi}, \quad (3.41)$$

where ∇_ρ denotes the covariant derivative with respect to $g_{\mu\nu}$. In order to have the full system of equations of motion, to these equations one should add the matter equation, $\Omega_m = 0$. As it can be seen these equations of motion are in general different from the equations of motion that result from varying the action (3.22) with respect to the original metric $g_{\mu\nu}$. We will call this new theory "mimetic" gravity and the transformation will be called mimetic disformal transformation. Note that the condition for the determinant of the system to be zero leads to exactly the same particular disformal transformation, Eq. (3.38), as the non-invertibility condition of the previous subsection.

3.4 Mimetic gravity from a Lagrange multiplier

In this section, we will show that the mimetic equations of motion that result after transforming the theory (3.22) via a mimetic disformal transformation (3.38) can also be obtained by variation of an action without performing any disformal transformation or introducing an auxiliary metric $\ell_{\mu\nu}$. For the case in which the original theory (3.22) is General Relativity and for conformal transformations this was first achieved in [52, 45].

Let us start with the very general action of the previous section where we add an additional term as

$$S_\lambda = \int d^4x \sqrt{-g} \mathcal{L}[g_{\mu\nu}, \partial_{\lambda_1} g_{\mu\nu}, \dots, \partial_{\lambda_1} \dots \partial_{\lambda_p} g_{\mu\nu}, \varphi, \partial_{\lambda_1} \varphi, \dots, \partial_{\lambda_1} \dots \partial_{\lambda_q} \varphi] + S_m[g_{\mu\nu}, \phi_m] + \int d^4x \sqrt{-g} \lambda (b(\varphi) g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - 1), \quad (3.42)$$

where λ is a Lagrange multiplier field which enforces the kinematical constraint. $b(\varphi)$ is a known potential function that defines the theory. The equations of motion that result from varying the action with respect to λ , φ , $g_{\mu\nu}$ and ϕ_m are respectively (after some simplification)

$$b(\varphi) g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - 1 = 0, \quad (3.43)$$

$$\Omega_\varphi + \sqrt{-g} \frac{\lambda}{b(\varphi)} \frac{db(\varphi)}{d\varphi} - 2\partial_\mu (\sqrt{-g} \lambda b(\varphi) g^{\mu\nu} \partial_\nu \varphi) = 0, \quad (3.44)$$

$$E^{\mu\nu} + T^{\mu\nu} - 2\lambda b(\varphi) \partial^\mu \varphi \partial^\nu \varphi = 0, \quad (3.45)$$

$$\Omega_m = 0, \quad (3.46)$$

with the same definitions of Section 3.3. Taking the trace of Eq. (3.45) and

after using Eq. (3.43), one obtains

$$2\lambda = E + T, \quad (3.47)$$

where $E = g_{\mu\nu}E^{\mu\nu}$ and $T = g_{\mu\nu}T^{\mu\nu}$. One can see that the Lagrange multiplier is given by the traces E and T and this can be used to eliminate λ from the equations of motion to obtain

$$b(\varphi)g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - 1 = 0, \quad (3.48)$$

$$\nabla_\mu [(E + T)b(\varphi)\partial^\mu\varphi] - \frac{\Omega_\varphi}{\sqrt{-g}} = \frac{E + T}{2} \frac{1}{b(\varphi)} \frac{db(\varphi)}{d\varphi}, \quad (3.49)$$

$$E^{\mu\nu} + T^{\mu\nu} = (E + T)b(\varphi)\partial^\mu\varphi\partial^\nu\varphi, \quad (3.50)$$

$$\Omega_m = 0. \quad (3.51)$$

These equations of motion are the same as the mimetic equations of motion in Subsection 3.3.2, i.e. (3.40), (3.41) and the matter equation. This shows that mimetic gravity can be formulated by action (3.42). The price to pay in this formulation is that one needs to introduce an additional scalar field, the Lagrange multiplier λ . It would be interesting to determine if it is possible to derive mimetic gravity from an action with no additional scalar fields like λ . We leave this for future work.

Let us take the covariant derivative of Eq. (3.50) and use $\nabla_\mu T^{\mu\nu} = 0$ ¹ to obtain

$$\begin{aligned} \nabla_\mu E^{\mu\nu} &= \nabla_\mu [(E + T)b(\varphi)\partial^\mu\varphi] \partial^\nu\varphi + (E + T)b(\varphi)\partial^\mu\varphi\nabla_\mu\partial^\nu\varphi \\ &= \partial^\nu\varphi \left[\nabla_\nu [(E + T)b(\varphi)\partial^\mu\varphi] - \frac{1}{2} \frac{1}{b(\varphi)} \frac{db(\varphi)}{d\varphi} (E + T) \right], \end{aligned} \quad (3.52)$$

¹The conservation of the energy-momentum tensor is a consequence of assuming that the action S_m can be written as a functional of the matter field and the metric $g_{\mu\nu}$ and by using the Horndeski identity, Eq. (3.53), applied to the matter action together with the equation of motion (3.51).

where in the second line we have used that $b(\varphi)\partial^\mu\varphi\partial_\mu\varphi = 1$ and that from its covariant derivative one obtains $b(\varphi)\nabla^\mu\nabla^\nu\varphi\partial_\mu\varphi = -\frac{1}{2}\frac{db(\varphi)}{d\varphi}\nabla^\nu\varphi\partial_\mu\varphi\partial^\mu\varphi$. It was shown by Horndeski [63] (see also references therein) that

$$\sqrt{-g}\nabla_\mu E^{\mu\nu} = \Omega_\varphi\nabla^\nu\varphi. \quad (3.53)$$

Using this and the fact that $\partial^\nu\varphi \neq 0$ at least for one index ν we can simplify Eq. (3.52) to

$$\nabla_\mu [(E + T)b(\varphi)\partial^\mu\varphi] - \frac{\Omega_\varphi}{\sqrt{-g}} = \frac{E + T}{2} \frac{1}{b(\varphi)} \frac{db(\varphi)}{d\varphi}. \quad (3.54)$$

This is exactly the same equation as (3.49). So we have managed to show that Eq. (3.49) results from taking the covariant derivative of Eq. (3.50) and use $\nabla_\mu T^{\mu\nu} = 0$ and Eqs. (3.48) and (3.51). This proof is independent of using the Lagrange multiplier method or not and shows that in order to solve the dynamics of the system it is sufficient to consider Eqs. (3.48), (3.50) and (3.51). These three equations, when written in terms of the metric $g_{\mu\nu}$, do not contain any more higher-order derivatives than the equations of motion that result from the non-mimetic theory defined by the Lagrangian \mathcal{L} . It is also worth noting that the action (3.42) does not contain any more higher-order derivatives than \mathcal{L} . However, the new theory (3.42) does contain a new field, the Lagrange multiplier λ . The three independent equations of motion when written in terms of the new metric $\ell_{\mu\nu}$ may contain higher-order derivatives. For concreteness, we can think of \mathcal{L} as being the Horndeski Lagrangian [63]² and we would be considering the ‘‘mimetic’’ Horndeski theory.

²One can also consider healthy extensions of Horndeski’s theory, like for instance the so-called G^3 theories [60, 61] or even their extensions [66].

3.5 Non-trivial examples of cosmology in the “mimetic” Horndeski model

As an application of the results of the preceding sections, in this section, we will present three simple examples of non-trivial cosmological solutions that arise in very simple “mimetic” Horndeski models.

In the next three subsections the actions of the models considered will be Eq. (3.42) with $\mathcal{L} = \mathcal{L}_H$ but with different choices for the free functions in each subsection.

Notice that for a general mimetic Horndeski model, the free function $b(\varphi)$ in the second term of Eq. (3.42) can be reabsorbed by defining a new field Φ as $d\Phi = \sqrt{|b|}d\varphi$. Because the Horndeski Lagrangian is form invariant under field redefinitions of this type, this transformation just amounts to consider a different starting Horndeski Lagrangian \mathcal{L}_H .

3.5.1 A very simple example

In our first simple example we will consider the mimetic theory of a canonical kinetic term scalar field with no potential coupled to Einstein’s gravity theory. The action of this model is Eq. (3.42) with $\mathcal{L} = \mathcal{L}_H$, $S_m = 0$ and with the choice

$$\begin{aligned} K(X, \varphi) &= c_2 X, & G_3(X, \varphi) &= 0, \\ G_4(X, \varphi) &= 1/2, & G_5(X, \varphi) &= 0, \end{aligned} \quad (3.55)$$

where c_2 is a constant which may have either sign. In the non-mimetic theory, if c_2 is negative it is well known that the scalar field has the wrong sign in

the kinetic term and is a ghost. In the present mimetic model it would be interesting to study perturbations and determine what are the conditions for the absence of ghost and other instabilities. We leave this work for the near future. With the same notation of the previous section and for a flat FLRW background, the objects that appear there are

$$E_{00} = -3H^2 + \frac{1}{2}c_2\dot{\varphi}^2, \quad (3.56)$$

$$E_{xx} = E_{yy} = E_{zz} = a^2(3H^2 + 2\dot{H} + \frac{1}{2}c_2\dot{\varphi}^2), \quad (3.57)$$

$$E = 12H^2 + 6\dot{H} + c_2\dot{\varphi}^2, \quad (3.58)$$

$$\Omega_\varphi = a^3(-3c_2H\dot{\varphi} - c_2\ddot{\varphi}), \quad (3.59)$$

where a is the scale factor, $H = \dot{a}/a$ and dot denotes derivative with respect to cosmic time. x, y, z denote the comoving spatial coordinates. The equations of motion for this simple mimetic model, i.e. Eq. (3.48), the time and spatial components of Eq. (3.50) and Eq. (3.49), are respectively

$$b(\varphi)\dot{\varphi}^2 + 1 = 0, \quad (3.60)$$

$$3H^2 = \frac{\dot{\varphi}^2}{2} \left[c_2 - 2b(\varphi) \left(12H^2 + 6\dot{H} + c_2\dot{\varphi}^2 \right) \right], \quad (3.61)$$

$$6H^2 + 4\dot{H} + c_2\dot{\varphi}^2 = 0, \quad (3.62)$$

$$b(\varphi) \left[-6H(6H^2\dot{\varphi} + 7\dot{\varphi}\dot{H} + 2H\ddot{\varphi}) - 6\dot{H}\ddot{\varphi} - 6\dot{\varphi}\ddot{H} - 3c_2\dot{\varphi}^2(H\dot{\varphi} + \ddot{\varphi}) \right] \\ + c_2(3H\dot{\varphi} + \ddot{\varphi}) + b'(\varphi) \left(\frac{1}{b(\varphi)} + 2\dot{\varphi}^2 \right) \left[-6H^2 - 3\dot{H} - \frac{c_2}{2}\dot{\varphi}^2 \right] = 0, \quad (3.63)$$

where prime denotes derivative with respect to the field φ . Eqs. (3.61) and (3.63) are not independent from Eqs. (3.60) and (3.62) because they can be derived from them.

It is easy to check that Eqs. (3.60) and (3.62) admit the following solution

$$\begin{aligned} a(t) &= t^{\frac{2}{3(1+\omega)}}, & \varphi(t) &= \pm \sqrt{-\frac{\alpha}{c_2}} \log \frac{t}{t_0}, \\ b(\varphi) &= -\frac{1}{\dot{\varphi}^2} = \frac{c_2}{\alpha} t^2 = \frac{c_2}{\alpha} t_0^2 e^{\pm 2\sqrt{-\frac{c_2}{\alpha}}\varphi}, \end{aligned} \quad (3.64)$$

where t_0 is an integration constant, the parameter α is $\alpha = -\frac{8\omega}{3(1+\omega)^2}$, where ω is a constant parameter. This expansion law is the same as the one given by a perfect fluid universe with a constant equation of state ω . If c_2 is positive then the equation of state ω has to be positive too. This shows that this simple mimetic scalar field model can mimic the background evolution of a perfect fluid universe with a constant equation of state. For different ω the value of α changes but the functional form of $b(\varphi)$ does not change. It is obvious that this new solution is not a solution of the Einstein plus Klein-Gordon (with zero potential) field theory. There ω is necessarily unity.

By adjusting the function $b(\varphi)$ accordingly (note that $b(\varphi) < 0$ for a time-like scalar velocity), this simple model can mimic the expansion history of almost any model. To be concrete, we can mimic the expansion history of a perfect fluid model with a fixed sign of the pressure. In that case, $6H^2 + 4\dot{H} = -2p$, where p is the pressure of the perfect fluid. So from the independent equation of motion (3.62) one can see that the pressure cannot change sign. The fact that one can have almost any expansion history desired is somewhat similar to the minimal extension of the original mimetic dark matter model proposed in [40]. See also [41] for an earlier work where models similar to our present one were considered.

3.5.2 Mimetic cubic Galileon

In this subsection, we will consider the mimetic cubic Galileon model as a further example of a simple mimetic Horndeski model. The mimetic cubic Galileon model with $c_2 = 0$ (and also including the other Galileon interactions) was previously studied in [73] but for the case of a constant $b(\varphi)$. Here we allow the function b to depend on φ . The action of the model is Eq. (3.42) with $\mathcal{L} = \mathcal{L}_H$, $S_m = 0$ and with the free functions chosen as

$$\begin{aligned} K(X, \varphi) &= c_2 X, & G_3(X, \varphi) &= 2c_3/\tilde{\Lambda}^3 X, \\ G_4(X, \varphi) &= 1/2, & G_5(X, \varphi) &= 0, \end{aligned} \quad (3.65)$$

where from now on we will set the cutoff scale $\tilde{\Lambda}$ to be $\tilde{\Lambda} = 1$ and c_3 is a new model parameter.

Analogously to the previous subsection, there are only two independent equations of motion, they can be chosen to be Eq. (3.48) and the spatial component of Eq. (3.50). They are respectively

$$b(\varphi)\dot{\varphi}^2 + 1 = 0, \quad (3.66)$$

$$6H^2 + 4\dot{H} + \dot{\varphi}^2(c_2 - 4c_3\ddot{\varphi}) = 0. \quad (3.67)$$

As in the preceding section by suitably choosing a function $b(\varphi)$ one can have almost any expansion history desired. Let us for instance concentrate on the expansion history of a universe filled with dark matter and a positive cosmological constant Λ . The scale factor solution for that universe is

$$a = a_* \sinh^{\frac{2}{3}}(\mathcal{C}t), \quad (3.68)$$

where $\mathcal{C} = \sqrt{3\Lambda/4}$. Eq. (3.67) can be integrated once to find

$$\frac{4c_3}{c_2} \left[-\arctan \left(\pm \sqrt{\frac{3c_2}{8\mathcal{C}^2}} \dot{\varphi} \right) \pm \sqrt{\frac{3c_2}{8\mathcal{C}^2}} \dot{\varphi} \right] = t. \quad (3.69)$$

In Fig. 3.1 we plot the time evolution of the scale factor $a(t)$, the time derivative of φ and by using Eq. (3.66) one can find the function $b(t)$. For illustration purposes we choose the model parameters as $\mathcal{C} = c_2 = c_3 = a_\star = 1$. For this choice, the matter-dominated era ends around $t = \mathcal{O}(1)$ and after that the universe becomes dominated by the energy density of the cosmological constant. For $\mathcal{C}t \gg 1$, the time derivative of φ is $\dot{\varphi} \propto t$ while for $\mathcal{C}t \ll 1$ it becomes $\dot{\varphi} \propto t^{1/3}$. The previous equations can be easily integrated to find the function $b(\varphi)$ as $b(\varphi) \propto -\varphi^{-1/2}$ for $\mathcal{C}t \ll 1$ and $b(\varphi) \propto -\varphi^{-1}$ for $\mathcal{C}t \gg 1$. By choosing a function $b(\varphi)$ with these asymptotic limits one can approximately reproduce the expansion history of a Λ -dark matter universe.

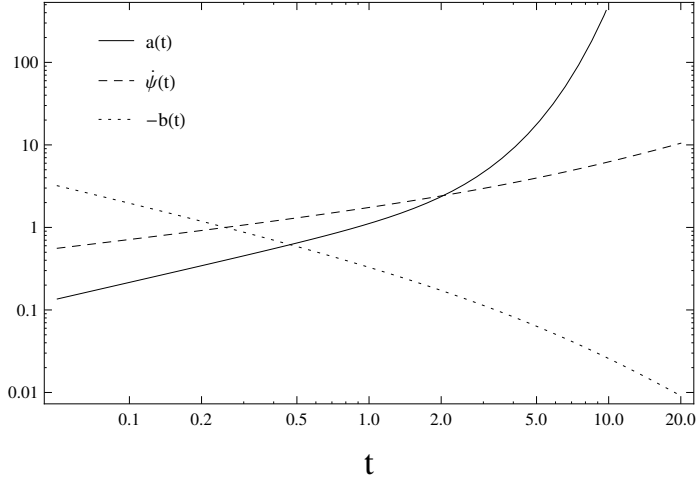


Figure 3.1: Plot of the scale factor $a(t)$ (solid line), the time derivative of field $\dot{\varphi}(t)$ (dashed line) and the function $-b(t)$ (dotted line) as functions of time t (in suitable units) for the parameter choice $\mathcal{C} = c_2 = c_3 = a_\star = 1$. This choice was made for illustration purposes only.

3.5.3 The case of minimal coupling to $\ell_{\mu\nu}$

The third example of non-trivial cosmological solutions that arises in the context of mimetic Horndeski models that we are going to present now involves promoting the auxiliary metric $\ell_{\mu\nu}$ to the physical metric. Say for instance, usual matter, like baryons, are minimally coupled with $\ell_{\mu\nu}$ instead of the more interesting case of minimal coupling with $g_{\mu\nu}$. The gravitational part of the action of this model is Eq. (3.22) with $S_m = 0$, where the fundamental metric variable is the metric $\ell_{\mu\nu}$, which is related to the metric $g_{\mu\nu}$ by a mimetic disformal transformation, i.e. a disformal transformation of the type (3.13) with the function B given by (3.21). Then we choose to minimally couple this gravitational theory for $\ell_{\mu\nu}$ and φ with (baryon) matter fields.

In the following discussion, we will restrict the mimetic disformal transformation to a particular type (see Eq. (3.71) below) so that we have a Weyl symmetry, that is, the gravitational part of the action will be invariant under a Weyl rescaling of the type $\ell_{\mu\nu} \rightarrow \Omega^2(x)\ell_{\mu\nu}$, where $\Omega(x)$ is a non-zero function. It is worth mentioning that in the case of a mimetic disformal transformation with $B(\varphi, w) = 0$ (the transformation is conformal) this implies that $A(\varphi, w) = b(\varphi)w$. It is then easy to see that the theory also has a Weyl symmetry which allows us to choose the gauge so that $g_{\mu\nu} = \ell_{\mu\nu}$ as it was done in [45]. Indeed, in [39, 40, 45], the authors used $B(\varphi, w) = 0$, $b(\varphi) = -1$ which leads to $A(\varphi, w) = -w$.

If the function $A(\varphi, w)$ is

$$A(\varphi, w) = (b(\varphi) - f(\varphi))w, \quad (3.70)$$

then the mimetic disformal transformation is

$$g_{\mu\nu} = (b(\varphi) - f(\varphi))w\ell_{\mu\nu} + f(\varphi)\partial_\mu\varphi\partial_\nu\varphi, \quad (3.71)$$

and the inverse metric transformation is

$$g^{\mu\nu} = \frac{\ell^{\mu\nu}}{(b(\varphi) - f(\varphi))w} - \frac{f(\varphi)}{w^2b(\varphi)(b(\varphi) - f(\varphi))}\ell^{\mu\rho}\partial_\rho\varphi\ell^{\nu\sigma}\partial_\sigma\varphi, \quad (3.72)$$

where one can easily see that they have the desired property of being invariant under a Weyl transformation $\ell_{\mu\nu} \rightarrow \Omega^2(x)\ell_{\mu\nu}$. For simplicity, we assume that the original scalar-tensor theory is actually just Einstein's General Relativity and in the following we also assume that the contribution of the baryons to the expansion can be neglected. The equations of motion of this model, Eqs. (3.40) and (3.41), are then

$$\begin{aligned} G_{\mu\nu} &= Gb(\varphi)\partial_\mu\varphi\partial_\nu\varphi, & 2\nabla_\rho(Gb\partial^\rho\varphi) &= G\frac{1}{b(\varphi)}\frac{db(\varphi)}{d\varphi}, \\ b(\varphi)g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi &= 1, \end{aligned} \quad (3.73)$$

where G denotes the trace of the Einstein tensor $G_{\mu\nu}$ as $G = g^{\mu\nu}G_{\mu\nu}$. We will look for cosmological solutions by setting the metric $\ell_{\mu\nu}$ to be equal to a flat FLRW metric and φ to be a function of time only. This implies that the non-zero components of the metric $g_{\mu\nu}$ (there is isotropy so the y and z component are equal to the x component) are

$$g_{tt} = b\dot{\varphi}^2, \quad g^{tt} = 1/(b\dot{\varphi}^2), \quad g_{xx} = a^2A, \quad g^{xx} = 1/(a^2A). \quad (3.74)$$

The tt and xx components of the previous generalized Einstein equations are

equal to each other and equal to

$$\begin{aligned}
& 4bA \left[A(3H^2 + 2\dot{H})\dot{\varphi} - 2AH\ddot{\varphi} + \dot{\varphi}^2(A_{,\varphi\varphi} - 4A_{,\varphi w}\ddot{\varphi} + 4A_{,ww}\ddot{\varphi}^2) \right. \\
& \left. + \dot{\varphi}^2(3HA_{,\varphi} - 2A_{,w}(\ddot{\varphi} + 3H\ddot{\varphi})) \right] - 2A\dot{\varphi}^2(2AH + \dot{\varphi}(A_{,\varphi} - 2A_{,w}\ddot{\varphi}))b' \\
& = b\dot{\varphi}^3(A_{,\varphi} - 2A_{,w}\ddot{\varphi})^2, \tag{3.75}
\end{aligned}$$

where prime means derivative with respect to φ and $A_{,\varphi}$, $A_{,w}$ and so on denote derivatives of A with respect to φ and w respectively. Eq. (3.75) contains higher-derivatives for φ and they disappear if A does not depend on w . Note that from the kinematical constraint in (3.73) we do not get any equation of motion if it is written in terms of the metric $\ell_{\mu\nu}$. One can also show that by combining Eq. (3.75) with its derivative one can find the equation of motion for φ .

Now let us consider a particular mimetic disformal transformation of the type (3.70), (3.71) and (3.72). In this case we have Weyl invariance and we can choose to fix the gauge as

$$A(\varphi, w) = (b(\varphi) - f(\varphi))w = 1. \tag{3.76}$$

In this new gauge, the independent generalized Einstein equation is

$$(3H^2 + 2\dot{H})\dot{\varphi} - 2H\ddot{\varphi} = H\dot{\varphi}^2 \frac{b'}{b}. \tag{3.77}$$

By doing a change of variables as

$$\frac{d\Phi}{d\varphi} = \sqrt{|b|}, \tag{3.78}$$

one can simplify the equation to

$$(3H^2 + 2\dot{H})\dot{\Phi} - 2H\ddot{\Phi} = 0. \quad (3.79)$$

The equation of motion (3.79) can be written as $\frac{d(\ln \dot{\Phi})}{dt} = \frac{3}{2} \frac{d(\ln a)}{dt} + \frac{d(\ln H)}{dt}$. So one can integrate it once to find $\dot{\Phi}(t) = \text{const} a(t)^{\frac{3}{2}} H(t)$, where const denotes a constant of integration. And integrating once more to find

$$\Phi(t) = \text{const}_1 a(t)^{\frac{3}{2}} + \text{const}_2, \quad (3.80)$$

where const_i with $i = 1, 2, 3, 4$ denote constants of integration. Using Eq. (3.74), the change of variables (3.78) and our gauge choice $A = 1$, the $g_{\mu\nu}$ metric components can be written as

$$g_{tt}(t) = \text{Sign}(b)\dot{\Phi}(t)^2, \quad g_{xx}(t) = a^2(t). \quad (3.81)$$

By doing a change of time variable as

$$\sqrt{-g_{tt}(t)} dt = d\tilde{t} \quad (3.82)$$

one can find that

$$a(t(\tilde{t})) = (\text{const}_3 \tilde{t} + \text{const}_4)^{\frac{2}{3}}, \quad (3.83)$$

which is a matter dominated universe for the $g_{\mu\nu}$ metric. This is always the case for any $\ell_{\mu\nu}$, $b(\varphi)$ or $f(\varphi)$. This result is consistent with the original findings of [39] that one only gets a matter universe as a solution for the metric $g_{\mu\nu}$ having started from mimetic General Relativity by doing a conformal transformation³. This result is expected in light of the findings

³If $f = 0$ then the disformal transformation Eq. (3.71) becomes simply a conformal transformation. Furthermore, the gauge condition Eq. (3.76) implies that $g_{\mu\nu} = \ell_{\mu\nu}$ and

of [42] that showed that the mimetic dark matter equations of motion, Eqs. (3.73), when written in terms of $g_{\mu\nu}$ are the same for any mimetic disformal transformation. However now in our case the physical metric is $\ell_{\mu\nu}$ so from Eq. (3.80) one can see that in this model we can have any expansion history desired. For a given scale factor solution $\ell_{xx} = a^2(t)$ the scalar field Φ adjusts according to Eq. (3.80) in order for this to be a solution of the equation of motion Eq. (3.79). The solution for the original field φ can be found once we specify the function $b(\varphi)$ by using Eq. (3.78). Finally the function $f(\varphi)$ is found by using the gauge condition, Eq. (3.76), which in the background can be written as $1 = -\dot{\varphi}^2(b-f) = -\dot{\Phi}^2(\text{Sign}(b) - f/|b|)$. In other words, by choosing a specific function f one can obtain the desired scale factor solution. For instance, de Sitter spacetime is a solution of (3.79) for $\Phi(t) \propto e^{3H/2t}$. A matter universe solution results from taking $\dot{\Phi} = \text{constant}$. The expansion history of a universe filled with a barotropic perfect fluid with a constant equations of state ω and a cosmological constant Λ is $a = a_* \sinh^{\frac{2}{3(1+\omega)}}(\mathcal{C}t)$, where $\mathcal{C} = \sqrt{3\Lambda/4(1+\omega)}$. The solution for Φ is $\Phi(t) = C_2 + C_1 \sinh^{\frac{1}{1+\omega}}(\mathcal{C}t)$, where C_1 and C_2 are integration constants.

3.6 Conclusions

In this chapter, we showed that a very general scalar-tensor theory of the type (3.22) is generically invariant under a disformal transformation of the kind (3.13), irrespective of whether the scalar field in the action is the same or different than the scalar field involved in the transformation. We also showed that there is a special subset of those disformal transformations for which the previous result is not valid. We call those special disformal transforma-

that $\dot{\Phi}$ is a constant. This singles out the matter-dominated universe solution from the set of all possible solutions of Eq. (3.80). This case is nothing more than the result of [39].

tions as mimetic disformal transformations because they give origin to a new scalar-tensor theory of gravity that is a generalization of the “mimetic” dark matter proposal [39]. These mimetic disformal transformations are given by Eq. (3.36). They basically are a subset of (3.13) where the two free functions $A(\varphi, X)$ and $B(\varphi, X)$ are related as (3.36). These results generalize the findings of [42] that were obtained for Einstein’s General Relativity. We also showed that the reason why a simple change of variables as in a mimetic disformal transformation leads to a new physical theory is because we are doing a non-invertible change of variables. If the change of variables is invertible then the physical theory does not change as expected. The derived non-invertibility or mimetic condition is the same for any general scalar-tensor theory, as it is the property of the disformal transformation of the kind of Eq. (3.13). We have shown that the mimetic equations of motion of the new scalar-tensor theory can be derived from an action containing an extra scalar field playing the role of a Lagrange multiplier that imposes the kinematical constraint (3.20) to be satisfied throughout the dynamics. Again this generalizes some results in [52, 45] to a general scalar-tensor theory context.

As an application of some of our findings, we have presented a simple toy model of the mimetic Horndeski theory where a canonically normalized scalar field with no potential (in the original theory) can be used to mimic the background expansion history of a universe filled with a barotropic perfect fluid with a constant equations of state. Actually, we showed that in this simple scalar-tensor model one can have almost any (the restriction is that the effective pressure cannot change sign) desired background expansion history by suitably choosing the “potential” function $b(\varphi)$ in the action (3.42). We have generalized the previous simple model to include a cubic Galileon interaction and as an example we showed that this model can easily mimic the back-

ground expansion history of a universe filled with dark matter and a positive cosmological constant.

We also presented an example where instead of minimally coupling baryons with $g_{\mu\nu}$ we coupled them minimally with the new metric $\ell_{\mu\nu}$. In this case, for the original theory, we took simply the disformally transformed Einstein-Hilbert action. We again found that, for a cosmological background, the metric $\ell_{\mu\nu}$ can have any expansion history desired by suitably choosing the free functions $b(\varphi)$ and $f(\varphi)$ in the mimetic disformal transformation (3.71). Finally, we also showed that the mimetic theory, when written in terms of the metric $g_{\mu\nu}$, does not contain any more derivatives than the scalar-tensor theory that originated it. This may not be the case for the mimetic theory when written in terms of the new metric $\ell_{\mu\nu}$.

Chapter 4

Cosmological perturbations in mimetic Horndeski gravity

4.1 Introduction

In the previous Chapter 3 (Ref.[2]), we have shown that such general theories are invariant under generalised disformal transformations. However, for a small subset of those transformations, when they are not invertible, the resulting theory is a generalisation of the original mimetic gravity theory. We have proposed two simple toy models within the mimetic Horndeski class and showed that they possess interesting cosmological solutions. For instance, the simplest mimetic model is able to mimic the cosmological background evolution of a flat FLRW model with a barotropic perfect fluid with any constant equation of state (see also [41] for an earlier work). Actually by appropriately choosing the function $b(\varphi)$ in the transformation one can mimic almost any desired expansion history.

The stability of mimetic gravity against negative energy states, i.e. ghosts, was studied in [45], where it was shown that ghosts are absent if the energy

density of the effective fluid is positive. Ref. [52] (see also [45] and [40]) showed that the original mimetic gravity can be derived from an action with a constraint imposed by a Lagrange multiplier without the need to invoke a disformal transformation. In this chapter we will follow this complementary approach of using a Lagrange multiplier.

In the original mimetic model [39] and its generalisation to include a potential [40], it was shown that the sound speed of scalar perturbations is exactly zero (independently of the desired expansion history) and consequently this model cannot describe a successful inflationary model because quantum fluctuations cannot be defined as usual. To circumvent this problem, it was proposed to introduce higher-derivatives terms in the action [40]. In this way a non-zero sound speed can be generated. These higher-derivative terms help to suppress power for large momentum and it has been argued that this can be relevant for the small-scale problems of cold dark matter [44].

The main purposes of this work are to study linear scalar perturbations in mimetic Horndeski gravity and to determine the corresponding value of the sound speed. These results will determine the growth of structure in mimetic Horndeski models.

In the meantime, there have been many works studying different aspects of the original mimetic theory and generalisations. For example, the Hamiltonian analysis was performed in [74, 75], cosmological perturbations were further analyzed in [76], extensions to $f(R)$ type models were presented in [77, 59], [78] studied the energy conditions and a generalization, a mimetic theory including a vector field was proposed in [45], cosmology in mimetic Galileon models studied in [73, 79], and the imperfect fluid nature induced by higher-derivative terms was further discussed in [43].

This chapter is organised as follows. In the next section, we introduce the

model and some notation. We also show the independence of general equations of motion of scalar-tensor mimetic gravity. In section 4.3, we discuss linear scalar perturbations of mimetic Horndeski in the Poisson gauge excluding other matter fields. We will compute the background equations of motion, then the first-order equations of motion for the Newtonian potential which we solve for the two toy models introduced in [2]. Section 4.4 is devoted to the initial value formulation of the problem and to the discussion on the sound speed in general cosmological backgrounds. Section 4.5 presents the conclusions of the chapter. The chapter has 4 appendices. In appendix B.1 we present the explicit expressions for the background equations of motion. Appendix B.2 contains the expressions for the functions defined in the main text and that enter the first-order equations of motion. In appendix B.3, we present the background and linear equations of motion for the mimetic Horndeski model including matter in the form of a fluid that may have anisotropic stress. Finally in appendix B.4, we compute the sound speed in a theory beyond mimetic Horndeski. We call this theory mimetic G^3 theory as it is the mimetic version of the so-called G^3 theory [60, 61].

4.2 The model and notation

In this section we will start by using the EOM of a very general mimetic scalar-tensor theory of gravity including a term with a Lagrange multiplier, which was derived in Chapter-3 (or in [2]). In the following sections, where we will present explicit results for linear cosmological perturbations, we will restrict the very general mimetic scalar-tensor theory to the mimetic Horndeski theory. Horndeski's theory [63] is the most general 4D covariant theory of scalar-tensor gravity that is derived from an action and gives rise to

second-order equations of motion (in all gauges and in any background) for both the metric and the scalar field. This useful property guarantees that the mimetic theory is free from higher-derivative ghosts because, as shown in [2], if the original theory is free from these ghosts then also the mimetic theory that it originates is free from them. However, we might have to put the right constraint over the free parameters in order to guarantee positive kinetic energy.

The set of equations, Eq. (3.48), (3.49), (3.50) and (3.51) are the equations of motion for the very general action for scalar-tensor mimetic gravity (3.42). However not all the equations in the set are independent from each other. As shown in [2], Eq. (3.49) can be derived from the other equations. Also as we will now show, the 0 – 0 component of Eqs. (3.50) can be derived from Eq. (3.43) and the remaining components of Eqs. (3.50).

Let us start with the constraint equation

$$\begin{aligned} b(\varphi)g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi &= b(\varphi)g^{00}(\varphi')^2 + 2b(\varphi)g^{0i}(\varphi')\partial_i\varphi + b(\varphi)g^{ij}\partial_i\varphi\partial_j\varphi \\ &= 1, \end{aligned} \quad (4.1)$$

where ' denotes the derivative with respect to the time coordinate (which in the next section we choose to be conformal time). Multiply both sides of the previous equation by $E + T$ to obtain

$$\begin{aligned} (E + T)b(\varphi)g^{00}(\varphi')^2 + 2(E + T)b(\varphi)g^{0i}(\varphi')\partial_i\varphi + (E + T)b(\varphi)g^{ij}\partial_i\varphi\partial_j\varphi \\ = g^{00}(E_{00} + T_{00}) + 2g^{0i}(E_{0i} + T_{0i}) + g^{ij}(E_{ij} + T_{ij}), \end{aligned} \quad (4.2)$$

where Latin indexes run from one to three only. By using the other compo-

nents of Eqs. (3.50), i.e.

$$E_{ij} + T_{ij} = (E + T)b(\varphi)\partial_i\varphi\partial_j\varphi, \quad E_{0i} + T_{0i} = (E + T)b(\varphi)\varphi'\partial_i\varphi, \quad (4.3)$$

one can show that Eq. (4.2) simplifies to

$$(E + T)b(\varphi)g^{00}(\varphi')^2 = g^{00}(E_{00} + T_{00}). \quad (4.4)$$

Because $g^{00} \neq 0$ we have the desired result that Eqs. (4.3) together with the constraint equation imply

$$E_{00} + T_{00} = (E + T)b(\varphi)(\varphi')^2. \quad (4.5)$$

This is a non-perturbative result and it will be important when counting the number of perturbation variables and their equations in the next section.

We will only consider a particular subset of theories of the form (3.42) known as mimetic Horndeski theory. We refer to the Section 3.2 and Section 3.4 for the action of Horndeski and, respectively, mimetic Horndeski gravity.

4.3 Linear scalar perturbations

This section is devoted to the study of cosmological linear scalar perturbations in the mimetic Horndeski gravity. Here we will assume that there is no matter in the model, i.e. $S_m = 0$. We expect this to be a good approximation during the time when the effective energy density of the mimetic scalar field is much larger than the other usual components of the total energy density like radiation or cold dark matter. In appendix B.3 we present the equations of motion of the mimetic Horndeski model including a matter source in the

form of a fluid which may have anisotropic stress as it would be the case for free-streaming neutrinos. Before that, in the next subsection we will present well-known (see for instance [65, 48]) results for linear scalar perturbations in Horndeski gravity, as a warm up.

We will work in the Poisson gauge. Because we are only interested in scalar perturbations, we will neglect vector and tensor perturbations. At linear order and in the flat FLRW background that we will assume, these different type of perturbations are all decoupled.

The metric is perturbed as

$$g_{00} = -a^2(\tau) (1 + 2\Phi), \quad g_{0i} = 0, \quad g_{ij} = a^2(\tau) (1 - 2\Psi) \delta_{ij}, \quad (4.6)$$

where a is the FLRW scale factor that depends on the conformal time τ , Φ denotes the generalised Newtonian (Bardeen) potential and Ψ the curvature perturbation. The inverse metric is

$$g^{00} = -a^{-2}(\tau) (1 - 2\Phi), \quad g^{0i} = 0, \quad g^{ij} = a^{-2}(\tau) (1 + 2\Psi) \delta_{ij}. \quad (4.7)$$

The scalar field is expanded as $\varphi(\tau, \mathbf{x}) = \varphi_0(\tau) + \delta\varphi(\tau, \mathbf{x})$, where φ_0 denotes the background field value and $\delta\varphi$ is the field perturbation.

4.3.1 Linear scalar perturbations in Horndeski

We will study linear perturbations of Horndeski gravity only in this subsection. The theory is defined by the action (3.8). The tensor $E^{\mu\nu}$ introduced in the previous section will be the same for both Horndeski and mimetic Horndeski gravity as it is clear from its definition.

Because we assume that there are no matter sources and the equation of motion for φ is not independent from the metric equations of motion as it is

well-known¹, the equations of motion are simply

$$E_{\mu\nu} = 0. \quad (4.8)$$

At the background level they reduce to $E_{\mu\nu}^{(0)} = 0$, where the superscript (0) denotes background quantities and the explicit expressions for $E_{\mu\nu}^{(0)}$ in terms of the Horndeski functions and their derivatives can be found in appendix B.1.

At first order (denoted by the superscript (1)) the tensor $E_{\mu\nu}$ can be written as

$$E_{00}^{(1)} = f_1\Psi' + f_2\delta\varphi' + f_3\Phi + f_4\delta\varphi + f_5\partial^2\Psi + f_6\partial^2\delta\varphi, \quad (4.9)$$

$$E_{ij}^{(1)} = \partial_i\partial_j(f_7\Psi + f_8\delta\varphi + f_9\Phi) + \delta_{ij}\left(-f_7\partial^2\Psi - f_8\partial^2\delta\varphi - f_9\partial^2\Phi + f_{10}\Psi'' + f_{11}\delta\varphi'' + f_{12}\Psi' + f_{13}\delta\varphi' + f_{14}\Phi' + f_{15}\Psi + f_{16}\delta\varphi + f_{17}\Phi\right), \quad (4.10)$$

$$E_{0i}^{(1)} = \partial_i(f_{18}\Psi' + f_{19}\delta\varphi' + f_{20}\delta\varphi + f_{21}\Phi), \quad (4.11)$$

where $'$ denotes derivative with respect to conformal time, the functions f_i , $i = 1, \dots, 21$ are linear functions of K, G_3, G_4, G_5 and their derivatives evaluated on the background, therefore the f_i are functions of time only. Their explicit expressions are given in appendix B.2. These functions are not all independent from each other and they obey certain relations also given in appendix B.2.

¹This well-known fact can be simply understood to be a consequence of Horndeski's identity [63] (see also references therein), i.e., $\sqrt{-g}\nabla_\mu E^{\mu\nu} = \Omega_\varphi\nabla^\nu\varphi$. For a general scalar-tensor theory defined by the first line of Eq. (3.42), which includes Horndeski's theory as a particular case, the equation of motion for the scalar field is $\Omega_\varphi = 0$, which implies, by using the previous identity, $\nabla_\mu E^{\mu\nu} = 0$. The previous equation is the generalization of the usual equation for the conservation of the energy-momentum tensor. Eq. (4.8) automatically implies that the equation of motion for the scalar field is satisfied.

For a standard kinetic term scalar field coupled to Einstein gravity, at first order, it is well known that Eqs. (4.8) are not all independent. By taking the time derivative of the $E_{0i}^{(1)} = 0$ equation and using it again together with the background equations it is possible to obtain the evolution part of the $E_{ij}^{(1)} = 0$ equation (which corresponds to the second line of Eq. (4.10)). In Horndeski theory, something similar should also happen as we will discuss below. By taking the traceless part of $E_{ij}^{(1)} = 0$ one can see that the first line of $E_{ij}^{(1)}$ vanishes. In other words the traceless part of $E_{ij}^{(1)} = 0$ implies that

$$f_7\Psi + f_8\delta\varphi + f_9\Phi = 0. \quad (4.12)$$

The physical implications of the previous equation are that the anisotropic stress is in general not zero and also that at least one of the fields is not a new dynamical degree of freedom. This equation will also be valid in the mimetic Horndeski case.

Let us now count the number of variables and equations of motion. We have three variables, $\delta\varphi$, Φ and Ψ to be determined by the equations of motion, $E_{\mu\nu}^{(1)} = 0$ which can be written as

$$f_1\Psi' + f_2\delta\varphi' + f_3\Phi + f_4\delta\varphi + f_5\partial^2\Psi + f_6\partial^2\delta\varphi = 0, \quad (4.13)$$

$$f_7\Psi + f_8\delta\varphi + f_9\Phi = 0, \quad (4.14)$$

$$f_{10}\Psi'' + f_{11}\delta\varphi'' + f_{12}\Psi' + f_{13}\delta\varphi' + f_{14}\Phi' + f_{15}\Psi + f_{16}\delta\varphi + f_{17}\Phi = 0, \quad (4.15)$$

$$f_{18}\Psi' + f_{19}\delta\varphi' + f_{20}\delta\varphi + f_{21}\Phi = 0. \quad (4.16)$$

Naively one has three variables and four equations, however, one can show that there are only three independent equations, as expected. Eq. (4.15) can be derived from Eqs. (4.14) and (4.16) and by using some of the identities

in appendix B.2 ².

4.3.2 Linear scalar perturbations in mimetic Horndeski

We now turn to the main goal of this chapter, i.e., to study linear scalar perturbations in mimetic Horndeski gravity. As we have explained in Sec. 4.2 the independent equations of motion for the model reduce to (assuming that there is no matter; see appendix B.3 for the case when matter is present)

$$b(\varphi)g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - 1 = 0, \quad E_{\mu i} = Eb(\varphi)\partial_\mu\varphi\partial_i\varphi. \quad (4.17)$$

At zeroth order on a flat FLRW background, they simplify to

$$-a^{-2}b_0(\varphi'_0)^2 = 1, \quad (4.18)$$

$$E_{ij}^{(0)} = 0, \quad (4.19)$$

where b_0 denotes $b(\varphi_0)$.

The first-order equations of motion are

$$2b_0\delta\varphi' + \varphi'_0 b_{,\varphi}\delta\varphi - 2b_0\varphi'_0\Phi = 0, \quad (4.20)$$

$$E_{ij}^{(1)} = 0, \quad (4.21)$$

$$E_{0i}^{(1)} = E^{(0)}b_0\varphi'_0\partial_i\delta\varphi, \quad (4.22)$$

²One can follow a brute force procedure: use Eq. (4.14) to write Ψ in terms of the other variables. Use Eq. (4.16) and solve it for Φ' . Take the time derivative of Eq. (4.16) and sum to it a term $T \times \Phi'$ and then subtract from it the same term $T \times \Phi'$ but where now Φ' is replaced with the previously found expression and $T = 4\mathcal{H}f_9^2/f_7$, where \mathcal{H} is defined as $\mathcal{H} = a'/a$. The equation obtained differs from Eq. (4.15) in terms proportional to Φ and $\delta\varphi$ only. However, by using the identities in appendix B.2 one can show that actually the coefficients of Φ and $\delta\varphi$ terms are exactly the ones in Eq. (4.15).

where $E^{(0)}$ denotes the zeroth-order trace of $E_{\mu\nu}$, $b_{,\varphi} = b_{,\varphi}(\varphi_0)$ and the subscript $_{,\varphi}$ denotes derivative with respect to the field φ . As mentioned in the previous subsection the $E_{\mu\nu}$ tensor is equal to the one defined for the Horndeski's theory whose explicit expressions are given by Eqs. (4.9)-(4.11). Eq. (4.21) implies

$$f_7\Psi + f_8\delta\varphi + f_9\Phi = 0, \quad (4.23)$$

$$f_{10}\Psi'' + f_{11}\delta\varphi'' + f_{12}\Psi' + f_{13}\delta\varphi' + f_{14}\Phi' + f_{15}\Psi + f_{16}\delta\varphi + f_{17}\Phi = 0, \quad (4.24)$$

and Eq. (4.22) implies

$$f_{18}\Psi' + f_{19}\delta\varphi' + \left(f_{20} + \frac{a^2 E^{(0)}}{\varphi_0'}\right)\delta\varphi + f_{21}\Phi = 0, \quad (4.25)$$

where we have used the zeroth-order constraint. As before, Eq. (4.23) can be used for example to eliminate Ψ from all the equations in favor of the pair $\delta\varphi, \Phi$. In other words, Ψ is not a new degree of freedom with respect to the pair $\delta\varphi, \Phi$. One can also see that in general $\Psi \neq \Phi$ for mimetic Horndeski (i.e., there is some non-zero effective anisotropic stress).

It is important to note that because Horndeski's theory is form-invariant under a field redefinition one can without loss of generality set $b(\varphi) = -1$ in mimetic Horndeski. In that case, $b_{,\varphi} = 0$ and then the first-order constraint implies $\Phi = \frac{\delta\varphi'}{\varphi_0'}$.

At this point, we have four equations of motion, Eqs. (4.20), (4.23), (4.24) and (4.25), and only three variables, Ψ , Φ and $\delta\varphi$. However, following a similar procedure to the one in the previous subsection one can show that (4.24) can be derived from the other two equations, i.e., Eqs. (4.23) and (4.25), where this time to complete the proof one also needs to use Eq.

(4.20). In summary, the independent first-order equations of motion for the mimetic Horndeski model that we will use from now on are

$$2b_0\delta\varphi' + \varphi_0' b_{,\varphi}\delta\varphi - 2b_0\varphi_0'\Phi = 0, \quad (4.26)$$

$$f_7\Psi + f_8\delta\varphi + f_9\Phi = 0, \quad (4.27)$$

$$f_{18}\Psi' + f_{19}\delta\varphi' + \left(f_{20} + \frac{a^2 E^{(0)}}{\varphi_0'}\right)\delta\varphi + f_{21}\Phi = 0. \quad (4.28)$$

Because in this system of equations there are no spatial derivatives one can anticipate that the sound speed for the dynamical scalar degree of freedom will be exactly zero.

Indeed, from the previous three equations one can find an evolution equation for the Newtonian potential Φ as

$$\begin{aligned} \Phi'' &+ \left(\frac{B_2}{B_3} + \left(\ln \frac{B_3}{B_1}\right)' + \mathcal{H} - \frac{\varphi_0''}{\varphi_0'}\right)\Phi' \\ &+ \left(\frac{B_1}{B_3}\varphi_0' + \frac{B_1}{B_3}\left(\frac{B_2}{B_1}\right)' + \frac{B_2}{B_3}\left(\mathcal{H} - \frac{\varphi_0''}{\varphi_0'}\right)\right)\Phi = 0, \end{aligned} \quad (4.29)$$

where the B_i functions are defined as

$$\begin{aligned} B_1 &= f_{20} + \frac{f_{10}f_8f_7'}{f_7^2} + f_{11}\left(-\mathcal{H} + \frac{\varphi_0''}{\varphi_0'}\right) \\ &\quad - \frac{f_{10}}{f_7}\left(f_8' + f_8\left(-\mathcal{H} + \frac{\varphi_0''}{\varphi_0'}\right)\right) + a^2\frac{E^{(0)}}{\varphi_0'}, \end{aligned} \quad (4.30)$$

$$B_2 = f_{14} + \frac{f_{10}f_9f_7'}{f_7^2} + f_{11}\varphi_0' - \frac{f_{10}(f_9' + f_8\varphi_0')}{f_7}, \quad (4.31)$$

$$B_3 = 2\frac{f_9^2}{f_7}. \quad (4.32)$$

There is no spatial Laplacian term so this means that the sound speed of the perturbations is exactly zero as anticipated. In appendix B.4, we show that the conclusion that the sound speed is exactly zero also applies to a

scalar-tensor theory more general than mimetic Horndeski, the one which is built starting from G^3 theories [60, 61]. Notice also that in Eq. (4.29) there is no source term on the right-hand side, which is usually associated to the presence of entropy perturbation modes, see, e.g., [80]. This will be also confirmed below when deriving the equation of motion for the comoving curvature perturbation. After solving Eq. (4.29) for Φ one can use the constraint equation, Eq. (4.26), for a given function $b(\varphi)$, to solve for $\delta\varphi$. Finally to find Ψ one can solve

$$\delta\varphi = \left(-2\mathcal{H}f_8 + f_{11}\varphi'_0 \frac{b_{,\varphi}}{2b_0} - f_{20} - \frac{a^2 E^{(0)}}{\varphi'_0} \right)^{-1} (f_{10}\Psi' + 2\mathcal{H}f_7\Psi). \quad (4.33)$$

It is convenient to introduce a new variable, the comoving curvature perturbation ζ (the comoving gauge is defined by $\delta\varphi = 0$ and in that gauge ζ is related (to first order) to the 3D curvature as $R^{(3)} = -4a^{-2}\partial^2\zeta$), which is defined as

$$-\zeta = \Psi + \frac{\mathcal{H}}{\varphi'_0}\delta\varphi. \quad (4.34)$$

One can show that the set of equations of motion of the model, Eqs. (4.26)-(4.28), is equivalent to (using the background equations of motion)

$$2b_0\delta\varphi' + \varphi'_0 b_{,\varphi}\delta\varphi - 2b_0\varphi'_0\Phi = 0, \quad (4.35)$$

$$-f_7\zeta + \left(f_8 - \frac{\mathcal{H}}{\varphi'_0}f_7 \right)\delta\varphi + f_9\Phi = 0, \quad (4.36)$$

$$\zeta' = 0. \quad (4.37)$$

Note that the comoving curvature perturbation ζ has a first-order equation of motion with solution $\zeta = \text{constant}$ on all scales (and vanishing intrinsic entropy perturbations, see, e.g. [81]).

For the particular case when $G_4(X, \varphi) = 1/2$ and $G_5(X, \varphi) = 0$ (and the

other functions of the Horndeski theory, i.e. K and G_3 , are kept general) we were able to simplify the evolution equation, Eq. (4.29), to obtain

$$\Phi'' + \Phi' (3\mathcal{H} + \tilde{\Gamma}) + \Phi (\mathcal{H}^2 + 2\mathcal{H}' + \tilde{\Gamma}\mathcal{H}) = 0, \quad (4.38)$$

where the variable $\tilde{\Gamma}$ is defined as

$$\tilde{\Gamma} = \frac{-\mathcal{H}'' + \mathcal{H}\mathcal{H}' + \mathcal{H}^3}{\mathcal{H}' - \mathcal{H}^2}. \quad (4.39)$$

The quantity $\tilde{\Gamma}$ can be seen as a correction to the perturbation equation of standard pressureless dust that arises in these mimetic models. This equation was first derived and solved in [41] for the case when the function $G_3(X, \varphi)$ was zero. What we found is that this equation is still valid even if $G_3(X, \varphi) \neq 0$. Let us note that the particular mimetic models for which Eq. (4.38) is valid include the two models studied in [2] that showed very interesting cosmological behaviour (e.g. they can reproduce exactly the Λ CDM background expansion). It is important to note that the quantity $\tilde{\Gamma}$ is written in a geometrical way and that it exactly vanishes for a Λ CDM expansion history. Indeed, the differential equation $\tilde{\Gamma} = 0$ has the three solutions: $a(t) \propto \exp^{\mathcal{C}t}$, $a(t) \propto t^{2/3}$ and $a(t) \propto \sinh^{\frac{2}{3}}(\mathcal{C}t)$. In the limit of a Λ CDM background expansion history, corresponding to the latter solution, the perturbations in these particular mimetic Horndeski models will behave exactly in the same way as the perturbations in a Λ CDM universe. For the particular case when $G_4(X, \varphi) = 1/2$ and $G_5(X, \varphi) = 0$, the relation between ζ and Φ is

$$\zeta = -\frac{2\mathcal{H}^2 - \mathcal{H}'}{\mathcal{H}^2 - \mathcal{H}'}\Phi - \frac{\mathcal{H}}{\mathcal{H}^2 - \mathcal{H}'}\Phi', \quad (4.40)$$

and $\Psi = \Phi$.

To find the solutions of Eq. (4.38) let us use the results of [41]. By using the number of e-folds, $N = \ln a$, as the time variable and using a new variable as

$$Q = \sqrt{\frac{a}{-H_{,N}}} \Phi, \quad (4.41)$$

where the subscript $,N$ denotes derivative with respect to N , the Hubble rate is defined as $H = da/(adt)$, where t denotes cosmic time, one can write the evolution equation, Eq. (4.38), as

$$Q_{,NN} - \frac{\Theta_{,NN}}{\Theta} Q = 0, \quad (4.42)$$

where the variable Θ is defined as

$$\Theta = \frac{H}{\sqrt{-aH_{,N}}}. \quad (4.43)$$

It is immediate to show that $Q \propto \Theta$ is a solution of Eq. (4.42). The other solution can be found using the Wronskian method and is

$Q \propto \sqrt{\frac{a}{-H_{,N}}} \left(1 - \frac{H}{a} \int \frac{da}{H}\right)$. This implies that the two solutions for the Newtonian potential are

$$\Phi_1 = \frac{H}{a}, \quad \Phi_2 = 1 - \frac{H}{a} \int \frac{da}{H}. \quad (4.44)$$

The general solution for Φ is a linear combination of Φ_1 and Φ_2 , with $C_1(\mathbf{x})$ and $C_2(\mathbf{x})$ being the integration constants, as

$$\Phi(\tau, \mathbf{x}) = C_1(\mathbf{x}) + \frac{H}{a} C_2(\mathbf{x}) - C_1(\mathbf{x}) \frac{H}{a} \int \frac{da}{H}. \quad (4.45)$$

The same solutions were found in [41] for a model with $G_4(X, \varphi) = 1/2$ and $G_3(X, \varphi) = G_5(X, \varphi) = 0$. Here we show that the form of the solutions is the

same even if $G_3(X, \varphi) \neq 0$. The solution Φ_1 corresponds to $\zeta = 0$ while the solution Φ_2 corresponds to $\zeta = \text{constant} \neq 0$. If the scale factor is $a \propto t^{\frac{2}{3(1+w)}}$, where w is the equation of state, then $H/a \propto a^{\frac{-5-3w}{2}}$ which decays for an expanding universe if $w > -5/3$ and therefore Φ_1 is the decaying mode in that case (Φ_2 is constant).

4.4 The Cauchy problem and the sound speed

In this section, we will follow the method of [41] to show that, without assuming any background but for a non-dynamical (“external”) metric, the sound speed is exactly zero in a general mimetic Horndeski theory (without additional matter fields). We will assume that the four-velocity is time-like because we have cosmology applications in mind.

The constraint equation, Eq. (3.48), which is the equation of motion for the Lagrange multiplier, implies that

$$b(\varphi)X = -\frac{1}{2}, \quad (4.46)$$

where we assume that $b(\varphi) < 0$. For a mimetic Horndeski theory, one can redefine the scalar field as to absorb $b(\varphi)$, or in other words, one can choose $b(\varphi) = -1$ without losing generality [2]. From now on in this section we will set $b(\varphi) = -1$. The constraint then implies $X = 1/2$. One can define the four-velocity as

$$u_\nu = \frac{\nabla_\nu \varphi}{\sqrt{2X}} = \nabla_\nu \varphi, \quad (4.47)$$

which satisfies the constraint $u_\nu u^\nu = -1$ and in the last equality we have used the constraint $X = 1/2$. In this section we will use the notation $\dot{(\)} = u^\nu \nabla_\nu (\)$ to denote the derivative along u^ν . It is easy to see that the four-acceleration,

a^ν , is always zero, i.e.

$$a^\nu = \dot{u}^\nu = u^\mu \nabla_\mu u^\nu = 0, \quad (4.48)$$

because of the constraint equation, Eq. (4.46). This means the flow is always geodesic. The constraint equation becomes very simple, it reads

$$\dot{\varphi} = -1. \quad (4.49)$$

Let us decompose the covariant derivative of u_ν in its symmetric and skew-symmetric parts as

$$\nabla_\mu u_\nu = \theta_{\mu\nu} + \omega_{\mu\nu}, \quad (4.50)$$

where

$$\begin{aligned} \theta_{\mu\nu} &= \nabla_{(\mu} u_{\nu)}, & \omega_{\mu\nu} &= \nabla_{[\mu} u_{\nu]}, & \theta_{\mu\nu} &= \theta_{(\mu\nu)} = \frac{1}{2}(\theta_{\mu\nu} + \theta_{\nu\mu}), \\ \omega_{\mu\nu} &= \omega_{[\mu\nu]} = \frac{1}{2}(\omega_{\mu\nu} - \omega_{\nu\mu}). \end{aligned} \quad (4.51)$$

The tensor $\theta_{\mu\nu}$ is called the expansion tensor and the tensor $\omega_{\mu\nu}$ is called the vorticity tensor. These satisfy $\theta_{\mu\nu} u^\mu = \omega_{\mu\nu} u^\mu = 0$. In the present case of a mimetic scalar field, the vorticity tensor is zero because $\nabla_\mu u_\nu = \nabla_\nu u_\mu$. Let us further decompose the expansion tensor in its trace and trace-free parts as

$$\theta_{\mu\nu} = \sigma_{\mu\nu} + \frac{1}{3}\theta h_{\mu\nu}, \quad (4.52)$$

where $h_{\mu\nu}$ is defined as $h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$. θ is called the volume expansion and $\sigma_{\mu\nu}$ is called the shear tensor. These satisfy $\sigma_{\mu\nu} u^\mu = \sigma_\nu^\nu = 0$ where

$\sigma_\nu^\nu = g^{\mu\nu}\sigma_{\mu\nu}$ and $\sigma_{(\mu\nu)} = \sigma_{\mu\nu}$. One can then find

$$\theta = g^{\mu\nu}\theta_{\mu\nu} = \nabla_\nu u^\nu, \quad \sigma_{\mu\nu} = \nabla_\mu u_\nu - \frac{1}{3}\theta(g_{\mu\nu} + u_\mu u_\nu). \quad (4.53)$$

θ satisfies the well-known Landau-Raychaudhuri equation

$$\dot{\theta} = -\sigma_{\mu\nu}\sigma^{\mu\nu} - \frac{\theta^2}{3} - R_{\mu\nu}u^\mu u^\nu. \quad (4.54)$$

The evolution equation for the shear tensor is

$$\dot{\sigma}_{\mu\nu} = -\sigma_\mu^\lambda \sigma_{\lambda\nu} - \frac{2}{3}\theta\sigma_{\mu\nu} - \frac{1}{3}h_{\mu\nu} \left(\dot{\theta} + \frac{1}{3}\theta^2 \right) - R_{\alpha\lambda\beta\rho}u^\lambda u^\rho. \quad (4.55)$$

In deriving the previous two equations we have used several times Eq. (4.48), $(\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha)V^\mu = R^\mu{}_{\nu\alpha\beta}V^\nu$ for a vector V^μ , $R_{\mu\nu} = R^\lambda{}_{\mu\lambda\nu}$ and the general properties of the Riemann tensor $R_{\mu\nu\alpha\beta}$.

The equation of motion for the field φ can be written as

$$2\dot{\lambda} + 2\theta\lambda + \frac{\Omega_\varphi}{\sqrt{-g}} = 0, \quad (4.56)$$

where λ is the Lagrange multiplier field introduced in Eq. (3.42), Ω_φ , defined in Eq. (3.24), can be written as $\Omega_\varphi = \sqrt{-g} \sum_{i=2}^5 \left(P_\varphi^{(i)} - \nabla^\mu J_\mu^{(i)} \right)$ and the explicit (long) expressions of $P_\varphi^{(i)}$ and $J_\mu^{(i)}$ can be found in Appendix B of [65]. For the mimetic Horndeski theory that we are interested in, those expressions can be written as

$$P_\varphi^{(2)} - \nabla^\mu J_\mu^{(2)} = K_{,\varphi} - (K_{,X\varphi} - K_{,X}\theta), \quad (4.57)$$

$$P_\varphi^{(3)} - \nabla^\mu J_\mu^{(3)} = -G_{3,\varphi\varphi} - \left[-2G_{3,\varphi\varphi} + (2G_{3,\varphi} - G_{3,X\varphi})\theta + G_{3,X}(\theta^2 + \dot{\theta}) \right], \quad (4.58)$$

$$\begin{aligned}
P_\varphi^{(4)} - \nabla^\mu J_\mu^{(4)} &= G_{4,\varphi}R + G_{4,X\varphi} \left(\frac{2}{3}\theta^2 - \sigma_{\mu\nu}\sigma^{\mu\nu} \right) \\
&\quad - \left[G_{4,X} \left(-R\theta + 2R_{\mu\nu} \left(\sigma^{\mu\nu} + \frac{1}{3}\theta h^{\mu\nu} \right) \right) \right. \\
&\quad - G_{4,XX} \left(\frac{4}{3}\theta\dot{\theta} - 2\sigma_{\mu\nu}\dot{\sigma}^{\mu\nu} + \left(\frac{2}{3}\theta^2 - \sigma_{\mu\nu}\sigma^{\mu\nu} \right) \theta \right) \\
&\quad + G_{4,X\varphi} \left(R + 2u^\mu u^\nu R_{\mu\nu} - 2\dot{\theta} - 2\theta^2 \right) \\
&\quad \left. + G_{4,XX\varphi} \left(\frac{2}{3}\theta^2 - \sigma_{\mu\nu}\sigma^{\mu\nu} \right) + 2G_{4,X\varphi\varphi}\theta \right], \quad (4.59)
\end{aligned}$$

$$\begin{aligned}
P_\varphi^{(5)} - \nabla^\mu J_\mu^{(5)} &= -G_{5,\varphi\varphi}u^\mu u^\nu G_{\mu\nu} - \frac{1}{6}G_{5,X\varphi} \left(2\sigma_{\mu\nu}\sigma^{\nu\lambda}\sigma_\lambda^\mu - \theta\sigma_{\mu\nu}\sigma^{\mu\nu} + \frac{2}{9}\theta^3 \right) \\
&\quad - \left[\left(\frac{1}{2}G_{5,X\varphi} \left(\frac{2}{3}\theta^2 - \sigma_{\mu\nu}\sigma^{\mu\nu} \right) \right. \right. \\
&\quad + \frac{1}{6}G_{5,XX} \left(2\sigma_{\mu\nu}\sigma^{\nu\lambda}\sigma_\lambda^\mu - \theta\sigma_{\mu\nu}\sigma^{\mu\nu} + \frac{2}{9}\theta^3 \right) \left. \right. \\
&\quad + \frac{1}{6}G_{5,XX}\theta \left(2\sigma_{\mu\nu}\sigma^{\nu\lambda}\sigma_\lambda^\mu - \theta\sigma_{\mu\nu}\sigma^{\mu\nu} + \frac{2}{9}\theta^3 \right) \\
&\quad - G_{5,\varphi} \left(2R_{\mu\nu}\sigma^{\mu\nu} - \frac{1}{3}R\theta + \frac{2}{3}\theta u^\mu u^\nu R_{\mu\nu} \right) \\
&\quad - G_{5,X\varphi} \left(-R_{\mu\nu}\sigma^{\mu\nu} + \frac{1}{6}\theta R + \frac{1}{3}\theta u^\mu u^\nu R_{\mu\nu} - u^\mu u^\nu R_{\nu\lambda}\sigma_\mu^\lambda \right. \\
&\quad \quad \left. - u^\mu u^\nu \sigma^{\alpha\beta} R_{\alpha\mu\beta\nu} - \frac{1}{3}\theta^3 + \frac{1}{2}\sigma_{\mu\nu}\sigma^{\mu\nu}\theta \right) \\
&\quad - G_{5,X} \left(R_{\mu\nu}\dot{\sigma}^{\mu\nu} - \frac{1}{6} \left(\dot{\theta} + \frac{1}{3}\theta^2 \right) R \right. \\
&\quad \quad + \frac{1}{3}R_{\mu\nu} \left(2\theta\sigma^{\mu\nu} + \left(\theta^2 + \dot{\theta} \right) u^\mu u^\nu \right) - R_{\mu\nu}\sigma^{\mu\lambda}\sigma_\lambda^\nu \\
&\quad \quad - \sigma^{\alpha\beta} R_{\alpha\mu\beta\nu} \left(\sigma^{\mu\nu} + \frac{2}{3}\theta u^\mu u^\nu \right) - R_{\mu\nu}R_\lambda^\mu u^\nu u^\lambda \\
&\quad \left. \left. - u^\nu u_\lambda R_{\alpha\mu\beta\nu} R^{\mu\beta\alpha\lambda} \right) - G_{5,\varphi\varphi} (2u^\mu u^\nu R_{\mu\nu} + R) \right], \quad (4.60)
\end{aligned}$$

where to obtain the previous equations we have used the Bianchi identity, $\nabla_\mu G^{\mu\nu} = 0$ and the second Bianchi identity $\nabla_\lambda R_{\alpha\beta\mu\nu} + \nabla_\mu R_{\alpha\beta\nu\lambda} + \nabla_\nu R_{\alpha\beta\lambda\mu} =$

0. Using the previous expressions and Eqs. (4.54) and (4.55), one can see that Eq. (4.56) does not contain derivatives higher than two in both the metric and the scalar field φ as expected. This is because the mimetic theory does not change the number of derivatives of the original theory [2] and in the previous case the original theory was Horndeski's theory whose solution is well-known to contain no derivatives higher than two in the equations of motion. Note also that the equations of motion for λ and φ , Eqs. (4.49) and (4.56) respectively, are first-order ordinary differential equations. Following [41] one can argue that the Cauchy problem has a unique solution locally that depends only on the two initial conditions for φ and λ . Furthermore from Eqs. (4.49) and (4.56) one can see that the solutions evolve along time-like geodesics and neighbouring space points do not "communicate" with each other, this implies that the sound speed is identically zero (for any cosmological background but with a non-dynamical metric) as we wanted to show. The Cauchy problem may become ill-defined for some time in the future for initial condition that give origin to caustics. This would be a problem for this model which is beyond the scope of this chapter. Caustics are known to appear in other theories with non-canonical scalar fields, see for example [82, 83].

4.5 Conclusions

In this chapter we have studied linear scalar perturbations around a flat FLRW background in mimetic Horndeski gravity. This work is an important first step in the study of the evolution of cosmological perturbations in this very general class of mimetic models. We have found that, in the absence of matter, the first-order equations of motion take a simple form given by Eqs.

(4.26)-(4.28) for the Bardeen potentials or equivalently Eqs. (4.35)-(4.37) using the comoving curvature perturbation. Just like in Horndeski's theory, in mimetic Horndeski gravity there is an effective anisotropic stress even in the absence of matter.

The (generalised) Newtonian potential was shown to satisfy a second-order ordinary differential equation with no spatial derivatives which implies that the sound speed for scalar perturbations is exactly zero for a flat FLRW background. We have explicitly solved this equation for mimetic Horndeski models which include the so-called cubic Galileon term. This case includes the cosmological models proposed in our previous work [2], where we showed that simple mimetic models can essentially mimic any desired expansion history. For this particular case, the form of the solutions that we found is the same as in the so-called $\lambda\varphi$ fluids [41] which are a generalisation of the mimetic dark matter scenario [39, 40]. We have shown that in these models, if the background expansion history is exactly equal to the Λ CDM expansion history then also the perturbations will evolve in exactly the same way.

The equation of motion for the comoving curvature perturbation is a first order ordinary differential equation which can be easily solved to find that the comoving curvature perturbation in general mimetic Horndeski gravity is exactly constant on all scales.

These results show that in these models there are no wave-like propagating scalar degrees of freedom.

We have shown that the conclusion that the sound speed of scalar perturbations is exactly zero around a flat FLRW background also applies to a mimetic theory beyond mimetic Horndeski, i.e. mimetic G^3 theories. (Mimetic) G^3 theories are interesting because they contain only one extra scalar degree of freedom in addition to the usual two polarizations of the graviton. On the

other hand it is well-known that if one starts with a theory with higher-derivative equations of motion and which contains additional scalar degrees of freedom then the mimetic theory may have a non-zero sound speed [40]. These results indicate that there might be a relation between the value of the scalar sound speed and the number of degrees of freedom of the original theory. We leave the detailed investigation of this relation for future work. Finally, for a non-dynamical metric, we have shown that the scalar sound speed is exactly zero for all cosmological backgrounds. For future work, we leave the study of the Hamiltonian formulation of mimetic Horndeski gravity and the issue of whether caustics will develop or not and if so how one can interpret them.

Chapter 5

Vector mimetic gravity theory

5.1 Introduction

In the previous chapters, we have discussed properties and applications of scalar disformal transformations in the mimetic gravity scenario. Sanders [84] suggested to replace the derivative of the scalar field by a vector field in the disformal transformation to explain the degree of light deflection observed in distant clusters of galaxies by the dynamical effect of dark matter. He proposed the relation of the metrics,

$$g_{\mu\nu} = e^{-2\varphi}\ell_{\mu\nu} - 2\sinh(2\varphi)V_\mu V_\nu. \quad (5.1)$$

The V_μ is a priori a non-dynamical vector field, $g^{\mu\nu}V_\mu V_\nu = -1$. Bekenstein has further explored this particular type of disformal transformation in his TeVeS article [85]. It is interesting to notice that, in the above disformal transformation, the physical metric is related to the auxiliary metric by a vector field as well by a scalar field. The conformal and disformal factors are only functions of the scalar field.

Jacobson and Mattingly proposed a covariant model in which local Lorentz invariance is broken by a dynamical unit timelike vector, called ‘‘aether’’, V_μ [86, 87]. This theory has been further generalized by them in the subsequent articles and is known as Einstein-Aether (EA) gravity theory [88, 89]. The two-derivative action for Einstein-Aether-theory has the form

$$S = \frac{1}{2} \int d^4x \sqrt{-g} (R + L), \quad (5.2)$$

where, $L = (c_1 L_1 + c_2 L_2 + c_3 L_3 + c_4 L_4) + \lambda(-V_\mu V^\mu - 1)$, where the c_i ’s are dimensionless coupling constants, and

$$L_1 = (\nabla_\mu V_\nu)(\nabla^\mu V^\nu), \quad (5.3)$$

$$L_2 = (\nabla_\mu V^\mu)^2, \quad (5.4)$$

$$L_3 = (\nabla_\mu V_\nu)(\nabla^\nu V^\mu), \quad (5.5)$$

$$L_4 = (V^\mu \nabla_\mu V^\alpha)(V^\nu \nabla_\nu V_\alpha). \quad (5.6)$$

The Lagrange multiplier incorporates the timelike constraint of aether.

The Lagrangian can also be rewritten as,

$$L_1 + L_2 + L_3 + L_4 = \frac{1}{3} c_\theta \theta^2 + c_\sigma \sigma^2 + c_\omega \omega^2 + c_a a^2, \quad (5.7)$$

where, θ , $\sigma_{\mu\nu}$, $\omega_{\mu\nu}$ and a_μ are expansion, shear, twist and acceleration. They are defined by,

$$\begin{aligned} \theta &= \nabla_\mu V^\mu, & \sigma_{\alpha\beta} &= h_{(\alpha}^\mu h_{\beta)}^\nu (\nabla_\mu V_\nu - \frac{1}{3} \theta h_{\mu\nu}), \\ \omega_{\alpha\beta} &= h_{[\alpha}^\mu h_{\beta]}^\nu \nabla_\mu V_\nu, & a_\nu &= V^\mu \nabla_\mu V_\nu. \end{aligned} \quad (5.8)$$

where $h_{\mu\nu}$ is defined as $g_{\mu\nu} = V_\mu V_\nu - h_{\mu\nu}$, [90, 89] and the coupling constants

are

$$\begin{aligned}
c_\theta &= c_1 + c_3 + 3c_2, \\
c_\sigma &= c_1 + c_3, \\
c_\omega &= c_1 - c_3, \\
c_a &= c_1 + c_4.
\end{aligned}
\tag{5.9}$$

The particular type (mixture of a scalar and a vector field) of the disformal transformation in Eq. (5.1) was used in Einstein-Aether theory. In [34, 91], the authors have shown that TeVeS (akin to Einstein-Aether theories) can be written as a single metric theory with a timelike vector field of unfixed norm by using the aforementioned disformal transformation. A generalization of TeVeS theory, g-TeVeS was studied in Ref. [92], where the authors also have recovered General Relativity from g-TeVeS by using a Galileon induced Vainshtein mechanism. A scalar version of Einstein-Aether theory and its Newtonian limit was studied in Ref. [93], where the vector field was replaced by the derivative of a scalar field.

In the twist-free limit, Einstein-Aether theory becomes the Hořava-Lifshitz gravity [90]. On the other hand, the IR limit of Hořava-Lifshitz gravity can mimic general relativity plus cold dark matter [94], which has been explored later and called “dust of dark energy” [41].

However, the complete vector field disformal transformation given in Eq. (5.12) where the conformal and disformal functions are also functions of the vector field, have not been studied so far in the literature. A generic vector can be split into two parts, a pure vector field and the derivative of a scalar

field,

$$V_\mu = \tilde{V}_\mu + \partial_\mu \varphi, \quad (5.10)$$

$$V_\mu(k) = (V_\mu(k))_t + (V_\mu(k))_l, \quad (5.11)$$

where k represents the momentum space. \tilde{V}_μ is the pure vector, i.e., solenoidal part. This is a divergence-free component, which is also called transverse component. $\partial_\mu \varphi$ is called irrotational part. This is a curl-free component of a vector field, which is also called longitudinal component.

In this chapter, we will introduce the non-invertibility condition of the vector disformal transformation. Then we shall apply it to vector Einstein-Aether action with non-zero acceleration and rotation and show that vector Einstein-Aether theory is a class of mimetic gravity theories. After that, we shall study the system that can be recovered in the weak coupling limit. In the next section, we shall apply the non-invertible vector disformal transformation on the generalized ghost-free vector field action and formulate the generalized mimetic vector field gravity.

5.2 Non-invertibility condition of a vector disformal transformation

We consider the following vector disformal transformation,

$$g_{\mu\nu} = A(w)\ell_{\mu\nu} + B(w)V_\mu V_\nu. \quad (5.12)$$

A and B are arbitrary functions of the vector field, $w = \ell^{\mu\nu}V_\mu V_\nu$ corresponding to the vector field V_μ that defines the transformation. $g_{\mu\nu}$ is the original

metric, and $\ell_{\mu\nu}$ is an auxiliary new metric.

We shall apply such a disformal transformation on a particular type of vector field action,

$$S = \frac{1}{16\pi G_0} \int d^4x \sqrt{-g}(R + L), \quad (5.13)$$

where, $L = (c_1 L_1 + c_2 L_2 + c_3 L_3 + c_4 L_4)$ and L_1, L_2, L_3 and L_4 are given in Eq. (5.3), (5.4), (5.5) and (5.6).

Similar to Section 3.3.1, one can show that the Jacobian of the transformation of the disformal transformation of type Eq. (5.12) is

$$\Delta = w^2 A \frac{\partial}{\partial w} \left(B + \frac{A}{w} \right). \quad (5.14)$$

The transformation, Eq. (5.12), is non-invertible, if the Jacobian, $\Delta = 0$. Therefore we obtain,

$$B = -\frac{A}{w} + b, \quad (5.15)$$

where b is an integration constant. One could also derive the same constraint by the similar alternative method used in Section 3.3.2, where we applied the disformal transformation directly on the action (here Eq. (5.13)) and found the non-invertibility condition. The alternative form of the above equation, Eq. (5.15) would be,

$$bg^{\mu\nu} V_\mu V_\nu = 1, \quad (5.16)$$

which we may apply as a constraint on the action via a Lagrange multiplier, λ . Therefore, the above action, Eq. (5.13) will be

$$S = \frac{1}{2} \int d^4x \sqrt{-g}(R + L) + \int d^4x \sqrt{-g} \lambda (bg^{\mu\nu} V_\mu V_\nu - 1). \quad (5.17)$$

The above action is known as vector Einstein-Aether action if we set the integration constant $b = -1$, i.e., we may absorb b into the vector field by a vector redefinition. Therefore we recover the vector Einstein-Aether action from the non-invertible vector disformal transformation of the particular vector field action.

5.3 Weak coupling limit

In this section, we will consider a minimal Einstein-Aether theory as a toy model [86], where $L = F^2 = F^{\mu\nu}F_{\mu\nu}$ in Eq. (5.17). One can derive the equations of motion of the system as,

$$bg^{\mu\nu}V_\mu V_\nu - 1 = 0, \tag{5.18}$$

$$2\nabla_\nu F^{\nu\mu} + \frac{1}{2\kappa}(G - \kappa T)bg^{\mu\nu}V_\nu = 0, \tag{5.19}$$

$$-\frac{1}{2\kappa}(G^{\mu\nu} - \kappa T^{\mu\nu}) - 2(F^{\nu\rho}F^\mu{}_\rho - \frac{1}{4}F^2g^{\mu\nu}) + \frac{1}{2\kappa}(G - \kappa T)bV^\mu V^\nu = 0. \tag{5.20}$$

In this section we shall study the weak coupling limit on the system, i.e., $\sqrt{b} \rightarrow 0$. On the other hand, the $\sqrt{b} \rightarrow \infty$ limit corresponds to infinite strong coupling. In this limit, nonlinear terms would become important and the perturbative expansion would break down. Quantum correction would also dominate over the classical action.

For simplicity, we replace $\sqrt{b} = \epsilon$ for taking the weak coupling limit. In the

new redefinition, Eq. (5.18 to 5.20) will look like

$$\epsilon^2 g^{\mu\nu} V_\mu V_\nu - 1 = 0, \quad (5.21)$$

$$2\nabla^\nu F_{\nu\mu} + \frac{1}{2\kappa}(G - \kappa T)\epsilon^2 g_{\mu\nu} V^\nu = 0 \quad (5.22)$$

$$-\frac{1}{2\kappa}(G_{\mu\nu} - \kappa T_{\mu\nu}) - 2(F_{\nu\rho} F_\mu{}^\rho - \frac{1}{4}F^2 g_{\mu\nu}) + \frac{1}{2\kappa}(G - \kappa T)\epsilon^2 V_\mu V_\nu = 0. \quad (5.23)$$

One may expand the metric and vector field for small ϵ as,

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \epsilon \delta g_{\mu\nu}^{(1)} + \epsilon^2 \frac{1}{2} \delta g_{\mu\nu}^{(2)} + \epsilon^3 \frac{1}{6} \delta g_{\mu\nu}^{(3)}, \quad (5.24)$$

$$V_\mu = \frac{1}{\epsilon} V_\mu^{(-1)} + \bar{V}_\mu + \epsilon V_\mu^{(1)}. \quad (5.25)$$

The bar represents the zeroth order and the number in the superscript represent the particular order of the field expansions. The contravariant form of those quantities are

$$\begin{aligned} g^{\mu\nu} &= \bar{g}^{\mu\nu} - \epsilon \delta g^{(1)\mu\nu} + \epsilon^2 \frac{1}{2} (2\delta g_\alpha^{(1)\nu} \delta g^{(1)\mu\alpha} - \delta g^{(2)\mu\nu}) \\ &+ \epsilon^3 \frac{1}{6} (-6\delta g_\alpha^{(1)\beta} \delta g_\beta^{(1)\nu} \delta g^{(1)\mu\alpha} + 3\delta g^{(1)\mu\nu} \delta g_\gamma^{(2)\nu} + 3\delta g_\rho^{(1)\nu} \delta g^{(2)\mu\rho} - \delta g^{(3)\mu\nu}), \end{aligned} \quad (5.26)$$

$$\begin{aligned} V^\mu &= g^{\mu\nu} V_\nu \\ &= \frac{1}{\epsilon} V_\nu^{(-1)} \bar{g}^{\mu\nu} + (\bar{g}^{\mu\nu} \bar{V}_\nu - \delta g^{(1)\mu\nu} V_\mu^{(-1)}) \\ &+ \epsilon (\bar{g}^{\mu\nu} V_\nu^{(1)} - \delta g^{(1)\mu\nu} \bar{V}_\nu + V_\nu^{(-1)} (2\delta g_\alpha^{(1)\nu} \delta g^{(1)\mu\alpha} - 2g^{(2)\mu\nu})). \end{aligned} \quad (5.27)$$

We expand Eq. (5.21), (5.22) and (5.23) for small b , then neglect $\mathcal{O}(\epsilon^2)$ and set $\mathcal{O}(\epsilon) \rightarrow 0$. ϵ^0 or lower order will survive.

Eq. (5.21) will be turned into,

$$\bar{g}^{\mu\nu} \delta V_\mu^{(-1)} \delta V_\nu^{(-1)} - 1 = 0. \quad (5.28)$$

Eq. (5.22) will be turned into,

$$\frac{1}{\epsilon} \bar{g}^{\nu\rho} \nabla_\rho F_{\mu\nu}^{(-1)} + [\bar{g}^{\nu\rho} \nabla_\rho \bar{F}_{\mu\nu} + \delta g^{(1)\nu\rho} \nabla_\rho F_{\mu\nu}^{(1)}] = 0, \quad (5.29)$$

where $F_{\mu\nu}^{(order)} = \partial_\mu V_\nu^{(order)} - \partial_\nu V_\mu^{(order)}$.

After contraction with $\bar{g}^{\nu\rho}$,

$$\frac{1}{\epsilon} \bar{\nabla}^\nu F_{\mu\nu}^{(-1)} + [\bar{\nabla}^\nu \bar{F}_{\mu\nu} + \delta g^{(1)\nu\rho} \nabla_\rho F_{\mu\nu}^{(-1)}] = 0. \quad (5.30)$$

Eq. (5.23) will be turned into,

$$\begin{aligned} & -\frac{1}{2\kappa} (\bar{G}_{\mu\nu} - \kappa \bar{T}_{\mu\nu}) + \frac{1}{2\kappa} (\bar{G} - \kappa \bar{T}) V_\mu^{(-1)} V_\nu^{(-1)} \\ & -2 \left(\left(\frac{1}{\epsilon^2} \bar{g}^{\rho\sigma} F_{\nu\rho}^{(-1)} F_{\mu\sigma}^{(-1)} + \frac{1}{\epsilon} [\bar{g}^{\rho\sigma} (F_{\nu\rho}^{(-1)} \bar{F}_{\mu\sigma} + \bar{F}_{\nu\rho} F_{\mu\sigma}^{(-1)}) - \delta g^{(1)\rho\sigma} F_{\nu\rho}^{(-1)} F_{\mu\sigma}^{(-1)}] \right) \right. \\ & + \epsilon^0 [\bar{g}^{\rho\sigma} (\bar{F}_{\nu\rho} \bar{F}_{\mu\sigma} + F_{\nu\rho}^{(-1)} F_{\mu\sigma}^{(1)} + F_{\nu\rho}^{(1)} F_{\mu\sigma}^{(-1)}) - \delta g^{(1)\rho\sigma} (F_{\nu\rho}^{(-1)} \bar{F}_{\mu\sigma} + \bar{F}_{\nu\rho} F_{\mu\sigma}^{(-1)}) \\ & \left. + \frac{1}{2} (2\delta g_\alpha^{(1)\sigma} g^{(1)\rho\alpha} - g^{(2)\rho\sigma}) F_{\nu\rho}^{(-1)} F_{\mu\sigma}^{(-1)} \right] \\ & -\frac{1}{4} \left(\frac{1}{\epsilon^2} \bar{g}_{\mu\nu} \bar{g}^{\rho\alpha} \bar{g}^{\sigma\beta} F_{\rho\sigma}^{(-1)} F_{\alpha\beta}^{(-1)} + \frac{1}{\epsilon} [\bar{g}_{\mu\nu} \bar{g}^{\rho\alpha} \bar{g}^{\sigma\beta} (F_{\rho\sigma}^{(-1)} \bar{F}_{\alpha\beta} + \bar{F}_{\rho\sigma} F_{\alpha\beta}^{(-1)}) \right. \\ & + (\delta g_{(1)\mu\nu} \bar{g}^{\rho\alpha} \bar{g}^{\sigma\beta} - \bar{g}_{\mu\nu} \delta g^{(1)\rho\alpha} \bar{g}^{\sigma\beta} - \bar{g}_{\mu\nu} \bar{g}^{\rho\alpha} \delta g^{(1)\sigma\beta}) F_{\rho\sigma}^{(-1)} F_{\alpha\beta}^{(-1)} \\ & + \epsilon^0 [\bar{g}_{\mu\nu} \bar{g}^{\rho\alpha} \bar{g}^{\sigma\beta} (\bar{F}_{\rho\sigma} \bar{F}_{\alpha\beta} + F_{\rho\sigma}^{(-1)} F_{\alpha\beta}^{(1)} + F_{\rho\sigma}^{(1)} F_{\alpha\beta}^{(-1)}) \\ & + (\delta g_{(1)\mu\nu} \bar{g}^{\rho\alpha} \bar{g}^{\sigma\beta} - \bar{g}_{\mu\nu} \delta g^{(1)\rho\alpha} \bar{g}^{\sigma\beta} - \bar{g}_{\mu\nu} \bar{g}^{\rho\alpha} \delta g^{(1)\sigma\beta}) (F_{\rho\sigma}^{(-1)} \bar{F}_{\alpha\beta} + \bar{F}_{\rho\sigma} F_{\alpha\beta}^{(-1)}) \\ & + (\bar{g}_{\mu\nu} \delta g^{(1)\rho\alpha} \delta g^{(1)\sigma\beta} - \delta g_{\mu\nu}^{(1)} \delta g^{(1)\rho\alpha} \bar{g}^{\sigma\beta} - \delta g_{\mu\nu}^{(1)} \bar{g}^{\rho\alpha} \delta g^{(1)\sigma\beta} + \bar{g}_{\mu\nu} \delta g^{(1)\rho\alpha} \delta g^{(1)\sigma\gamma} \delta g_\gamma^{(1)\beta} \\ & + \bar{g}_{\mu\nu} \delta g^{(1)\rho\gamma} \delta g^{(1)\sigma\beta} \delta g_\gamma^{(1)\alpha} + \frac{1}{2} \bar{g}_{\mu\nu}^{(2)} \bar{g}^{\rho\alpha} \bar{g}^{\sigma\beta} - \frac{1}{2} \bar{g}_{\mu\nu} \bar{g}^{(2)\rho\alpha} \bar{g}^{\sigma\beta} \\ & \left. \left. - \frac{1}{2} \bar{g}_{\mu\nu} \bar{g}^{\rho\alpha} \bar{g}^{(2)\sigma\beta} F_{\rho\sigma}^{(-1)} F_{\alpha\beta}^{(-1)} \right) F_{\rho\sigma}^{(-1)} F_{\alpha\beta}^{(-1)} \right] \right) = 0. \quad (5.31) \end{aligned}$$

At this point it is worth listing the following comments:

1. The $1/\epsilon^2$ order of Eq. (5.31) shows that,

$$\bar{g}^{\rho\sigma} F_{\nu\rho}^{(-1)} F_{\mu\sigma}^{(-1)} - \frac{1}{4} \bar{g}_{\mu\nu} \bar{g}^{\rho\alpha} \bar{g}^{\sigma\beta} F_{\rho\sigma}^{(-1)} F_{\alpha\beta}^{(-1)} = 0. \quad (5.32)$$

According to the Einstein's equivalence principle, there is always a frame where there is no curvature, i.e., the metric is Minkowski. So we can replace $g_{\mu\nu}$ by $\eta_{\mu\nu}$ in the electromagnetic stress-energy tensor. In Minkowski, the energy density, $T_{00} = -\frac{1}{2}(E^2 + B^2)$. This quantity will be zero only when $E = B = 0$. Therefore $F_{\mu\nu}^{(-1)} = 0$. If $F_{\mu\nu}^{(-1)}$ vanishes in Minkowski space, then it will vanish in every coordinate system, as we can always transform $F_{\mu\nu}^{(-1)}$ like a tensor.

Therefore, vanishing Maxwell stress-energy tensor (or energy-momentum tensor) also implies vanishing curvature $F_{\mu\nu}^{(-1)}$, i.e., $V_\mu^{(-1)}$ is rotation-free.

2. If we apply $F_{\mu\nu}^{(-1)} = 0$, Eq. (5.31) (contracted by $\bar{g}^{\mu\nu}$) reduces to,

$$\begin{aligned} -\frac{1}{2\kappa}(\bar{G}_{\mu\nu} - \kappa\bar{T}_{\mu\nu}) + \frac{1}{2\kappa}(\bar{G} - \kappa\bar{T})V_\mu^{(-1)}V_\nu^{(-1)} \\ -2\left(\bar{F}_{\nu\rho}\bar{F}_\mu^\rho - \frac{1}{4}\bar{g}_{\mu\nu}\bar{F}_{\rho\sigma}\bar{F}^{\rho\sigma}\right) = 0. \end{aligned} \quad (5.33)$$

That represents the Einstein equation for $\bar{g}_{\mu\nu}$ sourced by the Maxwell field \bar{V}_μ and the dust $V_\mu^{(-1)}$.

3. If we apply $F_{\mu\nu}^{(-1)} = 0$, Eq. (5.29) reduces to $\bar{g}^{\nu\rho}\bar{\nabla}_\rho\bar{F}_{\mu\nu} = \bar{\nabla}^\nu\bar{F}_{\mu\nu} = 0$: Maxwell equation for \bar{V}_μ .
4. The zeroth order of Eq. (5.28) implies that $V_\mu^{(-1)}$ has a unit norm.

The gradient of Eq. (5.28) give us

$$V^\mu(\nabla_\mu V_\nu) = V^\mu F_{\mu\nu}. \quad (5.34)$$

By using the definition given in Eq. (5.8), the acceleration of time like vector field, V_μ , is $a_\nu = V^\mu F_{\mu\nu}$. Therefore, the right hand side of the Eq. (5.34) represents the acceleration for any order of the vector field, V_μ . Considering, $F_{\mu\nu}^{(-1)} = 0$, the acceleration of the $V_\mu^{(-1)}$ is

$$a_\nu^{(-1)} = V^{(-1)\mu} F_{\mu\nu}^{(-1)}, \quad (5.35)$$

$$= 0. \quad (5.36)$$

Therefore, $V_\mu^{(-1)}$ is acceleration-free.

Of course, the full V_μ is not acceleration-free because the full $F_{\mu\nu}$ is not zero at all orders.

In summary, we showed that in the weak limit, the above vector mimetic action becomes rotation and acceleration-free and behaves as the scalar mimetic theory, and the sound speed will be the same as a scalar field mimetic theory as explained in the previous chapter. However, away from this limit, the sound speed squared will be positive.

5.4 Mimetic generalized Proca theories

In Ref. [95], authors described the generalized Proca theories, which is claimed to provide second-order equations of motion in curved space-time. If we perform the non-invertible vector disformal transformation of type, Eq. (5.17) on this action, it will leave the same constraint as Eq. 5.16. The

Mimetic of the generalized Proca theories is

$$S = \int d^4x \sqrt{-g} (\mathcal{L} + \mathcal{L}_M) + \int d^4x \sqrt{-g} \lambda (bg^{\mu\nu} V_\mu V_\nu - 1), \quad (5.37)$$

$$\text{with } \mathcal{L} = \mathcal{L}_F + \sum_{i=2}^5 \mathcal{L}_i, \quad (5.38)$$

where \mathcal{L} represents the generalized Proca action and \mathcal{L}_M is the matter Lagrangian, and

$$\mathcal{L}_F = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \quad (5.39)$$

$$\mathcal{L}_2 = G_2(X), \quad (5.40)$$

$$\mathcal{L}_3 = G_3(X) \nabla_\mu V^\mu, \quad (5.41)$$

$$\mathcal{L}_4 = G_4(X) R + G_{4,X}(X) \left[(\nabla_\mu V^\mu)^2 + c_2 \nabla_\rho V_\sigma \nabla^\rho V^\sigma - (1 + c_2) \nabla_\rho V_\sigma \nabla^\sigma V^\rho \right], \quad (5.42)$$

$$\mathcal{L}_5 = G_5(X) G_{\mu\nu} \nabla^\mu V^\nu - \frac{1}{6} G_{5,X}(X) \left[(\nabla_\mu V^\mu)^3 - 3d_2 \nabla_\mu V^\mu \nabla_\rho V_\sigma \nabla^\rho V^\sigma - 3(1 - d_2) \nabla_\mu V^\mu \nabla_\rho V_\sigma \nabla^\sigma V^\rho + (2 - 3d_2) \nabla_\rho V_\sigma \nabla^\gamma V^\rho \nabla^\sigma V_\gamma + 3d_2 \nabla_\rho V_\sigma \nabla^\gamma V^\rho \nabla_\gamma V^\sigma \right]. \quad (5.43)$$

Here, V_μ is a vector field with $F_{\mu\nu} = \nabla_\mu V_\nu - \nabla_\nu V_\mu$; ∇_μ is the covariant derivative operator. c_2, d_2 are constants, $G_{2,3,4,5}$ are arbitrary functions of $X = -\frac{1}{2} V_\mu V^\mu$ and $G_{i,X} = \partial G_i / \partial X$.

The propagating degrees of freedom up to three with second-order equations of motion for the defined Lagrangians $\mathcal{L}_{2,3,4,5}$. There are two transverse polarizations for the vector field, like in the standard massless Maxwell theory. The choice of the functions, $G_2(X) = m^2 X$ with $G_{3,4,5} = 0$ will correspond to the Proca theory, in which case the introduction of the mass term m breaks the $U(1)$ gauge symmetry. This gives rise to an additional degree of freedom

in the longitudinal direction.

We leave the cosmological implications of this mimetic vector gravity for our ongoing project.

Chapter 6

Conclusions and future outlook

Since mimetic dark matter [39], a modification of General Relativity (GR) leading to a scalar-tensor type theory, has attracted considerable attention in the cosmology community. The main reason is that the theory possesses some very attractive features. For example, it was shown that the original theory [39] contains an extra scalar mode (of gravitational origin) which can mimic the behaviour of cold dark matter even in the absence of any form of matter. Soon after it was realised that, with a small generalisation of the original theory, the scalar mode could be used to mimic the behaviour of almost any type of matter and in this way one can have almost any desired expansion history of the universe [40].

The mimetic scalar field was introduced in GR by doing a non-invertible conformal transformation in the Einstein-Hilbert action of the type $g_{\mu\nu} = -w\ell_{\mu\nu}$, where the physical metric is $g_{\mu\nu}$, the auxiliary metric is $\ell_{\mu\nu}$, w is defined in terms of a scalar field φ as $w = \ell^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi$ [39, 40, 45]. Soon after it was realized [42] that the type of metric transformation that leads to mimetic gravity can be further generalised from the previous transformation to include also a disformal term [46], as $g_{\mu\nu} = A(\varphi, w)\ell_{\mu\nu} + B(\varphi, w)\partial_\mu\varphi\partial_\nu\varphi$,

where A and B are free functions of two variables and they must obey some conditions (see [47], [51] and also Section 2.3.1, where the conditions for disformally coupled theories to have a so-called Jordan frame were discussed), so that the Lorentzian signature is preserved, the transformation is causal and regular, $g^{\mu\nu}$ exists and A and B are related as $B = -A/w + b$, where b is an arbitrary function of φ only, and it should not cross zero. If A and B are arbitrary functions and do not obey the previous relation then the equations of motion that one obtains are just Einstein's equations [42].

In this thesis, we have proposed and explored the “generalized mimetic gravity”, as an alternative to the Λ CDM model.

In Chapter 3, we applied a full disformal transformation to very general single scalar-tensor theory of gravity [2]. We have shown that the very general scalar-tensor theories of gravity are generically invariant under disformal transformations. However, there exists a special subset, which is non-invertible under the transformation which yields generalized “mimetic” gravity theories, i.e., the scalar field mimics almost any desired expansion history (including Dark Energy and Dark Matter dominated universes). We call it “generalized mimetic gravity”.

The generalized mimetic equation of motion (EOM) can also be derived using the Lagrange multiplier method. The general mimetic scalar-tensor theory has the same number of derivatives in the EOM as the original scalar-tensor theory. As an application, the simplest model of the mimetic Horndeski theory was built, where a canonically normalized scalar field with no potential (in the original theory) is able to mimic the cosmological background expansion history of a flat FLRW model filled with a barotropic perfect fluid with any (constant) equation of state.

In Chapter 4, we studied linear scalar perturbations around a flat FLRW

background in mimetic Horndeski gravity [3]. In the absence of matter, we have shown that the Newtonian potential satisfies a second-order differential equation with no spatial derivatives. The sound speed is zero for all backgrounds and, therefore, the system does not have any wave-like scalar degrees of freedom.

In Chapter 5, we have also shown and explained how the vector Einstein-Aether theory is also a type of mimetic gravity theories. In that context, we have shown that the non-invertibility condition of a vector disformal transformation leads to the Einstein-Aether theory. In the weak limit, the vector Einstein-Aether theory becomes irrotational and acceleration-free and behaves as the scalar field mimetic theory. Furthermore, we have proposed a generalized vector mimetic gravity action. This is an ongoing project. We leave the analysis and cosmological implications to the near future.

Interestingly, the simple mimetic matter scenario with a higher-derivative term arises as a particular (IR) limit [96] of projectable Hořava-Lifshitz gravity, which have been shown to be renormalizable [97] and a candidate for the theory of quantum gravity.

As part of the work leading up to my Master's thesis, I have attempted to solve the cosmological constant problem by breaking general covariance in the unimodular theory of gravity [98], and have explored the cosmological implementation in a consecutive paper [99]. Unimodular gravity tries to solve the CC problem by breaking general covariance in the form of a constraint on the metric determinant. In the present case of mimetic gravity a constraint is applied to the scalar field instead and the theory was proposed as a possible solution to the dark matter problem. If we impose both constraints at the same time, then the theory that one obtains is called unimodular-mimetic gravity, and it could solve both the DE and DM problems [100, 101].

After having the background and perturbation check of the model “generalized mimetic gravity” [2, 3], it would be appropriate to test, constrain and analyze to confirm this novel hypothesis by using the existing data and future upcoming high-quality data. EUCLID is of particular interest, as it will allow to constrain a large number of dark energy and modified gravity models with high precision and accuracy, and much of this proposal’s tests are dedicated to preparing the model for confronting future EUCLID data. Given the proliferation of high-precision and pertinent experimental data, “mimetic gravity” and other theories of modified gravity are highly relevant in the coming decade.

The model may contain a build-in screening mechanism that may imply that effectively GR is recovered on small scales in which we do experiments (e.g. table-top laboratory experiments) or astronomical observations in the solar system. This is an interesting possibility which deserves further study and I leave it for future work. It would also be interesting to simulate and test the nonlinear evolution of the mimetic scalar field up to galaxy and cluster scales. We leave these works for our near future.

Appendices

Appendix A

Appendix-Mimetic Horndeski

A.1 Disformal transformation with a new scalar field

In Section 3.3 of the main text we have considered the theory that results from performing a disformal transformation on a very general scalar-tensor theory where the scalar field in the action was the same as the scalar field involved in the transformation. In this Appendix, we consider the case when the scalar field in the transformation is not the same as the scalar field present in the action of the theory.

The action of the model is Eq. (3.22). The disformal transformation that we are considering is

$$g_{\mu\nu} = A(\Phi, Y) \ell_{\mu\nu} + B(\Phi, Y) \partial_\mu \Phi \partial_\nu \Phi, \quad \text{where } Y \equiv \ell^{\rho\sigma} \partial_\rho \Phi \partial_\sigma \Phi, \quad (\text{A.1})$$

and, as before, the arbitrary disformal functions, A and B , depends on both the scalar field Φ and its kinetic term Y . Using Eq. (A.1) one can find the

variation of $g_{\mu\nu}$ as

$$\begin{aligned}
\delta g_{\mu\nu} &= A \delta \ell_{\mu\nu} - \left(\ell_{\mu\nu} \frac{\partial A}{\partial Y} + \partial_\mu \Phi \partial_\nu \Phi \frac{\partial B}{\partial Y} \right) \\
&\quad [(\ell^{\alpha\rho} \partial_\alpha \Phi) (\ell^{\beta\sigma} \partial_\beta \Phi) \delta \ell_{\rho\sigma} - 2\ell^{\rho\sigma} (\partial_\rho \Phi) (\partial_\sigma \delta \Phi)] \\
&\quad + \left(\ell_{\mu\nu} \frac{\partial A}{\partial \Phi} + \partial_\mu \Phi \partial_\nu \Phi \frac{\partial B}{\partial \Phi} \right) \delta \Phi \\
&\quad + B [(\partial_\mu \Phi) (\partial_\nu \delta \Phi) + (\partial_\nu \Phi) (\partial_\mu \delta \Phi)]. \tag{A.2}
\end{aligned}$$

The modified Einstein equations of motion, $\delta S / \delta \ell_{\mu\nu} = 0$, are

$$A(E^{\mu\nu} + T^{\mu\nu}) = \left(\alpha_1 \frac{\partial A}{\partial Y} + \alpha_3 \frac{\partial B}{\partial Y} \right) (\ell^{\mu\rho} \partial_\rho \Phi) (\ell^{\nu\sigma} \partial_\sigma \Phi). \tag{A.3}$$

The equation of motion for the scalar field Φ , $\delta S / \delta \Phi = 0$, is

$$\begin{aligned}
\frac{1}{\sqrt{-g}} \partial_\rho \left\{ \sqrt{-g} \partial_\sigma \Phi \left[B(E^{\rho\sigma} + T^{\rho\sigma}) + \left(\alpha_1 \frac{\partial A}{\partial Y} + \alpha_3 \frac{\partial B}{\partial Y} \right) \ell^{\rho\sigma} \right] \right\} \\
= \frac{1}{2} \left(\alpha_1 \frac{\partial A}{\partial \Phi} + \alpha_3 \frac{\partial B}{\partial \Phi} \right), \tag{A.4}
\end{aligned}$$

where we have defined the new quantities α_1 and α_3 as

$$\alpha_1 \equiv (E^{\rho\sigma} + T^{\rho\sigma}) \ell_{\rho\sigma}, \quad \alpha_3 \equiv (E^{\rho\sigma} + T^{\rho\sigma}) \partial_\rho \Phi \partial_\sigma \Phi, \tag{A.5}$$

and the equation for the scalar field Ψ , $\delta S / \delta \Psi = 0$, is $\Omega_\Psi = 0$. For the matter field we have the same equation of motion as before, i.e. $\delta S / \delta \phi_m = 0$ or $\Omega_m = 0$. It is important to note that the modified Einstein equations of motion, Eq. (A.3), have the same structure as in subsection 3.3.2 of the main text. Following the same procedure as before one can find different solutions for the resulting system depending on its determinant being zero or not. We will now consider these two cases separately and show that they lead to different physical theories exactly as in the case studied in the main text.

The generic case

If the determinant of the system of linear equations that results from contracting Eq. (A.3) with $\ell_{\mu\nu}$ and $\partial_\mu\Phi\partial_\nu\Phi$ is non-zero then the only solution is $\alpha_1 = \alpha_3 = 0$. Hence, Eqs. (A.3) and (A.4) imply

$$A(E^{\mu\nu} + T^{\mu\nu}) = 0, \quad (\text{A.6})$$

$$\partial_\rho [\sqrt{-g} \partial_\sigma \Phi B(E^{\rho\sigma} + T^{\rho\sigma})] = 0. \quad (\text{A.7})$$

Eq. (A.7) is empty after considering the modified Einstein equation (A.6). Therefore the full equations of motion of this theory are

$$E^{\mu\nu} + T^{\mu\nu} = 0, \quad \Omega_\Psi = 0, \quad \Omega_m = 0. \quad (\text{A.8})$$

In terms of the original metric $g_{\mu\nu}$, these equations are exactly the same as the equations of motion for the theory (3.22) if we take the variation with respect to the original fields $g_{\mu\nu}$, Ψ and ϕ_m instead of taking the variation with respect to the new fields $\ell_{\mu\nu}$, Φ , Ψ and ϕ_m as we did to arrive at Eqs. (A.8). This shows that a very general scalar-tensor theory of the type (3.22) is invariant under a generic disformal transformation of the type (A.1) even if the scalar field defining the transformation is not the same as the scalar field in the action.

Mimetic gravity

If the determinant of the system is zero, as we showed in the main text, it implies that the transformation functions A and B should be related as

$$B(\Phi, Y) = -\frac{A(\Phi, Y)}{Y} + b(\Phi), \quad (\text{A.9})$$

where $b(\Phi)$ again arises as an integration constant in Y which we assume to be non-zero. Inserting this expression in Eqs. (A.3) and (A.4) one obtains

$$E_{\mu\nu} + T_{\mu\nu} = (E + T) b \partial_\mu \Phi \partial_\nu \Phi, \quad \nabla_\rho [(E + T) b \partial^\rho \Phi] = \frac{1}{2} (E + T) \frac{1}{b} \frac{db}{d\Phi}. \quad (\text{A.10})$$

Following [42], one can redefine the scalar field Φ in terms of a new scalar field Θ as $d\Theta/d\Phi = \sqrt{|b|}$ in order to eliminate the function $b(\Phi)$ from the equations of motion to finally obtain

$$E_{\mu\nu} + T_{\mu\nu} = \epsilon (E + T) \partial_\mu \Theta \partial_\nu \Theta, \quad \nabla_\rho [(E + T) \partial^\rho \Theta] = 0, \quad (\text{A.11})$$

where $\epsilon = \text{Sign}(b) = \pm 1$ depending on the sign of the norm of $\partial_\mu \Phi$, i.e. $g^{\mu\nu} \partial_\mu \Theta \partial_\nu \Theta = \epsilon$. The full set of equations of motion includes in addition to the previous two equations also the equations of motion for Ψ and the matter field ϕ_m . They are

$$\Omega_\Psi = 0, \quad \Omega_m = 0. \quad (\text{A.12})$$

Appendix B

Appendix-Sound speed

B.1 The background equations of motion

In this appendix, we provide the expressions for the tensor $E_{\mu\nu}$ on a flat FLRW background. The same expressions can be used for the Horndeski and mimetic Horndeski models. The non-zero components read

$$\begin{aligned} E_{00}^{(0)} = & -a^2 K - 6G_4 \mathcal{H}^2 - 6G_{4,\varphi} \mathcal{H} \varphi'_0 \\ & + (\varphi'_0)^2 \left(\frac{12G_{4,X} \mathcal{H}^2 - 9G_{5,\varphi} \mathcal{H}^2}{a^2} - G_{3,\varphi} + K_{,X} \right) \\ & + (\varphi'_0)^3 \left(\frac{5G_{5,X} \mathcal{H}^3}{a^4} + \frac{3G_{3,X} \mathcal{H} - 6G_{4,X\varphi} \mathcal{H}}{a^2} \right) \\ & + (\varphi'_0)^4 \frac{(6G_{4,XX} \mathcal{H}^2 - 3G_{5,X\varphi} \mathcal{H}^2)}{a^4} \\ & + (\varphi'_0)^5 \frac{G_{5,XX} \mathcal{H}^3}{a^6}, \end{aligned}$$

$$\begin{aligned}
E_{ij}^{(0)} = & \delta_{ij} \left[a^2 K + 2G_4 \mathcal{H}^2 + 4G_4 \mathcal{H}' \right. \\
& + \varphi_0' \left(\frac{4G_{5,\varphi} \mathcal{H} \varphi_0'' - 4G_{4,X} \mathcal{H} \varphi_0''}{a^2} + 2G_{4,\varphi} \mathcal{H} \right) \\
& + (\varphi_0')^2 \left(-G_{3,\varphi} + 2G_{4,\varphi\varphi} - \frac{3G_{5,X} \mathcal{H}^2 \varphi_0''}{a^4} \right) \\
& + (\varphi_0')^2 \left(\frac{-G_{3,X} \varphi_0'' + 2G_{4,X} (\mathcal{H}^2 - 2\mathcal{H}') + 2G_{4,X\varphi} \varphi_0'' + G_{5,\varphi} (2\mathcal{H}' - 3\mathcal{H}^2)}{a^2} \right) \\
& + (\varphi_0')^3 \left(\frac{-4G_{4,XX} \mathcal{H} \varphi_0'' + 2G_{5,X\varphi} \mathcal{H} \varphi_0'' + 3G_{5,X} \mathcal{H}^3 - 2G_{5,X} \mathcal{H} \mathcal{H}'}{a^4} \right) \\
& + (\varphi_0')^3 \left(\frac{G_{3,X} \mathcal{H} - 6G_{4,X\varphi} \mathcal{H} + 2G_{5,\varphi\varphi} \mathcal{H}}{a^2} \right) \\
& + (\varphi_0')^4 \left(-\frac{G_{5,XX} \mathcal{H}^2 \varphi_0''}{a^6} + \frac{4G_{4,XX} \mathcal{H}^2 - 3G_{5,X\varphi} \mathcal{H}^2}{a^4} \right) \\
& \left. + \frac{G_{5,XX} \mathcal{H}^3 (\varphi_0')^5}{a^6} + 2G_{4,\varphi} \varphi_0'' \right]. \tag{B.1}
\end{aligned}$$

The zeroth-order trace $E^{(0)}$ can be easily computed from the previous equations by using $E^{(0)} = -a^{-2} E_{00}^{(0)} + a^{-2} \delta^{ij} E_{ij}^{(0)}$.

B.2 The explicit expressions of the f_i functions

In this appendix, we give the explicit expressions for the functions f_i , $i = 1, \dots, 21$ defined in the main text. These expressions can be used for both the Horndeski and mimetic Horndeski models because no equations of motion were used.

They read

$$\begin{aligned}
f_1 = & 12G_4 \mathcal{H} + 6G_{4,\varphi} \varphi_0' \\
& + \frac{(18G_{5,\varphi} \mathcal{H} - 24G_{4,X} \mathcal{H})}{a^2} (\varphi_0')^2 \\
& + (\varphi_0')^3 \left(\frac{6G_{4,X\varphi} - 3G_{3,X}}{a^2} - \frac{15G_{5,X} \mathcal{H}^2}{a^4} \right) \\
& + \frac{(6G_{5,X\varphi} \mathcal{H} - 12G_{4,XX} \mathcal{H})}{a^4} (\varphi_0')^4 - \frac{3G_{5,XX} \mathcal{H}^2}{a^6} (\varphi_0')^5,
\end{aligned}$$

$$\begin{aligned}
f_2 = & -6G_{4,\varphi}\mathcal{H} + \varphi'_0 \left(\frac{18G_{4,X}\mathcal{H}^2 - 18G_{5,\varphi}\mathcal{H}^2}{a^2} - 2G_{3,\varphi} + K_{,X} \right) \\
& + (\varphi'_0)^2 \left(\frac{15G_{5,X}\mathcal{H}^3}{a^4} + \frac{9G_{3,X}\mathcal{H} - 24G_{4,X\varphi}\mathcal{H}}{a^2} \right) \\
& + (\varphi'_0)^3 \left(\frac{36G_{4,XX}\mathcal{H}^2 - 21G_{5,X\varphi}\mathcal{H}^2}{a^4} + \frac{K_{,XX} - G_{3,X\varphi}}{a^2} \right) \\
& + (\varphi'_0)^4 \left(\frac{10G_{5,XX}\mathcal{H}^3}{a^6} + \frac{3G_{3,XX}\mathcal{H} - 6G_{4,XX\varphi}\mathcal{H}}{a^4} \right) \\
& + \frac{(6G_{4,XXX}\mathcal{H}^2 - 3G_{5,XX\varphi}\mathcal{H}^2)}{a^6} (\varphi'_0)^5 + \frac{G_{5,XXX}\mathcal{H}^3}{a^8} (\varphi'_0)^6,
\end{aligned}$$

$$\begin{aligned}
f_3 = & -2a^2K + (\varphi'_0)^2 \left(\frac{18G_{5,\varphi}\mathcal{H}^2 - 18G_{4,X}\mathcal{H}^2}{a^2} + K_{,X} \right) \\
& + (\varphi'_0)^3 \left(\frac{18G_{4,X\varphi}\mathcal{H} - 6G_{3,X}\mathcal{H}}{a^2} - \frac{20G_{5,X}\mathcal{H}^3}{a^4} \right) \\
& + (\varphi'_0)^4 \left(\frac{21G_{5,X\varphi}\mathcal{H}^2 - 36G_{4,XX}\mathcal{H}^2}{a^4} + \frac{G_{3,X\varphi} - K_{,XX}}{a^2} \right) \\
& + (\varphi'_0)^5 \left(\frac{6G_{4,XX\varphi}\mathcal{H} - 3G_{3,XX}\mathcal{H}}{a^4} - \frac{11G_{5,XX}\mathcal{H}^3}{a^6} \right) \\
& + (\varphi'_0)^6 \frac{(3G_{5,XX\varphi}\mathcal{H}^2 - 6G_{4,XXX}\mathcal{H}^2)}{a^6} - \frac{G_{5,XXX}\mathcal{H}^3}{a^8} (\varphi'_0)^7,
\end{aligned}$$

$$\begin{aligned}
f_4 = & -a^2K_{,\varphi} - 6G_{4,\varphi}\mathcal{H}^2 - 6G_{4,\varphi\varphi}\mathcal{H}\varphi'_0 \\
& + (\varphi'_0)^2 \left(\frac{12G_{4,X\varphi}\mathcal{H}^2 - 9G_{5,\varphi\varphi}\mathcal{H}^2}{a^2} - G_{3,\varphi\varphi} + K_{,X\varphi} \right) \\
& + (\varphi'_0)^3 \left(\frac{5G_{5,X\varphi}\mathcal{H}^3}{a^4} + \frac{3G_{3,X\varphi}\mathcal{H} - 6G_{4,X\varphi\varphi}\mathcal{H}}{a^2} \right) \\
& + \frac{(6G_{4,XX\varphi}\mathcal{H}^2 - 3G_{5,X\varphi\varphi}\mathcal{H}^2)}{a^4} (\varphi'_0)^4 + \frac{G_{5,XX\varphi}\mathcal{H}^3}{a^6} (\varphi'_0)^5,
\end{aligned}$$

$$f_5 = -4G_4 + \frac{(4G_{4,X} - 2G_{5,\varphi})(\varphi'_0)^2}{a^2} + \frac{2G_{5,X}\mathcal{H}(\varphi'_0)^3}{a^4},$$

$$\begin{aligned}
f_6 &= 2G_{4,\varphi} + \frac{\varphi'_0(4G_{5,\varphi}\mathcal{H} - 4G_{4,X}\mathcal{H})}{a^2} \\
&\quad + (\varphi'_0)^2 \left(\frac{2G_{4,X\varphi} - G_{3,X}}{a^2} - \frac{3G_{5,X}\mathcal{H}^2}{a^4} \right) + \frac{(2G_{5,X\varphi}\mathcal{H} - 4G_{4,XX}\mathcal{H})}{a^4} (\varphi'_0)^3 \\
&\quad - \frac{G_{5,XX}\mathcal{H}^2 (\varphi'_0)^4}{a^6}, \\
f_7 &= -2G_4 + (\varphi'_0)^2 \left(\frac{G_{5,X}\varphi''_0}{a^4} + \frac{G_{5,\varphi}}{a^2} \right) - \frac{G_{5,X}\mathcal{H}}{a^4} (\varphi'_0)^3, \\
f_8 &= 2G_{4,\varphi} \\
&\quad + (\varphi'_0)^2 \left(\frac{(G_{5,X\varphi} - 2G_{4,XX})\varphi''_0}{a^4} + \frac{2G_{5,X}\mathcal{H}^2 - G_{5,X}\mathcal{H}'}{a^4} + \frac{G_{5,\varphi\varphi} - 2G_{4,X\varphi}}{a^2} \right) \\
&\quad + (\varphi'_0)^3 \left(\frac{2G_{4,XX}\mathcal{H} - 2G_{5,X\varphi}\mathcal{H}}{a^4} - \frac{G_{5,XX}\mathcal{H}\varphi''_0}{a^6} \right) \\
&\quad + \frac{G_{5,XX}\mathcal{H}^2}{a^6} (\varphi'_0)^4 - \frac{2G_{5,X}\mathcal{H}}{a^4} \varphi'_0 \varphi''_0 + \frac{(2G_{5,\varphi} - 2G_{4,X})}{a^2} \varphi''_0, \\
f_9 &= 2G_4 + \frac{(G_{5,\varphi} - 2G_{4,X})}{a^2} (\varphi'_0)^2 - \frac{G_{5,X}\mathcal{H}}{a^4} (\varphi'_0)^3, \\
f_{11} &= 2G_{4,\varphi} + \frac{(4G_{5,\varphi}\mathcal{H} - 4G_{4,X}\mathcal{H})}{a^2} \varphi'_0 \\
&\quad + \left(\frac{2G_{4,X\varphi} - G_{3,X}}{a^2} - \frac{3G_{5,X}\mathcal{H}^2}{a^4} \right) (\varphi'_0)^2 \\
&\quad + \frac{(2G_{5,X\varphi}\mathcal{H} - 4G_{4,XX}\mathcal{H})}{a^4} (\varphi'_0)^3 - \frac{G_{5,XX}\mathcal{H}^2 (\varphi'_0)^4}{a^6}, \quad (\text{B.2}) \\
f_{12} &= -8G_4\mathcal{H} + \varphi'_0 \left(\frac{(4G_{4,X} - 4G_{5,\varphi})\varphi''_0}{a^2} - 4G_{4,\varphi} \right) \\
&\quad + (\varphi'_0)^2 \left(\frac{6G_{5,X}\mathcal{H}\varphi''_0}{a^4} + \frac{4G_{4,X}\mathcal{H}}{a^2} \right) \\
&\quad + (\varphi'_0)^3 \left(\frac{(4G_{4,XX} - 2G_{5,X\varphi})\varphi''_0}{a^4} + \frac{2G_{5,X}\mathcal{H}' - 4G_{5,X}\mathcal{H}^2}{a^4} + \frac{4G_{4,X\varphi} - 2G_{5,\varphi\varphi}}{a^2} \right) \\
&\quad + (\varphi'_0)^4 \left(\frac{2G_{5,XX}\mathcal{H}\varphi''_0}{a^6} + \frac{4G_{5,X\varphi}\mathcal{H} - 4G_{4,XX}\mathcal{H}}{a^4} \right) - \frac{2G_{5,XX}\mathcal{H}^2}{a^6} (\varphi'_0)^5,
\end{aligned}$$

$$\begin{aligned}
f_{13} &= 2G_{4,\varphi}\mathcal{H} \\
&+ \varphi'_0 \left(\frac{2G_{4,X}(3\mathcal{H}^2 - 2\mathcal{H}') - 2G_{5,\varphi}(3\mathcal{H}^2 - 2\mathcal{H}')}{a^2} \right) + \\
&\varphi'_0 \left(\varphi_0'' \left(\frac{6G_{4,X\varphi} - 2G_{3,X}}{a^2} - \frac{6G_{5,X}\mathcal{H}^2}{a^4} \right) - 2G_{3,\varphi} + 4G_{4,\varphi\varphi} + K_{,X} \right) \\
&+ (\varphi'_0)^2 \left(\frac{\varphi_0''(10G_{5,X\varphi}\mathcal{H} - 16G_{4,XX}\mathcal{H})}{a^4} + \frac{9G_{5,X}\mathcal{H}^3 - 6G_{5,X}\mathcal{H}\mathcal{H}'}{a^4} \right) + \\
&(\varphi'_0)^2 \left(\frac{3G_{3,X}\mathcal{H} - 16G_{4,X\varphi}\mathcal{H} + 6G_{5,\varphi\varphi}\mathcal{H}}{a^2} \right) \\
&+ (\varphi'_0)^3 \left(\frac{18G_{4,XX}\mathcal{H}^2 - 4G_{4,XX}\mathcal{H}' - 15G_{5,X\varphi}\mathcal{H}^2 + 2G_{5,X\varphi}\mathcal{H}'}{a^4} \right) + \\
&(\varphi'_0)^3 \left(\frac{2G_{4,X\varphi\varphi} - G_{3,X\varphi}}{a^2} + \varphi_0'' \left(\frac{2G_{4,XX\varphi} - G_{3,XX}}{a^4} - \frac{7G_{5,XX}\mathcal{H}^2}{a^6} \right) \right) \\
&+ (\varphi'_0)^4 \left(\frac{\varphi_0''(2G_{5,XX\varphi}\mathcal{H} - 4G_{4,XX}\mathcal{H})}{a^6} + \frac{8G_{5,XX}\mathcal{H}^3 - 2G_{5,XX}\mathcal{H}\mathcal{H}'}{a^6} \right) + \\
&(\varphi'_0)^4 \left(\frac{G_{3,XX}\mathcal{H} - 6G_{4,XX\varphi}\mathcal{H} + 2G_{5,X\varphi\varphi}\mathcal{H}}{a^4} \right) \\
&+ (\varphi'_0)^5 \left(\frac{4G_{4,XX}\mathcal{H}^2 - 3G_{5,XX\varphi}\mathcal{H}^2}{a^6} - \frac{G_{5,XX}\mathcal{H}^2\varphi_0''}{a^8} \right) \\
&+ \frac{G_{5,XX}\mathcal{H}^3(\varphi'_0)^6}{a^8} + \frac{\varphi_0''(4G_{5,\varphi}\mathcal{H} - 4G_{4,X}\mathcal{H})}{a^2}, \\
f_{14} &= -4G_4\mathcal{H} - 2G_{4,\varphi}\varphi'_0 + (\varphi'_0)^2 \frac{(8G_{4,X}\mathcal{H} - 6G_{5,\varphi}\mathcal{H})}{a^2} \\
&+ (\varphi'_0)^3 \left(\frac{5G_{5,X}\mathcal{H}^2}{a^4} + \frac{G_{3,X} - 2G_{4,X\varphi}}{a^2} \right) \\
&+ (\varphi'_0)^4 \frac{(4G_{4,XX}\mathcal{H} - 2G_{5,X\varphi}\mathcal{H})}{a^4} + \frac{G_{5,XX}\mathcal{H}^2(\varphi'_0)^5}{a^6},
\end{aligned}$$

$$\begin{aligned}
f_{16} = & a^2 K_{,\varphi} + 2G_{4,\varphi} \mathcal{H}^2 + 4G_{4,\varphi} \mathcal{H}' + \varphi_0' \left(\frac{\varphi_0''(4G_{5,\varphi\varphi} \mathcal{H} - 4G_{4,X\varphi} \mathcal{H})}{a^2} + 2G_{4,\varphi\varphi} \mathcal{H} \right) \\
& + (\varphi_0')^2 \left(\frac{2G_{4,X\varphi}(\mathcal{H}^2 - 2\mathcal{H}') + G_{5,\varphi\varphi}(2\mathcal{H}' - 3\mathcal{H}^2)}{a^2} \right) + \\
& (\varphi_0')^2 \left(\varphi_0'' \left(\frac{2G_{4,X\varphi\varphi} - G_{3,X\varphi}}{a^2} - \frac{3G_{5,X\varphi} \mathcal{H}^2}{a^4} \right) - G_{3,\varphi\varphi} + 2G_{4,\varphi\varphi\varphi} \right) \\
& + (\varphi_0')^3 \left(\frac{\varphi_0''(2G_{5,X\varphi\varphi} \mathcal{H} - 4G_{4,XX\varphi} \mathcal{H})}{a^4} + \frac{3G_{5,X\varphi} \mathcal{H}^3 - 2G_{5,X\varphi} \mathcal{H} \mathcal{H}'}{a^4} \right) + \\
& (\varphi_0')^3 \left(\frac{G_{3,X\varphi} \mathcal{H} - 6G_{4,X\varphi\varphi} \mathcal{H} + 2G_{5,\varphi\varphi\varphi} \mathcal{H}}{a^2} \right) \\
& + (\varphi_0')^4 \left(\frac{4G_{4,XX\varphi} \mathcal{H}^2 - 3G_{5,X\varphi\varphi} \mathcal{H}^2}{a^4} - \frac{G_{5,XX\varphi} \mathcal{H}^2 \varphi_0''}{a^6} \right) + \\
& \frac{G_{5,XX\varphi} \mathcal{H}^3 (\varphi_0')^5}{a^6} + 2G_{4,\varphi\varphi} \varphi_0'',
\end{aligned}$$

$$\begin{aligned}
f_{17} = & -4G_4\mathcal{H}^2 - 8G_4\mathcal{H}' + \varphi_0' \left(\frac{\varphi_0''(16G_{4,X}\mathcal{H} - 16G_{5,\varphi}\mathcal{H})}{a^2} - 4G_{4,\varphi}\mathcal{H} \right) \\
& + (\varphi_0')^2 \left(+ 2G_{3,\varphi} - 4G_{4,\varphi\varphi} - K_{,X} + \right. \\
& \left. - \frac{10G_{4,X}\mathcal{H}^2 + 12G_{4,X}\mathcal{H}' + 12G_{5,\varphi}\mathcal{H}^2 - 8G_{5,\varphi}\mathcal{H}'}{a^2} \right. \\
& \left. + \varphi_0'' \left(\frac{18G_{5,X}\mathcal{H}^2}{a^4} + \frac{4G_{3,X} - 10G_{4,X\varphi}}{a^2} \right) \right) \\
& + (\varphi_0')^3 \left(\frac{\varphi_0''(28G_{4,XX}\mathcal{H} - 16G_{5,X\varphi}\mathcal{H})}{a^4} + \frac{12G_{5,X}\mathcal{H}\mathcal{H}' - 18G_{5,X}\mathcal{H}^3}{a^4} \right) + \\
& (\varphi_0')^3 \left(\frac{-4G_{3,X}\mathcal{H} + 22G_{4,X\varphi}\mathcal{H} - 8G_{5,\varphi\varphi}\mathcal{H}}{a^2} \right) \\
& + (\varphi_0')^4 \left(\frac{-26G_{4,XX}\mathcal{H}^2 + 4G_{4,XX}\mathcal{H}' + 21G_{5,X\varphi}\mathcal{H}^2 - 2G_{5,X\varphi}\mathcal{H}'}{a^4} + \right. \\
& \left. \frac{G_{3,X\varphi} - 2G_{4,X\varphi\varphi}}{a^2} + \varphi_0'' \left(\frac{11G_{5,XX}\mathcal{H}^2}{a^6} + \frac{G_{3,XX} - 2G_{4,XX\varphi}}{a^4} \right) \right) \\
& + (\varphi_0')^5 \left(\frac{\varphi_0''(4G_{4,XXX}\mathcal{H} - 2G_{5,XX\varphi}\mathcal{H})}{a^6} + \frac{2G_{5,XX}\mathcal{H}\mathcal{H}' - 11G_{5,XX}\mathcal{H}^3}{a^6} \right) + \\
& (\varphi_0')^5 \left(\frac{-G_{3,XX}\mathcal{H} + 6G_{4,XX\varphi}\mathcal{H} - 2G_{5,X\varphi\varphi}\mathcal{H}}{a^4} \right) \\
& + (\varphi_0')^6 \left(\frac{G_{5,XXX}\mathcal{H}^2\varphi_0''}{a^8} + \frac{3G_{5,XX\varphi}\mathcal{H}^2 - 4G_{4,XXX}\mathcal{H}^2}{a^6} \right) \\
& - \frac{G_{5,XXX}\mathcal{H}^3}{a^8} (\varphi_0')^7 - 4G_{4,\varphi}\varphi_0'', \\
f_{20} = & -2G_{4,\varphi}\mathcal{H} + \left(\frac{6G_{4,X}\mathcal{H}^2 - 6G_{5,\varphi}\mathcal{H}^2}{a^2} - 2G_{3,\varphi} + 2G_{4,\varphi\varphi} + K_{,X} \right) \varphi_0' \\
& + \left(\frac{3G_{5,X}\mathcal{H}^3}{a^4} + \frac{3G_{3,X}\mathcal{H} - 10G_{4,X\varphi}\mathcal{H} + 2G_{5,\varphi\varphi}\mathcal{H}}{a^2} \right) (\varphi_0')^2 \\
& + \frac{(6G_{4,XX}\mathcal{H}^2 - 4G_{5,X\varphi}\mathcal{H}^2)}{a^4} (\varphi_0')^3 + \frac{G_{5,XX}\mathcal{H}^3}{a^6} (\varphi_0')^4.
\end{aligned} \tag{B.3}$$

The functions f_i obey the following identities:

$$\begin{aligned}
f_{10} = f_{18}, \quad f_{11} = f_{19}, \quad f_{21} = f_{14}, \quad f_9 = -\frac{f_{10}}{2}, \\
\frac{f_{10}' - f_{12}}{f_{10}} = -2\mathcal{H},
\end{aligned} \tag{B.4}$$

$$f_{11}(f'_{10} - f_{12}) + f_{10}(f_{13} - f_{20} - f'_{11}) = 0,$$

$$f_{14} - \mathcal{H}f_{10} + \varphi'_0 f_{11} = 0, \quad (\text{B.5})$$

$$f_{17} - \frac{f_{15}f_9}{f_7} - \frac{f_{14}}{f_{10}}(f_{12} - f'_{10}) - f'_{14} + 3E_{ij}^{(0)} + E_{00}^{(0)} + \alpha E_{ij}^{(0)} = 0, \quad (\text{B.6})$$

$$\left(f_{16} - \frac{f_8 f_{15}}{f_7} - \frac{f_{20}}{f_{10}}(f_{12} - f'_{10}) - f'_{20} \right) \varphi'_0 +$$

$$E_{00}^{(0)'} + \mathcal{H}(3E_{ij}^{(0)} + E_{00}^{(0)}) + \beta E_{ij}^{(0)} = 0, \quad (\text{B.7})$$

$$\left(\frac{\mathcal{H}}{\varphi'_0} \right)' f_{10} + \left(\frac{\varphi''_0}{(\varphi'_0)^2} - \frac{\mathcal{H}}{\varphi'_0} \right) f_{14} - f_{20} - a^2 \frac{E^{(0)}}{\varphi'_0} - 4 \frac{E_{ij}^{(0)}}{\varphi'_0} = 0,$$

$$2E_{ij}^{(0)} = -f_{15}, \quad (\text{B.8})$$

where

$$\alpha = -2 - 2 \frac{f_9}{f_7} = -\frac{2(\varphi'_0)^2 G_{4,X}}{a^2 G_4} +$$

$$\frac{1}{a^2 G_4 (2a^4 G_4 - a^2 G_{5,\varphi}(\varphi'_0)^2 + G_{5,X}(\varphi'_0)^2 (\mathcal{H}\varphi'_0 - \varphi''_0))} \times$$

$$2(\varphi'_0)^2 \left((a^2 G_{5,\varphi} - \mathcal{H}G_{5,X}\varphi'_0) (2a^2 G_4 - G_{4,X}(\varphi'_0)^2) \right.$$

$$\left. + G_{5,X} (a^2 G_4 - G_{4,X}(\varphi'_0)^2) \varphi''_0 \right), \quad (\text{B.9})$$

and

$$\begin{aligned}
\beta &= -2\varphi'_0 \frac{f_8}{f_7} \\
&= \frac{2\varphi'_0 (a^4 G_{4,\varphi} + G_{4,XX}(\varphi'_0)^2 (\mathcal{H}\varphi'_0 - \varphi''_0) - a^2 (G_{4,X\varphi}(\varphi'_0)^2 + G_{4,X}\varphi''_0))}{a^4 G_4} \\
&\quad + \frac{2\varphi'_0}{a^4 G_4 (2a^4 G_4 - a^2 G_{5,\varphi}(\varphi'_0)^2 + G_{5,X}(\varphi'_0)^2 (\mathcal{H}\varphi'_0 - \varphi''_0))} \\
&\quad \times \left[-G_{4,XX} G_{5,X} (\varphi'_0)^4 (-\mathcal{H}\varphi'_0 + \varphi''_0)^2 + \right. \\
&\quad a^6 \left((G_{4,\varphi} G_{5,\varphi} + G_4 G_{5,\varphi\varphi}) (\varphi'_0)^2 + 2G_4 G_{5,\varphi} \varphi''_0 \right) + a^2 (\varphi'_0)^2 (\mathcal{H}\varphi'_0 - \varphi''_0) \\
&\quad \left(G_4 \mathcal{H} G_{5,XX} \varphi'_0 + (G_{4,X\varphi} G_{5,X} + G_{4,XX} G_{5,\varphi}) (\varphi'_0)^2 + G_{4,X} G_{5,X} \varphi''_0 \right) + \\
&\quad a^4 \varphi'_0 \left(\varphi'_0 \left(G_4 G_{5,X} (2\mathcal{H}^2 - \mathcal{H}') - \mathcal{H} (G_{4,\varphi} G_{5,X} + 2G_4 G_{5,X\varphi}) \varphi'_0 \right. \right. \\
&\quad \left. \left. - G_{4,X\varphi} G_{5,\varphi} (\varphi'_0)^2 \right) \right. \\
&\quad \left. + \left(-2G_4 \mathcal{H} G_{5,X} + (G_{4,\varphi} G_{5,X} + G_4 G_{5,X\varphi} - G_{4,X} G_{5,\varphi}) \varphi'_0 \right) \varphi''_0 \right], \tag{B.10}
\end{aligned}$$

where $E_{ij}^{(0)}$ in the previous equations denotes the coefficient of δ_{ij} in $E_{ij}^{(0)}$ (i.e. the expression inside square brackets in Eq. (B.1)). The explicit expressions for the f_i functions with $i = 10, 15, 18, 19, 21$ can be found easily from the previous identities and the provided expressions for the other f_i functions.

B.3 Linear equations of motion in mimetic Horndeski gravity coupled to matter

In this appendix we briefly summarize well-known expressions for the linear scalar perturbations of the energy-momentum tensor in the Poisson gauge, see for example the review [102], and then we present the linear equations

of motion in mimetic Horndeski gravity coupled to fluid matter. We assume a general energy-momentum tensor of matter that may contain anisotropic stress. We also assume that there is no direct coupling between this matter fluid and the mimetic scalar field φ .

The energy-momentum tensor of the fluid that we consider has the form

$$T_{\mu\nu} = (\rho + P)u_\mu u_\nu + P g_{\mu\nu} + \pi_{\mu\nu}, \quad (\text{B.11})$$

where ρ is the energy density, P the pressure and $\pi_{\mu\nu}$ is the anisotropic stress tensor. $\pi_{\mu\nu}$ vanishes for a perfect fluid or a minimally coupled scalar field, however it is non-zero for a non-minimally coupled scalar field and free-streaming neutrinos (or radiation). u^μ is the 4-velocity and note that the 4-velocity in this appendix is not related with the 4-velocity introduced in Sec. 4.4. $\pi_{\mu\nu}$ obeys $\pi_{\mu\nu}u^\mu = 0$ and $\pi^\mu{}_\mu = 0$. We assume that the anisotropic stress is a first-order quantity and that the 4-velocity is defined so that $\pi_{00} = \pi_{0i} = 0$ [103, 6] (this is the so-called energy frame), while the spatial part of $\pi_{\mu\nu}$ can be decomposed as

$$\pi_{ij} = a^2 \left(\partial_i \partial_j \Pi - \frac{1}{3} \delta_{ij} \partial^2 \Pi + \frac{1}{2} (\partial_i \Pi_j + \partial_j \Pi_i) + \Pi_{ij} \right), \quad (\text{B.12})$$

where the vector Π_i obeys $\partial^i \Pi_i = 0$ and the tensor Π_{ij} obeys $\Pi^i{}_i = \partial^i \Pi_{ij} = 0$ (where the indices are raised with δ^{ij}). From now on we will neglect the vector and tensor parts of the anisotropic stress tensor. The 4-velocity obeys the constraint $u_\mu u^\mu = -1$ and can be expanded as

$$u^0 = a^{-1}(1 - \Phi), \quad u^i = a^{-1}v^i, \quad (\text{B.13})$$

where the velocity v^i , a first-order quantity, can be decomposed in a scalar and intrinsic vector parts as $v^i = \delta^{ij} \partial_j v + v_{\text{vec}}^i$, where $\partial_i v_{\text{vec}}^i = 0$. From now on we will also neglect v_{vec}^i . The zeroth-order components of the energy-momentum tensor are

$$T_{00}^{(0)} = a^2 \rho_0, \quad T_{0i}^{(0)} = 0, \quad T_{ij}^{(0)} = a^2 P_0 \delta_{ij}, \quad (\text{B.14})$$

where ρ_0 and P_0 denote the zeroth-order energy density and pressure respectively. The trace is $T^{(0)} = -\rho_0 + 3P_0$. At first order we have

$$\begin{aligned} T_{00}^{(1)} &= a^2 (\delta\rho + 2\rho_0\Phi), & T_{0i}^{(1)} &= -a^2 (\rho_0 + P_0) \partial_i v, \\ T_{ij}^{(1)} &= a^2 \left((\delta P - 2P_0\Psi)\delta_{ij} + \partial_i\partial_j\Pi - \frac{1}{3}\delta_{ij}\partial^2\Pi \right), \end{aligned} \quad (\text{B.15})$$

where $\delta\rho$ and δP denote the energy density and pressure perturbations respectively. The trace is $T^{(1)} = -\delta\rho + 3\delta P$. The conservation of the energy-momentum tensor, $\nabla^\mu T_{\mu\nu} = 0$ implies at zeroth order

$$\rho'_0 + 3\mathcal{H}(\rho_0 + P_0) = 0, \quad (\text{B.16})$$

and at first order

$$\delta\rho' + 3\mathcal{H}(\delta\rho + \delta P) - 3(\rho_0 + P_0)\Psi' + (\rho_0 + P_0)\partial^2 v = 0, \quad (\text{B.17})$$

$$((\rho_0 + P_0)v)' + \delta P + \frac{2}{3}\partial^2\Pi + 4\mathcal{H}(\rho_0 + P_0)v + (\rho_0 + P_0)\Phi = 0. \quad (\text{B.18})$$

The previous results are all well-known in the literature and now we will present the equations of motion of mimetic Horndeski gravity coupled with this fluid.

The equations of motion of the mimetic Horndeski model including matter are Eqs. (3.51) where the $E_{\mu\nu}$ tensor is computed from the Horndeski Lagrangian. They read

$$b(\varphi)g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - 1 = 0, \quad (\text{B.19})$$

$$E^{\mu\nu} + T^{\mu\nu} = (E + T)b(\varphi)\partial^\mu\varphi\partial^\nu\varphi, \quad (\text{B.20})$$

$$\nabla_\mu T^{\mu\nu} = 0, \quad (\text{B.21})$$

where we dropped the field equation because it is redundant and replaced the equation $\Omega_m = 0$ with the equivalent equation $\nabla_\mu T^{\mu\nu} = 0$. As shown in section 4.2, the "time-time" component of the generalized Einstein equations is also redundant. In the background the previous equations of motion reduce

to

$$-a^{-2}b(\varphi_0)(\varphi'_0)^2 = 1, \quad E_{ij}^{(0)} = -a^2 P_0 \delta_{ij}, \quad \rho'_0 + 3\mathcal{H}(\rho_0 + P_0) = 0. \quad (\text{B.22})$$

At first order they are

$$2b_0 \delta\varphi' + \varphi'_0 b_{,\varphi} \delta\varphi - 2b_0 \varphi'_0 \Phi = 0, \quad (\text{B.23})$$

$$f_7 \Psi + f_8 \delta\varphi + f_9 \Phi + a^2 \Pi = 0, \quad (\text{B.24})$$

$$f_{10} \Psi'' + f_{11} \delta\varphi'' + f_{12} \Psi' + f_{13} \delta\varphi' + f_{14} \Phi' + f_{15} \Psi + f_{16} \delta\varphi + f_{17} \Phi + \frac{2}{3} a^2 \partial^2 \Pi + a^2 (\delta P - 2P_0 \Psi) = 0, \quad (\text{B.25})$$

$$f_{10} \Psi' + f_{11} \delta\varphi' + \left(f_{20} + \frac{a^2 (E^{(0)} + T^{(0)})}{\varphi'_0} \right) \delta\varphi + f_{14} \Phi - a^2 (\rho_0 + P_0) v = 0, \quad (\text{B.26})$$

$$\delta\rho' + 3\mathcal{H}(\delta\rho + \delta P) - 3(\rho_0 + P_0)\Psi' + (\rho_0 + P_0)\partial^2 v = 0, \quad (\text{B.27})$$

$$((\rho_0 + P_0)v)' + \delta P + \frac{2}{3} \partial^2 \Pi + 4\mathcal{H}(\rho_0 + P_0)v + (\rho_0 + P_0)\Phi = 0. \quad (\text{B.28})$$

Similarly to the case discussed in the main text, one can show that the third equation of the previous set can be derived from the other equations (one does not need to use the fifth equation to show that) and using the background equations of motion. In conclusion the set of independent equations of motion in mimetic gravity with matter is given by Eqs. (B.23), (B.24), (B.26), (B.27) and (B.28). Using the variable ζ defined in subsection 4.3.2 one can write Eq. (B.26) in the previous set as

$$\zeta' = a^2 \frac{\rho_0 + P_0}{f_{10}} \left(\frac{\delta\varphi}{\varphi'_0} - v \right). \quad (\text{B.29})$$

B.4 The sound speed in the mimetic G^3 theory

Horndeski's theory is the most general 4D covariant scalar-tensor theory that can be derived from an action and contains only second order equations of

motion. However it is known [60, 61, 66] that there are theories that include Horndeski's theory, can be derived from an action and are more general than Horndeski's theory. In some cases, these theories have been shown to propagate exactly the same number of degrees of freedom as Horndeski's theory and therefore are free from higher-derivative ghosts despite having covariant higher-order equations of motion. The theories presented in [60, 61] are also known as G^3 theories.

In this appendix we show that even in a mimetic G^3 theory (without matter) at first order around a flat FLRW background, the speed of sound of scalar perturbations is still exactly zero. The beyond Horndeski theory of [60, 61] is defined by adding to the Horndeski action the following action

$$\begin{aligned}
S_{G^3} = \int d^4x \sqrt{-g} & \left[A_1(X, \varphi) \left(-2X ((\square\varphi)^2 - \nabla_\mu \nabla_\nu \varphi \nabla^\mu \nabla^\nu \varphi) \right. \right. \\
& \left. \left. -2 (\nabla^\mu \varphi \nabla^\nu \varphi \nabla_\mu \nabla_\nu \varphi \square\varphi - \nabla^\mu \varphi \nabla_\mu \nabla_\nu \varphi \nabla_\lambda \varphi \nabla^\lambda \nabla^\nu \varphi) \right) \right. \\
& + A_2(X, \varphi) \left(-2X ((\square\varphi)^3 - 3\square\varphi \nabla_\mu \nabla_\nu \varphi \nabla^\mu \nabla^\nu \varphi \right. \\
& \quad + 2\nabla_\mu \nabla_\nu \varphi \nabla^\nu \nabla^\rho \varphi \nabla^\mu \nabla_\rho \varphi) - 3((\square\varphi)^2 \nabla_\mu \varphi \nabla^\mu \nabla^\nu \varphi \nabla_\nu \varphi \\
& \quad - 2\square\varphi \nabla_\mu \varphi \nabla^\mu \nabla^\nu \varphi \nabla_\nu \nabla_\rho \varphi \nabla^\rho \varphi - \nabla_\mu \nabla_\nu \varphi \nabla^\mu \nabla^\nu \varphi \nabla_\rho \varphi \nabla^\rho \nabla^\lambda \varphi \nabla_\lambda \varphi \\
& \quad \left. \left. + 2\nabla_\mu \varphi \nabla^\mu \nabla^\nu \varphi \nabla_\nu \nabla_\rho \varphi \nabla^\rho \nabla^\lambda \varphi \nabla_\lambda \varphi) \right) \right], \tag{B.30}
\end{aligned}$$

where the functions A_1 and A_2 are free functions of their two arguments and together with the four free functions present in an Horndeski theory they define the G^3 theory.

The first-order components of the new $E_{\mu\nu}$ tensor coming from the previous

action are of the form

$$\tilde{E}_{00}^{(1)} = g_1 \Psi' + g_2 \delta\varphi' + g_3 \delta\varphi + g_4 \Phi + g_5 \partial^2 \delta\varphi, \quad (\text{B.31})$$

$$\begin{aligned} \tilde{E}_{ij}^{(1)} = \partial_i \partial_j (g_6 \delta\varphi' + g_7 \delta\varphi) + \delta_{ij} \Big(& -g_6 \partial^2 \delta\varphi' - g_7 \partial^2 \delta\varphi \\ & + g_8 \Psi'' + g_9 \delta\varphi'' + g_{10} \Psi' + g_{11} \delta\varphi' + g_{12} \Phi' \\ & + g_{13} \Psi + g_{14} \delta\varphi + g_{15} \Phi \Big), \end{aligned} \quad (\text{B.32})$$

$$\tilde{E}_{0i}^{(1)} = \partial_i (g_{16} \Psi' + g_{17} \delta\varphi' + g_{18} \delta\varphi + g_{19} \Phi), \quad (\text{B.33})$$

where the g_i with $i = 1, \dots, 19$ are functions of A_1 , A_2 and their derivatives. We do not write the explicit expressions for these functions because they are rather long and they are not important for our discussion regarding the value of the sound speed.

The first-order equations of motion for the mimetic G^3 model are determined only by $\tilde{E}_{ij}^{(1)}$ and $\tilde{E}_{0i}^{(1)}$ and their counterparts for the remaining Horndeski terms. In the absence of matter, the $i - j$ equation of motion is simply $E_{ij}^{(1)} + \tilde{E}_{ij}^{(1)} = 0$. This implies two equations as

$$f_7 \Psi + (f_8 + g_7) \delta\varphi + f_9 \Phi + g_6 \delta\varphi' = 0, \quad (\text{B.34})$$

$$\begin{aligned} & (f_{10} + g_8) \Psi'' + (f_{11} + g_9) \delta\varphi'' + (f_{12} + g_{10}) \Psi' + (f_{13} + g_{11}) \delta\varphi' \\ & + (f_{14} + g_{12}) \Phi' + (f_{15} + g_{13}) \Psi + (f_{16} + g_{14}) \delta\varphi + (f_{17} + g_{15}) \Phi = 0. \end{aligned} \quad (\text{B.35})$$

In a general mimetic theory (and in particular also for mimetic G^3) the equation $E_{00}^{(1)} + \tilde{E}_{00}^{(1)} = 0$ is replaced by the mimetic constraint, Eq. (4.26), which at first order does not contain any spatial derivatives. Because also Eqs. (B.34) and (B.35) do not contain spatial derivatives, the sound speed of the mimetic G^3 model has to be zero as in the mimetic Horndeski model.

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