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Representable functionals and derivations on Banach quasi *-algebras

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To my (grand)mum Nina

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Abstract

Locally convex quasi *-algebras, in particular Banach quasi *-algebras, have been deeply investigated by many mathematicians in the last decades in order to describe quantum physical phenomena (see [7, 8, 9, 15, 21, 35, 46, 47, 61, 68, 70]).

Banach quasi *-algebras constitute the framework of this thesis. They form a special family of locally convex quasi *-algebras, whose topology is generated by a single norm, instead of a separating family of seminorms (see, for instance, [14, 19, 20, 22]).

The first part of the work concerns the study of representable functionals and their properties. The analysis is carried through the key notions of *fully representability* and **-semisimplicity*, appeared in the literature in [9, 14, 20, 38]. In the case of Banach quasi *-algebras, these notions are equivalent up to a certain *positivity condition*. This is shown in [3], by proving first that every sesquilinear form associated to a representable functional is everywhere defined and continuous. In particular, Hilbert quasi *-algebras are always fully representable.

The aforementioned result about sesquilinear forms allows one to select *well behaved* Banach quasi *-algebras where it makes sense to reconsider in a new framework classical problems that are relevant in applications (see [13, 25, 44, 49, 58, 69, 72, 73, 74]). One of them is certainly that of derivations and of the related automorphisms groups (for instance see [4, 6, 12, 17, 26]). Definitions of course must be adapted to the new situation and for this reason we introduce weak *-derivations and weak automorphisms in [4]. We study conditions for a weak *-derivation to be the generator of such a group. An infinitesimal generator of a continuous one-parameter group of uniformly bounded weak *-automorphisms is shown to be closed and to have certain properties on its spectrum, whereas, to acquire such a group starting with a certain closed * derivation, extra regularity conditions on its domain are required. These results are then applied to a concrete example of weak *-derivations, like inner qu*-derivation occurring in physics.

Another way to study representations of a Banach quasi *-algebra is to

construct new objects starting from a finite number of them, like *tensor products* (see [5, 36, 37, 41, 43, 52, 53, 59]). In [2] we construct the tensor product of two Banach quasi *-algebras in order to obtain again a Banach quasi *-algebra tensor product. We are interested in studying their capacity to preserve properties of their factors concerning representations, like the aforementioned full representability and *-semisimplicity. It has been shown that a fully representable (resp. *-semisimple) tensor product Banach quasi *-algebra passes its properties of representability to its factors. About the viceversa, it is true if only the pre-completion is considered, i.e. if the factors are fully representable (resp. *-semisimple), then the tensor product pre-completion normed quasi *-algebra is fully representable (resp. *-semisimple).

Several examples are investigated from the point of view of Banach quasi *-algebras.

Contents

Introduction

In the last century, many mathematicians put their effort in describing quantum systems with rigorous mathematical models. Among them, in a celebrated paper [40] about algebraic formulation of Quantum theories, R. Haag and D. Kastler employed C*-algebras as suitable tools in order to describe physical phenomena. Despite this, there are quantum models not fitting in this formulation. For instance, in certain spin lattice system with long range interactions, the thermodynamical limit does not converge in any C*-topology (see [9, 21, 50, 51]).

In order to give a rigourous mathematical formulation of this kind of problems G. Lassner introduced and studied locally convex quasi*-algebras in [50, 51]. The simplest example is given by the completion of a locally convex *-algebra with separately continuous multiplication [9, 15, 35]. Clearly, in this case the multiplication *is not necessarily everywhere defined*.

For what concerns representations of locally convex quasi *-algebras, bounded operators are not enough, despite they have nice properties and they can be handled without any trouble. For this aim, the family of $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ is employed. It is made of closable operators with the same domain $\mathcal{D}$, i.e. a dense subspace of a Hilbert space $\mathcal{H}$, such that the domain of the Hilbertian adjoint contains $\mathcal{D}$. This family of *unbounded* operators can be made into a *partial *-algebra* by defining a partial product between operators. The latter were introduced by J.-P. Antoine and K. Karwowski in [8] and then extensively studied by many authors (see [9]).

This thesis aim to present results about continuity of representable functionals, i.e. those functionals that admit a GNS-like construction, and their applications to derivations arising as infinitesimal generators of *-automorphisms groups and topological tensor products in the special context of Banach quasi *-algebras.

Representations constitute an important tool to look at abstract structures (see [3, 9, 16, 17, 23, 24, 38, 71, 65]). In the case of C*-algebras, *-representations have a deep link with positive functionals, because these can be regarded as "blocks" used in the process of building *-representations, namely the *GNS*-construction. The lack of an everywhere defined multiplication makes it impossible to deal with positive functionals. However the notion of *representable functional*, introduced in [65] plays a similar role in this context. A representable functional is positive on the *core* *-algebra and some appropriate conditions guarantee the existence of a GNS-like triple, as in the classical case.

In spite of this reasonable behaviour, complete results on the continuity of representable functionals are still missing and no example of a discontinuous representable functional is known so far, whereas examples of continuos functionals that are not representable do exist. (see, for instance, [9, 38]).

Prior to investigation, Chapter 1 is devoted to background material needed for the ongoing work in the thesis. Chapter 2 concerns representable functionals on Banach quasi *-algebras and some related concepts like full representability and *-semisimplicity, devoting a special attention to the case of Hilbert quasi *-algebras, i.e. completions of Hilbert algebras under the norm defined by their inner product. The investigation of the problem concerning continuity starts looking at sesquilinear forms associated to representable and continuous functionals. These forms turn out to be everywhere defined and *bounded*, hence the notion of full representability reduces to the sufficiency of the family of these functionals, in the sense that they distinguish points in the Banach quasi *-algebra. In the case of a *-semisimple Banach quasi *-algebra, the family of representable and continuous functionals is shown to be always sufficient, thus *-semisimple Banach quasi *-algebras are *always* fully representable. The converse is true under the condition of positivity, satisfied in many examples.

Having a representable functional at hand, it is possible to associate to it a second sesquilinear form defined through the GNS-representation of the functional (see [3, 65]). This form is everywhere defined and, in the case the functional is also continuous, it coincides with the closure of the above sesquilinear form defined through the funcitonal. This remark suggests that these sesquilinear forms might be useful to characterize continuity. Indeed, it has been shown that every representable functional is continuous if, and only if, there exists another representable and *continuous* functional less or equal to the given representable functional. Nonetheless, the results gives no algorithm to construct such a functional.

Our investigation continues focusing on the case of Hilbert quasi *-algebras, that turn out to be fully representable. In this situation, representable and continuous functionals are in 1-1 correspondence with *bounded and weakly positive* elements of the Hilbert quasi *-algebra. The definition of weakly positive element is indeed a generalization of the notion given in [38].

Positivity plays an important role in studying the continuity for representable functionals. Indeed, the existence of a continuous module function, i.e. a sort of generalization of the absolute value in the case of Hilbert quasi *-algebras, owning certain invariance properties, guarantees the continuity of representable functionals that are positive on the set of all weakly positive elements. The latter condition is difficult to verify though, hence we examine in details the Hilbert space of square integrable functions, that is a Hilbert quasi *-algebra over continuous functions first and then over essentially bounded functions. In these examples, it is shown that every representable functional is continuous.

The previous chapters show the particular role of *-semisimple and fully representable Banach quasi *-algebras. These properties motivate the study of specific problems usually treated in the context of C*-algebras, in particular those more relevant for applications such as derivations and the generation of groups of automorphisms (see [10, 11, 18]). For this reason in Chapter 3, we examine derivations obtained as infinitesimal generators of automorphisms groups. We employ sesquilinear forms in order to define what a derivation on a Banach quasi *-algebra is. In this framework, the main point is to define a suitable Leibnitz rule for the derivation, since a priori its image doesn't belong to the universal multipliers, i.e. those elements for which the left and right multiplication operators are everywhere defined.

Our first step is to look at densely defined derivations on a Banach quasi *-algebra $(\mathfrak{A}, \mathfrak{A}_0)$, starting our investigation from inner qu*-derivations, namely those that can be written as $\delta_h(x) = i[h, x]$ for $x \in \mathfrak{A}_0$ and fixed $h \in \mathfrak{A}$. In the *-semisimple case, it is shown that every inner qu*-derivation is closable, independently by the nature of $h \in \mathfrak{A}$. The closure, in general, is not again a derivation in the classical sense, because the domain is not a quasi *-algebra over $\mathfrak{A}_0$. Therefore, we need to weaken the Leibnitz rule through the employment of sesquilinear forms, achieving a more general kind of derivation, i.e. a *weak *-derivation.*

As for weak *-derivations, we need a suitable notion of *-automorphisms, namely *weak *-automorphism*, in order to extend the well known result of Bratteli-Robinson about the 1-1 correspondence between certain closed *-derivations and continuous *-automorphism groups in a C*-algebra (see [25, 26]). In our case, in order to get a closed weak *-derivation as infinitesimal generator, we have to ask the group to be made of *uniformly bounded* weak *-automorphisms, condition automatically verified for C*-algebras. On the other hand, for a weak *-derivation to be the generator of a weak *-automorphisms group as before, stronger conditions have to be required, as for instance the domain should consist of bounded elements $\mathfrak{A}_b$. Despite that, these extra conditions are verified in some classical examples, thus appear to be reasonable for our work.

The last step consists of computing one parameter group generated by inner qu*-derivations and give a physical examples motivating our choice to examine derivations in a more general context when the implementing element is *unbounded*.

In the last chapter, Chapter 4, we construct the *tensor product Banach quasi *-algebra* and we explore its properties in relation with its factors. There is few literature about tensor products of unbounded operator algebras, despite the wide applications of topological tensor products (see [36, 37, 41]). This construction aims to study representations of the factors through the tensor product.

We first analyse the algebraic candidate for the tensor product Banach quasi *-algebra. A quasi *-algebra $\mathfrak{A}$ over $\mathfrak{A}_0$ can be regarded as a bimodule over $\mathfrak{A}_0$. The problem is that two quasi *-algebras are bimodules over different rings. In order to solve this problems, one might sum the rings and construct a bimodule structure of the direct sum, but this leads to a trivial tensor product, if one of the factor is unital. Then, we suppose the existence of an embedding between the *-algebras involved in the tensor product. In this way, both the quasi *-algebras are bimodules over the same ring. Moreover, if we extend the scalars and compute the tensor product, what we obtain is the same structure obtained constructing the tensor product on the smallest *-algebra.

Having at our hands a notion of tensor product quasi *-algebra, the definition of topological tensor product of normed (resp. Banach) quasi *-algebra is given. We endow the tensor product quasi *-algebra with an *admissible* norm, for instance the injective or the projective norm, in order to get a tensor product normed quasi *-algebra. The completion of the latter will be for us the *tensor product Banach quasi *-algebra*.

At this point, the existence and the relation between representations of a tensor product normed (resp. Banach) quasi *-algebra and those of the tensor factors are explored. If the tensor product Banach quasi *-algebra possesses *-representations, hence representable functionals, then also the tensor factors do. Although, the converse is true if we consider the pre-completion, i.e. the tensor product normed quasi *-algebra possesses *-representations if the factors Banach quasi *-algebras admit them.

Chapter 1

Brief review on quasi *-algebras and their representations

1.1 Partial *-algebras of operators

Partial *-algebras of *unbounded operators* play a relevant role in representation theory. We recall here the basic definitions and facts; for further details, see [8, 9].

Let $\mathcal{D}$ be a dense subspace of a Hilbert space $\mathcal{H}[\langle\cdot|\cdot\rangle]$. Denote with $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ the set of all closable linear operators $X:\mathcal{D}\to\mathcal{H}$ for which $\mathcal{D}(X^*)\supset\mathcal{D}$, where X^* indicates the adjoint of X. In symbols,

$$\mathcal{L}^\dagger(\mathcal{D},\mathcal{H}) := \{X:\mathcal{D}\to\mathcal{H} : \mathcal{D}(X^*)\supset\mathcal{D}\}.$$

$\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ is a complex vector space with respect to sum and scalar product defined in the canonical way.

In $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$, it is possible to identify the following subspace

$$\mathcal{L}^\dagger(\mathcal{D}) = \{X:\mathcal{D}\to\mathcal{D} : \mathcal{D}(X^*)\supset\mathcal{D}, X^*\mathcal{D}\subset\mathcal{D}\}.$$

If we define an involution as $X\mapsto X^\dagger\equiv X^*_{\restriction\mathcal{D}}$ and a *partial* multiplication $X\Box Y := X^{\dagger*}Y$ whenever $Y\mathcal{D}\subset\mathcal{D}(X^{\dagger*})$ and $X^\dagger\mathcal{D}\subset\mathcal{D}(Y^*)$ on $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$, then $\mathcal{L}^\dagger(\mathcal{D})$ equipped with the involution † is a *-algebra, whereas $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ is a *partial *-algebra* in sense of the following definition

Definition 1.1.1 A *partial *-algebra* is a complex vector space $\mathfrak{A}$ endowed with an involution such that $(x+\lambda y)^* = x^*+\overline{\lambda}y^*$ for $\lambda\in\mathbb{C}$ and $x^{**}=x$, coupled with a subset $\Gamma\in\mathfrak{A}\times\mathfrak{A}$ such that

(i) $(x,y)\in\Gamma$ if, and only if, $(y^*,x^*)\in\Gamma$;

(ii) $(x, y) \in \Gamma$ and $(x, z) \in \Gamma$ then $(x, \lambda y + \mu z) \in \Gamma$ for all $\lambda, \mu \in \mathbb{C}$;

(iii) whenever $(x, y) \in \Gamma$, there exists an element $x \cdot y \in \mathfrak{A}$ that satisfies the following

$$x \cdot (y + \lambda z) = x \cdot y + \lambda(x \cdot z) \text{ and } (x \cdot y)^* = y^* \cdot x^*$$

for $(x, y), (x, z) \in \Gamma$ and $\lambda \in \mathbb{C}$.

Definition 1.1.2 A †-invariant subspace $\mathfrak{O}$ of $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ such that $X \square Y \in \mathfrak{O}$ whenever $X, Y \in \mathfrak{O}$ and X is a left multiplier of Y is called *partial O^*-algebra.*

If $\mathfrak{F}$ is a †-invariant family of operators of $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$, the *weak commutant* $(\mathfrak{F}, \mathcal{D})'_w$ of $\mathfrak{F}$ is defined as follows

$$(\mathfrak{F}, \mathcal{D})'_w = \{B \in \mathcal{B}(\mathcal{H}) : \langle BX\xi|\eta\rangle = \langle B\xi|X^\dagger\eta\rangle, \forall X \in \mathfrak{F}, \xi, \eta \in \mathcal{D}\}. \quad (1.1)$$

$(\mathfrak{F}, \mathcal{D})'_w$ is stable under involution and it is weakly closed, but it is not an algebra, in general.

A rich collection of examples of partial *-algebras of interest might be analysed. For this aim, provide $\mathcal{D}$ of a locally convex topology τ finer than the topology induced by the Hilbert norm, then the conjugate dual $\mathcal{D}^\times$ of $\mathcal{D}[\tau]$, i.e. the linear space of al continuous conjugate linear functionals on $\mathcal{D}[\tau]$, contains a linear space isomorphic to $\mathcal{H}$.

If we endow $\mathcal{D}^\times$ with the strong dual topology $\tau^\times = \beta(\mathcal{D}^\times, \mathcal{D})$ generated by the family for semi-norms

$$\mathcal{D}^\times \ni f \mapsto \sup_{\xi \in B} |f(\xi)|$$

for B running over the family of all bounded subsets of $\mathcal{D}[\tau]$, then $\mathcal{H}$ is dense in $\mathcal{D}^\times[\tau^\times]$ and we achieve a *rigged Hilbert space*

$$\mathcal{D}[\tau] \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{D}^\times[\tau^\times],$$

where all the inclusions are continuous and have dense range.

A familiar example of rigged Hilbert space is given by triplets of distribution spaces; for further details, see [9].

Consider now the following family of operators

$$\mathfrak{L}(\mathcal{D}, \mathcal{D}^\times) = \{X : \mathcal{D}[\tau] \to \mathcal{D}^\times[\tau^\times] : X \text{ is continuous}\}.$$

In general, $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$, as well as $\mathcal{L}^\dagger(\mathcal{D})$, is not contained in $\mathfrak{L}(\mathcal{D}, \mathcal{D}^\times)$, because the operators $X \in \mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ need not to be continuous when $\mathcal{D}$ is endowed with its original topology. Hence we denote by $\mathfrak{L}^\dagger(\mathcal{D},\mathcal{H})$, respectively $\mathfrak{L}^\dagger(\mathcal{D})$, the subspaces of those operators that are continuous.

If $X \in \mathfrak{L}(\mathcal{D},\mathcal{D}^\times)$, then X can be interpreted as a separately continuous sesquilinear form $\beta_X : \mathcal{D}\times\mathcal{D} \to \mathbb{C}$ defined as $\beta_X(\xi,\eta) = \langle X\xi|\eta\rangle$ for $\xi,\eta \in \mathcal{D}$. In particular, if $X \in \mathfrak{L}^\dagger(\mathcal{D})$, then β_X is jointly continuous.

With the above identification, it is easy to see that $(\mathfrak{L}(\mathcal{D},\mathcal{D}^\times), \mathfrak{L}^\dagger(\mathcal{D}))$ is a special partial *-algebra, named *quasi *-algebra*.

Definition 1.1.3 A partial *-algebra $\mathfrak{A}$ containing a *-algebra $\mathfrak{A}_0$ is called *quasi *-algebra* with distinguished $\mathfrak{A}_0$ (or shortly over $\mathfrak{A}_0$) whenever $(x,y) \in \Gamma$ if, and only if, $x \in \mathfrak{A}_0$ or $y \in \mathfrak{A}_0$. Since the structure is determined by $\mathfrak{A}$ and $\mathfrak{A}_0$, we will denote by $(\mathfrak{A},\mathfrak{A}_0)$ a quasi *-algebra.

If $\mathcal{D}$ is a dense subspace of a Hilbert space $\mathcal{H}$, we have seen the partial *-algebra $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$. The mentioned partial *-algebra of operators can be provided of several *locally convex topologies.*

Among them, we introduce the *weak* topology τ_w, the *strong* topology τ_s, the *strong* *-topology τ_{s^*}.

These topologies are generated respectively by the following families of semi-norms

τ_w: $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H}) \ni X \mapsto p_{\xi,\eta}(X) = |\langle X\xi|\eta\rangle|$ for $\xi,\eta \in \mathcal{D}$;

τ_s: $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H}) \ni X \mapsto p_{\xi}(X) = \|X\xi\|$ for $\xi \in \mathcal{D}$;

τ_{s^*}: $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H}) \ni X \mapsto p_{\xi}(X) = \|X\xi\| + \|X^\dagger\xi\|$ for $\xi \in \mathcal{D}$.

If we denote with $\mathcal{L}^\dagger_b(\mathcal{D})$ the *bounded part* of $\mathcal{L}^\dagger(\mathcal{D})$, i.e.

$$\mathcal{L}^\dagger(\mathcal{D})_b := \{X \in \mathcal{L}^\dagger(\mathcal{D}) : \overline{X} \in \mathcal{B}(\mathcal{H})\},$$

then the couples $(\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})[\tau_w], \mathcal{L}^\dagger(\mathcal{D})_b)$ and $(\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})[\tau_{s^*}], \mathcal{L}^\dagger_b(\mathcal{D}))$ are *locally convex quasi *-algebras.*

Definition 1.1.4 Let $(\mathfrak{A},\mathfrak{A}_0)$ be a quasi *-algebra and τ a locally convex topology on $\mathfrak{A}$. $(\mathfrak{A},\mathfrak{A}_0)$ is called *locally convex quasi *-algebra* if

(i) the map $a \mapsto a^*$ is continuous;

(ii) for every $x \in \mathfrak{A}_0$, the multiplication operators $a \mapsto ax$ and $a \mapsto xa$ are continuous in $\mathfrak{A}[\tau]$;

(iii) $\mathfrak{A}_0$ is τ-dense in $\mathfrak{A}$.

If $\mathfrak{A}_0[\tau]$ is a locally convex *-algebra, then the completion $\widetilde{\mathfrak{A}_0}[\tau]$ is a locally convex quasi *-algebra, if the multiplication is not jointly continuous.

1.2 Concrete examples of quasi *-algebras

In this section we give some further examples to give the reader the flavour of (locally convex) quasi *-algebras.

Example 1.2.1 Let $\mathcal{S}(\mathbb{R})$ be the Schwartz space, i.e. the space of all rapidly decreasing $\mathcal{C}^\infty$-functions on $\mathbb{R}$. If we provide $\mathcal{S}(\mathbb{R})$ of the locally convex topology generated by the family of semi-norms

$$p_{k,r}(f) = \sup_{x\in\mathbb{R}} |x^k D^r f(x)|, \quad f \in \mathcal{S}(\mathbb{R}); k, r \in \mathbb{N},$$

then its topological dual $\mathcal{S}'(\mathbb{R})$ is the space of tempered distributions. $\mathcal{S}'(\mathbb{R})$ endowed with the strong dual topology can be thought as a locally convex quasi *-algebra over $\mathcal{S}(\mathbb{R})$ with partial multiplication defined as

$$(F \cdot f)(g) = (f \cdot F)(g) = F(fg), \quad F \in \mathcal{S}'(\mathbb{R}); f, g \in \mathcal{S}(\mathbb{R}).$$

Example 1.2.2 Consider the commutative von Neumann algebra $L^\infty(I, d\lambda)$ for I to be a compact interval of the real line and λ the Lebesgue measure. Define on $L^\infty[0, 1]$ the usual norm p for $p \geq 1$ as

$$\|f\|_p := \left(\int_I |f|^p d\lambda\right)^{\frac{1}{p}}, \quad f \in L^\infty(I, d\lambda).$$

In this situation, $L^p(I, d\lambda) = \widetilde{L^\infty(I, d\lambda)}[\|\cdot\|_p]$ and $(L^p(I, d\lambda), L^\infty(I, d\lambda))$ is a proper CQ*-algebra for every $p \geq 1$ (see Definition 1.3.3).

We want to stress that the Banach space $L^p(I, d\lambda)$ can be coupled with several *-algebras of functions to obtain a Banach quasi *-algebra, for instance, the space $\mathcal{C}^\infty(I)$ of all smooth functions over the interval I, the Sobolev space $W^{1,p}(I)$ of all L^p-functions admitting a first weak derivative in $L^p(I, d\lambda)$ or the space $\mathcal{C}(I)$ of all continuous functions over I.

Example 1.2.3 Let $\mathfrak{M}$ be a general von Neumann algebra and φ a normal faithful finite trace defined on $\mathfrak{M}^+$, the cone of all positive operators in $\mathfrak{M}$. For each $p \geq 1$, let us define now a norm through φ in the following way

$$\|X\|_p := \varphi(|X|^p)^{\frac{1}{p}}, \quad X \in \mathfrak{M}.$$

We indicate with $L^p(\varphi)$ the completion of $\mathfrak{M}$ with respect to $\|\cdot\|_p$ and with $L^\infty(\varphi) = \mathfrak{M}$. By [22, Proposition 2.1], $(L^p(\varphi), L^\infty(\varphi))$ is a *Banach quasi *-algebra* and, then $(L^p(\varphi), L^\infty(\varphi))$ is a *proper CQ*-algebra*.

1.3 A special case: Banach quasi *-algebras

Some of the examples in Section 1.2 are *Banach quasi *-algebras*, a special case of locally convex quasi *-algebras in which the locally convex topology is generated by a single norm.

Definition 1.3.1 Let $(\mathfrak{A}, \mathfrak{A}_0)$ be a quasi *-algebra. $(\mathfrak{A}, \mathfrak{A}_0)$ is called *normed quasi *-algebra* if a norm $\|\cdot\|$ is defined on $\mathfrak{A}$ for which the following conditions hold

(i) the involution $a \mapsto a^*$ is isometric, i.e., $\|a\| = \|a^*\|$, for every $a \in \mathfrak{A}$;

(ii) $\mathfrak{A}_0$ is $\| \cdot \|$-dense in $\mathfrak{A}$;

(iii) the map $R_x : \mathfrak{A} \ni a \mapsto ax \in \mathfrak{A}$ is continuous.

If $(\mathfrak{A}, \| \cdot \|)$ is a Banach space, we will refer to $(\mathfrak{A}, \mathfrak{A}_0)$ to a *Banach quasi *-algebra*. The norm topology of $\mathfrak{A}$ will be denoted by τ_n.

Remark 1.3.2 Certainly, if the right multiplication operators R_x for elements $x \in \mathfrak{A}_0$ are continuous, then also the left multiplication operators $L_x : \mathfrak{A} \ni a \mapsto xa \in \mathfrak{A}$ are continuous.

If the Banach quasi *-algebra $(\mathfrak{A}, \mathfrak{A}_0)$ is *unital*, , i.e. there exists a unique element $\mathbb{1} \in \mathfrak{A}_0$ such that $a\mathbb{1} = a = \mathbb{1}a$ for every $a \in \mathfrak{A}$, then we can assume that $\|\mathbb{1}\| = 1$ without loss of generality. As well as for quasi *-algebras, it is always possible to embed a Banach quasi *-algebra without unit in a Banach quasi *-algebra with unit.

To avoid trivial situations, we assume that

$$\text{If } a \in \mathfrak{A} \text{ and } ax = 0 \text{ for every } x \in \mathfrak{A}_0, \text{ then } a = 0. \tag{A}$$

This is clearly true if $\mathbb{1} \in \mathfrak{A}$.

Let us define now a new norm on $\mathfrak{A}_0$: if $x \in \mathfrak{A}_0$, we put

$$\|x\|_0 := \max\{\|x\|, \|L_x\|, \|R_x\|\},$$

where $\|L_x\|$, $\|R_x\|$ are the familiar operator norms defined for bounded operators. Then $\mathfrak{A}_0[\| \cdot \|_0]$ is a normed *-algebra and

$$\|ax\| \leq \|a\|\|x\|_0, \quad \|xa\| \leq \|x\|_0\|a\|, \quad \forall a \in \mathfrak{A}; x \in \mathfrak{A}_0.$$

Definition 1.3.3 Let $(\mathfrak{A}, \mathfrak{A}_0)$ be a Banach quasi *-algebra. $(\mathfrak{A}, \mathfrak{A}_0)$ is called a *proper CQ*-algebra* if $\mathfrak{A}_0[\| \cdot \|_0]$ is a C*-algebra.

The definition of CQ*-algebra is more general than that given above. It involves two different *-algebras and $\mathfrak{A}_0$ usually denotes their intersection. In the proper case the two *-algebras coincide with $\mathfrak{A}_0$. For further information, see [9, 20].

Let $(\mathfrak{A}, \mathfrak{A}_0)$ be a normed quasi *-algebra. Suppose that the norm is induced by an inner product $\langle \cdot | \cdot \rangle$. In this case, we deal with a *Hilbert quasi *-algebra*.

Definition 1.3.4 A Hilbert algebra is a *-algebra $\mathfrak{A}_0$ which is also a pre-Hilbert space with inner product $\langle \cdot | \cdot \rangle$ such that

(i) the map $y \mapsto xy$ is continuous with respect to the norm defined by the inner product;

(ii) $\langle xy | z \rangle = \langle y | x^* z \rangle$ for all $x, y, z \in \mathfrak{A}_0$;

(iii) $\langle x | y \rangle = \langle y^* | x^* \rangle$ for all $x, y \in \mathfrak{A}_0$;

(iv) $\mathfrak{A}_0^2$ is total in $\mathfrak{A}_0$.

Let $\mathcal{H}$ denote the Hilbert space completion of $\mathfrak{A}_0$ with respect to the norm defined by the inner product, then $(\mathcal{H}, \mathfrak{A}_0)$ is a Banach quasi *-algebra named a *Hilbert quasi *-algebra.*

Remark 1.3.5 The property

(ii)' $\langle xy | z \rangle = \langle x | z y^* \rangle$ for all $x, y, z \in \mathfrak{A}_0$

is a consequence of properties (ii) and (iii) in Definition 1.3.4.

1.3.1 Bounded elements

In order to study the structure properties of Banach quasi *-algebras, we discuss *bounded elements*, i.e. well behaved elements with respect to the multiplication defined on $\mathfrak{A}$. For further details, we refer to [66, 67].

Definition 1.3.6 Let $(\mathfrak{A}, \mathfrak{A}_0)$ be a Banach quasi *-algebra and $a \in \mathfrak{A}$. We say that

- a is *left bounded* if there exists $\gamma_a > 0$ such that

$$\|ax\| \leq \gamma_a \|x\|, \quad \forall x \in \mathfrak{A}_0.$$

- a is *right bounded* if there exists $\gamma'_a > 0$ such that

$$\|xa\| \leq \gamma'_a \|x\|, \quad \forall x \in \mathfrak{A}_0.$$

- a is *bounded* if it is left and right bounded.

The collection of all bounded elements will be denoted by $\mathfrak{A}_b$.

Clearly every $x \in \mathfrak{A}_0$ belongs to $\mathfrak{A}_b$. If $a \in \mathfrak{A}$ is left-bounded (right-bounded respectively), then the multiplication operator $\overline{L}_a$ ($\overline{R}_a$ respectively) will be *bounded*. If $a \in \mathfrak{A}$ is a general element in $\mathfrak{A}$, we don't know a priori whether R_a and L_a are closable or not. In the case they are closable for all $a \in \mathfrak{A}$, we have the following

Definition 1.3.7 Let $(\mathfrak{A}, \mathfrak{A}_0)$ a Banach quasi *-algebra. $(\mathfrak{A}, \mathfrak{A}_0)$ is said to be *fully closable* if L_a is a closable operator for every $a \in \mathfrak{A}$.

Remark 1.3.8 If all the left multiplication operators L_a are closable, then all the right multiplication operators are closable as well.

In the case in which a is left bounded, it is possible to define the element $\overline{L}_a b$ for every $b \in \mathfrak{A}$. As well, if b is right bounded, we can compute the element $\overline{R}_b a$. In general these two elements do not coincide.

Definition 1.3.9 A Banach quasi *-algebra $(\mathfrak{A}, \mathfrak{A}_0)$ is said to be *normal* if $\overline{L}_a b = \overline{R}_b a$ for every $a, b \in \mathfrak{A}_b$. In this case, we can define a *weak multiplication* $\bullet$ between bounded elements as

$$a \bullet b := \overline{L}_a b = \overline{R}_b a, \quad \forall a, b \in \mathfrak{A}_b.$$

On $\mathfrak{A}_b$ we define the norm

$$\|a\|_b = \max\left\{\|a\|, \|\overline{L}_a\|, \|\overline{R}_a\|\right\}.$$

Proposition 1.3.10 *[66] Let $(\mathfrak{A}, \mathfrak{A}_0)$ be a Banach *-algebra. If $(\mathfrak{A}, \mathfrak{A}_0)$ is normal, then $(\mathfrak{A}_b, \|\cdot\|_b)$ is a Banach quasi *-algebra with respect to the multiplication $\bullet$.*

Bounded elements of a <u>normal</u> unital Banach quasi *-algebra $(\mathfrak{A}, \mathfrak{A}_0)$ can be characterized through their *spectrum*.

Definition 1.3.11 Let $(\mathfrak{A}, \mathfrak{A}_0)$ be a normal Banach quasi *-algebra with unit $\mathbb{1}$. For every $a \in \mathfrak{A}$, two important subsets of the complex plane can be defined

- the *resolvent set* $\rho(a) := \{\lambda \in \mathbb{C} : \exists b \in \mathfrak{A}_b$ such that $\overline{R}_b a = \overline{L}_b a = \mathbb{1}\}$;
- the *spectrum* $\sigma(a) := \mathbb{C} \setminus \rho(a)$.

Similar properties to the classical case hold for the spectrum and the resolvent in a normal unital Banach quasi *-algebra $(\mathfrak{A}, \mathfrak{A}_0)$.

Proposition 1.3.12 *[66] Let $a \in \mathfrak{A}$. Then:*

(i) the resolvent $\rho(a)$ is an open subset of the complex plane;

(ii) the resolvent function $R_\lambda(a) : \lambda \in \rho(a) \mapsto (a - \lambda \mathbb{1})^{-1}$ is $\| \cdot \|_{\mathrm{b}}$-analytic on each connected component of $\rho(a)$;

(iii) $\sigma(a)$ is non-empty set;

(iv) if $r(a) = \sup_{\lambda \in \sigma(a)} |\lambda|$ is the spectral radius of a, then $a \in \mathfrak{A}_{\mathrm{b}}$ if, and only if, $r(a) < \infty$.

1.4 *-Representations of quasi *-algebras

If $(\mathfrak{A}, \mathfrak{A}_0)$ is a quasi *-algebra, we wonder whether it might be represented as a certain class of operators introduced above. For this aim, a different notion of representation is needed.

Definition 1.4.1 A **-representation* of a quasi *-algebra $(\mathfrak{A}, \mathfrak{A}_0)$ is a *-homomorphism $\pi : \mathfrak{A} \to \mathcal{L}^\dagger(\mathcal{D}_\pi, \mathcal{H}_\pi)$, where $\mathcal{D}_\pi$ is a dense subspace of the Hilbert space $\mathcal{H}_\pi$ with the following properties

(i) $\pi(a^*) = \pi(a)^\dagger$ for all $a \in \mathfrak{A}$

(ii) if $a \in \mathfrak{A}$, $x \in \mathfrak{A}_0$, then $\pi(a) \square \pi(x)$ is well-defined and $\pi(a) \square \pi(x) = \pi(ax)$.

If $(\mathfrak{A}[\| \cdot \|], \mathfrak{A}_0)$ is a Banach quasi *-algebra, then we will say that π is a *$(\| \cdot \|$-$\tau)$-continuous* *-representation if π is continuous from $\mathfrak{A}[\| \cdot \|]$ into $\mathcal{L}^\dagger(\mathcal{D}_\pi, \mathcal{H}_\pi)[\tau]$.

Remark 1.4.2 In general, $\pi(\mathfrak{A}_0)$ is non a subspace of $\mathcal{L}^\dagger(\mathcal{D})$. However, it is always possible to construct an associated *-representation $\widehat{\pi}$ through π that verifies the pointed out condition. $\widehat{\pi}$ is called a *qu*-representation* (see [65]).

If $(\mathfrak{A}, \mathfrak{A}_0)$ has a unit $\mathbb{1}$, then $\pi(\mathbb{1}) = I_{\mathcal{D}}$ for every *-representation π of the quasi *-algebra $(\mathfrak{A}, \mathfrak{A}_0)$.

Definition 1.4.3 Let π be a *-representation of the quasi *-algebra $(\mathfrak{A}, \mathfrak{A}_0)$. π is said to be *faithful* if $\pi(a) = 0$ implies $a = 0$.

Definition 1.4.4 If π is a *-representation of the quasi *-algebra $(\mathfrak{A}, \mathfrak{A}_0)$, then π is said to be *cyclic* if there exists $\xi_0 \in \mathcal{D}_\pi$ such that $\pi(\mathfrak{A}_0)\xi_0$ is dense in $\mathcal{D}_\pi$, whereas π is said to be *ultra-cyclic* $\mathcal{D}_\pi = \pi(\mathfrak{A}_0)\xi_0$ for some $\xi_0 \in \mathcal{D}_\pi$.

In order to define the closure of a *-representation π, consider the *graph topology* τ_π defined by the family of semi-norms

$$\xi \ni \mathcal{D}_\pi \mapsto \|\pi(a)\xi\|, \quad a \in \mathfrak{A}.$$

Now, let us indicate the completion of $\mathcal{D}_\pi$ with respect to τ_π with $\widetilde{\mathcal{D}_\pi}[\tau_\pi]$. The *closure* $\widetilde{\pi}$ of π is defined as the restriction of $\overline{\pi(a)}$ to $\widetilde{\mathcal{D}_\pi}$, for all $a \in \mathfrak{A}$. If $\pi = \widetilde{\pi}$, then π is said to be *closed.*

In the case of a C*-algebra, a way to obtain *-representations is to build them through positive functionals (see Theorem A.3.8). Due to the structure of Banach quasi *-algebras, we need to introduce *representable functionals.*

Definition 1.4.5 Let $(\mathfrak{A}, \mathfrak{A}_0)$ be a quasi *-algebra. A linear functional ω on $\mathfrak{A}$ is said to be *representable* if the following conditions hold

(R.1) $\omega(x^*x) \geq 0$ for every $x \in \mathfrak{A}_0$;

(R.2) $\omega(y^*a^*x) = \overline{\omega(x^*ay)}$ for every $x, y \in \mathfrak{A}_0$, $a \in \mathfrak{A}$;

(R.3) for all $a \in \mathfrak{A}$, there exists $\gamma_a > 0$ such that

$$|\omega(a^*x)| \leq \gamma_a\, \omega(x^*x)^{\frac{1}{2}}, \quad \forall x \in \mathfrak{A}_0.$$

The family of all representable functionals of the quasi *-algebra $(\mathfrak{A}, \mathfrak{A}_0)$ is indicated with $\mathcal{R}(\mathfrak{A}, \mathfrak{A}_0)$. Furthermore, if $(\mathfrak{A}[\|\cdot\|], \mathfrak{A}_0)$ is a Banach quasi *-algebra, we denote with $\mathcal{R}_c(\mathfrak{A}, \mathfrak{A}_0)$ the family of all representable and continuous functionals on $\mathfrak{A}[\|\cdot\|]$. ω is said to be *continuous* if there exists a positive constant γ such that $|\omega(a)| \leq \gamma\|a\|$ for every $a \in \mathfrak{A}$.

Remark 1.4.6 If $\omega_1, \omega_2 \in \mathcal{R}(\mathfrak{A}, \mathfrak{A}_0)$, then we have $\omega_1 + \omega_2, \in \mathcal{R}(\mathfrak{A}, \mathfrak{A}_0)$ and $\lambda\omega_1 \in \mathcal{R}(\mathfrak{A}, \mathfrak{A}_0)$ for every $\lambda \geq 0$. If $y \in \mathfrak{A}_0$ and $\omega \in \mathcal{R}(\mathfrak{A}, \mathfrak{A}_0)$, then the linear functional $\omega_y(a) := \omega(y^*ay)$ belongs to $\mathcal{R}(\mathfrak{A}, \mathfrak{A}_0)$.

Theorem 1.4.7 *[65, Theorem 3.5] Let* $(\mathfrak{A}, \mathfrak{A}_0)$ *be a quasi *-algebra with unit* $\mathbb{1}$ *and let* ω *be a representable linear functional on* $(\mathfrak{A}, \mathfrak{A}_0)$. *Then, there exists a triple* $(\pi_\omega, \lambda_\omega, \mathcal{H}_\omega)$ *such that*

(i) π_ω *is an ultra-cyclic *-representation of* $\mathfrak{A}$ *with ultra-cyclic vector* ξ_ω*;*

(ii) λ_ω *is a linear map of* $\mathfrak{A}$ *into* $\mathcal{H}_\omega$ *with* $\lambda_\omega(\mathfrak{A}_0) = \mathcal{D}_{\pi_\omega}$, $\xi_\omega = \lambda_\omega(\mathbb{1})$ *and* $\pi_\omega(a)\lambda_\omega(x) = \lambda_\omega(ax)$*;*

(iii) $\omega(a) = \langle \pi_\omega(a)\xi_\omega | \xi_\omega \rangle$ *for every* $a \in \mathfrak{A}$*.*

By Theorem 1.4.7, if ω is a representable functional on the quasi *-algebra $(\mathfrak{A}, \mathfrak{A}_0)$, then it is associated with the triple $(\pi_\omega, \lambda_\omega, \mathcal{H}_\omega)$. Through the *-representation π_ω, it is possible to define a sesquilienar form Θ^ω on $\mathfrak{A}$ as

$$\Theta^\omega(a,b) := \langle \pi_\omega(a)\xi_\omega | \pi_\omega(b)\xi_\omega \rangle, \quad a, b \in \mathfrak{A}.$$

Θ^ω belongs to the family $\mathcal{Q}_{\mathfrak{A}_0}(\mathfrak{A})$ of all sesquilinear forms on $\mathfrak{A} \times \mathfrak{A}$ such that

(i) $\Theta^\omega(a,a) \geq 0$ for every $a \in \mathfrak{A}$;

(ii) $\Theta^\omega(ax, y) = \Theta(x, a^*y)$ for all $a \in \mathfrak{A}$, $x, y \in \mathfrak{A}_0$.

Through ω, another sesquilinear form can be defined in the following way

$$\varphi_\omega(x,y) := \omega(y^*x), \quad x, y \in \mathfrak{A}_0. \tag{1.2}$$

Clearly, Θ^ω extends φ_ω. Moreover, if ω is continuous, then the sesquilinear form φ_ω is *closable* (see [38]).

A positive sesquilinear form $\varphi : \mathfrak{A}_0 \times \mathfrak{A}_0 \to \mathbb{C}$ is said *closable* if for a sequence $\{x_n\}$ in $\mathfrak{A}_0$, the following condition is verified

$$x_n \xrightarrow{\|\cdot\|} 0 \quad \text{and} \quad \varphi(x_n - x_m, x_n - x_m) \to 0 \quad \Rightarrow \quad \varphi(x_n, x_n) \to 0.$$

If φ is closable, then the sequence $\{\varphi(x_n, x_n)\}$ is Cauchy. Therefore, φ can be extended to a positive sesquilinear form $\overline{\varphi}$ on $\mathcal{D}(\overline{\varphi}) \times \mathcal{D}(\overline{\varphi})$ through a limit procedure

$$\overline{\varphi}(a,a) := \lim_{n\to\infty} \varphi(x_n, x_n),$$

where $\mathcal{D}(\overline{\varphi})$ is given by

$$\mathcal{D}(\overline{\varphi}) = \{a \in \mathfrak{A} : \exists \{x_n\} \subset \mathfrak{A}_0 \text{ such that } x_n \xrightarrow{\|\cdot\|} a \\ \text{and } \varphi(x_n - x_m, x_n - x_m) \to 0\}.$$

Example 1.4.8 Not every Banach quasi *-algebra has non-trivial representable and continuous functionals. For instance, consider the Banach quasi *-algebra $(L^1(I, d\lambda), L^\infty(I, d\lambda))$, where $I = [0,1]$ and λ is the Lebesgue measure. Then, every representable and continuous functional on $L^1(I, d\lambda)$ would be of the form

$$\omega(f) = \int_0^1 f(x)w(x)dx, \quad f \in L^1(I, d\lambda),$$

where $w \in L^\infty(I, d\lambda)$, $w \geq 0$. In this case, the only way to satisfy the condition (R.3) is for $w \equiv 0$. Indeed, the condition (R.3) in Definition 1.4.5 translates into

$$|\omega(f^*\psi)| = \left| \int_0^1 \overline{f(x)}\psi(x)w(x)dx \right| \leq \gamma_f \left(\int_0^1 \left|\psi(x)^2\right| w(x)dx \right)^{\frac{1}{2}}, \quad \forall \psi \in L^\infty(I, d\lambda).$$

This condition has to be true for every $f \in L^1(I, d\lambda)$, hence $L^1(I, d\lambda) \subset L^2(I, wd\lambda)$ for $w \in L^\infty(I, d\lambda)$, $w \geq 0$. This is never true for $w \neq 0$.

In general, a continuous functional ω satisfying (R.1) and (R.2) need not to satisfy (R.3) and therefore it is not representable, as the next examples show.

Example 1.4.9 Let us consider the Banach quasi *-algebra $L^2(I, d\lambda)$ over $\mathcal{C}(I)$ (I and λ as in Example 1.4.8). Let $w \in L^2(I, d\lambda)$, $w \geq 0$. We define

$$\omega(f) = \int_I f(x)w(x)d\lambda(x), \quad f \in L^2(I, d\lambda).$$

It is clear that ω satisfies (R.1) and (R.2). Condition (R.3) requires that, for every $f \in L^2(I)$, there exists $\gamma_f > 0$ such that

$$\left| \int_I \overline{f(x)}\phi(x)w(x)d\lambda(x) \right| \leq \gamma_f \left(\int_I |\phi(x)|^2 w(x)d\lambda(x) \right)^{\frac{1}{2}}, \quad \forall \phi \in \mathcal{C}(I).$$

Since $w(x) > 0$ a.e., this inequality implies that $f \in L^2(I, wd\lambda)$. Were $L^2(I, d\lambda) \subseteq L^2(I, wd\lambda)$ then we should have that $f\sqrt{w} \in L^2(I, d\lambda)$, for every $f \in L^2(I, d\lambda)$. This in turn implies that $w \in L^\infty(I, d\lambda)$. So it suffices to pick $w \in L^2(I, d\lambda) \setminus L^\infty(I, d\lambda)$ to get the desired example.

Example 1.4.10 Let $\mathcal{D}$ be a dense domain in a Hilbert space $\mathcal{H}$ and $\|\cdot\|_1$ a norm on $\mathcal{D}$, stronger than the Hilbert norm $\|\cdot\|$. Let $\mathsf{B}(\mathcal{D}, \mathcal{D})$ denote the vector space of all *jointly* continuous sesquilinear forms on $\mathcal{D} \times \mathcal{D}$, with respect to $\|\cdot\|_1$. The map $\varphi \mapsto \varphi^*$ with

$$\varphi^*(\xi, \eta) = \overline{\varphi(\eta, \xi)},$$

defines an involution in $\mathsf{B}(\mathcal{D}, \mathcal{D})$ (see, [65, Example 3.8]).

We denote by $\mathfrak{L}^\dagger(\mathcal{D})$ the *-subalgebra of $\mathcal{L}^\dagger(\mathcal{D})$ consisting of all operators $A \in \mathcal{L}^\dagger(\mathcal{D})$ such that both A and $A^\dagger$ are continuous from $\mathcal{D}[\|\cdot\|_1]$ into itself.

Every $A \in \mathfrak{L}^\dagger(\mathcal{D})$ defines a sesquilinear form $\varphi_A \in \mathsf{B}(\mathcal{D}, \mathcal{D})$ by

$$\varphi_A(\xi, \eta) = \langle A\xi | \eta \rangle, \quad \xi, \eta \in \mathcal{D}.$$

Indeed, we have, for every $\xi, \eta \in \mathcal{D}$,

$$|\varphi_A(\xi,\eta)| = |\langle A\xi|\eta\rangle| \leq \|A\xi\|\|\eta\| \leq \gamma\|A\xi\|_1\|\eta\|_1 \leq \gamma'\|\xi\|_1\|\eta\|_1.$$

We put

$$\mathsf{B}^\dagger(\mathcal{D}) = \{\varphi_A : A \in \mathfrak{L}^\dagger(\mathcal{D})\}.$$

We have that $\varphi_A^* = \varphi_{A^\dagger}$, for every $A \in \mathfrak{L}^\dagger(\mathcal{D})$. Indeed, for every $\xi, \eta \in \mathcal{D}$,

$$\varphi_A^*(\xi,\eta) = \overline{\varphi_A(\eta,\xi)} = \overline{\langle A\eta|\xi\rangle} = \langle \xi|A\eta\rangle = \langle A^\dagger\xi|\eta\rangle = \varphi_{A^\dagger}(\xi,\eta).$$

For $\varphi \in \mathsf{B}(\mathcal{D},\mathcal{D})$, $\varphi_A \in \mathsf{B}^\dagger(\mathcal{D})$, the multiplications are defined as

$$(\varphi \circ \varphi_A)(\xi,\eta) = \varphi(A\xi,\eta), \quad \xi,\eta \in \mathcal{D}$$
$$(\varphi_A \circ \varphi)(\xi,\eta) = \varphi(\xi,A^\dagger\eta), \quad \xi,\eta \in \mathcal{D}.$$

With these operations and involution, $(\mathsf{B}(\mathcal{D},\mathcal{D}), \mathsf{B}^\dagger(\mathcal{D}))$ is a quasi *-algebra. A norm on $\mathsf{B}(\mathcal{D},\mathcal{D})$ is defined by

$$\|\varphi\| := \sup_{\|\xi\|_1=\|\eta\|_1=1} |\varphi(\xi,\eta)|.$$

Then the pair $(\overline{\mathsf{B}^\dagger(\mathcal{D})}, \mathsf{B}^\dagger(\mathcal{D}))$, where $\overline{\mathsf{B}^\dagger(\mathcal{D})}$ denotes the $\|\cdot\|$-closure of $\mathsf{B}^\dagger(\mathcal{D})$, is a Banach quasi *-algebra.

For every $\xi \in \mathcal{D}$, we define

$$\omega_\xi(\varphi) = \varphi(\xi,\xi), \quad \varphi \in \mathsf{B}(\mathcal{D},\mathcal{D}).$$

Then ω_ξ is a linear functional on $\mathsf{B}(\mathcal{D},\mathcal{D})$. Moreover,

$$\omega_\xi(\varphi_A^* \circ \varphi_A) = (\varphi_{A^\dagger} \circ \varphi_A)(\xi,\xi) = \langle A\xi|A\xi\rangle \geq 0.$$
$$\omega_\xi(\varphi_{B^\dagger} \circ \varphi \circ \varphi_A) = \varphi(A\xi, B\xi) = \overline{\omega_\xi(\varphi_{A^\dagger} \circ \varphi^* \circ \varphi_B)}.$$

Hence, ω_ξ satisfies (R.1) and (R.2) of Definition 1.4.5.

The functional ω_ξ is representable if, and only if, for every $\varphi \in \mathsf{B}(\mathcal{D},\mathcal{D})$, there exists $\gamma_\varphi > 0$, such that

$$|\varphi(A\xi,\xi)| \leq \gamma_\varphi\|A\xi\|, \quad \forall\, A \in \mathfrak{L}^\dagger(\mathcal{D}).$$

Indeed, ω_ξ satisfies (R.3) if, and only if, for every $\varphi \in \mathsf{B}(\mathcal{D},\mathcal{D})$, there exists $\gamma_\varphi > 0$ such that

$$|\omega_\xi(\varphi^* \circ \varphi_A)| = |(\varphi \circ \varphi_A)(\xi,\xi)| = |\varphi^*(A\xi,\xi)| = |\overline{\varphi(\xi,A\xi)}| = |\varphi(\xi,A\xi)|$$
$$\leq \gamma_\varphi \omega_\xi(\varphi_A^* \circ \varphi_A)^{\frac{1}{2}} \leq \gamma_\varphi\|A\xi\|.$$

The previous condition is clearly satisfied if, and only if, φ is bounded in the second variable on the subspace $\mathcal{M}_\xi = \{A\xi; A \in \mathfrak{L}^\dagger(\mathcal{D})\}$. If this is the case, then there exists $\zeta \in \mathcal{H}$, such that

$$\omega_\xi(\varphi \circ \varphi_A) = \langle A\xi|\zeta\rangle, \quad \forall\, A \in \mathfrak{L}^\dagger(\mathcal{D}).$$

Hence, every ω_ξ is continuous but it need not be representable.

1.4.1 Full representability

Example 1.4.8 suggests us the necessity of a new definition in order to discern those Banach quasi *-algebras that have *enough* representable and continuous functionals, i.e. *fully representable* Banach quasi *-algebras.

Definition 1.4.11 Let $(\mathfrak{A}, \mathfrak{A}_0)$ be a Banach quasi *-algebra. Let $\mathfrak{A}_0^+$ be the wedge of positive elements of the *-algebra $\mathfrak{A}_0$

$$\mathfrak{A}_0^+ = \left\{ \sum_{k=1}^{n} x_k^* x_k, x_k \in \mathfrak{A}_0, n \in \mathbb{N} \right\}.$$

We call $a \in \mathfrak{A}$ *positive* if there exists a sequence $\{y_n\}$ in $\mathfrak{A}_0^+$ that converges to a. The set of all positive elements in $\mathfrak{A}$ is denoted by $\mathfrak{A}^+ := \overline{\mathfrak{A}_0^+}^{\|\cdot\|}$.

The wedge $\mathfrak{A}^+$ is a *cone* if $\mathfrak{A}^+ \cap (-\mathfrak{A}^+) = \{0\}$.

Definition 1.4.12 Let $\omega : \mathfrak{A} \to \mathbb{C}$ be a linear functional. ω is called a *positive* functional if $\omega(a) \geq 0$ for every $a \in \mathfrak{A}^+$.

Positivity is preserved by certain *-representations and continuous representable functionals.

Proposition 1.4.13 *[38] Let* $(\mathfrak{A}, \mathfrak{A}_0)$ *be a Banach quasi *-algebra and consider* $a \in \mathfrak{A}^+$. *Then*

(i) $\pi(a)$ *is positive for every* $(\|\cdot\|$-$\tau_w)$*-continuous *-representation;*

(ii) $\omega \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{A}_0)$ *is a positive functional.*

Definition 1.4.14 Let $(\mathfrak{A}, \mathfrak{A}_0)$ be a Banach quasi *-algebra. A family of positive functionals $\mathcal{F}$ on $\mathfrak{A}$ is called *sufficient* if for every $0 \neq a \in \mathfrak{A}^+$ there exists a functional $\omega \in \mathcal{F}$ such that $\omega(a) > 0$.

We are interested in studying the sufficiency of the family $\mathcal{R}_c(\mathfrak{A}, \mathfrak{A}_0)$. This property can be characterized using positive elements.

Proposition 1.4.15 *[38] Let* $(\mathfrak{A}, \mathfrak{A}_0)$ *be a Banach quasi *-algebra. The following statements are equivalent:*

(i) $\mathfrak{A}^+$ *is a cone;*

(ii) $\mathcal{R}_c(\mathfrak{A}, \mathfrak{A}_0)$ *is sufficient.*

Definition 1.4.16 A Banach quasi *-algebra $(\mathfrak{A}, \mathfrak{A}_0)$ will be said *fully representable* if $\mathcal{R}_c(\mathfrak{A}, \mathfrak{A}_0)$ is sufficient and $\mathcal{D}(\overline{\varphi_\omega}) = \mathfrak{A}$ for every $\omega \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{A}_0)$.

If $\mathcal{R}_c(\mathfrak{A}, \mathfrak{A}_0)$ sufficient and a condition of positivity is valid, we can obtain the reverse statement of Proposition 1.4.13.

Proposition 1.4.17 *[38] Let $(\mathfrak{A}, \mathfrak{A}_0)$ be a Banach quasi *-algebra for which $\mathcal{R}_c(\mathfrak{A}, \mathfrak{A}_0)$ is sufficient. Suppose the following condition (P) holds*

$$b \in \mathfrak{A} \quad \textit{and} \quad \omega(x^* b x) \geq 0 \quad \forall \omega \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{A}_0);\ \forall x \in \mathfrak{A}_0 \quad \Rightarrow \quad b \in \mathfrak{A}^+. \tag{P}$$

Then the following are equivalent for $a \in \mathfrak{A}^+$

(i) $a \in \mathfrak{A}^+$;

(ii) $\omega(a) \geq 0$ for all $\omega \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{A}_0)$;

*(iii) $\pi(a)$ is positive for every $(\|\cdot\|$-$\tau_w)$-continuous *-representation.*

1.4.2 *-Semisimplicity

Another approach to the problem of finding *-representations and studying their continuity is to investigate a certain set of continuous sesquilinear forms.

Definition 1.4.18 Let $(\mathfrak{A}, \mathfrak{A}_0)$ be a Banach quasi *-algebra. We denote by $\mathcal{S}_{\mathfrak{A}_0}(\mathfrak{A})$ the family of all sesquilinear forms $\Theta : \mathfrak{A} \times \mathfrak{A} \to \mathbb{C}$ such that

(S.1) $\Theta(a, a) \geq 0$ for all $a \in \mathfrak{A}$;

(S.2) $\Theta(ax, y) = \Theta(x, a^* y)$ for all $a \in \mathfrak{A}$, $x, y \in \mathfrak{A}_0$;

(S.3) $\|\Theta\| = \sup_{\|a\|=1=\|b\|} |\Theta(a, b)| = 1$.

Remark 1.4.19 In other words, $\mathcal{S}_{\mathfrak{A}_0}(\mathfrak{A})$ is the subset of $\mathcal{Q}_{\mathfrak{A}_0}(\mathfrak{A})$ of all the sesquilinear forms with unit norms.

Lemma 1.4.20 *[20] Let $(\mathfrak{A}, \mathfrak{A}_0)$ be a Banach quasi *-algebra. Then the following sets*

$$\begin{aligned}
\mathfrak{R}_1 &= \{a \in \mathfrak{A} : \Theta(a, a) = 0, \forall \Theta \in \mathcal{S}_{\mathfrak{A}_0}(\mathfrak{A})\}\,; \\
\mathfrak{R}_2 &= \{a \in \mathfrak{A} : \Theta(ax, y) = 0, \forall \Theta \in \mathcal{S}_{\mathfrak{A}_0}(\mathfrak{A});\ \forall x, y \in \mathfrak{A}_0\} \\
\mathfrak{R}_3 &= \{a \in \mathfrak{A} : \Theta(ax, ay) = 0, \forall \Theta \in \mathcal{S}_{\mathfrak{A}_0}(\mathfrak{A});\ \forall x, y \in \mathfrak{A}_0\}
\end{aligned}$$

are equal, i.e. $\mathfrak{R}_1 = \mathfrak{R}_2 = \mathfrak{R}_3 =: \mathfrak{R}^*$.

*The set $\mathfrak{R}^*$ is called the *-radical of $\mathfrak{A}$.*

Definition 1.4.21 We call **-semisimple* any Banach quasi *-algebra $(\mathfrak{A}, \mathfrak{A}_0)$ for which $\mathfrak{R}^* = \{0\}$.

Definition 1.4.22 A *-semisimple Banach quasi *-algebra $(\mathfrak{A}, \mathfrak{A}_0)$ is said to be *regular* if

$$\|a\| := \sup_{\Theta \in \mathcal{S}_{\mathfrak{A}_0}(\mathfrak{A})} \Theta(a,a)^{\frac{1}{2}}, \quad \forall a \in \mathfrak{A}.$$

Example 1.4.23 [19] For $p \geq 2$, any Banach quasi *-algebra $L^p(I, d\lambda)$ over $L^\infty(I, d\lambda))$ is both fully representable and *-semisimple. Indeed, all the sesquilinear forms of the form

$$\Theta_\xi(f,g) := \|\xi\|_p^{2-p} \int_0^1 f(x)\overline{g(x)}|\xi(x)|^{p-2}dx, \quad f, g \in L^p(I, d\lambda)$$

for $\xi \in L^p(I, d\lambda)$, belong to $\mathcal{S}_{\mathfrak{A}_0}(\mathfrak{A})$ and constitute a sufficient subset in $\mathcal{S}_{\mathfrak{A}_0}(\mathfrak{A})$.

If $(\mathfrak{A}, \mathfrak{A}_0)$ is a *-semisimple Banach quasi *-algebra, then it is possible to introduce a partial multiplication on $\mathfrak{A}$.

Definition 1.4.24 [20] Let $(\mathfrak{A}, \mathfrak{A}_0)$ be a *-semisimple Banach quasi *-algebra and let $a, b \in \mathfrak{A}$. We define the *weak product* $a \Box b$ *of* a, b if there exists $c \in \mathfrak{A}$ such that

$$\Theta(bx, a^*y) = \Theta(cx, y), \quad \forall \Theta \in \mathcal{S}_{\mathfrak{A}_0}(\mathfrak{A}), \forall x, y \in \mathfrak{A}_0.$$

The element c, if it exists, is unique and $a \Box b := c$.

We will denote with $R_w(\mathfrak{A})$ (respectively $L_w(\mathfrak{A})$) the space of *universal weak multipliers of* $\mathfrak{A}$, i.e. the space of all $b \in \mathfrak{A}$ such that $a \Box b$ ($b \Box a$ respectively) is well-defined for every $a \in \mathfrak{A}$. Clearly, $\mathfrak{A}_0 \subseteq R_w(\mathfrak{A})$ and $\mathfrak{A}_0 \subseteq L_w(\mathfrak{A})$.

Proposition 1.4.25 *[20] Let* $(\mathfrak{A}, \mathfrak{A}_0)$ *be a *-semisimple Banach quasi *-algebra. Then* $(\mathfrak{A}, \mathfrak{A}_0)$ *provided with the weak multiplication* $\Box$ *is a partial *-algebra.*

It is worth to introduce some topologies originated by the sesquilinear forms in $\mathcal{S}_{\mathfrak{A}_0}(\mathfrak{A})$, in the case $(\mathfrak{A}, \mathfrak{A}_0)$ is *-semisimple. These topologies are similar to those introduced in Section 1.1, but the semi-norms are essentially given by the sesquilinear forms.

τ_w: $\mathfrak{A} \ni a \mapsto |\Theta(ax, y)|$ for all $\omega \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{A}_0)$, $x, y \in \mathfrak{A}_0$;

τ_s: $\mathfrak{A} \ni a \mapsto \Theta(a,a)^{\frac{1}{2}}$ for all $\omega \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{A}_0)$;

τ_{s^*}: $\mathfrak{A} \ni a \mapsto \max\{\Theta(a,a)^{\frac{1}{2}}, \Theta(a^*,a^*)^{\frac{1}{2}}\}$ for all $\omega \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{A}_0)$;

*-Semisimplicity has a plenty of interesting implications, that we collect in a unique proposition.

Proposition 1.4.26 *[20, 66] Let* $(\mathfrak{A}, \mathfrak{A}_0)$ *be a *-semisimple Banach quasi *-algebra. Then*

(i) $\mathfrak{A}_0$ *is *-semisimple;*

(ii) $(\mathfrak{A}, \mathfrak{A}_0)$ *is fully closable;*

(iii) $(\mathfrak{A}, \mathfrak{A}_0)$ *is normal.*

Theorem 1.4.27 *[15, 66] Let* $(\mathfrak{A}, \mathfrak{A}_0)$ *be a Banach quasi *-algebra with unit* $\mathbb{1}$*. The following statements are equivalent*

(i) There exists a faithful $(\|\cdot\|\text{-}\tau_{s^*})$*-continuous *-representation* π *of* $\mathfrak{A}$ *and* $\mathfrak{A}_{\mathrm{b}} = \{a \in \mathfrak{A} : \overline{\pi(a)} \in \mathcal{B}(\mathcal{H})\}$*;*

(ii) $(\mathfrak{A}, \mathfrak{A}_0)$ *is *-semisimple.*

Corollary 1.4.28 *Let* $(\mathfrak{A}, \mathfrak{A}_0)$ *be a *-semisimple Banach quasi *-algebra with unit* $\mathbb{1}$*. Then the Banach quasi *-algebra* $(\mathfrak{A}, \mathfrak{A}_0)$ *is continuously embedded into a locally convex quasi *-algebra of operators. Moreover, if* π *is the faithful *-representation that realizes the embedding, then*

$$\mathfrak{A}_{\mathrm{b}} = \left\{ a \in \mathfrak{A} : \overline{\pi(a)} \in \mathcal{B}(\mathcal{H}) \right\}.$$

Chapter 2

The continuity of representable functionals

In the case of a C*-algebra, the counterpart of representable functionals are positive functionals. The condition of positivity automatically implies the existence of non-trivial functionals and continuity of them.

Theorem 2.0.1 *[63] Let $\mathfrak{A}_0[\|\cdot\|_0]$ be a unital C*-algebra and let $x \in \mathfrak{A}_0$. Then there exists a positive functional ω such that $\omega(x) = \|x\|_0$. Moreover, ω is continuous and $\|\omega\| = \omega(\mathbb{1})$.*

For representable functionals on locally convex quasi *-algebras no similar theorem is known, neither for the special case of Banach quasi *-algebras. Our aim is to investigate when representable functionals on a Banach quasi *-algebra $(\mathfrak{A}, \mathfrak{A}_0)$ are automatically continuous.

Representable functionals on certain Banach quasi *-algebras are indeed continuous. The existence of a noncontinuous representable functional is still an open question.

2.1 Full representability vs *-semisimplicity

As we have seen in Chapter 1 in Theorem 1.4.27, the existence and the continuity of *-representations on Banach quasi *-algebras is related to the notion of *-semisimplicity.

All the examples of *-semisimple Banach quasi *-algebras are also fully representable, therefore we ask ourselves if there is any relationship between them.

It is know that if $\omega \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{A}_0)$ for a locally convex quasi *-algebra $(\mathfrak{A}, \mathfrak{A}_0)$, then φ_ω is closable by [38].

In the case $(\mathfrak{A}, \mathfrak{A}_0)$ is a Banach quasi *-algebra, we are able to show that φ_ω can be extended continuously to all $\mathfrak{A}$.

Proposition 2.1.1 *Let $(\mathfrak{A}, \mathfrak{A}_0)$ be a unital Banach quasi *-algebra, ω in $\mathcal{R}_c(\mathfrak{A}, \mathfrak{A}_0)$ and φ_ω the associated sesquilinear form on $\mathfrak{A}_0 \times \mathfrak{A}_0$ defined as $\varphi_\omega(x, y) = \omega(y^*x)$ for every $x, y \in \mathfrak{A}_0$. Then $\mathcal{D}(\overline{\varphi}_\omega) = \mathfrak{A}$; hence $\overline{\varphi}_\omega$ is everywhere defined and bounded.*

Proof. Since ω is representable, by Theorem 1.4.7, there exists a Hilbert space $\mathcal{H}_\omega$, a linear map $\lambda_\omega : \mathfrak{A}_0 \to \mathcal{H}_\omega$ and a *-representation π_ω with values in $\mathcal{L}^\dagger(\lambda_\omega(\mathfrak{A}_0), \mathcal{H}_\omega)$ such that

$$\omega(y^*ax) = \langle \pi_\omega(a)\lambda_\omega(x)|\lambda_\omega(y)\rangle, \quad \forall a \in \mathfrak{A},\ x, y \in \mathfrak{A}_0.$$

Then, by continuity of ω and the properties of the norm on $(\mathfrak{A}, \mathfrak{A}_0)$, for every $a \in \mathfrak{A}$ and $x, y \in \mathfrak{A}_0$,

$$\begin{aligned} |\langle \pi_\omega(a)\lambda_\omega(x)|\lambda_\omega(y)\rangle| &= |\omega(y^*ax)| \leq \gamma\|y^*ax\| \leq \gamma\|y^*\|_0\,\|ax\| \\ &\leq \gamma\|a\|\|x\|_0\|y\|_0. \end{aligned} \tag{2.1}$$

Now, consider the sesquilinear form Θ^ω defined as $\Theta^\omega(a, b) = \langle \pi(a)\xi_\omega|\pi(b)\xi_\omega\rangle$ for all $a, b \in \mathfrak{A}$. As already noticed in Section 1.4, Θ^ω extends φ_ω. It remains to show that Θ^ω is closable.

Suppose now that $\{a_n\}$ is a sequence in $\mathfrak{A}$ such that $\|a_n\| \to 0$ and $\Theta^\omega(a_n - a_m, a_n - a_m) = \|\pi_\omega(a_n - a_m)\xi_\omega\|^2 \to 0$. Then the sequence $\pi_\omega(a_n)\xi_\omega$ converges to a vector $\zeta \in \mathcal{H}_\omega$. Thus,

$$\langle \pi_\omega(a_n)\xi_\omega|\lambda_\omega(y)\rangle \to \langle \zeta|\lambda_\omega(y)\rangle, \quad \forall y \in \mathfrak{A}_0.$$

By Theorem 1.4.7, the *-representation π is ultra-cyclic, thus $\pi(\mathfrak{A}_0)\mathcal{D}_\pi = \mathcal{D}_\pi$ is dense in $\mathcal{H}_\omega$. Employing (2.1), we obtain $\zeta = 0$. Indeed

$$\begin{aligned} |\langle \zeta|\lambda_\omega(y)\rangle| &= |\lim_{n\to\infty} \langle \pi_\omega(a_n)\xi_\omega|\lambda_\omega(y)\rangle| = \lim_{n\to\infty} |\langle \pi_\omega(a_n)\xi_\omega|\lambda_\omega(y)\rangle| \\ &\leq \lim_{n\to\infty} \gamma\|a_n\|\,\|x\|_0\,\|y\|_0 = 0, \quad \forall y \in \mathfrak{A}_0. \end{aligned}$$

Hence $\Theta^\omega(a_n, a_n) \to 0$; i.e., Θ^ω is closable. Thus Θ^ω is closed and everywhere defined, hence bounded. We conclude that $\Theta^\omega = \overline{\varphi}_\omega$ by the uniqueness of continuous extension. Indeed, if $a \in \mathcal{D}(\overline{\varphi}_\omega)$, then there exists a sequence $\{x_n\}$ of elements in $\mathfrak{A}_0$ such that $\|x_n - a\| \to 0$ and

$$\overline{\varphi}_\omega(a, a) = \lim_{n\to\infty} \varphi(x_n, x_n) = \lim_{n\to\infty} \Theta^\omega(x_n, x_n) = \Theta^\omega(a, a). \qquad \square$$

Corollary 2.1.2 *Let* $(\mathfrak{A}, \mathfrak{A}_0)$ *be Banach quasi *-algebra with unit. Then* $(\mathfrak{A}, \mathfrak{A}_0)$ *is fully representable if, and only if,* $\mathcal{R}_c(\mathfrak{A}, \mathfrak{A}_0)$ *is sufficient.*

Proposition 2.1.1 suggests us a deep link between full representability and *-semisimplicity. Indeed, they differ only by the condition (P) introduced in Chapter 1.

Theorem 2.1.3 *Let* $(\mathfrak{A}, \mathfrak{A}_0)$ *be a Banach quasi *-algebra with unit* $\mathbb{1}$*. If the condition of positivity (P) holds, then* $(\mathfrak{A}, \mathfrak{A}_0)$ *is *-semisimple if, and only if* $(\mathfrak{A}, \mathfrak{A}_0)$ *is fully representable.*

Proof. Assume that the condition of positivity (P) is valid. Let $\mathcal{S}_{\mathfrak{A}_0}(\mathfrak{A})$ be as in Definition 1.4.18. First, we notice that every $\Theta \in \mathcal{S}_{\mathfrak{A}_0}(\mathfrak{A})$ can be written as $\overline{\varphi}_\omega$, for some $\omega \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{A}_0)$. If we put

$$\omega_\Theta(a) := \Theta(a, \mathbb{1}), \quad a \in \mathfrak{A},$$

then ω_Θ is continuous and representable. Indeed, (R.1), (R.2) in Definition 1.4.5 are given by the positivity and the invariance of Θ, i.e. by conditions (S.1) and (S.2) in Definition 1.4.18. For (R.3), consider $a \in \mathfrak{A}$

$$\begin{aligned}|\omega_\Theta(a^*x)| = |\Theta(a^*x, \mathbb{1})| = |\Theta(x, a)| &\leq \Theta(a, a)^{\frac{1}{2}}\Theta(x, x)^{\frac{1}{2}} \\ &\leq \gamma_a \Theta(x^*x, \mathbb{1})^{\frac{1}{2}} = \gamma_a \omega_\Theta(x^*x)^{\frac{1}{2}}, \quad \forall x \in \mathfrak{A}_0,\end{aligned}$$

where $\gamma_a := \Theta(a, a)^{\frac{1}{2}} + 1 > 0$. Therefore, $\omega_\Theta \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{A}_0)$ and

$$\overline{\varphi}_{\omega_\Theta}(x, x) = \omega_\Theta(x^*x) = \Theta(x, x), \quad \forall x \in \mathfrak{A}_0.$$

Therefore, by Proposition 2.1.1, $\varphi_{\omega_\Theta} = \Theta$.

On the other hand, consider a linear functional $0 \neq \omega \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{A}_0)$ and let $\overline{\varphi}_\omega$ be the sesquilinear form associated to it as before in (1.2).

By Proposition 2.1.1 $\mathcal{D}(\overline{\varphi}_\omega) = \mathfrak{A}$ and $\overline{\varphi}_\omega$ is bounded. If $\varphi'_\omega = \overline{\varphi}_\omega / \|\overline{\varphi}_\omega\|$, then $\varphi'_\omega \in \mathcal{S}_{\mathfrak{A}_0}(\mathfrak{A})$.

Assume that $(\mathfrak{A}, \mathfrak{A}_0)$ is fully representable. Let $a \in \mathfrak{A}$ be such that $\Theta(a, a) = 0$ for every $\Theta \in \mathcal{S}_{\mathfrak{A}_0}(\mathfrak{A})$. For what we have just shown, it is enough to prove that, if $\overline{\varphi}_\omega(a, a) = 0$, for every $\omega \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{A}_0)$, then $a = 0$. We have

$$|\omega(a)| = |\overline{\varphi}_\omega(a, \mathbb{1})| \leq \overline{\varphi}_\omega(\mathbb{1}, \mathbb{1})^{1/2}\overline{\varphi}_\omega(a, a)^{1/2} = 0$$

Then by Remark 3.10 of [38] employing condition (P), we get the statement.

Assuming now that $(\mathfrak{A}, \mathfrak{A}_0)$ is *-semisimple, it is possible to show that $(\mathfrak{A}, \mathfrak{A}_0)$ is fully representable with a similar argument used in [38] as follows.

For this aim, let $\Theta \in \mathcal{S}_{\mathfrak{A}_0}(\mathfrak{A})$ and define a functional $\omega_\Theta(a) := \Theta(a, \mathbb{1})$ for $a \in \mathfrak{A}$. For every $h \in \mathfrak{A}_+ \cap (-\mathfrak{A}^+)$, we have $\omega_\Theta(h) = 0$ for every $\Theta \in \mathcal{S}_{\mathfrak{A}_0}(\mathfrak{A})$. Indeed, $h \in \mathfrak{A}^+$, hence

$$\omega_\Theta(h) = \lim_{n\to\infty} \omega_\Theta(x_n) \geq 0$$

for $\{x_n\}$ a sequence in $\mathfrak{A}_0^+$ such that $\|x_n - h\| \to 0$ as $n \to \infty$. As well, $h \in (-\mathfrak{A}^+)$, therefore $\omega_\Theta(h) \leq 0$.

Consider now $x \in \mathfrak{A}_0$ such that $\|x\| \leq 1$. Then, $\Theta_x(a, b) := \Theta(ax, bx)$ for $a, b \in \mathfrak{A}$ belongs to $\mathcal{S}_{\mathfrak{A}_0}(\mathfrak{A})$ and $\omega_{\Theta_x}(h) = 0$. Hence $\Theta(hx, x) = 0$ for every $x \in \mathfrak{A}_0$ and then $\Theta(hx, y) = 0$ for all $x, y \in \mathfrak{A}_0$. We obtain

$$\Theta(h, h) = \lim_{n\to\infty} \Theta(h, x_n) = 0, \quad \forall \Theta \in \mathcal{S}_{\mathfrak{A}_0}(\mathfrak{A})$$

and, from the *-semisimplicity of $(\mathfrak{A}, \mathfrak{A}_0)$, $h = 0$.

We showed that $\mathfrak{A}^+$ is a cone and thus by Proposition 1.4.15, $\mathcal{R}_c(\mathfrak{A}, \mathfrak{A}_0)$ is sufficient. We conclude by Proposition 2.1.1. □

Remark 2.1.4 We stress that in Theorem 2.1.3, the condition of positivity (P) is used only in the proof that a fully representable $(\mathfrak{A}, \mathfrak{A}_0)$ is *-semisimple.

Example 2.1.5 As well as for the commutative case, for $p \geq 2$, the Banach quasi *-algebra $(L^p(\varphi), L^\infty(\varphi))$ for φ a finite faithful normal trace on the von Neumann algebra $L^\infty(\varphi)$ is *-semisimple by Proposition 2.6 in [22] . Hence, $(L^p(\varphi), L^\infty(\varphi))$ is also fully representable by Theorem 2.1.3.

We have just shown in Proposition 2.1.1 that for every representable and continuous functional ω on a Banach quasi *-algebra $(\mathfrak{A}, \mathfrak{A}_0)$ the sesquilinear form φ_ω associated to ω is everywhere defined and continuous.

For a representable functional, not surely continuous, nothing is known about the closability or continuity of φ_ω. Through the *-representation π_ω associated to ω, it is still possible to define Θ^ω, an everywhere defined sesquilinear form.

By means of Θ^ω, we define a partial order in $\mathcal{R}(\mathfrak{A}, \mathfrak{A}_0)$ in the following way: if $\omega, \psi \in \mathcal{R}(\mathfrak{A}, \mathfrak{A}_0)$ we say that

$$\psi \leq \omega \quad \text{if} \quad \Theta^\psi(a, a) \leq \Theta^\omega(a, a), \quad \forall a \in \mathfrak{A}.$$

Before demonstrating a result on the continuity of representable functionals through the partial order we introduced above, we need a characterization of representability.

Proposition 2.1.6 *Let* $(\mathfrak{A}, \mathfrak{A}_0)$ *be a quasi *-algebra with unit* $\mathbb{1}$ *and* ω *a linear functional on* $\mathfrak{A}$ *satisfying (R.1) and (R.2). The following statements are equivalent.*

(i) ω *is representable.*

*(ii) There exist a *-representation* π *defined on a dense domain* $\mathcal{D}_\pi$ *of a Hilbert space* $\mathcal{H}_\pi$ *and a vector* $\zeta \in \mathcal{D}_\pi$ *such that*

$$\omega(a) = \langle \pi(a)\zeta|\zeta\rangle, \quad \forall a \in \mathfrak{A}.$$

(iii) There exists a sesquilinear form $\Theta^\omega \in \mathcal{Q}_{\mathfrak{A}_0}(\mathfrak{A})$ *such that*

$$\omega(a) = \Theta^\omega(a, \mathbb{1}), \quad \forall a \in \mathfrak{A}.$$

Proof. (i) implies (ii) by Theorem 1.4.7. Suppose now that (ii) holds and define:

$$\Theta^\omega(a, b) := \langle \pi(a)\zeta|\pi(b)\zeta\rangle, \quad a, b \in \mathfrak{A}.$$

Then $\Theta^\omega(a, \mathbb{1}) = \langle \pi(a)\zeta|\zeta\rangle = \omega(a)$ and by the properties (R.1) and (R.2) of ω, $\Theta^\omega \in \mathcal{Q}_{\mathfrak{A}_0}(\mathfrak{A})$. This proves (iii).

Finally suppose that (iii) holds. Properties (R.1) and (R.2) clearly come from $\Theta^\omega \in \mathcal{Q}_{\mathfrak{A}_0}(\mathfrak{A})$. What remains to show is (R.3). For every $a \in \mathfrak{A}$

$$|\omega(a^*x)| = |\Theta^\omega(x, a)| \leq \Theta^\omega(a, a)^{\frac{1}{2}}\Theta^\omega(x, x)^{\frac{1}{2}} \leq \gamma_a \omega(x^*x)^{\frac{1}{2}}, \quad \forall x \in \mathfrak{A}_0,$$

where, for instance, $\gamma_a := 1 + \Theta^\omega(a, a)^{1/2}$. Hence, ω is representable. □

Lemma 2.1.7 *Let* $\omega, \psi \in \mathcal{R}(\mathfrak{A}, \mathfrak{A}_0)$ *with* $\omega \leq \psi$*. Then* $\psi - \omega \in \mathcal{R}(\mathfrak{A}, \mathfrak{A}_0)$*.*

Proof. The conditions (R.1) and (R.2) are satisfied by the properties of Θ^ψ and Θ^ω. What remains to check is (R.3). For every $a \in \mathfrak{A}$ and $x \in \mathfrak{A}_0$, using the Cauchy-Schwarz inequality for the positive sesquilinear form $\Theta^\psi - \Theta^\omega$, we get

$$\begin{aligned}
|(\psi - \omega)(a^*x)| &= |(\Theta^\psi - \Theta^\omega)(a^*x, \mathbb{1})| = |(\Theta^\psi - \Theta^\omega)(x, a)| \\
&\leq (\Theta^\psi - \Theta^\omega)(x, x)^{1/2}(\Theta^\psi - \Theta^\omega)(a, a)^{1/2} \\
&= (\psi - \omega)(x^*x)^{1/2}(\Theta^\psi - \Theta^\omega)(a, a)^{1/2} \\
&\leq \gamma_a(\psi - \omega)(x^*x)^{\frac{1}{2}},
\end{aligned}$$

where $\gamma_a := (\Theta^\psi - \Theta^\omega)(a, a)^{\frac{1}{2}} + 1 > 0$. □

Theorem 2.1.8 *Let $(\mathfrak{A}, \mathfrak{A}_0)$ be a Banach quasi *-algebra. The following statements are equivalent.*

(i) Every $\omega \in \mathcal{R}(\mathfrak{A}, \mathfrak{A}_0)$ is bounded; i.e., $\mathcal{R}(\mathfrak{A}, \mathfrak{A}_0) = \mathcal{R}_c(\mathfrak{A}, \mathfrak{A}_0)$.

*(ii) Every *-representation π of $(\mathfrak{A}, \mathfrak{A}_0)$ is weakly continuous from $\mathfrak{A}[\|\cdot\|]$ into $\mathcal{L}^\dagger(\mathcal{D}_\pi, \mathcal{H}_\pi)[\tau_w]$.*

(iii) For every $\omega \in \mathcal{R}(\mathfrak{A}, \mathfrak{A}_0)$, $\omega \neq 0$, there exists a non-zero $\psi \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{A}_0)$ such that $\psi \leq \omega$.

Proof. (i) ⇒ (ii): Let π be a *-representation of $(\mathfrak{A}, \mathfrak{A}_0)$. Then, for every $\xi \in \mathcal{D}_\pi$ the linear functional $\omega(a) = \langle \pi(a)\xi|\xi\rangle$ is representable by Proposition 2.1.6 and, therefore bounded. Hence $\pi : \mathfrak{A}[\|\cdot\|] \to \mathcal{L}^\dagger(\mathcal{D}_\pi, \mathcal{H}_\pi)[\tau_w]$ is continuous. Indeed, if $\{a_n\}$ is a sequence of elements in $\mathfrak{A}$ such that $\|a_n - a\| \to 0$ for $n \to \infty$, then

$$\langle \pi(a_n)\xi|\xi\rangle = \omega_{\pi,\xi}(a_n) \to \omega_{\pi,\xi}(a) = \langle \pi(a)\xi|\xi\rangle, \quad \forall \xi \in \mathcal{D}_\pi \tag{2.2}$$

where $\omega_{\pi,\xi}$ is the functional associated to the *-representation π and the vector $\xi \in \mathcal{D}_\pi$.

(ii) ⇒ (iii): Let $\omega \in \mathcal{R}(\mathfrak{A}, \mathfrak{A}_0)$ and π_ω the corresponding GNS-representation (which is weakly-continuous by assumption) with cyclic vector ξ_ω. Then for every $\xi, \eta \in \mathcal{D}_\omega$ there exists $\gamma_{\xi,\eta} > 0$ such that

$$|\langle \pi_\omega(a)\xi|\eta\rangle| \leq \gamma_{\xi,\eta}\|a\|, \quad \forall a \in \mathfrak{A}.$$

In particular, for the cyclic vector ξ_ω, we have

$$|\omega(a)| = |\langle \pi_\omega(a)\xi_\omega|\xi_\omega\rangle| \leq \gamma_{\xi_\omega,\xi_\omega}\|a\|, \quad \forall a \in \mathfrak{A}.$$

Then (iii) holds with the obvious choice of $\psi = \omega$.

(iii) ⇒ (i): By the assumption, the set $\mathcal{K}_\omega = \{\psi \in \mathcal{R}_c(\mathcal{H}, \mathfrak{A}_0) : \psi \leq \omega\}$ is a non-empty partially ordered (by $\leq$) set. Let $\mathcal{W}$ be a totally ordered subset of $\mathcal{K}_\omega$. Then

$$\lim_{\psi \in \mathcal{W}} \psi(a)$$

exists for every $a \in \mathfrak{A}$. Indeed, the set of numbers $\{\Theta^\psi(a, a); \psi \in \mathcal{W}\}$ is increasing and bounded from above by $\Theta^\omega(a, a)$. We set, for every $a \in \mathfrak{A}$,

$$\Lambda(a, a) = \lim_{\psi \in \mathcal{W}} \Theta^\psi(a, a).$$

Then Λ satisfies the equality

$$\Lambda(a + b, a + b) + \Lambda(a - b, a - b) = 2\Lambda(a, a) + 2\Lambda(b, b), \quad \forall a, b \in \mathfrak{A},$$

hence we can define Λ on $\mathfrak{A} \times \mathfrak{A}$ using the polarization identity. By the properties of $\Theta^\psi \in \mathcal{Q}_{\mathfrak{A}_0}(\mathfrak{A})$ for every $\psi \in \mathcal{K}_\omega$, then $\Lambda \in \mathcal{Q}_{\mathfrak{A}_0}(\mathfrak{A})$.

If we put $\omega^\circ(a) = \Lambda(a, \mathbb{1})$, then $\omega^\circ(a) = \lim_{\psi \in \mathcal{W}} \psi(a)$. We now prove that $\omega^\circ \in \mathcal{R}(\mathfrak{A}, \mathfrak{A}_0)$. It is clear that ω° is a linear functional on $\mathfrak{A}$ and $\omega^\circ \leq \omega$. The conditions (R.1) and (R.2) are satisfied by the properties of Λ. We prove (R.3). Let $a \in \mathfrak{A}$. Then

$$\begin{aligned}|\omega^\circ(a^*x)| &= \lim_{\psi \in \mathcal{W}} |\psi(a^*x)| \leq \lim_{\psi \in \mathcal{W}} (1 + \Theta^\psi(a,a)^{\frac{1}{2}}) \lim_{\psi \in \mathcal{W}} \psi(x^*x)^{\frac{1}{2}} \\ &\leq (1 + \Lambda(a,a)^{\frac{1}{2}}) \omega^\circ(x^*x)^{\frac{1}{2}}, \quad \forall x \in \mathfrak{A}_0.\end{aligned}$$

We show now that ω° is bounded. For every $a \in \mathfrak{A}$ the set $\{|\psi(a)|; \psi \in \mathcal{W}\}$ is bounded; indeed, for every $\psi \in \mathcal{W}$, we get

$$|\psi(a)| = |\Theta^\psi(a, \mathbb{1})| \leq \Theta^\psi(a,a)^{\frac{1}{2}} \Theta^\psi(\mathbb{1}, \mathbb{1})^{\frac{1}{2}} \leq \Theta^\omega(a,a)^{\frac{1}{2}} \Theta^\omega(\mathbb{1}, \mathbb{1})^{\frac{1}{2}}.$$

By the uniform boundedness principle, we conclude that there exists $\gamma > 0$ such that $|\psi(a)| \leq \gamma \|a\|$, for every $\psi \in \mathcal{W}$ and for every $a \in \mathfrak{A}$. Hence,

$$|\omega^\circ(a)| = \lim_{\psi \in \mathcal{W}} |\psi(a)| \leq \gamma \|a\|, \quad \forall a \in \mathcal{H}.$$

Then, $\mathcal{W}$ has an upper bound. Then, by Zorn's lemma, $\mathcal{K}_\omega$ has a maximal element $\omega^\bullet$. It remains to prove that $\omega = \omega^\bullet$. Assume, on the contrary that $\omega > \omega^\bullet$. Let us consider the functional $\omega - \omega^\bullet$, which is non-zero and representable by Lemma 2.1.7. Then, there exists $\sigma \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{A}_0)$ such that $\omega - \omega^\bullet \geq \sigma$. Hence, $\omega \geq \omega^\bullet + \sigma$, contradicting the maximality of $\omega^\bullet$. Then $\omega = \omega^\bullet$ and, therefore, ω is continuous. □

Remark 2.1.9 The equivalence of (i) and (ii) of the previous theorem holds also in the case when $(\mathfrak{A}, \mathfrak{A}_0)$ is only a normed quasi *-algebra. Indeed, no properties related to the completeness are needed to prove that (i) implies (ii). The other direction can be shown again through the computation in (2.2).

The proof of (iii) $\Rightarrow$ (i) is inspired by a well known result of the theory of Banach *-algebras [33, Lemma 5.5.5].

2.2 Representable functionals on a Hilbert quasi *-algebra

Theorem 2.1.8 provided us conditions for the continuity of representable functionals based on the existence of a certain continuous representable functional, but it does not provide any algorithm to build the mentioned functional.

This remark encourages us to continue our investigation about the problem of continuity, focusing on the particular case of Hilbert quasi *-algebras on which the structure is richer.

For the reader's convenience, we remind that we assume the following condition (A)

(A) If $\xi \in \mathcal{H}$ and $\xi x = 0$, for every $x \in \mathfrak{A}_0$, then $\xi = 0$.

As in Definition 1.3.7, we consider, for every $\xi \in \mathcal{H}$, the following operators:

$$L_\xi : \mathfrak{A}_0 \to \mathcal{H}, \; L_\xi x = \xi x$$

and

$$R_\xi : \mathfrak{A}_0 \to \mathcal{H}, \; R_\xi x = x\xi.$$

Lemma 2.2.1 *Every Hilbert quasi *-algebra $(\mathcal{H}, \mathfrak{A}_0)$ is fully closable and, in addition, $L_{\xi^*} \subset L_\xi^*$, $R_{\xi^*} \subset R_\xi^*$ for every $\xi \in \mathcal{H}$. Hence, $L_\xi \in \mathcal{L}^\dagger(\mathfrak{A}_0, \mathcal{H})$, for every $\xi \in \mathcal{H}$. Moreover, the map*

$$\xi \in \mathcal{H} \to L_\xi \in \mathcal{L}^\dagger(\mathfrak{A}_0, \mathcal{H})$$

is injective and, if $\eta \in \mathcal{D}(\overline{L}_\xi)$ then $\eta x \in \mathcal{D}(\overline{L}_\xi)$, for every $x \in \mathfrak{A}_0$.

Proof. In order to show that L_ξ is closable, let $\{x_n\}$ be a vanishing sequence in $\mathfrak{A}_0$ such that $\xi x_n \to \eta \in \mathcal{H}$. We want to prove that $\eta = 0$.

For every $y \in \mathfrak{A}_0$, $\|y x_n^*\| \to 0$. Therefore, on one hand we have

$$\langle \xi x_n | y \rangle \to \langle \eta | y \rangle \quad \forall y \in \mathfrak{A}_0$$

and on the other

$$\langle \xi x_n | y \rangle = \langle \xi | y x_n^* \rangle \to 0 \quad \forall y \in \mathfrak{A}_0.$$

We conclude that $\langle \eta | y \rangle = 0$ for every $y \in \mathfrak{A}_0$, hence $\eta = 0$.

If $x, y \in \mathfrak{A}_0$, we have

$$\langle L_\xi x | y \rangle = \langle \xi x | y \rangle = \langle x | \xi^* y \rangle = \langle x | L_{\xi^*} y \rangle.$$

This proves the inclusion $L_{\xi^*} \subset L_\xi^*$. Symmetrical arguments can be employed for R_ξ, $\xi \in \mathcal{H}$.

The injectivity of the map $\mathcal{H} \ni \xi \mapsto L_\xi \in \mathcal{L}^\dagger(\mathfrak{A}_0, \mathcal{H})$ comes from the condition A. □

Among multiplication operators, a distinguished role is played by those whose closure is *everywhere defined*.

2.2.1 Partial multiplication and bounded elements

If $(\mathcal{H}, \mathfrak{A}_0)$ is a Hilbert quasi *-algebra, then the sesquilinear form φ

$$\varphi(\xi, \eta) := \langle \xi | \eta \rangle, \quad \forall \xi, \eta \in \mathcal{H}$$

belongs to $\mathcal{S}_{\mathfrak{A}_0}(\mathfrak{A})$. Hence, every Hilbert quasi *-algebra is *-semisimple. Therefore, Definition 1.4.24 can be rephrased in the following way, due to Riesz representation theorem for bounded sesquilinear forms.

Definition 2.2.2 Let $\xi, \eta \in \mathcal{H}$. We say that ξ is a left-multiplier of η (or, η is a right-multiplier of ξ) if there exists $\zeta \in \mathcal{H}$ such that

$$\langle \eta x | \xi^* y \rangle = \langle \zeta x | y \rangle, \quad \forall x, y \in \mathfrak{A}_0.$$

In this case we put $\xi \square \eta := \zeta$.

Clearly, $\xi \square \eta$ is well-defined if and only if $\eta^* \square \xi^*$ is well defined. Moreover $(\xi \square \eta)^* = \eta^* \square \xi^*$.

If $\xi \square \eta$ is well-defined then $L_\xi \,\square\, L_\eta$ is well defined in $\mathcal{L}^\dagger(\mathfrak{A}_0, \mathcal{H})$ and we have $L_{\xi \square \eta} = L_\xi \,\square\, L_\eta$. For $\xi \in \mathcal{H}$, we denote by $L_w(\xi)$ (resp., $R_w(\xi)$) the set of left- (resp., right-) multipliers of ξ.

Proposition 2.2.3 *Let $(\mathcal{H}, \mathfrak{A}_0)$ be a Hilbert quasi *-algebra. Then $\mathcal{H}$ is a partial *-algebra with respect to the multiplication $\square$.*

Proof. By the above discussion, $(\mathcal{H}, \mathfrak{A}_0)$ is *-semisimple. Hence, apply Proposition 1.4.25. □

Remark 2.2.4 $\xi \square \eta$ is well defined if $L_\xi \,\square\, L_\eta$ is well defined in $\mathcal{L}^\dagger(\mathfrak{A}_0, \mathcal{H})$ and vice-versa. Moreover, $L_\xi \,\square\, L_\eta = L_\zeta$ for some $\zeta \in \mathcal{H}$. If $(\mathcal{H}, \mathfrak{A}_0)$ is unital, the second condition is automatically satisfied whenever $L_\xi \,\square\, L_\eta$ is well defined, since one can put $\zeta = (L_\xi \,\square\, L_\eta)\mathbb{1}$. In this case, $L_\mathcal{H} := \{L_\xi;\, \xi \in \mathcal{H}\}$ is a partial O*-algebra on $\mathfrak{A}_0$, according to Definition 1.1.2.

Proposition 2.2.5 *Let $(\mathcal{H}, \mathfrak{A}_0)$ be a Hilbert quasi *-algebra and $\xi \in \mathcal{H}$. The following statements hold.*

(i) $R_w(\xi^*) = \{\eta \in \mathcal{D}((L_\xi \restriction_{\mathfrak{A}_0^2})^*) : \eta y \in \mathcal{D}(L_\xi^*),\, \forall y \in \mathfrak{A}_0\}$.

(ii) If $(\mathcal{H}, \mathfrak{A}_0)$ is unital, then $R_w(\xi^*) = \mathcal{D}(L_\xi^*)$.

(iii) If $(\mathcal{H}, \mathfrak{A}_0)$ is unital and ξ^* is a universal right multiplier (i.e. $R_w(\xi^*)$ is equal to $\mathcal{H}$), then $\overline{L}_\xi$ and $\overline{L}_{\xi^*}$ are bounded operators.

Proof. (i): Let $\eta \in R_w(\xi^*)$. Then,

$$\langle L_\xi x|\eta y\rangle = \langle \xi x|\eta y\rangle = \langle x|(\xi^* \Box \eta)y\rangle.$$

Hence $\eta y \in \mathcal{D}(L_\xi^*)$ and $L_\xi^*(\eta y) = (\xi^* \Box \eta)y$, for every $y \in \mathfrak{A}_0$.

Moreover, since

$$\langle \xi x|\eta y\rangle = \langle \xi x y^*|\eta\rangle = \langle x|(\xi^* \Box \eta)y\rangle = \langle xy^*|\xi^* \Box \eta\rangle,$$

we also have that $\eta \in \mathcal{D}((L_\xi \upharpoonright_{\mathfrak{A}_0^2})^*)$.

Conversely, let $\eta \in \mathcal{D}((L_\xi \upharpoonright_{\mathfrak{A}_0^2})^*)$ be such that $\eta y \in \mathcal{D}(L_\xi^*)$, for every $y \in \mathfrak{A}_0$. Then,

$$\langle L_\xi x|\eta y\rangle = \langle x|L_\xi^*(\eta y)\rangle, \quad \forall x, y \in \mathfrak{A}_0$$

On the other hand,

$$\langle L_\xi x|\eta y\rangle = \langle L_\xi x y^*|\eta\rangle = \langle xy^*|(L_\xi \upharpoonright_{\mathfrak{A}_0^2})^*\eta\rangle = \langle x|((L_\xi \upharpoonright_{\mathfrak{A}_0^2})^*\eta)y\rangle.$$

Thus,

$$\langle \xi x|\eta y\rangle = \langle x|((L_\xi \upharpoonright_{\mathfrak{A}_0^2})^*\eta)y\rangle.$$

This implies that $\eta \in R_w(\xi^*)$ and $\xi^* \Box \eta = (L_\xi \upharpoonright_{\mathfrak{A}_0^2})^*\eta$.

(ii): This follows immediately from the closed graph theorem.

(iii): In this case, $\mathfrak{A}_0^2 = \mathfrak{A}_0$. Now if $\eta \in \mathcal{D}(L_\xi^*)$ and $y \in \mathfrak{A}_0$, then $\eta y \in \mathcal{D}(L_\xi^*)$. Indeed, we have

$$\langle L_\xi x|\eta y\rangle = \langle L_\xi(xy^*)|\eta\rangle = \langle xy^*|L_\xi^*\eta\rangle = \langle x|(L_\xi^*\eta)y\rangle.$$

Hence, $\eta y \in \mathcal{D}(L_\xi^*)$ and $L_\xi^*(\eta y) = (L_\xi^*\eta)y$. □

For some $\xi \in \mathcal{H}$ it may happen that the operator L_ξ (resp. R_ξ) is bounded on $\mathfrak{A}_0$. Then its closure $\overline{L}_\xi$ (resp. $\overline{R}_\xi$) is an everywhere defined bounded operator in $\mathcal{H}$. In this case, we say that ξ is a *left- (*resp. *right-) bounded element.*

The following result is inspired by [56, Proposition 11.7.5]:

Proposition 2.2.6 *Let $(\mathcal{H}, \mathfrak{A}_0)$ be a Hilbert quasi *-algebra. Then $\xi \in \mathcal{H}$ is left-bounded if and only if it is right bounded. Moreover, $\overline{L}_\xi x = (R_{\xi^*}x^*)^*$, for every left-bounded element $\xi \in \mathcal{H}$.*

Proof. Indeed, ξ is left-bounded if, and only if, ξ^* is right-bounded, since

$$\|x\xi^*\| = \|\xi x^*\| \leq \gamma\|x^*\| = \gamma\|x\|, \quad \forall x \in \mathfrak{A}_0.$$

On the other hand, by Lemma 2.2.1, ξ is left- (resp. right-) bounded if, and only if, ξ^* is left- (resp. right-) bounded. □

From now on, we speak only of *bounded elements*. The set of bounded elements is denoted by $\mathcal{H}_b$ (see Definition 1.3.6).

Let $\xi, \eta \in \mathcal{H}_b$. Then the multiplication $\xi \Box \eta$ is, in this case, well-defined. Indeed, for every $x, y \in \mathcal{H}$, we have

$$\begin{aligned}\langle \eta x | \xi^* y \rangle &= \langle L_\eta x | \xi^* y \rangle = \langle x | L_\eta^* (\xi^* y) \rangle = \langle x | \overline{L}_{\eta^*} (L_{\xi^*} y) \rangle \\ &= \langle x | (\overline{L}_{\eta^*} \overline{L}_{\xi^*}) y) \rangle = \langle (\overline{L}_\xi \eta) x | y) \rangle.\end{aligned}$$

Hence, $\xi \Box \eta = \overline{L}_\xi \eta$. Thus $\mathcal{H}_b$ is a *-algebra containing $\mathfrak{A}_0$. It is natural to define a norm in $\mathcal{H}_b$ by $\|\xi\|_b = \|L_\xi\|$ where the latter denotes the operator norm of $\mathcal{B}(\mathcal{H})$. We notice that there is no ambiguity in this choice, because, as it is easy to check, $\|L_\xi\| = \|R_\xi\|$, for every bounded element ξ. Indeed, by Lemma 2.2.1 $R_{\xi^*} = R_\xi^*$ for $\xi \in \mathcal{H}_b$, we have

$$\|L_\xi\| = \sup_{\|x\| \le 1} \|L_\xi x\| = \sup_{\|x\| \le 1} \|\xi x\| = \sup_{\|x^*\| \le 1} \|x^* \xi^*\| = \|R_{\xi^*}\| = \|R_\xi^*\| = \|R_\xi\|.$$

Proposition 2.2.7 *$\mathcal{H}_b$ is a pre C^*-algebra. If, in addition, $(\mathcal{H}, \mathfrak{A}_0)$ has a unit $\mathbb{1}$, then $\mathcal{H}_b[\|\cdot\|_b]$ is complete and, therefore, it is a C^*-algebra.*

From (iii) of Proposition 2.2.5 we get

Proposition 2.2.8 *If $(\mathcal{H}, \mathfrak{A}_0)$ has a unit $\mathbb{1}$, then the space $\mathcal{H}_b$ of bounded elements coincides with the set $R_w(\mathcal{H})(= L_w(\mathcal{H}))$ of the universal multipliers of $\mathcal{H}$.*

Remark 2.2.9 If $(\mathcal{H}, \mathfrak{A}_0)$ has no unit, then $\mathcal{H}_b[\|\cdot\|_b]$ need not be complete. As an example take $\mathfrak{A}_0 = \mathcal{C}_c(\mathbb{R})$ the algebra of continuous functions with compact support with the L^2- inner product. Then $\mathcal{H} = L^2(\mathbb{R}, d\lambda)$, where λ indicates the Lebesgue measure on the real line. In this case $\mathcal{H}_b = L^\infty(\mathbb{R}, d\lambda) \cap L^2(\mathbb{R}, d\lambda)$ which is not complete in the L^∞-norm. If $L^\infty(\mathbb{R}, d\lambda) \cap L^2(\mathbb{R}, d\lambda)$ was complete with respect to the L^∞-norm, this norm would be equivalent to the projective norm $\|\cdot\|_2 + \|\cdot\|_\infty$. This implies $\|\cdot\|_2 \le C\|\cdot\|_\infty$ on $L^2(\mathbb{R}, d\lambda) \cap L^\infty(\mathbb{R}, d\lambda)$ and this is not happening (consider, for instance, the case of characteristic functions $\chi_{[-n,n]}$ of the interval $[-n, n]$)

Remark 2.2.10 In [67] a notion of strongly bounded element was also introduced: an element $a \in \mathfrak{A}$ is called *strongly bounded* if there exists a sequence $\{x_n\} \subset \mathfrak{A}_0$ such that

$$\sup_n \|x_n\|_0 < \infty \quad \text{and} \quad \lim_{n \to \infty} \|x_n - a\| = 0.$$

In the case of a Hilbert quasi *-algebra $(\mathcal{H}, \mathfrak{A}_0)$ the two notions are equivalent: every bounded element of $\mathcal{H}$ is strongly bounded (see, [67], and [56, Proposition 11.7.9]).

Let us now consider the following two sets of bounded operators in $\mathcal{H}$.

$$\mathcal{L} := \{R_x; x \in \mathfrak{A}_0\}', \qquad \mathcal{R} := \{L_x; x \in \mathfrak{A}_0\}',$$

where $\mathcal{A}'$ denotes the usual commutant of the set $\mathcal{A}$ of bounded operators in $\mathcal{H}$. Then, as proved in [67, Section 3], one has the equalities

$$\mathcal{L}' = \mathcal{R}, \quad \mathcal{L}'' = \mathcal{L}, \quad \mathcal{R}'' = \mathcal{R},$$

Hence $\mathcal{L}$ and $\mathcal{R}$ are both von Neumann algebras and they are the commutant of each other. In fact, if we consider the closable multiplication operators L_ξ for $\xi \in \mathcal{H}$ and we compute the *weak commutant*, i.e.

$$\{L_\xi; \xi \in \mathcal{H}\}'_w := \{S \in \mathcal{B}(\mathcal{H}) : \langle SL_\xi x|y\rangle = \langle Sx|L_{\xi^*}y\rangle, \forall \xi \in \mathcal{H}, \forall x, y \in \mathfrak{A}_0\},$$

we obtain again a von Neumann algebra, as it is proved in the following

Proposition 2.2.11 *Let* $(\mathcal{H}, \mathfrak{A}_0)$ *be a Hilbert quasi *-algebra with unit* $\mathbb{1}$*. Then,* $\{L_\xi; \xi \in \mathcal{H}\}'_w = \mathcal{R}$*. Hence* $\{L_\xi; \xi \in \mathcal{H}\}'_w$ *is a von Neumann algebra.*

Proof. Let $T \in \mathcal{R}$ and $\{x_n\}$ a sequence in $\mathfrak{A}_0$ converging to ξ. Then,

$$\begin{aligned}\langle TL_\xi x|y\rangle &= \langle T(\xi x)|y\rangle = \lim_{n\to\infty}\langle T(x_n x)|y\rangle \\ &= \lim_{n\to\infty}\langle TL_{x_n}x)|y\rangle = \lim_{n\to\infty}\langle L_{x_n}Tx|y\rangle \\ &= \lim_{n\to\infty}\langle Tx|L_{x_n^*}y\rangle = \langle Tx|L_{\xi^*}y\rangle.\end{aligned}$$

Hence $T \in \{L_\xi; \xi \in \mathcal{H}\}'_w$. The converse inclusion is obvious. □

This result opens up the possibility of classifying Hilbert quasi *-algebras following the classification of their bounded part as von Neumann algebras. We leave this problem for further investigations.

2.2.2 Positive elements and representable functionals

A notion of *positivity* has already been introduced in Definition 1.4.11. In this subsection we introduce a weaker notion of positivity that helps us to characterize representable and continuous functionals on a Hilbert quasi *-algebra. A natural question is as to whether these two notions coincide.

Definition 2.2.12 Let $(\mathcal{H}, \mathfrak{A}_0)$ be a Hilbert quasi *-algebra and $\xi \in \mathcal{H}$. We say that ξ is *w-positive* if L_ξ (or, equivalently R_ξ) is a positive operator; i.e., if

$$\langle L_\xi x|x\rangle = \langle \xi x|x\rangle \geq 0, \forall x \in \mathfrak{A}_0.$$

If ξ is w-positive, then $\xi = \xi^*$. Moreover L_ξ is positive, if and only if, R_ξ is positive. The latter statement is due to the equalities

$$\langle L_\xi x|x\rangle = \langle \xi x|x\rangle = \langle x^*|x^*\xi\rangle \geq 0 \text{ if } L_\xi \geq 0 \text{ and } x \in \mathfrak{A}_0.$$

We put

$$\mathcal{H}_w^+ = \{\xi \in \mathcal{H} :\ \langle \xi x|x\rangle \geq 0, \forall x \in \mathfrak{A}_0\} = \{\xi, \in \mathcal{H} :\ \xi \text{ is w-positive}\}.$$

The wedge $\mathcal{H}_w^+$ defines in a standard way a partial order in the real space $\mathcal{H}_h = \{\xi \in \mathcal{H} : \xi = \xi^*\}$: if $\xi, \eta \in \mathcal{H}_h$ we write $\xi \leq \eta$ when $\eta - \xi \in \mathcal{H}_w^+$.

The notion of w-positive element plays a role in the description of representable and continuous functionals on $(\mathcal{H}, \mathfrak{A}_0)$.

By means of this new notion of w-positivity, we can show the 1-1 correspondence between representable and continuous functionals on $(\mathcal{H}, \mathfrak{A}_0)$ and bounded and w-positive elements in $(\mathcal{H}, \mathfrak{A}_0)$.

Theorem 2.2.13 *Let $(\mathcal{H}, \mathfrak{A}_0)$ be a Hilbert quasi *-algebra. Then, ω is in $\mathcal{R}_c(\mathcal{H}, \mathfrak{A}_0)$ if, and only if, there exists a unique w-positive bounded element $\eta \in \mathcal{H}$ such that*

$$\omega(\xi) = \langle \xi|\eta\rangle, \quad \forall \xi \in \mathcal{H}. \tag{2.3}$$

Proof. Let ω be a representable and continuous functional on $(\mathcal{H}, \mathfrak{A}_0)$. Then there exists a unique vector $\eta \in \mathcal{H}$ such that

$$\omega(\xi) = \langle \xi|\eta\rangle, \quad \forall \xi \in \mathcal{H}. \tag{2.4}$$

We want to show that η is w-positive and bounded. The condition (R.1) implies that η is w-positive, i.e. the operator R_η is positive (but not necessarily self-adjoint). Indeed,

$$0 \leq \omega(x^*x) = \langle x^*x|\eta\rangle = \langle x|x\eta\rangle = \langle x|R_\eta x\rangle, \quad \forall x \in \mathfrak{A}_0.$$

This in turn implies that $\eta = \eta^*$.

In the situation we are examining, the condition (R.3) reads as follows

$$\forall \xi \in \mathcal{H},\ \exists \gamma_\xi > 0 :\ |\langle \xi^* x|\eta\rangle| \leq \gamma_\xi \langle x^*x|\eta\rangle^{1/2},\ \forall x \in \mathfrak{A}_0.$$

By Proposition 2.1.1, $\varphi_\omega(x, x) = \langle x^*x|\eta\rangle$ has an everywhere defined closure $\overline{\varphi}_\omega$ in $\mathcal{H}$, therefore for every $\xi \in \mathcal{H}$ there exists $c_{\xi,\eta} > 0$ such that

$$|\langle \xi^* x|\eta\rangle| \leq c_{\xi,\eta}\|x\|, \quad \forall x \in \mathfrak{A}_0.$$

This implies that there exists a vector $\eta' \in \mathcal{H}$ such that

$$\langle \xi^* x | \eta \rangle = \langle x | \eta' \rangle, \quad \forall x \in \mathfrak{A}_0.$$

Hence, for every $x, y \in \mathfrak{A}_0$ we get

$$\langle \xi^* x | \eta y \rangle = \langle \xi^* x y^* | \eta \rangle = \langle x y^* | \eta' \rangle = \langle x | \eta' y \rangle.$$

Therefore, $\xi \square \eta$ is well defined for every $\xi \in \mathcal{H}$. Thus $L_w(\eta) = \mathcal{H}$ and, since $\eta = \eta^*$, $R_w(\eta) = \mathcal{H}$. Taking into account Lemma 2.2.1, both L_η and R_η are bounded operators on $\mathcal{H}$.

For the sufficiency, let $\eta \in \mathcal{H}$ be a w-positive bounded element and define the linear map $\omega(\xi) := \langle \xi | \eta \rangle$. We want to show that ω is representable and continuous.

(R.1) and (R.2) can be proven easily. As for (R.3), denoting by $\overline{R}_\eta$ the continuous extension of R_η to $\mathcal{H}$, we have, for every $\xi \in \mathcal{H}$,

$$\begin{aligned} |\omega(\xi^* x)| &= |\langle \xi^* x | \eta \rangle| = |\langle x | \overline{R}_\eta \xi \rangle| \leq \langle x | R_\eta x \rangle^{1/2} \langle \xi | \overline{R}_\eta \xi \rangle^{1/2} \\ &= \omega(x^* x)^{1/2} \langle \xi | \overline{R}_\eta \xi \rangle^{1/2}, \quad \forall x \in \mathfrak{A}_0 \end{aligned}$$

due to the generalized Cauchy-Schwarz inequality for positive operators. Thus (R.3) is fulfilled. □

Theorem 2.2.13 allows to compare the two sets of elements $\mathcal{H}^+$ and $\mathcal{H}^+_w$ related to the notion of positivity.

Lemma 2.2.14 *Let $(\mathcal{H}, \mathfrak{A}_0)$ be a Hilbert quasi *-algebra. Then, $\mathcal{H}^+ \subseteq \mathcal{H}^+_w$.*

Proof. Let $\eta \in \mathcal{H}^+$; i.e. there exists a sequence $\{x_n\}$ of elements of $\mathfrak{A}_0^+$ (thus each x_n has the form $x_n = \sum_{k=1}^N z_{nk}^* z_{nk}$, $z_{nk} \in \mathfrak{A}_0$) τ_n-converging to η. Then,

$$\begin{aligned} \langle L_\eta y | y \rangle &= \langle \eta y | y \rangle = \lim_{n\to\infty} \langle x_n y | y \rangle \\ &= \lim_{n\to\infty} \langle \sum_{k=1}^N z_{nk}^* z_{nk} y | y \rangle = \lim_{n\to\infty} \sum_{k=1}^N \langle z_{nk} y | z_{nk} y \rangle \geq 0. \quad \square \end{aligned}$$

Lemma 2.2.2 implies that the wedge $\mathcal{H}^+_w$ is actually a cone. Indeed, $(\mathcal{H}, \mathfrak{A}_0)$ is fully representable by the discussion in Subsection 2.2.1. By Corollary 2.1.2 $\mathcal{R}_c(\mathcal{H}, \mathfrak{A}_0)$ is sufficient and hence $\mathcal{H}^+$ is a cone, by Proposition 1.4.15. By Lemma 2.2.2, $\mathcal{H}^+ \subseteq \mathcal{H}^+_w$, thus $\mathcal{H}_w$ is also a cone.

Proposition 2.2.15 *Let $(\mathcal{H}, \mathfrak{A}_0)$ be a Hilbert quasi *-algebra. If the condition (P) holds and $\mathcal{H}^+_{wb} := \{\xi \in \mathcal{H}_b : \langle \xi x | x \rangle \geq 0, \forall x \in \mathfrak{A}_0\} \subseteq \mathcal{H}^+$, then $\mathcal{H}^+ = \mathcal{H}^+_w$.*

Proof. Suppose now that the condition (P) is valid and let $\eta \in \mathcal{H}_w^+$. We want to show that $\eta \in \mathcal{H}^+$.

By weak positivity, $\langle \eta y | y \rangle \geq 0$ for every $y \in \mathfrak{A}_0$. This implies that $\eta \in \mathcal{H}$ is w-positive if, and only if, $\langle \eta, h \rangle \geq 0$ for every $h \in \mathfrak{A}_0^+$, so by the continuity of the inner product, $\langle \eta, \xi \rangle \geq 0$ for every $\xi \in \mathcal{H}^+$.

For the assumption $\mathcal{H}_{wb}^+ \subseteq \mathcal{H}^+$, the previous claim is valid in particular for every $\chi \in \mathcal{H}_{wb}^+$ and, by Theorem 2.2.13, every χ corresponds to a functional $\omega \in \mathcal{R}_c(\mathcal{H}, \mathfrak{A}_0)$ defined as $\omega(\zeta) := \langle \zeta | \chi \rangle$, for every $\zeta \in \mathcal{H}$. Therefore, for every $\omega \in \mathcal{R}_c(\mathcal{H}, \mathfrak{A}_0)$, we have

$$\omega(\eta) = \langle \eta | \chi \rangle \geq 0.$$

By condition (P) it follows that $\eta \in \mathcal{H}^+$. □

2.2.3 Integrable Hilbert quasi *-algebras

Let $(\mathcal{H}, \mathfrak{A}_0)$ be a Hilbert quasi *-algebra. As we have seen, in general, if $\xi \in \mathcal{H}$ and ξ is not bounded, we have $\overline{L_\xi} \subseteq L_{\xi^*}^*$, but we do not know if the equality holds for every $\xi \in \mathcal{H}$. For this reason we introduce the following definition.

Definition 2.2.16 A Hilbert quasi *-algebra $(\mathcal{H}, \mathfrak{A}_0)$ is called *integrable* if $\mathfrak{A}_0$ is a core for L_ξ^*, for every $\xi \in \mathcal{H}$.

It is clear that, if $(\mathcal{H}, \mathfrak{A}_0)$ is integrable, then, for every $\xi \in \mathcal{H}$, $\overline{L_\xi} = L_{\xi^*}^*$, since $L_{\xi^*} \subseteq L_\xi^*$. In particular, if $\xi = \xi^*$, then L_ξ is essentially self-adjoint.

Definition 2.2.17 Let $(\mathcal{H}, \mathfrak{A}_0)$ be a Hilbert quasi *-algebra with unit $\mathbb{1}$. We say that $(\mathcal{H}, \mathfrak{A}_0)$ admits a *module function* if there exists a map $\mu : \mathcal{H} \to \mathcal{H}$ with the following properties:

(i) $\mu(\xi) \in \mathcal{H}_w^+$, for every $\xi \in \mathcal{H}$;

(ii) $\mu(\xi) = \xi$, for every $\mathcal{H}_w^+$;

(iii) $\|\mu(\xi)\| = \|\xi\|$, for every $\xi \in \mathcal{H}$.

Remark 2.2.18 Condition (iii) of Definition 2.2.17 obviously implies that μ is continuous at 0.

Proposition 2.2.19 *Let $(\mathcal{H}, \mathfrak{A}_0)$ be an integrable Hilbert quasi *-algebra with unit $\mathbb{1}$. Assume that for every $\xi \in \mathcal{H}$, the operator $H_{(\xi)} = (L_\xi^* \overline{L_\xi})^{1/2}$ has the following property:*

$$H_{(\xi)}(xy) = (H_{(\xi)}x)y \quad \forall x, y \in \mathfrak{A}_0. \tag{2.5}$$

Then $(\mathcal{H}, \mathfrak{A}_0)$ *admits a module function.*

Proof. $H_{(\xi)}$ is a positive self-adjoint operator with $D(H_{(\xi)}) = D(\overline{L}_\xi)$. By choosing $x = \mathbb{1}$ in (2.5), we get $H_{(\xi)}y = (H_{(\xi)}\mathbb{1})y$ for every $y \in \mathfrak{A}_0$; that is, $H_{(\xi)}y = L_{(H_{(\xi)}\mathbb{1})}y$, for every $y \in \mathfrak{A}_0$.

The module function is then defined everywhere in $\mathcal{H}$ by the following map

$$\mu : \mathcal{H} \ni \xi \mapsto \mu(\xi) := H_{(\xi)}\mathbb{1} \in \mathcal{H}.$$

We check the conditions (i)-(iii) of Definition 2.2.17. We have $\mu(\xi) \in \mathcal{H}_w^+$. Indeed,

$$\langle \mu(\xi)x|x\rangle = \langle (H_{(\xi)}\mathbb{1})x|x\rangle = \langle H_{(\xi)}x|x\rangle \geq 0, \quad \forall x \in \mathfrak{A}_0$$

As for (ii), for every $\xi \in \mathcal{H}_w^+$, the operator L_ξ is essentially self-adjoint and positive by the integrability. Hence $H_{(\xi)} = (L_\xi^*\overline{L}_\xi)^{1/2} = \overline{L}_\xi$. This implies that $\mu(\xi) = \xi$.

Finally, for every $\xi \in \mathcal{H}$,

$$\|\xi\|^2 = \langle \xi|\xi\rangle = \langle \xi\mathbb{1}|\xi\mathbb{1}\rangle = \langle L_\xi\mathbb{1}|L_\xi\mathbb{1}\rangle = \langle H_{(\xi)}\mathbb{1}|H_{(\xi)}\mathbb{1}\rangle = \| \mu(\xi) \|^2.$$

So (iii) holds. □

A case in which (2.5) verifies is the following. Assume that ω is trace, i.e. $\omega(xy) = \omega(yx)$ for every $x, y \in \mathfrak{A}_0$. Then, in Proposition 2.1.1, we proved that if $\omega \in \mathcal{R}_c(\mathcal{H}, \mathfrak{A}_0)$, then $\overline{\varphi}_\omega$ is everywhere defined and bounded. Therefore, there exists a bounded operator $B : \mathfrak{A} \to \mathfrak{A}$ such that

$$\overline{\varphi}_\omega(a, b) = \langle Ba|b\rangle, \quad \forall a, b \in \mathfrak{A}.$$

If $x, y, z \in \mathfrak{A}_0$, then we obtain

$$\begin{aligned}\langle B(xy)|z\rangle &= \varphi_\omega(xy, z) = \omega(z^*xy) = \langle \lambda_\omega(x)\lambda_\omega(y)|\lambda_\omega(z)\rangle \\ &= \langle \lambda_\omega(x)|\lambda_\omega(z)\lambda_\omega(y)^*\rangle = \varphi_\omega(x, zy^*) = \langle Bx|zy^*\rangle = \langle (Bx)y|z\rangle.\end{aligned}$$

The density of $\mathfrak{A}_0$ in $\mathcal{H}$ implies

$$B(xy) = (Bx)y, \quad \forall x, y \in \mathfrak{A}_0.$$

Hence B is verifying the assumption 2.5 in Proposition 2.2.19.

Assume now that $\mathfrak{A}_0$ has unit $\mathbb{1}$. Then, if we put $\xi := B\mathbb{1}$, we obtain $Bx = L_\xi x$ for every $x \in \mathfrak{A}_0$. This element $\xi \in \mathcal{H}$ is uniquely determined. Moreover, ξ is a bounded element by the fact that B is bounded.

Lemma 2.2.20 *Let* $(\mathcal{H}, \mathfrak{A}_0)$ *be a unital integrable Hilbert quasi *-algebra with a module function* μ. *Then, for every* $\xi \in \mathcal{H}$, *with* $\xi = \xi^*$, *there exists two elements* $\xi_+, \xi_- \in \mathcal{H}_w^+$ *such that* $\xi = \xi_+ - \xi_-$; $\mu(\xi) = \xi_+ + \xi_-$; $\xi_+ \square \xi_-$ *and* $\xi_- \square \xi_+$ *are both well-defined and* $\xi_+ \square \xi_- = \xi_- \square \xi_+ = 0$.

Proof. Let us define the following two operators on $\mathcal{D}(\overline{L}_\xi) = \mathcal{D}(H_{(\xi)})$:

$$P_{(\xi)} := \frac{1}{2}(H_{(\xi)} + \overline{L}_\xi), \qquad N_{(\xi)} := \frac{1}{2}(H_{(\xi)} - \overline{L}_\xi).$$

Put $\xi_+ = P_{(\xi)}\mathbb{1}$ and $\xi_- = N_{(\xi)}\mathbb{1}$. We want to show that these elements are in $\mathcal{H}_w^+$. Indeed, for every $x \in \mathfrak{A}_0$ we have

$$2\langle \xi_+ x | x \rangle = \langle (H_{(\xi)}\mathbb{1} + \overline{L}_\xi \mathbb{1})x | x \rangle = \langle \mu(\xi)x | x \rangle + \langle \xi x | x \rangle. \tag{2.6}$$

Our aim is to show that the above quantity (2.6) is nonnegative for every $x \in \mathfrak{A}_0$. The worst case happens when $\langle \xi x | x \rangle \leq 0$ for every $x \in \mathfrak{A}_0$, i.e. when $-\xi \in \mathcal{H}_w^+$, but in this case $\mu(\xi) = \mu(-\xi) = -\xi$. Hence, (2.6) becomes

$$\langle -\xi x | x \rangle + \langle \xi x | x \rangle = 0$$

and the claim is proved. The proof for ξ_- is similar and clearly $\xi = \xi_+ - \xi_-$ and $\mu(\xi) = \xi_+ + \xi_-$.

Let $x, y \in \mathfrak{A}_0$. We want to show that $\xi_+ \square \xi_- = 0$, i.e.

$$\langle (H_{(\xi)}\mathbb{1} - \overline{L}_\xi \mathbb{1})x | (H_{(\xi)}\mathbb{1} + \overline{L}_\xi \mathbb{1})y \rangle = 0.$$

Computing in the expression above, we obtain

$$\begin{aligned}\langle (H_{(\xi)}\mathbb{1})x | (H_{(\xi)}\mathbb{1})y \rangle + \langle (H_{(\xi)}\mathbb{1})x | \overline{L}_\xi y \rangle - \langle \overline{L}_\xi x | (H_{(\xi)}\mathbb{1})y \rangle - \langle \overline{L}_\xi x | \overline{L}_\xi y \rangle \\ = \langle \mu(\xi)x | \mu(\xi)y \rangle + \langle \mu(\xi)x | \xi y \rangle - \langle \xi x | \mu(\xi)y \rangle - \langle \xi x | \xi y \rangle.\end{aligned}$$

Now, we can show that $\|\mu(\xi)x\| = \|\xi x\|$, for every $x \in \mathfrak{A}_0$. Indeed,

$$\begin{aligned}\|\xi x\|^2 &= \langle \xi x | \xi x \rangle = \langle L_\xi x | L_\xi x \rangle \\ &= \langle H_{(\xi)}x | H_{(\xi)}x \rangle = \langle (H_{(\xi)}\mathbb{1})x | (H_{(\xi)}\mathbb{1})x \rangle \\ &= \| \mu(\xi)x \|^2, \quad \forall \xi \in \mathcal{H}.\end{aligned}$$

By the polarization formula, $\langle \mu(\xi)x | \mu(\xi)y \rangle = \langle \xi x | \xi y \rangle$ for every $x, y \in \mathfrak{A}_0$.

Notice that, by (ii) of Definition 2.2.17, $\mu^2 = \mu$, i.e. $\mu(\mu(\xi)) = \mu(\xi)$ for every $\xi \in \mathcal{H}$. Hence, if $\mu^2(\xi) := \mu(\mu(\xi))$,

$$\langle \mu(\xi)x | \xi y \rangle = \langle \mu^2(\xi)x | \mu(\xi)y \rangle = \langle \mu(\xi)x | \mu^2(\xi)y \rangle = \langle \xi x | \mu(\xi)y \rangle.$$

From what we have just shown, we have that $\xi_- \square \xi_+$ is well-defined and $\xi_- \square \xi_+ = 0$. Using the same argument, it is easily proven that $\xi_+ \square \xi_- = 0$. Then ξ_+ and ξ_- have the desired properties. $\square$

Remark 2.2.21 Suppose that $(\mathcal{H}, \mathfrak{A}_0)$ is integrable and unital. Suppose in addition that the Hilbert norm of $\mathcal{H}$ is increasing; i.e.,

$$0 \leq \xi \leq \eta \;\Rightarrow\; \|\xi\| \leq \|\eta\|.$$

and that

$$\mu(\mu(\xi) - \mu(\eta)) \leq \mu(\xi - \eta), \quad \forall \xi, \eta \in \mathcal{H}. \tag{2.7}$$

Then we get

$$\|\mu(\xi) - \mu(\eta)\| = \|\mu(\mu(\xi) - \mu(\eta))\| \leq \|\mu(\xi - \eta)\| = \|\xi - \eta\|$$

This obviously implies that μ is continuous.

Proposition 2.2.22 *Let $(\mathcal{H}, \mathfrak{A}_0)$ be a Hilbert quasi *-algebra with a continuous module function. Suppose that $\mu(\mathfrak{A}_0) \subseteq \mathfrak{A}_0^+$. Then every positive element is the limit of a sequence of elements of $\mathfrak{A}_0^+$; i.e., $\mathcal{H}^+ = \mathcal{H}_w^+$.*

Proof. Let $\xi \in \mathcal{H}_w^+$ then by density there exists a sequence $\{x_n\} \subset \mathfrak{A}_0$ such that $x_n \to \xi$. By the assumptions $\mu(x_n) \in \mathfrak{A}_0^+$ and $\mu(x_n) \to \mu(\xi) = \xi$. □

Proposition 2.2.23 *Let $(\mathcal{H}, \mathfrak{A}_0)$ be a unital integrable Hilbert quasi *-algebra with a τ_n-continuous module function. Suppose that $\mu(\mathfrak{A}_0) \subseteq \mathfrak{A}_0^+$. Every $\omega \in \mathcal{R}(\mathcal{H}, \mathfrak{A}_0)$ which is positive on $\mathcal{H}_w^+$ is continuous.*

Proof. As shown in [65], every positive functional on $\mathcal{H}_w^+ = \mathcal{H}^+$ (by Proposition 2.2.22) is bounded on positive elements. The statement then follows from Lemma 2.2.20. □

2.3 Intertwining operators and representable functionals

After describing the set $\mathcal{R}_c(\mathcal{H}, \mathfrak{A}_0)$, a second natural step consists in looking for conditions for $\omega \in \mathcal{R}(\mathcal{H}, \mathfrak{A}_0)$ to be continuous on $\mathcal{H}$, at least in some particular situations.

Before proceeding, we remind the reader of the definition of *critical eigenvalue*.

Definition 2.3.1 [62, Definition 3.1] A complex number μ is said to be a *critical eigenvalue* of a couple of operators (A, B) in a Banach space X if $(A - \mu I)X$ has infinite codimension and μ is an eigenvalue of B.

Critical eigenvalues are important in order to study the continuity of a linear operator T which intertwines a couple (A, B), i.e. $TA = BT$.

Proposition 2.3.2 *[62] Let A and B be normal operator on a Hilbert space $\mathcal{H}$. Then every linear operator T satisfying $TA = BT$ is continuous if, and only if, the couple (A, B) has no critical eigenvalues.*

Proposition 2.3.3 *Let $(\mathfrak{A}, \mathfrak{A}_0)$ be a quasi *-algebra with unit $\mathbb{1}$ and ω in $\mathcal{R}(\mathfrak{A}, \mathfrak{A}_0)$. Then there exists a linear operator $T_\omega : \mathfrak{A} \to \mathcal{H}_\omega$ such that*

$$\omega(a) = \langle T_\omega a | \xi_\omega \rangle, \quad \forall a \in \mathfrak{A},$$

where ξ_ω is the cyclic vector of the GNS representation associated to ω.

Proof. Let $\omega \in \mathcal{R}(\mathfrak{A}, \mathfrak{A}_0)$. Let $N_\omega = \{x \in \mathfrak{A}_0 : \omega(x^* x) = 0\}$ and $\mathcal{H}_\omega$ the Hilbert space completion of $\mathfrak{A}_0 / N_\omega$ with respect to the inner product

$$\langle \lambda_\omega(x) | \lambda_\omega(y) \rangle = \omega(y^* x), \; x, y \in \mathfrak{A}_0,$$

where $\lambda_\omega(x) = x + N_\omega$. $\mathcal{H}_\omega$ is nothing but the carrier space of the GNS representation constructed in [65].

For every $a \in \mathfrak{A}$, the linear functional F_a on $\lambda_\omega(\mathfrak{A}_0)$ defined by

$$F_a(\lambda_\omega(x)) = \omega(a^* x), \quad x \in \mathfrak{A}_0$$

is well-defined and bounded by (R.3). Indeed,

$$|F_a(\lambda_\omega(x))| = |\omega(a^* x) \leq \gamma_a \omega(x^* x)^{\frac{1}{2}} = \gamma_a \|\lambda_\omega(x)\|, \quad \forall x \in \mathfrak{A}_0$$

Hence there exists a unique $\xi(a) \in \mathcal{H}_\omega$ such that

$$F_a(\lambda_\omega(x)) = \omega(a^* x) = \langle \lambda_\omega(x) | \xi(a) \rangle, \quad \forall x \in \mathfrak{A}_0.$$

We define a linear map $T_\omega : \mathfrak{A} \to \mathcal{H}_\omega$ by $T_\omega a = \xi(a)$, $a \in \mathfrak{A}$. Then we have

$$\omega(a) = \overline{\omega(a^*)} = \overline{\langle \lambda_\omega(\mathbb{1}) | \xi(a) \rangle} = \langle \xi(a) | \xi_\omega \rangle = \langle T_\omega a | \xi_\omega \rangle, \quad \forall a \in \mathfrak{A},$$

where $\xi_\omega = \lambda_\omega(\mathbb{1})$ is the cyclic vector of the GNS representation associated to the functional ω. □

Proposition 2.3.4 *Let $(\mathcal{H}, \mathfrak{A}_0)$ be a commutative Hilbert quasi *-algebra with unit $\mathbb{1}$. Assume that $\mathfrak{A}_0$ is a Banach *-algebra with respect to $\| \cdot \|_0$ and that there exists an element x of $\mathfrak{A}_0$ such that the spectrum $\sigma(R_x)$ of the bounded operator R_x of right multiplication by x consists only of its continuous part $\sigma_c(R_x)$. If $\omega \in \mathcal{R}(\mathfrak{A}, \mathfrak{A}_0)$, then ω is bounded.*

Proof. By Proposition 2.3.3, there exists an operator $T_\omega : \mathcal{H} \to \mathcal{H}_\omega$ such that $\omega(\eta) = \langle T_\omega \eta | \xi_\omega \rangle$ for every $\eta \in \mathcal{H}$. Then, if we put

$$\lambda_\omega(x) \cdot \lambda_\omega(y) = \lambda_\omega(xy),$$

This multiplication $\cdot$ is well-defined on $\lambda_\omega(\mathfrak{A}_0)$ and makes it into a commutative *-algebra. Moreover, if $\mathfrak{A}_0$ is a Banach *-algebra with respect to $\|\cdot\|_0$, by Theorem A.3.5, we have

$$\|\lambda_\omega(xy)\|^2 = \omega(y^*x^*xy) \leq \|y\|_0 \omega(x^*x) = \|y\|_0 \|\lambda_\omega(x)\|^2. \tag{2.8}$$

The inequality (2.8) implies that the right multiplication operator $R_{\lambda_\omega(y)}$ by $\lambda_\omega(y)$ is bounded on $\lambda_\omega(\mathfrak{A}_0)$ and therefore it has a unique bounded extension to $\mathcal{H}_\omega$ denoted by the same symbol.

By the condition (R.3) on ω, the functional $F_\eta(z) = \omega(\eta z)$ for $z \in \mathfrak{A}_0$ is bounded on $\mathfrak{A}_0$ with respect to $\|\cdot\|_\omega$, so it can be extended on the whole $\mathcal{H}$ and there exists $\chi \in \mathcal{H}$ such that $\omega(\eta z) = \langle \lambda_\omega(z) | \chi \rangle$ for every $z \in \mathfrak{A}_0$.

On the other hand, $\omega(\eta z) = \langle T_\omega(\eta z) | \xi_\omega \rangle$. Therefore, for $z = \mathbb{1}$, we have

$$\langle \xi_\omega | \chi \rangle = \omega(\eta) = \langle T_\omega(\eta) | \xi_\omega \rangle$$

and so $\chi = (T_\omega \eta)^*$. By the definition of T_ω, $T_\omega \eta^* = (T_\omega \eta)^*$ for every $\eta \in \mathcal{H}$. Hence, for every $\eta \in \mathcal{H}$ and $z \in \mathfrak{A}_0$

$$\langle T_\omega(\eta z) | \xi_\omega \rangle = \omega(\eta z) = \langle \lambda(z) | T_\omega \eta^* \rangle = \langle \lambda(z) | (T_\omega \eta)^* \rangle = \langle T_\omega \eta \cdot \lambda_\omega(z) | \xi_\omega \rangle.$$

By the density of $\lambda_\omega(\mathfrak{A}_0)$, we have the following equalities

$$T_\omega(\eta z) = T_\omega \eta \cdot \lambda_\omega(z), \ \forall \eta \in \mathcal{H}, z \in \mathfrak{A}_0; \tag{2.9}$$

$$T_\omega z = \lambda_\omega(z), \ \forall z \in \mathfrak{A}_0. \tag{2.10}$$

In particular (2.9) reads as follows

$$T_\omega(R_z \eta) = R_{\lambda_\omega(z)} T_\omega \eta, \ \forall \eta \in \mathcal{H}, z \in \mathfrak{A}_0; \tag{2.11}$$

i.e., T_ω intertwines the couple $(R_z, R_{\lambda_\omega(z)})$, for every $z \in \mathfrak{A}_0$. Then the continuity of T_ω can be deduced from Proposition 2.3.2.

By the assumption, there exists $x \in \mathfrak{A}_0$ such that $\sigma(\overline{R}_x) = \sigma_c(\overline{R}_x)$. Hence, for every $\mu \in \mathbb{C}$, the range of the operator $R_x - \mu I$ is either $\mathcal{H}$ itself or a dense subspace of $\mathcal{H}$. Thus no critical eigenvalue for the couple $(R_x, R_{\lambda_\omega(z)})$ may exists. □

2.3.1 Two peculiar examples

Let I be a compact interval of the real line and λ the Lebesgue measure on it. In this section we will show that every representable functional over $(L^2(I, d\lambda), \mathfrak{A}_0)$, where $\mathfrak{A}_0 = \mathcal{C}(I)$ or $\mathfrak{A}_0 = L^\infty(I, d\lambda)$, is continuous, as application of Proposition 2.3.4. In this case, more can be said about representable functionals over these Hilbert quasi *-algebras.

$(L^2(I, d\lambda), \mathcal{C}(I))$ and $(L^2(I, d\lambda), L^\infty(I, d\lambda))$ are Hilbert quasi *-algebras and, as it is easy to see, both are integrable in the sense of Definition 2.2.16. A description of representable functionals on $(L^2(I, d\lambda), \mathcal{C}(I))$ is provided by the following representation theorem.

Proposition 2.3.5 *Let ω be a representable functional on $(L^2(I, d\lambda), \mathcal{C}(I))$. Then there exists a unique Borel measure μ on I and a unique bounded linear operator $T : L^2(I, d\lambda) \to L^2(I, d\mu)$ such that*

$$\omega(f) = \int_I (Tf) d\mu, \quad \forall f \in L^2(I, \lambda). \tag{2.12}$$

The operator T satisfies the following conditions:

$$T(f\phi) = (Tf)\phi = \phi(Tf) \quad \forall f \in L^2(I, d\lambda), \phi \in \mathcal{C}(I); \tag{2.13}$$

$$T\phi = \phi, \quad \forall \phi \in \mathcal{C}(I). \tag{2.14}$$

Thus, every representable functional ω on $(L^2(I, d\lambda), \mathcal{C}(I))$ is continuous. Moreover, μ is absolutely continuous with respect to λ.

Proof. By definition ω is positive on $\mathcal{C}(I)$. Therefore by the Riesz-Markov theorem, there exists a unique Borel measure μ on I such that

$$\omega(\phi) = \int_I \phi d\mu, \quad \forall \phi \in \mathcal{C}(I).$$

By condition (R.3), for every $f \in L^2(I, d\lambda)$, there exists γ_f such that

$$|\omega(f^*\phi)| \leq \gamma_f \omega(\phi^*\phi)^{\frac{1}{2}} = \gamma_f \left[\int_I |\phi|^2 d\mu \right]^{\frac{1}{2}} = \gamma_f \|\phi\|_{2,\mu}, \quad \forall \phi \in \mathcal{C}(I).$$

Hence, the linear functional L_f defined by $L_f(\phi) = \omega(f^*\phi)$, $\phi \in \mathcal{C}(I)$, is bounded on $\mathcal{C}(I)$, with respect to $\|\cdot\|_{2,\mu}$. Thus, it extends to a bounded linear functional on $L^2(I, d\mu)$ and there exists a unique function $h_f \in L^2(I, d\mu)$ such that

$$\omega(f^*\phi) = \int_I \phi \overline{h_f} d\mu, \quad \forall \phi \in \mathcal{C}(I). \tag{2.15}$$

We can define a linear map $T : L^2(I, d\lambda) \to L^2(I, d\mu)$ by putting $Tf = h_f$, $f \in L^2(I, d\lambda)$. With this definition we have

$$\omega(f) = \int_I (Tf) d\mu, \quad \forall f \in L^2(I, d\lambda).$$

It is straightforward to show that the map T has the following properties:

$$T(f\phi) = (Tf)\phi = \varphi(Tf) \quad \forall f \in L^2(I, \lambda), \phi \in \mathcal{C}(I); \tag{2.16}$$

$$T\phi = \phi, \quad \forall \phi \in \mathcal{C}(I). \tag{2.17}$$

In particular, (2.13) can be rewritten as follows

$$TR_\phi f = R'_\phi Tf, \quad \forall f \in L^2(I, \lambda), \phi \in \mathcal{C}(I),$$

where R_ϕ and R'_ϕ denote the multiplication operators by ϕ in $L^2(I, d\lambda)$ and $L^2(I, d\mu)$, respectively. This means that T intertwines the couple (R_ϕ, R'_ϕ) for every $\phi \in \mathcal{C}(I)$. The operator T is continuous if, and only if, there exists $\phi \in \mathcal{C}(I)$ such that the couple (R_ϕ, R'_ϕ) has no critical eigenvalues by Proposition 2.3.2.

In $L^2(I, d\lambda)$, (λ the Lebesgue measure) the operator R_ϕ has continuous spectrum. Indeed, this depends on the fact that, if ϕ is not constant, for every $z \in \mathbb{C}$, we have $\mathsf{Ran}(R_\phi - zI)^\perp = \{0\}$ where $\mathsf{Ran}(R_\phi - zI)$ is the range of $R_\phi - zI$. Hence, the couple (R_ϕ, R'_ϕ) has no critical eigenvalues unless ϕ is a constant function. Therefore the statement follows from Proposition 2.3.2.

Since T is bounded, it has an adjoint $T^* : L^2(I, d\mu) \to L^2(I, d\lambda)$. Hence, if we denote by u the unit function in $\mathcal{C}(I)$ (i.e., $u(x) = 1$, for every $x \in I$), we get

$$\begin{aligned} \omega(f) &= \int_I Tf d\mu = \int_I (Tf) u d\mu = \langle Tf | u \rangle_{2,\mu} = \langle f | T^* u \rangle_{2,\lambda} \\ &= \int_I f \cdot \overline{(T^* u)} d\lambda, \quad \forall f \in L^2(I, d\lambda). \end{aligned} \tag{2.18}$$

It is easily seen that T^*u is a nonnegative function and by (2.18) it follows also that ω is necessarily positive on positive elements of $L^2(I, d\lambda)$.

Put $w = T^*u$. From (2.18), we get, in particular,

$$\omega(\phi) = \int_I \phi d\mu = \int_I \phi w d\lambda, \quad \forall \phi \in \mathcal{C}(I).$$

The previous equality implies by the uniqueness of the measure associated to a positive linear functional on $\mathcal{C}(I)$ us that $d\mu = w d\lambda$; i.e., μ is λ-absolutely continuous with Radon-Nikodym derivative w. □

Let us now consider the Banach quasi *-algebra $(L^2(I, d\lambda), L^\infty(I, d\lambda))$. In this case we have more information about the measure that allows us to represent the functional.

Proposition 2.3.6 *Let ω be a representable functional on the Banach quasi *-algebra $(L^2(I, d\lambda), L^\infty(I, d\lambda))$. Then there exists a unique bounded finitely additive measure ν on I which vanish on subsets of I of zero λ-measure and a unique bounded linear operator $S : L^2(I, d\lambda) \to L^2(I, d\nu)$ such that*

$$\omega(f) = \int_I (Sf) d\nu, \quad \forall f \in L^2(I, d\lambda). \tag{2.19}$$

The map S has the following properties:

$$S(f\phi) = (Sf)\phi = \phi(Sf) \quad \forall f \in L^2(I, d\lambda), \phi \in L^\infty(I, d\lambda), ; \tag{2.20}$$
$$S\phi = \phi, \quad \forall \phi \in L^\infty(I, d\lambda). \tag{2.21}$$

Proof. The proof of this statement is essentially the same as that of Proposition 2.3.5. Indeed, by [32, Theorem IV.8.16], there exists a complex valued measure ν absolutely continuous with respect to λ, for which ω has the following form

$$\omega(\phi) = \int_I \phi d\nu \qquad \phi \in L^\infty(I, d\lambda).$$

Since the functional ω is positive, i.e. $\omega(\phi^*\phi) \geq 0$, for every $\phi \in L^\infty(I, d\lambda)$, the measure ν is positive.

Following the proof of Proposition 2.3.5, one can show that there exists a unique linear map $S : L^2(I, d\lambda) \to L^2(I, d\nu)$ such that

$$\omega(f) = \int_I (Sf) d\nu, , \quad \forall f \in L^2(I, d\lambda).$$

The boundedness of S follows directly from the analogous statement used for T in the proof of Proposition 2.3.5, taking into account that S intertwines the multiplication operators for a function $\phi \in L^\infty(I, \lambda)$. In particular ϕ can be chosen to be continuous. □

It is clear that the continuity of the operators T and S implies the continuity of the corresponding representable functionals. Thus, we can conclude with the following

Corollary 2.3.7 *Every representable functional over the Banach quasi *-algebras $(L^2(I, d\lambda), \mathcal{C}(I))$ or $(L^2(I, d\lambda), L^\infty(I, d\lambda))$, where I is a compact interval of the real line and λ be the Lebesgue measure on it, is continuous.*

Remark 2.3.8 (The Banach case) It would be desirable to extend the results about the continuity of representable functionals we had before for Hilbert quasi *-algebras to the case of Banach quasi *-algebras, using the same strategy of intertwining operators.

In the case of a Hilbert quasi *-algebra, we applied a result of intertwining operators for Hilbert spaces for which the operators of the intertwining couple had to be normal and no eigenvalues.

For the Banach case, beside other assumptions like that on critical eigenvalues, the second operator of the intertwining couple has to posses *countable spectrum*, in order to apply Theorem 4.1 of [62] and get continuity for representable functionals.

Let us consider for a moment the case of Banach quasi *-algebras of functions, for instance $(L^p(I, d\lambda), \mathcal{C}(I))$ and $(L^p(I, d\lambda), L^\infty(I, d\lambda))$ for $p > 2$, where I is the unitary interval of the real line and λ is the Lebesgue measure. In this case, we deal with commutative Banach quasi *-algebras and this allows us to define a multiplication on the Hilbert space associated to the representable functional through the GNS construction.

What we search for is a continuous function ψ such that the spectrum of $R_{\lambda_\omega(\psi)}$ is countable and ψ doesn't posses any eigenvalue, i.e. it is not constant function. The spectrum of ψ is the essential image of ψ, that cannot be countable unless ψ is constant. In the latter case, ψ would posses critical eigenvalues. Hence, requiring a countable spectrum would mean to allow critical eigenvalues and viceversa.

This is an evidence of the difficulty of finding an example that verifies all the assumptions of Theorem 4.1 of [62]. This suggests that that the approach with interwining operators is not appropriate, in general.

Chapter 3

Unbounded derivations and *-automorphism groups

As application of the study of continuity performed in Chapter 2, we investigate (unbounded) derivations on Banach quasi *-algebras, focusing our attention in particular on those arising as infinitesimal generators of one-parameter groups of *-automorphisms.

For the case of Banach *-algebras, a derivation is a linear map for which the Leibnitz rule holds. In this case, having a partial multiplication, the Leibnitz rule needs to be adapted or weakened.

When $\mathfrak{A}_0$ is a C*-algebra, Bratteli and Robinson proved in 1975 the existence of a deep relationship between certain closed derivations and *continuous* one parameter group of *-automorphism $\{\beta_t\}_{t\in\mathbb{R}}$ (see [25, 26, 27]).

Definition 3.0.1 [26] A *derivation* δ *of a C*-algebra* $\mathfrak{A}$ is a linear mapping from a dense *-subalgebra $\mathcal{D}(\delta)(\mathfrak{A})$, named the domain, to a subspace $R(\delta)|subset\mathfrak{A}$, called the range, satisfying the two properties

1. $\delta(xy) = \delta(x)y + x\delta(y)$, $x, y \in \mathcal{D}(\delta)$,
2. $\delta(x^*) = -\delta(x)^*$, $x \in \mathcal{D}(\delta)$.

Theorem 3.0.2 *[26] Let δ be a derivation of a C*-algebra $\mathfrak{A}_0[\|\cdot\|_0]$. The following are equivalent*

1. *δ is the infinitesimal generator of a strongly continuous one-parameter group of *-automorphisms of $\mathfrak{A}$.*
2. *δ is closed, its resolvent set $\rho(\delta)$ contains $\mathbb{R}\setminus\{0\}$ and*

$$\|\delta(x) - zx\| \geq |Im\, z|\, \|x\|_0, \quad x \in \mathcal{D}(\delta).$$

Other criteria for closability can be given. Among them, there is one that involves states of C*-algebras.

Theorem 3.0.3 *[26] Let δ be a derivation of a C*-algebra $\mathfrak{A}$. Assume that a state ω generates a faithful cyclic representation $(\mathcal{H}_\omega, \pi_\omega, \xi_\omega)$ and also satisfies*

$$\omega(\delta(x)) = 0, \quad x \in \mathcal{D}(\delta).$$

It follows that δ is closable and there exists a symmetric operator H_δ on $\mathcal{H}_\omega$ such that

$$\mathcal{D}(H_\delta) = \{\psi; \psi = \pi_\omega(x)\xi_\omega, x \in \mathcal{D}(\delta)\}$$
$$\pi_\omega(\delta(x))\psi = [H_\delta, \pi_\omega(x)]\psi, \quad x \in \mathcal{D}(\delta), \psi \in \mathcal{D}(H_\delta).$$

These two theorems of Bratteli and Robinson published in the mentioned paper of 1975 highlight the relationship between unbounded derivations, one-parameter groups of automorphisms and states on a C*-algebra. Our target would be to study in depth how these theorems can be generalized to the case of Banach quasi *-algebras.

In the previous Chapter, we already studied representable functionals and sesquilienar forms associated to them. At this time, our aim is to investigate how the definition of derivation can be adapted to a framework in which the multiplication is only partially defined and therefore examining its properties.

3.1 Densely defined derivations

In this section we are interested in studying derivations *defined on* $\mathfrak{A}_0$ with values in the $\mathfrak{A}_0$-*bimodule* $\mathfrak{A}$. Indeed, it is clear that the properties required in the definition of quasi *-algebra (Definition 1.1.3) endow $\mathfrak{A}$ with a structure of a bimodule over $\mathfrak{A}_0$ where the left and right actions are given respectively by left and right multiplication.

Definition 3.1.1 Let $(\mathfrak{A}, \mathfrak{A}_0)$ be a quasi *-algebra and δ a linear map of $\mathfrak{A}_0$ into $\mathfrak{A}$. We say that δ is a *qu*-derivation* of $(\mathfrak{A}, \mathfrak{A}_0)$ if

(i) $\delta(x^*) = \delta(x)^*, \ \forall x \in \mathfrak{A}_0$

(ii) $\delta(xy) = \delta(x)y + x\delta(y), \ \forall x, y \in \mathfrak{A}_0$

Example 3.1.2 The easiest example of a qu*-derivation on a quasi *-algebra is provided by the *commutator*; i.e., if $h = h^* \in \mathfrak{A}$ we put

$$\delta_h(x) = i[h, x] := i(hx - xh).$$

Motivated by this example we give the following

Definition 3.1.3 Let $(\mathfrak{A}, \mathfrak{A}_0)$ be a quasi *-algebra and δ a qu*-derivation of $(\mathfrak{A}, \mathfrak{A}_0)$. We say that δ is *inner* if there exists $h = h^* \in \mathfrak{A}$ such that

$$\delta_h(x) = i[h, x].$$

Qu*-derivations are indeed a special case of derivations from a *-algebra $\mathfrak{A}_0$ with values on a $\mathfrak{A}_0$-bimodule. Hence, as a consequence of a celebrated theorem of J. Ringrose (Theorem 2 of [57]) every qu*-derivation δ is *continuous* if $(\mathfrak{A}, \mathfrak{A}_0)$ is a proper CQ*-algebra and we provide $\mathfrak{A}_0$ the norm $\| \cdot \|_0$ instead of the induced norm $\| \cdot \|$.

Therefore, from now on we will be interested in studying densely defined qu*-derivations $\delta : \mathfrak{A}_0[\| \cdot \|] \to \mathfrak{A}[\| \cdot \|]$.

A natural question is whether a qu*-derivation could be extended to a larger domain $\mathcal{D} \supset \mathfrak{A}_0$. If it is possible to extend δ beyond $\mathfrak{A}_0$, then we wonder about the closability of δ.

If $\delta : \mathfrak{A}_0[\| \cdot \|] \to \mathfrak{A}[\| \cdot \|]$ is closable as a linear map, then the closure is defined in the usual way by

$$\overline{\delta}(a) := \lim_{n \to \infty} \delta(x_n), \quad a \in \mathcal{D}(\overline{\delta}),$$

where $\mathcal{D}(\overline{\delta})$ is the following set

$$\mathcal{D}(\overline{\delta}) = \{a \in \mathfrak{A} : \exists \{x_n\} \subset \mathfrak{A}_0, w \in \mathfrak{A} \text{ s.t. } \|a - x_n\| \to 0 \text{ and } \|\delta(x_n) - w\| \to 0\}.$$

In order to have a well-defined Leibnitz rule, we should have some regularity property on $\mathcal{D}(\overline{\delta})$, i.e. $(\mathcal{D}(\overline{\delta}), \mathfrak{A}_0)$ should be a *quasi *-algebra*. This is certainly true if $\mathcal{D}(\overline{\delta})$ is made of *bounded* elements, as in the following example.

Example 3.1.4 Consider the Banach quasi *-algebra $(L^p(\mathbb{R}), \mathcal{C}_c^\infty(\mathbb{R}))$. For $p \geq 2$ $(L^p(\mathbb{R}), \mathcal{C}_c^\infty(\mathbb{R}))$ is a *-semisimple Banach quasi *-algebra.

Define on $\mathcal{C}_c^\infty(\mathbb{R})$ the derivation $\delta(f) = f'$ for every $f \in \mathcal{C}_c^\infty(\mathbb{R})$, where f' is the classical derivative of f. Then δ is closable and its closure is the *weak derivative in* $W^{1,p}(\mathbb{R})$. In this case the Leibnitz rule is still valid because $W^{1,p}(\mathbb{R})$ is made of essentially bounded functions.

The use of representations (and/or representable functionals) allows us to get the following first result of a purely algebraic nature.

Let $(\mathfrak{A}, \mathfrak{A}_0)$ be a quasi *-algebra, δ a qu*-derivation of $(\mathfrak{A}, \mathfrak{A}_0)$ and π a *-representation of $(\mathfrak{A}, \mathfrak{A}_0)$. Assume that

$$\text{whenever } x \in \mathfrak{A}_0 \text{ is such that } \pi(x) = 0, \text{ then } \pi(\delta(x)) = 0. \tag{3.1}$$

Under this assumption, the linear map

$$\delta_\pi(\pi(x)) := \pi(\delta(x)), \quad x \in \mathfrak{A}_0$$

is well defined on $\pi(\mathfrak{A}_0)$ with values in $\pi(\mathfrak{A})$ and it is easily checked that δ_π is a qu*-derivation of $\mathfrak{A}_0$ named *induced by* π.

Definition 3.1.5 Let $(\mathfrak{A}, \mathfrak{A}_0)$ be a quasi *-algebra and δ a qu*-derivation of $(\mathfrak{A}, \mathfrak{A}_0)$. Furthermore, let π be a cyclic *-representation of $(\mathfrak{A}, \mathfrak{A}_0)$ with cyclic vector ξ_0 satisfying the assumption (3.1). The induced qu*-derivation δ_π is *spatial* if there exists $H = H^\dagger \in \mathcal{L}(\mathcal{D}_\pi, \mathcal{H}_\pi)$ such that

$$\delta_\pi(\pi(x)) = i[H, \pi(x)], \quad x \in \mathfrak{A}_0.$$

Proposition 3.1.6 *Let $(\mathfrak{A}, \mathfrak{A}_0)$ be a quasi *-algebra with unit $\mathbb{1}$ and let δ be a qu*-derivation of $(\mathfrak{A}, \mathfrak{A}_0)$. Suppose that there exists a representable linear functional ω with $\omega(\delta(x)) = 0$ for $x \in \mathfrak{A}_0$ and let $(\mathcal{H}_\omega, \pi_\omega, \lambda_\omega)$ the GNS-construction associated to ω. Then there exists an element $H = H^\dagger$ of $\mathcal{L}^\dagger(\lambda_\omega(\mathfrak{A}_0))$ such that*

$$\pi_\omega(\delta(x)) = -i[H, \pi_\omega(x)], \quad \forall x \in \mathfrak{A}_0.$$

Proof. Define H on $\lambda_\omega(\mathfrak{A}_0)$ by

$$H\lambda_\omega(x) = i\pi_\omega(\delta(x))\xi_\omega, \quad x \in \mathfrak{A}_0$$

where $\xi_\omega = \lambda_\omega(\mathbb{1})$. We first prove that H is well defined. We have

$$\begin{aligned}
\langle \pi_\omega(\delta(x))\xi_\omega | \pi_\omega(y)\xi_\omega \rangle &= \langle \pi_\omega(y^*)\pi_\omega(\delta(x))\xi_\omega | \xi_\omega \rangle \\
&= \langle \pi_\omega(y^*\delta(x))\xi_\omega | \xi_\omega \rangle \\
&= \langle \pi_\omega(\delta(y^*x) - \delta(y^*)x)\xi_\omega | \xi_\omega \rangle \\
&= -\langle \pi_\omega(\delta(y^*)x)\xi_\omega | \xi_\omega \rangle \\
&= -\langle \pi_\omega(x)\xi_\omega | \pi_\omega(\delta(y))\xi_\omega \rangle.
\end{aligned}$$

Hence if $\lambda_\omega(x) = \pi_\omega(x)\xi_\omega = 0$, it follows that $\langle \pi_\omega(\delta(x))\xi_\omega | \pi_\omega(y)\xi_\omega \rangle = 0$, for every $y \in \mathfrak{A}_0$. This in turn implies that $\pi_\omega(\delta(x))\xi_\omega = 0$.

The above computation shows also that H is symmetric. Indeed,

$$\begin{aligned}\langle H\lambda_\omega(x)|\lambda_\omega(y)\rangle &= i\langle \pi_\omega(\delta(x))\xi_\omega|\pi_\omega(y)\xi_\omega\rangle \\ &= -i\langle \pi_\omega(x)\xi_\omega|\pi_\omega(\delta(y))\xi_\omega\rangle \\ &= \langle \lambda_\omega(x)|H\lambda_\omega(y)\rangle.\end{aligned}$$

Finally, if $x \in \mathfrak{A}_0$,

$$\begin{aligned}\pi_\omega(\delta(x))\lambda_\omega(y) &= \pi_\omega(\delta(x)) \,\square\, \pi_\omega(y)\xi_\omega \\ &= \pi_\omega(\delta(xy))\xi_\omega - \pi_\omega(x) \,\square\, \pi_\omega(\delta(y))\xi_\omega \\ &= -iH\pi_\omega(x)\lambda_\omega(y) + i\pi_\omega(x)H\lambda_\omega(y) \\ &= -i[H, \pi_\omega(x)]\lambda_\omega(y), \quad \forall y \in \mathfrak{A}_0. \quad \square\end{aligned}$$

Remark 3.1.7 In Proposition 3.1.6, if $(\mathfrak{A}, \mathfrak{A}_0)$ is a Banach quasi *-algebra and ω is a representable and continuous such that the *-representation π_ω in the GNS construction is faithful, then it is possible to show that δ is a *closable* qu*-derivation (see [1]).

3.2 Extension of a qu*-derivation

In the framework of Banach quasi *-algebras, we have a reasonable definition of derivation at hand. We are interested in studying them, in particular the question concerning the closability.

The simplest case to start with is that of *inner qu*-derivations*. Not surprisingly, $\overline{\delta}_h$ is continuous whenever the element $h \in \mathfrak{A}$ that generates the qu*-derivation δ_h is bounded in the sense of Definition 1.3.6. Indeed it is possible to write δ_h as difference of the bounded operators L_h and R_h in the following way

$$\delta_h(x) = i[h, x] = i(L_h - R_h)(x), \quad \forall x \in \mathfrak{A}_0.$$

We wonder if it is possible to remove the hypothesis of boundedness on the generating element $h \in \mathfrak{A}$. It turns out that if $(\mathfrak{A}, \mathfrak{A}_0)$ is a *-semisimple Banach quasi *-algebra in the sense of Definition 1.4.21, we get the following

Proposition 3.2.1 *Let $(\mathfrak{A}, \mathfrak{A}_0)$ be a *-semisimple Banach quasi *-algebra. Let $h \in \mathfrak{A}$ be a fixed element in $\mathfrak{A}$ such that $h = h^*$ and δ_h the qu*-derivation defined as $\delta_h(x) := i[h, x]$ for $x \in \mathfrak{A}_0$. Then δ_h is closable.*

Proof. Let $\{x_n\} \subset \mathfrak{A}_0$ be a sequence that vanishes as $n \to \infty$ and such that $\delta_h(x_n)$ is $\|\cdot\|$-Cauchy, i.e. there exists $w \in \mathfrak{A}$ such that $\|\delta_h(x_n) - w\| \to 0$ as $n \to +\infty$. We want to show that $w = 0$.

On one hand, for every $\Theta \in \mathcal{S}_{\mathfrak{A}_0}(\mathfrak{A})$ and for every $u, v \in \mathfrak{A}_0$,

$$\begin{aligned}\Theta(\delta_h(x_n)u, v) &= i\Theta(hx_n u, v) - i\Theta(x_n hu, v) \\ &= i\Theta(x_n u, hv) - i\Theta(hu, x_n^* v) \to 0.\end{aligned}$$

On the other hand, by the hypotheses $\Theta(\delta_h(x_n)u, v) \to \Theta(wu, v)$, for every $\Theta \in \mathcal{S}_{\mathfrak{A}_0}(\mathfrak{A})$ and for every $u, v \in \mathfrak{A}_0$. We conclude by Lemma 1.4.20 and the arbitrary choice of $\Theta \in \mathcal{S}_{\mathfrak{A}_0}(\mathfrak{A})$. □

3.2.1 Weak multiplication and weak topologies

Proposition 3.2.1 highlights that the existence of certain sesquilinear forms to work with is *crucial* when dealing with this problem. For this reason, we will assume that $(\mathfrak{A}, \mathfrak{A}_0)$ is a **-semisimple* Banach quasi *-algebra.

Sesquilinear forms $\Theta \in \mathcal{S}_{\mathfrak{A}_0}(\mathfrak{A})$ are employed in the definition of a weak multiplication (see Definition 1.4.24), and locally convex topologies coarser than the norm topology (see Section 1.4.2).

As we shall see below, the weak multiplication in Definition 1.4.24 can be characterized through some closedness properties with respect to the mentioned topologies defined by means of the sesquilinear forms $\Theta \in \mathcal{S}_{\mathfrak{A}_0}(\mathfrak{A})$.

Remark 3.2.2 From the continuity of $\Theta \in \mathcal{S}_{\mathfrak{A}_0}(\mathfrak{A})$ it follows that all the topologies τ_w, τ_s, (and also τ_{s^*}, if the involution is τ-continuous) are coarser than the initial norm topology of $\mathfrak{A}$.

Proposition 3.2.3 *The following statements are equivalent.*

(i) The weak product $a \Box b$ is well defined.

(ii) There exists a sequence $\{y_n\}$ of elements in $\mathfrak{A}_0$ such that $\|y_n - b\| \to 0$ and $ay_n \xrightarrow{\tau_w} a \Box b \in \mathfrak{A}$.

(iii) There exists a sequence $\{x_n\}$ of elements in $\mathfrak{A}_0$ such that $\|x_n - a\| \to 0$ and $x_n b \xrightarrow{\tau_w} a \Box b \in \mathfrak{A}$.

Proof. We prove only that (i) ⇔ (ii). The proof of (i) ⇔ (iii) is very similar. Assume that $a \Box b$ is defined. By the $\|\cdot\|$-density of $\mathfrak{A}_0$, there exists a sequence $\{y_n\}$ in $\mathfrak{A}_0$ approximating b. Then for every $z, z' \in \mathfrak{A}_0$ and for all $\Theta \in \mathcal{S}_{\mathfrak{A}_0}(\mathfrak{A})$

$$\Theta((ay_n)z, z') = \Theta(y_n z, a^* z') \to \Theta(bz, a^* z') = \Theta((a \Box b)z, z'),$$

i.e. $ay_n \xrightarrow{\tau_w} a \Box b$. Conversely, assume the existence of a sequence $\{y_n\}$ in $\mathfrak{A}_0$ approximating b such that $ay_n \xrightarrow{\tau_w} c \in \mathfrak{A}$. Then, for every $z, z' \in \mathfrak{A}_0$ we have

$$\Theta(bz, a^* z') = \lim_{n\to\infty} \Theta(y_n z, a^* z') = \lim_{n\to\infty} \Theta((ay_n)z, z') = \Theta(cz, z'),$$

i.e. $a \Box b$ is defined. □

Remark 3.2.4 In Proposition 3.2.3, if $a, b \in \mathfrak{A}$ are such that $a \Box b$ is well-defined, then *every* sequence $\{y_n\}$ in $\mathfrak{A}_0$ such that $\|y_n - b\| \to 0$ verifies condition (ii). Indeed, let $\{y_n\}$ be a generic sequence in $\mathfrak{A}_0$ that verifies $\|y_n - b\| \to 0$. Then we show that $ay_n \to a \Box b$ for $n \to \infty$: for every $z, z' \in \mathfrak{A}_0$ and $\Theta \in \mathcal{S}_{\mathfrak{A}_0}(\mathfrak{A})$

$$\lim_{n\to\infty} \Theta(ay_n z, z') = \lim_{n\to\infty} \Theta(y_n z, a^* z) = \Theta(bx, a^* z) = \Theta((a \Box b)z, z').$$

Likewise, the same holds for a sequence $\{x_n\}$ is $\mathfrak{A}_0$ such that $\|x_n - a\| \to 0$.

Let $(\mathfrak{A}, \mathfrak{A}_0)$ be a Banach quasi *-algebra. To every $a \in \mathfrak{A}$ there corresponds the linear maps L_a and R_a defined as

$$L_a : \mathfrak{A}_0 \to \mathfrak{A} \quad L_a x = ax \quad \forall x \in \mathfrak{A}_0 \tag{3.2}$$

$$R_a : \mathfrak{A}_0 \to \mathfrak{A} \quad R_a x = xa \quad \forall x \in \mathfrak{A}_0, \tag{3.3}$$

as seen in Section 1.3.

If $(\mathfrak{A}, \mathfrak{A}_0)$ is a *-semisimple Banach quasi *-algebra, then the weak multiplication $\Box$ allows us to extend L_a, (resp., R_a) to $R_w(a)$ (resp., $L_w(a)$). Let us denote by $\widehat{L}_a$, (resp. $\widehat{R}_a$) these extensions. Then $\widehat{L}_a b = a \Box b$, for every $b \in R_w(a)$ and $\widehat{R}_a c = c \Box a$, for every $c \in L_w(a)$.

Lemma 3.2.5 *If $(\mathfrak{A}, \mathfrak{A}_0)$ is a *-semisimple unital Banach quasi-*algebra, the set $\mathfrak{A}_b$ of bounded elements is a *semisimple Banach algebra. Moreover, $\mathfrak{A}_b$ coincides with the set $R_w(\mathfrak{A}) \cap L_w(\mathfrak{A})$.*

Proof. The first statement and the inclusion $\mathfrak{A}_b \subset R_w(\mathfrak{A}) \cap L_w(\mathfrak{A})$ were shown in [9]. Let $a \in R_w(\mathfrak{A}) \cap L_w(\mathfrak{A})$ then $R_w(a) = L_w(a) = \mathfrak{A}$. Thus $\widehat{L}_a$ (resp., $\widehat{R}_a$) is closed and everywhere defined. Hence both L_a and R_a are bounded. □

Remark 3.2.6 Lemma 3.2.5 shows that the set of bounded elements with respect to the weak multiplication $\Box$ (in the sense of Definition 1.4.24) coincide with the set of universal multipliers for the multiplication $\bullet$ (Definition 1.3.9), in the case of a *-semisimple Banach quasi *-algebra. Therefore, there is only one notion of boundedness, no matter which weak multiplication is considered, $\Box$ or $\bullet$.

3.2.2 Inner qu*-derivations

Let us now assume that δ is a closable qu*-derivation. We consider the question as to whether its closure $\overline{\delta}$ is a *-derivation in some weaker sense; i.e.; if a sort of Leibniz rule still holds.

Proposition 3.2.7 *Let $(\mathfrak{A}, \mathfrak{A}_0)$ be a *-semisimple Banach quasi *-algebra with $\mathfrak{A}[\tau_w]$ sequentially complete. Let δ be a closable qu*-derivation of $(\mathfrak{A}, \mathfrak{A}_0)$ with closure $\overline{\delta}$. Then, if $a, b \in \mathcal{D}(\overline{\delta})$ and $a \Box b$ is well-defined, there exists an element $\overline{\delta}_w(a \Box b) \in \mathfrak{A}$ such that*

$$\Theta(\overline{\delta}_w(a \Box b)u, v) = \Theta(bu, \overline{\delta}(a)^* v) + \Theta(\overline{\delta}(b)u, a^* v) \quad \forall u, v \in \mathfrak{A}_0, \Theta \in \mathcal{S}_{\mathfrak{A}_0}(\mathfrak{A}).$$

Proof. Suppose that δ is a closable qu*-derivation and let $\overline{\delta}$ be its closure. Let $a, b \in \mathcal{D}(\overline{\delta})$, then there exist sequences $\{x_n\}, \{y_n\}$ of elements in $\mathfrak{A}_0$ such that $\|x_n - a\| \to 0$, $\|y_n - b\| \to 0$ and $\|\delta(x_n) - \overline{\delta}(a)\| \to 0$, $\|\delta(y_n) - \overline{\delta}(b)\| \to 0$.

The sequence $\{x_n y_n\} \subset \mathfrak{A}_0$ is τ_w-convergent to $a \Box b$ and thus $\{\delta(x_n y_n)\}$ is τ_w-Cauchy. Indeed,

$$\begin{aligned}
\delta(x_n y_n) - \delta(x_m y_m) &= \delta(x_n) y_n + x_n \delta(y_n) - \delta(x_m) y_m - x_m \delta(y_m) \\
&= \delta(x_n)(y_n - y_m) + (x_n - x_m)\delta(y_m) \\
&+ (\delta(x_n) - \delta(x_m)) y_m + (x_n - x_m)\delta(y_m)
\end{aligned}$$

Hence, for every $z, z' \in \mathfrak{A}_0$ and for all $\Theta \in \mathcal{S}_{\mathfrak{A}_0}(\mathfrak{A})$,

$$\begin{aligned}
&\Theta\left((\delta(x_n y_n) - \delta(x_m y_m))z, z'\right) = \Theta\left((y_n - y_m)z, \delta(x_n)^* z'\right) \\
&\quad + \Theta\left(\delta(y_m)z, (x_n - x_m)^* z'\right) + \Theta\left(y_m z, (\delta(x_n) - \delta(x_m))^* z'\right) \\
&\quad + \Theta\left(\delta(y_m)z, (x_n - x_m)^* z'\right) \to 0
\end{aligned}$$

By the sequential completeness of $\mathfrak{A}[\tau_w]$, there exists $c \in \mathfrak{A}$ such that $\delta(x_n y_n) \xrightarrow{\tau_w} c$. Computing the τ_w-limit

$$\begin{aligned}
\Theta(\delta(x_n y_n)u, v) &= \Theta(\delta(x_n) y_n u, v) + \Theta(x_n \delta(y_n)u, v) \\
&= \Theta(y_n u, \delta(x_n)^* v) + \Theta(\delta(y_n)u, x_n^* v) \\
&\to \Theta(bu, \overline{\delta}(a)^* v) + \Theta(\overline{\delta}(b)u, a^* v),
\end{aligned}$$

for every $u, v \in \mathfrak{A}_0$, for all $\Theta \in \mathcal{S}_{\mathfrak{A}_0}(\mathfrak{A})$, we obtain

$$\overline{\delta}_w(a \Box b) := c = \tau_w - \lim_{n \to \infty} \delta(x_n y_n)$$

and therefore

$$\Theta(\overline{\delta}_w(a \Box b)u, v) = \Theta(bu, \overline{\delta}(a)^* v) + \Theta(\overline{\delta}(b)u, a^* v) \quad \forall \Theta \in \mathcal{S}_{\mathfrak{A}_0}(\mathfrak{A}), u, v \in \mathfrak{A}_0. \quad \Box$$

Remark 3.2.8 Let $(\mathfrak{A}, \mathfrak{A}_0)$ be a Banach quasi *-algebra as in Proposition 3.2.7. Let $h \in \mathfrak{A}$ and $\delta_h(x) := [h, x]$ for $x \in \mathfrak{A}_0$. If $a \in \mathcal{D}(\overline{\delta}_h)$ and $x \in \mathfrak{A}_0$, then there exists an element of $\mathfrak{A}$, denoted by $\hat{\delta}_h(ax)$ such that

$$\Theta(\hat{\delta}_h(a \Box b)u, v) = \Theta(bu, \overline{\delta}_h(a)^* v) + \Theta(\overline{\delta}_h(b)u, a^* v) \quad \forall u, v \in \mathfrak{A}_0, \Theta \in \mathcal{S}_{\mathfrak{A}_0}(\mathfrak{A}).$$

Let $\{x_n\} \subset \mathfrak{A}_0$ be a sequence approximating a fixed element $a \in \mathcal{D}(\overline{\delta}_h)$ such that $\overline{\delta}_h(a) = \lim_{n\to\infty} \delta_h(x_n)$.

By the classical Leibnitz rule we have $\delta_h(x_n x) = ih(x_n x) - i(x_n x)h$, for all $x \in \mathfrak{A}_0$ and for every $n \in \mathbb{N}$. Then, for every $u, v \in \mathfrak{A}_0$ and $\Theta \in \mathcal{S}_{\mathfrak{A}_0}(\mathfrak{A})$ we have

$$\Theta\left(\delta_h(x_n x)u, v\right) \to i\left[\Theta(axu, hv) - \Theta(hu, x^* a^* v)\right] =: \Theta(\delta_h(ax)u, v).$$

If $a \Box h$ is well defined, $\overline{\delta}_h(a)$ is given by $\overline{\delta}_h(a) = i(h \Box a - a \Box h)$.

We want to stress that if $a, b \in \mathcal{D}(\overline{\delta})$ and the products $a \Box h$, $b \Box h$, $a \Box b$ are well-defined, then it is not true a priori that the products $(a \Box b)\Box h$ and $h \Box (a \Box b)$ are well-defined. Even if the mentioned products are well-defined, the associative law for the weak multiplication may fall.

In the case $\mathfrak{A}$ is sequentially complete and h is bounded, then

$$\Theta(h \Box (a \Box b)u, x) = \Theta(bu, (h \Box a)^* v), \quad \forall \Theta \in \mathcal{S}_{\mathfrak{A}_0}(\mathfrak{A}), \forall u, v \in \mathfrak{A}_0$$

applying Proposition 3.2.3. Therefore, for h bounded we have

$$\hat{\delta}_h(a \Box b) = \overline{\delta}_h(a) \Box b + a \Box \overline{\delta}_h(b),$$

for every $a, b \in \mathcal{D}(\overline{\delta})$ for which $a \Box b$ is well-defined.

3.2.3 Derivations as infinitesimal generators

Closed densely defined *-derivations on a C*-algebra $\mathfrak{A}_0$ often occur as infinitesimal generators of norm continuous *-automorphisms one parameter groups. We first need a suitable definition of *-automorphism in the case of Banach quasi *-algebras.

Definition 3.2.9 Let $(\mathfrak{A}, \mathfrak{A}_0)$ be a *-semisimple Banach quasi *-algebra and $\psi : \mathfrak{A} \to \mathfrak{A}$ a linear bijection. We say that ψ is a weak *-automorphism of $(\mathfrak{A}, \mathfrak{A}_0)$ if

(i) $\psi(a^*) = \psi(a)^*$, for every $a \in \mathfrak{A}$;

(ii) $\psi(a) \Box \psi(b)$ is well defined if, and only if, $a \Box b$ is well defined and, in this case,
$$\psi(a \Box b) = \psi(a) \Box \psi(b).$$

By the previous definition it follows that if ψ is a weak *-automorphism, then ψ^{-1} is a weak *-automorphism too.

Lemma 3.2.10 *If ψ is a weak*-automorphism of a *-semisimple Banach quasi *-algebra $(\mathfrak{A}, \mathfrak{A}_0)$, then $\psi(\mathfrak{A}_b) = \mathfrak{A}_b$.*

Proof. Let $a \in \mathfrak{A}_b$, then $a \Box b$ is well defined for every $b \in \mathfrak{A}$ by the boundness of $a \in \mathfrak{A}$. Hence, $\psi(a) \in R_w(\psi(\mathfrak{A})) = R_w(\mathfrak{A})$. Similarly, $\psi(a) \in L_w(\mathfrak{A})$. Thus $\psi(a) \in \mathfrak{A}_b$, by Lemma 3.2.5. Applying this result to ψ^{-1} one gets the converse inclusion. □

In the case of a C*-algebra, every *-automorphism is automatically continuous and furthermore isometric (see Theorem A.3.3). In the case of a CQ*-algebra, it is unknown whether this statement is true. However, we conjecture that similar strategies used to study the continuity of representable functionals on Banach quasi *-algebras could be employed in the study of continuity for weak *-automorphism, as the following example shows.

Example 3.2.11 Let us consider the Banach quasi *-algebras $L^2(I, d\lambda)$ over $\mathcal{C}(I)$ or $L^\infty(I, d\lambda)$, where, for instance, $I = [0, 1]$ and λ is the Lebesgue measure.

As in [3], by [62, Theorem 5.5], if we find a couple of continuous operators (R_1, R_2) intertwining with the weak *-automorphism ψ such that (R_1, R_2) have no critical eigenvalues, then ψ is automatically continuous.

$\psi : L^2(I, d\lambda) \to L^2(I, d\lambda)$ is an intertwining operator with the couple $(R_\eta, R_{\psi(\eta)})$, i.e. $\psi \circ R_\eta = R_{\psi(\eta)} \circ \psi$ for every $\eta \in \mathcal{C}(I)$. By the fact that ψ is a weak *-automorphism, $\sigma(R_\eta) = \sigma(R_{\psi(\eta)})$ for every $\eta \in \mathcal{C}(I)$.

It remains to show that $R_{\psi(\eta)}$ is continuous. By Lemma 3.2.10 $\psi(\mathfrak{A}_b) = \mathfrak{A}_b$ so the operator $\overline{R_{\psi(\eta)}}$ is everywhere defined. To prove that it is continuous it suffices to show that it is closed. Thus, if $\{\eta_n\}$ is a sequence in $\mathcal{C}(I)$ that vanishes such that $R_{\psi(\eta)}(\eta_n) = \eta_n\psi(\eta)$ converges to $f \in L^2(I, d\lambda)$, then

$$\Theta(\eta_n\psi(\eta)u, v) = \Theta(\psi(\eta)u, \eta_n^* v) \to 0$$

and, on the other hand,

$$\Theta(\eta_n\psi(\eta)u, v) \to \Theta(fu, v)$$

for every $\Theta \in \mathcal{S}_{\mathcal{C}(I)}(L^2(I, d\lambda))$ and for all $u, v \in \mathcal{C}(I)$. By the *-semisimplicity of $(L^2(I, d\lambda), \mathcal{C}(I))$, we conclude that $f = 0$.

In order to apply Theorem 5.5 of [62], choose $\widehat{\eta} \in \mathcal{C}(I)$ such that $\widehat{\eta}$ has only continuous spectrum in $L^2(I, d\lambda)$. Hence ψ is continuous by the aforementioned theorem.

To show a similar statement $(L^2(I, d\lambda), L^\infty(I, d\lambda))$, it is enough to employ the same argument used for $(L^2(I, d\lambda), \mathcal{C}(I))$.

Definition 3.2.12 Let $(\mathfrak{A}, \mathfrak{A}_0)$ be a *-semisimple Banach quasi *-algebra. Suppose that for every fixed $t \in \mathbb{R}$, β_t is a weak *-automorphism of $\mathfrak{A}$. If

(i) $\beta_0(a) = a$, $\forall a \in \mathfrak{A}$

(ii) $\beta_{t+s}(a) = \beta_t(\beta_s(a))$, $\forall a \in \mathfrak{A}$

then we say that β_t is a *one-parameter group of weak *-automorphisms of* $(\mathfrak{A}, \mathfrak{A}_0)$. If τ is a topology on $\mathfrak{A}$ and the map $t \mapsto \beta_t(a)$ is τ-continuous, for every $a \in \mathfrak{A}$, we say that β_t is a τ-continuous weak *-automorphism group.

The definition of the infinitesimal generator of β_t is now quite natural. If β_t is τ-continuous, we set

$$\mathcal{D}(\delta_\tau) = \left\{ a \in \mathfrak{A} : \lim_{t \to 0} \frac{\beta_t(a) - a}{t} \text{ exists in } \mathfrak{A}[\tau] \right\}$$

and

$$\delta_\tau(a) = \tau - \lim_{t \to 0} \frac{\beta_t(a) - a}{t}, \quad a \in \mathcal{D}(\delta_\tau).$$

If the involution $a \mapsto a^*$ is τ-continuous, then $a \in \mathcal{D}(\delta_\tau)$ implies a^* in $\mathcal{D}(\delta_\tau)$ and $\delta(a^*) = \delta(a)^*$. Clearly, $\mathcal{D}(\delta_{\tau_n}) \subseteq \mathcal{D}(\delta_{\tau_{s^*}}) \subseteq \mathcal{D}(\delta_{\tau_w})$.

What we expect is $\mathcal{D}(\delta_\tau)$ to be a partial *-algebra and δ_τ a *-derivation in a sense to be specified. Hence we should decide which form of Leibniz rule must be taken to define conveniently derivations on a partial *-algebra. The following proposition suggests an answer to that question.

Proposition 3.2.13 *Let $(\mathfrak{A}, \mathfrak{A}_0)$ be a *-semisimple Banach quasi *-algebra and β_t a τ_{s^*}-continuous weak *-automorphism group of $(\mathfrak{A}, \mathfrak{A}_0)$. Then the following statements hold.*

(i) $\delta_{\tau_{s^*}}(a^*) = \delta_{\tau_{s^*}}(a)^*$*;*

(ii) If $a, b \in \mathcal{D}(\delta_{\tau_{s^*}})$ and $a \square b$ is well defined, then $a \square b \in \mathcal{D}(\delta_{\tau_w})$ and

$$\Theta(\delta_{\tau_w}(a \square b)x, y) = \Theta(bx, \delta_{\tau_{s^*}}(a)^* y) + \Theta(\delta_{\tau_{s^*}}(b)x, a^* y),$$
$$\forall a, b \in \mathcal{D}(\delta_{\tau_{s^*}}),\ a \in L_w(b); x, y \in \mathfrak{A}_0.$$

*(iii) If $\mathcal{D}(\delta_{\tau_w}) = \mathcal{D}(\delta_{\tau_n})$ then $\mathcal{D}(\delta_{\tau_n})$ is a partial *-algebra with respect to the weak multiplication.*

Proof. We start proving (i). By definition of $\delta_{\tau_{s^*}}$ and the τ_{s^*}-continuity of the involution, we have

$$\delta_{\tau_{s^*}}(a*) = \tau_{s^*} - \lim_{t \to 0} \frac{\beta_t(a^*) - a^*}{t} = \tau_{s^*} - \lim_{t \to 0} \frac{\beta_t(a^*) - a^*}{t}$$

$$= \left(\tau_{s^*} - \lim_{t\to 0} \frac{\beta_t(a) - a}{t} \right)^* = \delta_{\tau_{s^*}}(a)^*.$$

Let us now prove (ii). Let $a, b \in \mathcal{D}(\delta_{\tau_{s^*}})$, with $a \in L_w(b)$. If $x, y \in \mathfrak{A}_0$, then

$$\begin{aligned}
\lim_{t\to 0} \Theta\left(\frac{\beta_t(a \Box b) - a \Box b}{t} x, y \right) &= \lim_{t\to 0} \Theta\left(\frac{\beta_t(a) \Box \beta_t(b) - a \Box b}{t} x, y \right) \\
&= \lim_{t\to 0} \frac{1}{t} \left[\Theta((\beta_t(a) \Box \beta_t(b))x, y) - \Theta(\beta_t(b)x, a^* y) \right] \\
&+ \lim_{t\to 0} \frac{1}{t} \left[\Theta(\beta_t(b)x, a^* y) - \Theta(bx, a^* y) \right]
\end{aligned}$$

Now, for the first term on the right hand side, we have

$$\begin{aligned}
&\left| \frac{1}{t} [\Theta((\beta_t(a) \Box \beta_t(b))x, y) - \Theta(\beta_t(b)x, a^* y)] - \Theta(bx, \delta_{\tau_{s^*}}(a)^* y) \right| \\
&\leq \left| \Theta\left(\beta_t(b)x, \frac{\beta_t(a)^* - a^*}{t} y \right) - \Theta(\beta_t(b)x, \delta_{\tau_{s^*}}(a)^* y) \right| \\
&+ \left| \Theta(\beta_t(b)x, \delta_{\tau_{s^*}}(a)^* y) - \Theta(bx, \delta_{\tau_{s^*}}(a)^* y) \right| \\
&\leq \Theta(\beta_t(b)x, \beta_t(b)x)^{1/2} \Theta\left(\frac{\beta_t(a)^* - a^*}{t} y - \delta_{\tau_{s^*}}(a)^* y, \frac{\beta_t(a)^* - a^*}{t} y - \delta_{\tau_{s^*}}(a)^* y \right)^{1/2} \\
&+ \Theta((\beta_t(b) - b)x, (\beta_t(b) - b)x)^{1/2} \Theta(\delta_{\tau_{s^*}}(a)^* y, \delta_{\tau_{s^*}}(a)^* y)^{1/2} \to 0.
\end{aligned}$$

because of the τ_{s^*}-continuity of β_t and of the involution. As for the second term we have, taking into account that $b \in \mathcal{D}(\delta_{\tau_{s^*}})$,

$$\lim_{t\to 0} \frac{1}{t} \left[\Theta(\beta_t(b)x, a^* y) - \Theta(bx, a^* y) \right] = \Theta(\delta_{\tau_{s^*}}(b)x, a^* y).$$

This proves at once that if $a, b \in \mathcal{D}(\delta_{\tau_{s^*}})$ and $a \Box b$ is well-defined, then $a \Box b \in \mathcal{D}(\delta_{\tau_w})$ and

$$\Theta(\delta_{\tau_w}(a \Box b)x, y) = \Theta(bx, \delta_{\tau_{s^*}}(a)^* y) + \Theta(\delta_{\tau_{s^*}}(b)x, a^* y), \quad \forall x, y \in \mathfrak{A}_0.$$

For (iii), let $a, b \in \mathcal{D}(\delta_{\tau_n})$ such that $a \Box b$ is well defined. By (ii), $a \Box b \in \mathcal{D}(\delta_{\tau_w})$. We conclude by the hypothesis that $\mathcal{D}(\delta_{\tau_n}) = \mathcal{D}(\delta_{\tau_n})$. □

Proposition 3.2.13 suggests the following definition inspired by the one given in [11, 12] for partial *-algebras of unbounded operators.

Definition 3.2.14 Let $(\mathfrak{A}, \mathfrak{A}_0)$ be a *-semisimple Banach quasi *-algebra and δ a linear map of $\mathcal{D}(\delta)$ into $\mathfrak{A}$, where $\mathcal{D}(\delta)$ is a partial *-algebra with respect to the weak multiplication $\Box$. We say that δ is a *weak *-derivation* of $(\mathfrak{A}, \mathfrak{A}_0)$ if

(i) $\mathfrak{A}_0 \subset \mathcal{D}(\delta)$

(ii) $\delta(x^*) = \delta(x)^*, \ \forall x \in \mathfrak{A}_0$

(iii) if $a, b \in \mathcal{D}(\delta)$ and $a \Box b$ is well defined, then $a \Box b \in \mathcal{D}(\delta)$ and

$$\Theta(\delta(a \Box b)x, y) = \Theta(bx, \delta(a)^* y) + \Theta(\delta(b)x, a^* y),$$

for all $\Theta \in \mathcal{S}_{\mathfrak{A}_0}(\mathfrak{A})$, for every $x, y \in \mathfrak{A}_0$.

Clearly, every qu*-derivation is a weak *-derivation with the assumption $\mathcal{D}(\delta) = \mathfrak{A}_0$.

Example 3.2.15 The space $L^p(\mathbb{R})$, $p \geq 1$, can be coupled with many *-algebras of functions (for instance, $\mathcal{C}_c^\infty(\mathbb{R})$, $\mathcal{C}_o(\mathbb{R}) \cap L^p(\mathbb{R})$, $W^{1,2}(\mathbb{R})$) to obtain a Banach quasi *-algebra. For $p \geq 2$, $(L^p(\mathbb{R}), \mathcal{C}_c^\infty(\mathbb{R}))$ is a *-semisimple Banach quasi *-algebra: the corresponding set $\mathcal{S}_{\mathfrak{A}_0}(\mathfrak{A})$ is given by the form Θ_w defined for $w \in L^{\frac{p}{p-2}}(\mathbb{R})$ (for $p = 2$, $\frac{p}{p-2} = \infty$), $w \geq 0$,

$$\Theta_w(f, g) = \int_{\mathbb{R}} f(x)\overline{g(x)} w(x) dx.$$

The weak multiplication $f \Box g$ is well defined if, and only if, $fg \in L^p(\mathbb{R})$. Let us define for $v \in \mathbb{R}$, $\beta_t(f) = f_t$ where $f_t(x) = f(x+t)$, $f \in L^p(\mathbb{R})$. Then β_t is a weak *-automorphisms group. Its infinitesimal generator is, formally, the derivative operator with domain $W^{1,2}(\mathbb{R})$. If we change the *-algebra taking for instance $\mathcal{C}_o(\mathbb{R}) \cap L^p(\mathbb{R})$ we see that the domain of δ does not contain $\mathfrak{A}_0$, in general.

3.3 Integrability of weak *-derivations

In the previous Section we gave a suitable definition of *-derivation and *-automorphism in the framework of *-semisimple Banach quasi *-algebras. This investigation allows us to prove analogous results about closability of δ as in the celebrated Bratteli - Robinson Theorem in [26], for a *-semisimple Banach quasi *-algebra.

First we prove a technical lemma useful for the results we are going to prove (see [55]).

Lemma 3.3.1 *Let $(\mathfrak{A}, \mathfrak{A}_0)$ be a *-semisimple Banach quasi *-algebra and let $\{\beta_t\}_{t\in\mathbb{R}}$ be a uniformly bounded τ_n-continuous group of weak *-automorphisms of $(\mathfrak{A}, \mathfrak{A}_0)$. Let δ be the infinitesimal generator of $\{\beta_t\}_{t\in\mathbb{R}}$. Then*

1. *for* $a \in \mathfrak{A}$
$$\|\cdot\| - \lim_{h\to 0} \frac{1}{h}\int_t^{t+h} \beta_s(a)ds = \beta_t(a);$$

2. *for* $a \in \mathfrak{A}$, $\int_0^t \beta_s(a)ds \in \mathcal{D}(\delta)$ *and*
$$\delta\left(\int_0^t \beta_s(a)ds\right) = \beta_t(a) - a;$$

3. *for* $a \in \mathcal{D}(\delta)$, $\beta_t(a) \in \mathcal{D}(\delta)$ *and*
$$\frac{d}{dt}\beta_t(a) = \delta(\beta_t(a)) = \beta_t(\delta(a));$$

4. *for* $a \in \mathcal{D}(\delta)$
$$\beta_t(a) - \beta_s(a) = \int_s^t \beta_r(\delta(a))dr = \int_s^t \delta(\beta_r(a))dr.$$

Proof. By norm continuity of $t \mapsto \beta_t(a)$ for every $a \in \mathfrak{A}$, we have
$$\lim_{h\to 0}\int_t^{t+h} \beta_s(a)ds = \beta_t(a).$$
This proves (1). Moreover, if $h > 0$, by the properties of uniform boundedness in t and norm continuity of β_t for a fixed t, then we have
$$\begin{aligned}\frac{\beta_t - I}{h}\left(\int_0^t \beta_s(a)ds\right) &= \frac{1}{h}\int_0^t \left(\beta_{s+h}(a) - \beta_s(a)\right)ds \\ &= \frac{1}{h}\int_t^{t+h} \beta_s(a)ds - \frac{1}{h}\int_0^h \beta_s(a)ds \to \beta_t(a) - a\end{aligned}$$
as $h \to 0$. By the previous argument, (2) is proved.

In the same setting as above,
$$\frac{\beta_h - I}{h}\left(\beta_t(a)\right) = \beta_t\left(\frac{\beta_h - I}{h}\right)(a) \to \beta_t\left(\delta(a)\right)$$
as $h \to 0$. Thus $\beta_t(a) \in \mathcal{D}(\delta)$ and $\delta\left(\beta_t(a)\right) = \beta_t\left(\delta(a)\right)$.

The above computation shows also that the right derivative of $\beta_t(a)$ exists and it is equal to $\beta_t\left(\delta(a)\right)$. We want to show that also the left derivative exists and it is equal to the right one. Indeed, for $h > 0$,
$$\lim_{h\to 0}\left[\frac{\beta_t(a) - \beta_{t-h}(a)}{h} - \beta_t\left(\delta(a)\right)\right]$$

$$= \lim_{h\to 0} \beta_{t-h}\left[\frac{\beta_h(a)-a}{h} - \delta(a)\right] + \lim_{h\to 0}\left(\beta_{t-h}(\delta(a)) - \beta_t(\delta(a))\right) \to 0$$

by the uniformly boundedness of $\{\beta_t\}_{t\in\mathbb{R}}$ in t, $a \in \mathcal{D}(\delta)$ and the norm continuity of $t \mapsto \beta_t(a)$. Hence we deduce that

$$\frac{d}{dt}\beta_t(a) = \delta(\beta_t(a)) = \beta_t(\delta(a))$$

for $a \in \mathcal{D}(\delta)$. This way we obtain (3) and then integrating the previous we have

$$\beta_t(a) - \beta_s(a) = \int_s^t \beta_r(\delta(a))dr = \int_s^t \delta(\beta_r(a))dr.$$

This gives us (4). □

The proof of the following theorem is inspired by the proof of Theorem 3.0.2.

Theorem 3.3.2 *Let $\delta : \mathcal{D}(\delta) \to \mathfrak{A}$ be a weak *-derivation on a *-semisimple Banach quasi *-algebra $(\mathfrak{A}, \mathfrak{A}_0)$. Suppose that δ is the infinitesimal generator of a uniformly bounded, τ_n-continuous group of weak *-automorphisms of $(\mathfrak{A}, \mathfrak{A}_0)$. Then δ is closed; its resolvent set $\rho(\delta)$ contains $\mathbb{R}\setminus\{0\}$ and*

$$\|\delta(a) - \lambda a\| \geq |\lambda|\,\|a\|, \quad a \in \mathcal{D}(\delta), \lambda \in \mathbb{R}. \tag{3.4}$$

Proof. By (1) and (2) of Lemma 3.3.1, if $a \in \mathfrak{A}$, we define $a_t := \frac{1}{t}\int_0^t \beta_s(a)ds$, then $a_t \in \mathcal{D}(\delta)$ for $t \in \mathbb{R}$ and $a_t \to a$ as $t \to 0$. We conclude $\overline{\mathcal{D}(\delta)} = \mathfrak{A}$.

In order to prove that δ is closed, let $\{a_n\}$ in $\mathcal{D}(\delta)$ such that $a_n \to a$ and $\delta(a_n) \to w$ as $n \to \infty$. By (4) of Lemma 3.3.1,

$$\beta_t(a_n) - a_n = \int_0^t \beta_s(\delta(a_n))ds.$$

Considering the limit on both sides of the equality and using again (4) of Lemma 3.3.1, we obtain

$$\beta_t(a) - a = \int_0^t \beta_s(w)ds.$$

Dividing by $t \neq 0$ and taking the limit as $t \to 0$, we conclude by (1) of Lemma 3.3.1 that $a \in \mathcal{D}(\delta)$ and $\delta(a) = w$, i.e. δ is closed.

If $\lambda = 0$, the inequality is obvious. Now we consider $\lambda > 0$ and define the operator

$$R_\lambda(a) := \int_0^\infty \mathrm{e}^{-\lambda t}\beta_t(a)dt.$$

The continuity of $t \mapsto \beta_t(a)$ for every $a \in \mathfrak{A}$ and the uniform boundedness of β_t in t for every $t \in \mathbb{R}$ guarantee that the above operator is well-defined and

$$\|R_\lambda(a)\| \leq \frac{1}{\lambda}\|a\|.$$

Moreover, $(\lambda I - \delta)(R_\lambda(a)) = a$, for every $a \in \mathfrak{A}$ and $R_\lambda((\lambda I - \delta)(a)) = a$, for every $a \in \mathcal{D}(\delta)$. Indeed, the right hand side of the following

$$\begin{aligned}\frac{\beta_h - I}{h}(R_\lambda(a)) &= \frac{1}{h}\int_0^\infty \mathsf{e}^{-\lambda t}\left[\beta_{t+h}(a) - \beta_t(a)\right]dt \\ &= \frac{\mathsf{e}^{\lambda h} - I}{h}\int_0^\infty \mathsf{e}^{-\lambda t}\beta_t(a)dt - \frac{\mathsf{e}^{\lambda h}}{h}\int_0^h \mathsf{e}^{-\lambda t}\beta_t(a)\end{aligned}$$

tells us that $R_\lambda(a) \in \mathcal{D}(\delta)$ and it converges to $\lambda R_\lambda(a) - a$ for every $a \in \mathfrak{A}$ and $\lambda > 0$. Thus, $(\lambda I - \delta)R_\lambda = I$.

By the closedness and again by Lemma 3.3.1, we obtain also the other equality. Indeed,

$$\begin{aligned}R_\lambda(\delta(a)) &= \int_0^\infty \mathsf{e}^{-\lambda t}\beta_t(\delta(a))dt = \int_0^\infty \mathsf{e}^{-\lambda t}\delta(\beta_t(a))dt \\ &= \lim_{y\to\infty}\int_0^y \mathsf{e}^{-\lambda t}\delta(\beta_t(a))dt = \lim_{y\to\infty}\delta\left(\int_0^y \mathsf{e}^{-\lambda t}\beta_t(a)dt\right) \\ &= \delta\left(\int_0^\infty \mathsf{e}^{-\lambda t}\beta_t(a)dt\right) = \delta(R_\lambda(a)),\end{aligned}$$

where the last passage is justified by the following computations

$$\begin{aligned}\delta\left(\int_0^y \mathsf{e}^{-\lambda t}\beta_t(a)dt\right) &= \lim_{h\to 0}\frac{\beta_h - I}{h}\left(\int_0^y \mathsf{e}^{-\lambda t}\beta_t(a)dt\right) \\ &= \lim_{h\to 0}\int_0^y \mathsf{e}^{-\lambda t}\frac{\beta_{t+h}(a) - \beta_t(a)}{h}dt \\ &= \int_0^y \mathsf{e}^{-\lambda t}\frac{d}{dt}\beta_t(a)dt = \int_0^y \mathsf{e}^{-\lambda t}\delta(\beta_t(a))dt.\end{aligned}$$

Hence, R_λ is the inverse of $\lambda I - \delta$ and the conditions on the spectrum are verified.

The case when $\lambda < 0$ can be handled in very similar way, by defining the operator $R_\lambda(a)$ as

$$R_\lambda(a) := \int_0^\infty \mathsf{e}^{\lambda t}\beta_{-t}(a)dt. \quad \square$$

In order to prove that a closed weak *-derivation is the infinitesimal generator of uniformly bounded, τ_n-continuous group of weak *-automorphisms further assumptions on δ are needed.

Theorem 3.3.3 *Let $\delta : \mathcal{D}(\delta) \subset \mathfrak{A}_{\mathrm{b}} \to \mathfrak{A}$ be a closed weak *-derivation on a *-semisimple Banach quasi *-algebra $(\mathfrak{A}, \mathfrak{A}_0)$. Suppose that the resolvent set of δ, denoted as $\rho(\delta)$, contains $\mathbb{R} \setminus \{0\}$ and*

$$\|\delta(a) - \lambda a\| \geq |\lambda| \, \|a\|, \quad a \in \mathcal{D}(\delta), \lambda \in \mathbb{R}. \tag{3.5}$$

*Moreover, assume that $\mathfrak{A}_0$ is a core for every multiplication operator $\widehat{L}_a$ for $a \in \mathfrak{A}$, i.e. $\widehat{L}_a = \overline{L}_a$. Then δ is the infinitesimal generator of a uniformly bounded, τ_n-continuous group of weak *-automorphisms of $(\mathfrak{A}, \mathfrak{A}_0)$.*

Proof. We want to show that the norm limit

$$\beta_t(a) := \lim_{n\to\infty} \left(I - \frac{t}{n}\delta\right)^{-1} (a)$$

gives us a uniformly bounded, τ_n-continuous group of weak *-automorphisms of $(\mathfrak{A}, \mathfrak{A}_0)$.

This limit exists by applying the theory of C_0-semigroups in Banach spaces [44, Chapter 12]. Moreover, the map $t \in \mathbb{R} \to \beta_t(a)$ is norm continuous since [44, p. 362] the convergence is uniform in every finite interval $[0, t_0]$.

By the condition on the spectrum of δ, β_t is, for every $t \in \mathbb{R}$, a bounded operator in $\mathfrak{A}$ and all its powers are well defined. By the condition (3.4), we obtain, for every $n \in \mathbb{N}^*$ and for every $a \in \mathfrak{A}$

$$\left\| \left(I - \frac{t}{n}\delta\right)^{-n} (a) \right\| = \left\| \left[\frac{t}{n}\left(\frac{n}{t} - \delta\right)\right]^{-n} (a) \right\| \leq \left|\frac{n}{t}\right|^{n} \cdot \left|\frac{n}{t}\right|^{-n} \|a\| = \|a\|$$

Hence passing to the limit we have $\|\beta_t(a)\| \leq \|a\|$ for every $a \in \mathfrak{A}$.

Let $t \in \mathbb{R}$ be fixed. Then β_t is a continuous linear and bijective operator. Moreover, β_t preserves the involution, i.e. $\beta_t(a)^* = \beta_t(a^*)$ for every $a \in \mathfrak{A}$. Indeed,

$$\beta_t(a)^* = \lim_{n\to\infty} \left(\left(1 - \frac{t}{n}\delta\right)^{-n} (a)\right)^* = \lim_{n\to+\infty} \left(1 - \frac{t}{n}\delta\right)^{-n} (a^*) = \beta_t(a^*).$$

Further δ commutes with all its negative powers, so for every $a \in \mathfrak{A}$ $\beta_t(\delta(a)) = \delta(\beta_t(a))$. Indeed, for every $a \in \mathfrak{A}$, we have

$$\begin{aligned}\left[\frac{t}{n}\left(\frac{n}{t} - \delta\right)\right]^{-n} (\delta(a)) &= \left[\frac{n}{t}\left(\frac{n}{t} + \delta\right)^{-1}\right]^{n-1} \left(\frac{n}{t} + \delta\right)^{-1} (\delta(a)) \\ &= \left[\frac{n}{t}\left(\frac{n}{t} + \delta\right)^{-1}\right]^{n-1} \delta\left(\frac{n}{t} + \delta\right)^{-1} (a)\end{aligned}$$

$$= \dots$$
$$= \left(\frac{n}{t}\right)^n \delta \left(\frac{n}{t} - \delta\right)^{-n} (a).$$

$\beta_t(a)$ is the solution of the Cauchy problem $\beta_t'(a) = \beta_t(\delta(a))$ with initial condition $\beta_0(a) = a$, hence we achieve the group property, i.e. we have $\beta_{t+s}(a) = \beta_t(\beta_s(a))$ for every $a \in \mathfrak{A}$, $t, s \in \mathbb{R}$.

The set of analytic elements, i.e. the set of all elements $a \in \mathcal{D}(\delta^n)$, for every $n \in \mathbb{N}$, such that the power series

$$z \in \mathbb{C} \mapsto \sum_{n=0}^{\infty} \frac{z^n}{n!} \delta^n(a) \in \mathfrak{A}$$

is well defined and analytic on a neighborhood of the origin, is dense in $\mathfrak{A}$ by [55, Theorem 2.7].

The last property we are going to prove is that β_t is a weak *-automorphism, i.e. $\beta_t(a) \square \beta_t(b)$ is well defined if, and only if, $a \square b$ is well defined and, in this case, $\beta_t(a \square b) = \beta_t(a) \square \beta_t(b)$.

By the hypotheses, $\mathcal{D}(\delta) \subset \mathfrak{A}_{\rm b}$ is a partial *-algebra with respect to the weak multiplication $\square$. By the boundedness of a, b, we can rewrite the weak Leibnitz rule as

$$\delta(a \square b) = \delta(a) \square b + a \square \delta(b).$$

Now suppose that a, b are analytic elements. Therefore $a, b \in \mathcal{D}(\delta^k)$ and $\delta^k(a), \delta^k(b) \in \mathfrak{A}_{\rm b}$ for every $k \in \mathbb{N}$. Indeed, $\delta^k(a) \in \mathcal{D}(\delta^{k+1}) \subset \mathcal{D}(\delta) \subset \mathfrak{A}_{\rm b}$. Hence, all the products $\delta^m(a) \square \delta^n(b)$ are well-defined for every $n, m \in \mathbb{N}$.

By the above argument, by induction we show that

$$\delta^n(a \square b) = \sum_{k=0}^{n} \binom{n}{k} \delta^{n-k}(a) \square \delta^k(b)$$

for every a, b analytic elements. Indeed, for $n = 1$, it is the Leibnitz rule. Suppose it is true for n and prove it for $n + 1$

$$\begin{aligned} \delta^{n+1}(a \square b) = \delta(\delta^n(a \square b)) &= \sum_{k=0}^{n} \binom{n}{k} \delta(\delta^{n-k}(a) \square \delta^k(b))) \\ &= \sum_{k=0}^{n} \binom{n}{k} \left[\delta^{n-k+1}(a) \square \delta^k(b) + \delta^{n-k}(a) \square \delta^{k+1}(b)\right] \\ &= \delta^{n+1}(a) \square b + \delta^n(a) \square \delta(b) + \binom{n}{1} \delta^n(a) \square \delta(b) + \dots \\ &+ \dots + \binom{n}{k-1} \delta^{n-k-1}(a) \square \delta^k(b) \end{aligned}$$

$$
\begin{aligned}
&+\binom{n}{k}\delta^{n-k+1}(a)\Box\delta^{k}(b)+\binom{n}{k}\delta^{n-k}(a)\Box\delta^{k+1}(b)\\
&+\binom{n}{k+1}\delta^{n-k}(a)\Box\delta^{k+1}(b)+\dots\\
&+\dots+\binom{n}{n-1}\delta(a)\Box\delta^{n}(b)+\delta(a)\Box\delta^{n}(b)+a\Box\delta^{n+1}(b)\\
&=\sum_{k=0}^{n+1}\binom{n+1}{k}\delta^{n-k+1}(a)\Box\delta^{k}(b).
\end{aligned}
$$

In a very standard way we achieve the weak *-automorphism property in the case a, b are analytic elements. For a, b analytic we have

$$
\begin{aligned}
\beta_t(a\Box b) &= \sum_{n\geq 0}\frac{t^n}{n!}\delta^n(a\Box b) = \sum_{n\geq 0}\frac{t^n}{n!}\sum_{k=0}^{n}\binom{n}{k}\delta^{n-k}(a)\Box\delta^{k}(b)\\
&= \sum_{n\geq 0}\frac{t^n}{n!}\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}\delta^{n-k}(a)\Box\delta^{k}(b)\\
&= \sum_{n\geq 0}\sum_{k=0}^{n}\frac{t^n}{n!}\frac{n!}{k!(n-k)!}\delta^{n-k}(a)\Box\delta^{k}(b)\\
&= \sum_{n\geq 0}\sum_{k=0}^{n}\left(\frac{t^{n-k}}{(n-k)!}\delta^{n-k}(a)\right)\Box\left(\frac{t^k}{k!}\delta^{k}(b)\right)\\
&= \left(\sum_{n\geq 0}\frac{t^n}{n!}\delta^n(a)\right)\Box\left(\sum_{m\geq 0}\frac{t^m}{m!}\delta^m(b)\right) = \beta_t(a)\Box\beta_t(b).
\end{aligned}
$$

Using the density of the set of analytic elements and the boundedness of the elements one proves the equality

$$\beta_t(a\Box b) = \beta_t(a)\Box\beta_t(b), \quad a, b \in \mathfrak{A}_{\mathrm{b}}.$$

Suppose first that $a \in \mathfrak{A}_{\mathrm{b}}$ and b is analytic. Then there exists a sequence $\{a_n\}$ of analytic elements that approximates a and, by continuity of β_t, we have

$$\beta_t(a\Box b) = \lim_{n\to\infty}\beta_t(a_n\Box b) = \lim_{n\to\infty}\beta_n(a_n)\Box\beta_t(b) = \beta_t(a)\Box\beta_t(b),$$

recalling that $\beta_t(b)$ is a bounded element by Lemma 3.2.10. In the case a, b are both bounded, the conclusion can be obtained with the same argument.

Let $a \in \mathfrak{A}$ and $b \in \mathfrak{A}_{\mathrm{b}}$. Approximating an unbounded element a through a sequence a_n of bounded elements, the weak product $a\Box b$ can be approximated by the sequence $a_n\Box b$ and we get

$$\beta_t(a\Box b) = \beta_t(a)\Box\beta_t(b) \quad \text{for } a \in \mathfrak{A}, b \in \mathfrak{A}_{\mathrm{b}}.$$

Suppose now that both $a, b \in \mathfrak{A}$ are unbounded. By hypothesis, $\mathfrak{A}_{\rm b}$ is a core for every L_a, then there exists a sequence $\{b_n\} \in \mathfrak{A}_{\rm b}$ that norm converges to b such that $\|a \Box b_n - a \Box b\|$ vanishes as n increases. By norm continuity of β_t we achieve the weak automorphism property for β_t, i.e.

$$\beta_t(a \Box b) = \beta_t(a) \Box \beta_t(b) \quad \forall a, b \in \mathfrak{A}. \quad \Box$$

Remark 3.3.4 The additional hypotheses of Theorem 3.3.3 are satisfied by the *weak derivative in* $L^p(I, d\lambda)$, where $I = [0, 1]$ and λ is the Lebesgue measure.

In this case $\mathcal{D}(\delta) = W^{1,p}(I, d\lambda)$ and it is well known (see [28, Theorem 8.8]) that if $u \in W^{1,p}(I, d\lambda)$ then $u \in L^\infty(I, d\lambda)$ and there exists $c > 0$ such that

$$\|u\|_\infty \leq c\|u\|_{1,p}$$

3.4 Examples and applications

In this section we present some examples of weak *-derivations and one-parameter groups generated by them.

Let $(\mathfrak{A}, \mathfrak{A}_0)$ be a regular Banach quasi *-algebra (see Definition 1.4.22) and consider again the example of inner qu*-derivations; i.e., $\delta : \mathfrak{A}_0 \to \mathfrak{A}$ is a densely defined derivation determined as $\delta_h(x) := i[h, x]$ for a *self-adjoint* element $h \in \mathfrak{A}$, i.e. $h = h^*$ and $\sigma(h) \subset \mathbb{R}$.

3.4.1 Commutators for h self-adjoint

Case 1: Suppose first that h is a *bounded element.* As we have already seen, in this case $\overline{\delta}_h(x)$ is *continuous.*

Like in the classical case, what we would expect is a derivation that generates a one-parameter group $\{\beta_t\}_{t\in\mathbb{R}}$ of weak *-automorphisms of $(\mathfrak{A}, \mathfrak{A}_0)$ of the form

$$\beta_t(a) = \mathsf{e}^{ith} \Box a \Box \mathsf{e}^{-ith} \quad \text{for all } t \in \mathbb{R}.$$

Suppose that $(\mathfrak{A}, \mathfrak{A}_0)$ is a *-semisimple Banach quasi *-algebra with unit $\mathbb{1}$. Then we define the Taylor series e^{ith} as

$$\mathsf{e}^{ith} := \sum_{n=0}^{\infty} \frac{(ith)^n}{n!},$$

where the series on the right hand side converges with respect to $\| \cdot \|_b$. We stress the fact that h^n is the weak product of h with itself n times. The above series is well defined, the exponential $\mathsf{e}^{ith} \in \mathfrak{A}_{\mathrm{b}}$ and all the known properties remain valid.

For each $t \in \mathbb{R}$, $\beta_t(a) := \mathsf{e}^{ith} \Box a \Box \mathsf{e}^{-ith}$ is a weak *-automorphism of $(\mathfrak{A}, \mathfrak{A}_0)$. We notice that by the separate continuity of multiplication and the *-semisimplicity of $(\mathfrak{A}, \mathfrak{A}_0)$ the use of brackets is not needed.

If we fix $t \in \mathbb{R}$, then it is routine to prove that β_t is a linear map preserving the weak multiplication when defined. Its inverse is given by the expression $\beta_t^{-1}(a) = \mathsf{e}^{-ith} \Box a \Box \mathsf{e}^{ith} = \beta_{-t}(a)$ and $\beta : t \mapsto \beta_t$ is in fact a weak *-automorphism group of $(\mathfrak{A}, \mathfrak{A}_0)$ for every $t \in \mathbb{R}$.

Self-adjoint elements of a regular Banach quasi *-algebra can be characterized as those elements such that $\|\mathsf{e}^{ith}\|_{\mathrm{b}} = 1$. In this case, by Theorem 3.6 in [66], $\mathfrak{A}_{\mathrm{b}}$ is a C*-algebra with respect to the norm introduced in Definition 1.3.9 (see also Remark 3.2.6). Following [33, Proposition 2.4.12], we have that, for a *-semisimple Banach quasi *-algebra $(\mathfrak{A}, \mathfrak{A}_0)$ and $h \in \mathfrak{A}_{\mathrm{b}}$ such that $h = h^*$, $\sigma(h) \subset \mathbb{R}$ if, and only if, $r_{\mathrm{b}}(\mathsf{e}^{ith}) = 1$.

Note that $(\mathsf{e}^{ith})^* = \mathsf{e}^{-ith}$ and $\mathsf{e}^{ith} \Box \mathsf{e}^{-ith} = \mathbb{1} = \mathsf{e}^{-ith} \Box \mathsf{e}^{ith}$. Hence, $\mathsf{e}^{ith} \in \mathfrak{A}_{\mathrm{b}}$ is normal and then $r(\mathsf{e}^{ith}) = 1 = \|\mathsf{e}^{ith}\|_{\mathrm{b}}$. Therefore we conclude that $\{\beta_t\}_{t\in\mathbb{R}}$ is uniformly bounded in t by $\|\beta_t(a)\| \leq \|\mathsf{e}^{ith}\|_{\mathrm{b}}^2 \|a\| = \|a\|$.

By standard computations it is easy to check that $\{\beta_t\}_{t\in\mathbb{R}}$ is really a norm continuous one-parameter group, i.e. $\beta_0(a) = a = \mathrm{Id}(a)$, $\beta_{t+s}(a) = \beta_t \circ \beta_s(a)$ and $\|\beta_t(a) - a\|$ vanishes as $t \to 0$, for every $a \in \mathfrak{A}$.

We now compute the infinitesimal generator of $\{\beta_t\}_{t\in\mathbb{R}}$. What we expect is the closure of the inner qu*-derivation δ_h for $h \in \mathfrak{A}_{\mathrm{b}}$. Indeed, it is straightforward to prove that $\frac{d}{dt}_{|t=0} \mathsf{e}^{ith} = ih$, so

$$\lim_{t\to 0} \frac{\beta_t(a) - a}{t} = \lim_{t\to 0} \frac{\mathsf{e}^{ith} \Box a \Box \mathsf{e}^{-ith} - a}{t} = ih \Box a - ia \Box h = \overline{\delta}_h(a)$$

for every $a \in \mathfrak{A} = \mathcal{D}(\overline{\delta}_h)$. Therefore $\overline{\delta}_h$ is everywhere defined and continuous.

Remark 3.4.1 Note that $\overline{\delta}_h$ is everywhere defined, that is $\mathcal{D}(\overline{\delta}_h) \not\subset \mathfrak{A}_{\mathrm{b}}$. Hence the hypothesis on the boundedness of $\mathcal{D}(\delta)$ is sufficient, but not necessary, to obtain a uniformly bounded norm continuous one-parameter group of weak *-automorphisms.

Case 2: We now consider the case in which h is self-adjoint, as before, but *unbounded*, i.e. $h \in \mathfrak{A} \setminus \mathfrak{A}_{\mathrm{b}}$.

It is easy to check that $\lambda \in \rho(h)$ if, and only if, $\lambda \in \rho(\overline{L}_h) \cap \rho(\overline{R}_h)$. We suppose that the element h verifies the following condition

$$\|(h + i\gamma)^{-1}\|_{\mathrm{b}} \leq \frac{1}{|\gamma|}, \quad \gamma \in \mathbb{R}.$$

This, in turn, implies that

$$\|(\overline{L}_h + i\gamma I)^{-1}\|_{\mathcal{B}(\mathfrak{A})} \leq \frac{1}{|\gamma|}, \quad \gamma \in \mathbb{R}$$
$$\|(\overline{R}_h + i\gamma I)^{-1}\|_{\mathcal{B}(\mathfrak{A})} \leq \frac{1}{|\gamma|}, \quad \gamma \in \mathbb{R}.$$

For every $t \in \mathbb{R}$, $it \in \rho(h)$ implies $it \in \rho(\overline{L}_h) \cap \rho(\overline{R}_h)$. Then there exists $\{U_L(t)\}_{t\in\mathbb{R}}$ strongly operator continuous one-parameter group such that $\|U_L(t)\|_{\mathcal{B}(\mathfrak{A})} \leq 1$ for every $t \in \mathbb{R}$ and $\overline{L}_h$ is the infinitesimal generator of $\{U_L(t)\}_{t\in\mathbb{R}}$ (see [48]).

In the same way, there exists a strongly continuous one-parameter group $\{U_R(t)\}_{t\in\mathbb{R}}$ such that $\|U_R(t)\|_{\mathcal{B}(\mathfrak{A})} \leq 1$ for every $t \in \mathbb{R}$ and $\overline{R}_h$ is the infinitesimal generator of $\{U_R(t)\}_{t\in\mathbb{R}}$

Let us define

$$u_L(t) := U_L(t)(\mathbb{1}) \quad \text{and} \quad u_R(t) := U_R(t)(\mathbb{1}).$$

Since both $u_L(t)$ and u_R are solution of the differential equation $\frac{du}{dt} = ihu$ with boundary condition $u(0) = \mathbb{1}$ in $\mathfrak{A}$, $u_L(t) = u_R(t)$ for every $t \in \mathbb{R}$.

Hence we define

$$\mathsf{e}^{ith} := u_L(t) = u_R(t), \quad t \in \mathbb{R}.$$

The exponential is a bounded element of $(\mathfrak{A}, \mathfrak{A}_0)$. Indeed, by [70, Lemma 2.5.3], it is easy to check that

$$\left(I - \frac{it}{n}\overline{L}_h\right)^{-n}(x) = \left(\left(I - \frac{it}{n}\overline{L}_h\right)^{-n}\mathbb{1}\right)x, \quad \forall x \in \mathfrak{A}_0.$$

The element $(I - \frac{it}{n}\overline{L}_h)^{-n}\mathbb{1}$ is *left-bounded* with the bound not depending neither on n nor on the element $\mathbb{1}$. Therefore it is possible to extend the above equality for generic elements in $\mathfrak{A}$

$$\left(I - \frac{it}{n}\overline{L}_h\right)^{-n}(a) = \left(\left(I - \frac{it}{n}\overline{L}_h\right)^{-n}\mathbb{1}\right)\Box a, \quad \forall a \in \mathfrak{A}.$$

Hence, by the strong continuity of $U_L(t)$, we achieve

$$U_L(t)a := \lim_{n\to\infty}\left(I - \frac{it}{n}\overline{L}_h\right)^{-n}a = \lim_{n\to\infty}\left(\left(I - \frac{it}{n}\overline{L}_h\right)^{-n}\mathbb{1}\right)\Box a$$

and $\|U_L(t)a\| \leq \|a\|$ for every $a \in \mathfrak{A}$.

Analogously, $U_R(t)\mathbb{1}$ is *right-bounded*, $U_R(t)a = U_R(t)\mathbb{1}\Box a$ for every $a \in \mathfrak{A}$ and $\|U_R(t)a\| \leq \|a\|$. Then we conclude that e^{ith} is *bounded* and $\|\mathsf{e}^{ith}\|_{\mathrm{b}} \leq 1$.

By the previous properties, we obtain $\mathsf{e}^{ith}\Box\mathsf{e}^{ish} = \mathsf{e}^{i(t+s)h}$ for every $t, s \in \mathbb{R}$, i.e. the group property for $\{\beta_t\}_{t\in\mathbb{R}}$.

Since

$$\left(\left(I - \frac{it}{n}\overline{L}_k\right)^{-1}(a)\right)^* = \left(1 + \frac{it}{n}\overline{R}_h\right)^{-1}(a^*), \quad \forall a \in \mathfrak{A},$$

$u_L(t)^* = u_r(-t) = u_L(-t)$. Then $\|\mathsf{e}^{ith}\|_{\mathrm{b}} = 1$.

Defining $\beta_t(a) := \mathsf{e}^{ith}\Box a\Box\mathsf{e}^{-ith}$, we already know that $\{\beta_t\}_{t\in\mathbb{R}}$ is uniformly bounded in t τ_n-continuous one-parameter group of continuous weak *-automorphisms. The infinitesimal generator is given by the weak *-derivation

$$\overline{\delta}_h(a) = \lim_{t\to 0}\frac{\beta_t(a) - a}{t} = \frac{\mathsf{e}^{ith}\Box a\Box\mathsf{e}^{-ith} - a}{t} = i(h\Box a - a\Box h)$$

when a is *bounded*.

3.4.2 A physical example: quantum lattice systems

The study of derivations and automorphisms is important for physical applications to quantum systems with infinitely many degrees of freedom as, for instance, spin lattice systems. Without giving full details (for which we refer to [9, 14, 19, 64]) we give an outline of their mathematical description and show how the ideas developed here may give some help when dealing with them.

Let V is a finite region of a d-dimensional lattice and $\mathfrak{A}_V$ the C*-algebra generated by the Pauli operators $\vec{\sigma}_p = (\sigma_p^1, \sigma_p^2, \sigma_p^3)$ at each point p of the finite region V (the number of points of V is indicated by $|V|$) and by the identity matrix $I_p \in M_2(\mathbb{C})$. It is easy to show that $\mathfrak{A}_V$ is isomorphic to $M_{2^{|V|}}(\mathcal{H}_V)$, where $\mathcal{H}_V = \otimes_{p\in V}\mathbb{C}_p^2$, and $\mathbb{C}_p^2$ is the 2-dimensional space at $p \in V$.

If $V \subset V'$, then there exists a natural embedding $\mathfrak{A}_V \hookrightarrow \mathfrak{A}_{V'}$, defined in obvious way. Hence $\mathfrak{A}_0 := \cup_V\mathfrak{A}_V$ is a C*-algebra, called the C*-algebra of local observables; its norm is denoted by $\|\cdot\|_0$.

To any infinite sequence $\{n\} = \{n_i\}_{i=1}^{\infty}$ of unit vectors in $\mathbb{R}^3$ there corresponds a state $|\{n\}\rangle$, constructed as in [9, Section 11.3.1]. This state determines (GNS construction) a *-representation of $\mathfrak{A}_0$ defined on the domain $\mathcal{D}^0_{\{n\}} = \mathfrak{A}_0|\{n\}\rangle$ whose completion is denoted by $\mathcal{H}_{\{n\}}$. Then one can define a family of vectors

$$\left\{ |\{m\},\{n\}\rangle = \otimes_p |m_p, n_p\rangle; m_p = 0, 1, \sum_p m_p < \infty \right\}$$

which constitutes an orthonormal basis of $\mathcal{H}_{\{n\}}$. Each vector $(|\{m\},\{n\}\rangle$ is obtained by flipping a finite number of spins in the *ground state* $|\{n\}\rangle$.

Then, an unbounded self-adjoint operator M acting on $\mathcal{H}_{\{n\}}$ is defined by

$$M|\{m\},\{n\}\rangle = \left(\sum_p m_p\right)|\{m\},\{n\}\rangle.$$

Roughly speaking, M counts the number of flipped spins in $|\{m\},\{n\}\rangle$ with respect to the ground state $|\{n\}\rangle$.

Note that M is strictly depending on the chosen sequence $\{n\}$. We set $\pi_{\{n\}} : \mathfrak{A}_0 \to \mathcal{L}^\dagger(\mathcal{D}_{\{n\}})$ to be the GNS *-representation defined by $\{n\}$ and we suppose that $\pi_{\{n\}}$ is faithful. The operator M is a number operator. Therefore, the operator e^M is a densely defined self-adjoint operator. Let $\mathcal{D}$ denote its domain. Then $\mathcal{D}$ can be made into a Hilbert space, denoted by $\mathcal{H}_M$, in canonical way. The norm in $\mathcal{H}_M$ is given by $\|f\|_M = \|\mathsf{e}^M f\|$, $f \in \mathcal{H}_M$.

Let us assume that both $\|\mathsf{e}^M \pi_{\{n\}}(x)\mathsf{e}^{-M}\|_0$ and $\|\mathsf{e}^{-M}\pi_{\{n\}}(x)\mathsf{e}^M\|_0$ are finite for every $x \in \mathfrak{A}_0$. Then the completion $\mathfrak{A}$ of $\mathfrak{A}_0$ with respect to the norm

$$\|x\| := \|\mathsf{e}^{-M}\pi_{\{n\}}(x)\mathsf{e}^{-M}\|$$

is a *-semisimple Banach quasi*-algebra. Assuming that h_V is the Hamiltonian of the finite volume system, then $h_V \in \mathfrak{A}_0$ and then $\mathsf{e}^{ith_V} \in \mathfrak{A}_0$. Now we define

$$\delta_V(x) := i[h_V, x] \quad \forall x \in \mathfrak{A}_0.$$

By Proposition 3.2.1, δ_V is closable. Moreover, $h_V \in \mathfrak{A}_0$, thus δ_V is actually continuous. Hence δ_V is infinitesimal generator of uniformly bounded norm continuous one parameter group of norm continuous weak *-automorphisms

$$\alpha_t^V(a) = \mathsf{e}^{ith_V} a \mathsf{e}^{-ith_V} \quad \forall a \in \mathfrak{A}.$$

The interesting point comes when considering the so-called thermodynamical limit of the local dynamics; i.e. the $\lim_{|V|\to\infty} \delta_V$. This limit, in general, fails to exists in the C*-algebra topology of $\mathfrak{A}_0$. It is here that the Banach quasi *-algebra structure plays a role, by taking the completion with respect to the norm $\|\cdot\|$ of $\mathfrak{A}$. As shown in [9, 19], under certain conditions, this limit exists and defines a weak *-derivation δ of $(\mathfrak{A}, \mathfrak{A}_0)$ which generates a one parameter group of *-automorphisms.

Chapter 4

Tensor products of Banach quasi *-algebras

In this chapter, we are interested in studying the structure of tensor products constructed from two Banach quasi *-algebras $(\mathfrak{A}, \mathfrak{A}_0)$ and $(\mathfrak{B}, \mathfrak{B}_0)$ having certain properties. We want the algebraic tensor product to be again a Banach quasi *-algebra carrying structural properties of the factors, especially those related to *-representations and representable functionals.

Our first step consists of finding the best choice of algebraic tensor product of two quasi *-algebras (see [29, 45]). We will use the fact that if $\mathfrak{A}$ is a quasi *-algebra over $\mathfrak{A}_0$, $\mathfrak{A}$ can be seen as a bimodule over $\mathfrak{A}_0$.

Once we have the algebraic tensor product, we should decide which topology has to be furnished to the tensor product in the way the resulting tensor product would be a normed quasi *-algebra. For further details on topological tensor products, see [31, 34, 39, 52, 54, 60, 63]..

After having a good notion of tensor product Banach quasi *-algebra, we investigate the existence and the relation between representations of a tensor product Banach quasi *-algebra and those of the tensor factors.

4.1 Algebraic construction

Let $(\mathfrak{A}, \mathfrak{A}_0)$ and $(\mathfrak{B}, \mathfrak{B}_0)$ be given quasi *-algebras. By definition, $\mathfrak{A}$ and $\mathfrak{B}$ are bimodules over $\mathfrak{A}_0$ and $\mathfrak{B}_0$ respectively. $\mathfrak{A}_0$ and $\mathfrak{B}_0$ are *-algebras, hence in particular they are rings. Thus, if R is a ring, X and Y are right and left R-modules respectively, the tensor product $X \otimes_R Y$ is well defined and it is a uniquely defined R-module.

In our case, $\mathfrak{A}$ and $\mathfrak{B}$ are bimodules over different rings $\mathfrak{A}_0$ and $\mathfrak{B}_0$, that have no relations a priori.

For this reason, we consider the direct sum $\mathfrak{A}_0 \oplus \mathfrak{B}_0$. It is still a ring and it is possible to extend the action of $\mathfrak{A}_0$ and $\mathfrak{B}_0$ on $\mathfrak{A}$ and $\mathfrak{B}$ respectively to an action of the direct sum, in the following way

$$\begin{aligned} (x,y)\cdot a &:= xa \quad \text{and} \quad a\cdot(x,y) := ax \\ (x,y)\cdot b &:= yb \quad \text{and} \quad b\cdot(x,y) := by, \end{aligned} \tag{4.1}$$

for every $(x,y) \in \mathfrak{A}_0 \oplus \mathfrak{B}_0$, $a \in \mathfrak{A}$ and $b \in \mathfrak{B}$. It is straightforward to show that $\mathfrak{A}$ and $\mathfrak{B}$ are bimodules over $\mathfrak{A}_0 \oplus \mathfrak{B}_0$.

Now, $\mathfrak{A}$ and $\mathfrak{B}$ are bimodules over the same ring $\mathfrak{A}_0 \oplus \mathfrak{B}_0$. To construct their tensor product and have again a bimodule over the same ring, we need the concept of a *balanced bilinear map* (see [42, p. 104, Definition 3.1]).

Definition 4.1.1 If X, Y are right and left modules over a ring R and E is a vector space, a bilinear map $\Psi : X \times Y \to E$ is called *balanced*, if

$$\Psi(x\cdot r, y) = \Psi(x, r\cdot y), \quad \forall\, x \in X,\ y \in Y,\ r \in R.$$

In this respect, [42, p. 104, Definition 3.2], the pair (G, Φ), with G a vector space and $\Phi : X \times Y \to G$ a balanced bilinear map, is a tensor product of X and Y, if the following condition is valid:

the pair (G, Φ) has the universal property, with respect to all balanced bilinear maps from $X \times Y$ in some vector space. (4.2)

In particular, the pair (G, Ψ) exists and it is unique, in the sense that if (G', Ψ') is another tensor product of the modules X, Y as before, then there is an algebraic isomorphism $i : G \to G'$, such that $i \circ \Psi = \Psi'$ (see again [42, pp. 104, 105, Theorems 3.3, 3.5, respectively]). The resulting module tensor product of X, Y will be denoted by $X \otimes_R Y$.

In our case, if $(\mathfrak{A}, \mathfrak{A}_0), (\mathfrak{B}, \mathfrak{B}_0)$ are given quasi *-algebras, then $\mathfrak{A} \otimes_{\mathfrak{A}_0 \oplus \mathfrak{B}_0} \mathfrak{B}$ is a bimodule over the ring (*-algebra) $\mathfrak{A}_0 \oplus \mathfrak{B}_0$.

The tensor map $\Phi : \mathfrak{A} \times \mathfrak{B} \to \mathfrak{A} \otimes_{\mathfrak{A}_0 \oplus \mathfrak{B}_0} \mathfrak{B}$, as well as its restriction $\Phi_0 : \mathfrak{A}_0 \times \mathfrak{B}_0 \to \mathfrak{A}_0 \otimes \mathfrak{B}_0$, is a balanced bilinear map, therefore (see also (4.1))

$$\begin{aligned} \Phi(a\cdot(x,y), b) &= \Phi(a, (x,y)\cdot b) \\ &\Leftrightarrow\ \Phi(ax, b) = \Phi(a, yb) \\ &\Leftrightarrow\ ax \otimes b = a \otimes yb, \end{aligned} \tag{4.3}$$

for all (x,y) in $\mathfrak{A}_0 \oplus \mathfrak{B}_0$ and (a,b) in $\mathfrak{A} \times \mathfrak{B}$. Similarly, one obtains

$$xa \otimes b = a \otimes by, \quad \forall\, (x,y) \in \mathfrak{A}_0 \oplus \mathfrak{B}_0,\ (a,b) \in \mathfrak{A} \times \mathfrak{B}. \tag{4.4}$$

The algebraic tensor product $\mathfrak{A}_0 \otimes \mathfrak{B}_0$ is a *-algebra. Indeed, an arbitrary element z in $\mathfrak{A}_0 \otimes \mathfrak{B}_0$ has the form $z = \sum_{i=1}^{n} x_i \otimes y_i$, where $x_i \otimes y_i$ are elementary tensors for $i = 1, \dots, n$. Let $z' = \sum_{j=1}^{m} x_j \otimes y_j$ be another arbitrary element in $\mathfrak{A}_0 \otimes \mathfrak{B}_0$ and set

$$zz' := \sum_{i=1}^{n} \sum_{j=1}^{m} x_i x_j' \otimes y_i y_j'.$$

Then, zz' is a well defined (associative) product on $\mathfrak{A}_0 \otimes \mathfrak{B}_0$. The involution is defined by $(x \otimes y)^* = x^* \otimes y^*$, for all $(x, y) \in \mathfrak{A}_0 \times \mathfrak{B}_0$.

$\mathfrak{A}$ and $\mathfrak{B}$ are complex vector spaces, hence the tensor product $\mathfrak{A} \otimes \mathfrak{B}$ carries a natural structure of complex vector space defining the scalar multiplication in the following way

$$\lambda \cdot z := \lambda \sum_{k=1}^{n} a_k \otimes b_k = \sum_{k=1}^{n} a_k \otimes \lambda b_k,$$

for every $\lambda \in \mathbb{C}$ and every $z = \sum_{k=1}^{n} a_k \otimes b_k \in \mathfrak{A} \otimes \mathfrak{B}$. Note that, if $(\mathfrak{A}, \mathfrak{A}_0)$ and $(\mathfrak{B}, \mathfrak{B}_0)$ are unital, the linear space structure is automatically given identifying $\mathbb{C}$ with $\mathbb{C}\mathbb{1}_{\mathfrak{A}_0 \oplus \mathfrak{B}_0}$.

Since $\mathfrak{A}$, $\mathfrak{B}$ carry an involution extending the involutions of $\mathfrak{A}_0$, $\mathfrak{B}_0$ respectively, an involution is also defined on $\mathfrak{A} \otimes_{\mathfrak{A}_0 \oplus \mathfrak{B}_0} \mathfrak{B}$, in a similar way as before, extending the involution of $\mathfrak{A}_0 \otimes \mathfrak{B}_0$. On the other hand, the module multiplications in $\mathfrak{A} \otimes_{\mathfrak{A}_0 \oplus \mathfrak{B}_0} \mathfrak{B}$, defined by

$$(x \otimes y)(a \otimes b) := xa \otimes yb, \quad \text{resp.} \quad (a \otimes b)(x \otimes y) := ax \otimes by,$$

for all (x, y) in $\mathfrak{A}_0 \times \mathfrak{B}_0$ and (a, b) in $\mathfrak{A} \times \mathfrak{B}$, satisfy the requirements of Definition 1.1.3. Hence, $(\mathfrak{A} \otimes_{\mathfrak{A}_0 \oplus \mathfrak{B}_0} \mathfrak{B}, \mathfrak{A}_0 \otimes \mathfrak{B}_0)$ *is a quasi *-algebra.*

Proposition 4.1.2 *Let* $(\mathfrak{A}, \mathfrak{A}_0)$ *and* $(\mathfrak{B}, \mathfrak{B}_0)$ *be quasi *-algebras. If* $(\mathfrak{A}, \mathfrak{A}_0)$ *or* $(\mathfrak{B}, \mathfrak{B}_0)$ *is unital, then the quasi *-algebra* $(\mathfrak{A} \otimes_{\mathfrak{A}_0 \oplus \mathfrak{B}_0} \mathfrak{B}, \mathfrak{A}_0 \otimes \mathfrak{B}_0)$ *is trivial.*

Proof. Let $a \otimes b$ be an arbitrary elementary tensor in $\mathfrak{A} \otimes \mathfrak{B}$ and let $(\mathfrak{A}, \mathfrak{A}_0)$ have an identity $\mathbb{1}$. Then, we have (see also (4.1) and (4.3))

$$a \otimes b = \mathbb{1}a \otimes b = [(\mathbb{1}, 0) \cdot a] \otimes b = a \otimes [b \cdot (\mathbb{1}, 0)] = a \otimes b0 = 0. \quad \square$$

Summing up, the construction of the quasi *-algebra $(\mathfrak{A} \otimes_{\mathfrak{A}_0 \oplus \mathfrak{B}_0} \mathfrak{B}, \mathfrak{A}_0 \otimes \mathfrak{B}_0)$ has an interest, only in the case, where $(\mathfrak{A}, \mathfrak{A}_0)$ and $(\mathfrak{B}, \mathfrak{B}_0)$ carry no identity elements.

It may happen that we are given two quasi *-algebras $(\mathfrak{A}, \mathfrak{A}_0), (\mathfrak{B}, \mathfrak{B}_0)$, with $\mathfrak{A}_0 = \mathfrak{B}_0$. This time $\mathfrak{A}$ and $\mathfrak{B}$ are bimodules over the same ring $\mathfrak{A}_0$. Hence $(\mathfrak{A} \otimes_{\mathfrak{A}_0} \mathfrak{B}, \mathfrak{A}_0 \otimes \mathfrak{B}_0)$ is constructed as before, without considering the direct sum of the rings. The left and right actions of $\mathfrak{A}_0$ on $\mathfrak{A} \otimes_{\mathfrak{A}_0} \mathfrak{B}$ are defined as follows (see, e.g., (4.3), (4.4))

$$x \cdot (a \otimes b) := xa \otimes b = a \otimes xb, \quad \text{resp.} \quad (a \otimes b) \cdot x = ax \otimes b = a \otimes bx,$$

for all x in $\mathfrak{A}_0$ and (a, b) in $\mathfrak{A} \times \mathfrak{B}$.

In this case, even if $(\mathfrak{A}, \mathfrak{A}_0)$ is unital, $\mathfrak{A} \otimes \mathfrak{B}$ is not trivial, in general. Take, for instance, the tensor product of the quasi *-algebra $(L^p(I, d\lambda), \mathcal{C}(I))$ with itself, where p is fixed such that $p \geq 1$, $I = [0, 1]$ and λ is the Lebesgue measure. The tensor product quasi *-algebra obtained is $(L^p(I) \otimes_{\mathcal{C}(I)} L^p(I), \mathcal{C}(I))$ and it is not trivial because it contains $L^p(I, d\lambda) \simeq \mathcal{C}(I) \otimes_{\mathcal{C}(I)} L^p(I, d\lambda)$.

In the sequel, we consider the more general case, where $\mathfrak{A}_0$ is embedded in $\mathfrak{B}_0$, but not equal to it. This will be denoted by $\mathfrak{A}_0 \hookrightarrow \mathfrak{B}_0$. Such an example is evidently provided by the quasi *-algebras $(L^p(I, d\lambda), \mathcal{C}(I))$, $(L^p(I, d\lambda), L^\infty(I, d\lambda))$. Then, it is possible to build the tensor product quasi *-algebra $(\mathfrak{A} \otimes_{\mathfrak{A}_0} \mathfrak{B}, \mathfrak{A}_0 \otimes_{\mathfrak{A}_0} \mathfrak{B}_0)$ in a similar way as before, because $\mathfrak{B}$ is also a bimodule over $\mathfrak{A}_0$. In this case $\mathfrak{A}_0 \otimes_{\mathfrak{A}_0} \mathfrak{B}_0 \simeq \mathfrak{B}_0$, hence we obtained another quasi *-algebra over the same *-algebra $\mathfrak{B}_0$.

Note that $\mathfrak{A} \otimes_{\mathfrak{B}_0} \mathfrak{B}$ may not be defined. The crucial point here is that $\mathfrak{A}$ is not necessarily a $\mathfrak{B}_0$-module. In this case, we may construct a $\mathfrak{B}_0$-module on which $\mathfrak{A}$ is embedded, i.e., we may construct the tensor product of $\mathfrak{B}_0$ and $\mathfrak{A}$ over $\mathfrak{A}_0$.

Having now two $\mathfrak{B}_0$-bimodules, $\mathfrak{B}_0 \otimes_{\mathfrak{A}_0} \mathfrak{A}$ and $\mathfrak{B}$, we build the tensor product quasi *-algebra $((\mathfrak{B}_0 \otimes_{\mathfrak{A}_0} \mathfrak{A}) \otimes_{\mathfrak{B}_0} \mathfrak{B}, (\mathfrak{B}_0 \otimes_{\mathfrak{A}_0} \mathfrak{A}_0) \otimes_{\mathfrak{B}_0} \mathfrak{B}_0)$. Using known tensor product properties, we have

$$\begin{aligned}(\mathfrak{B}_0 \otimes_{\mathfrak{A}_0} \mathfrak{A}) \otimes_{\mathfrak{B}_0} \mathfrak{B} &= \mathfrak{B} \otimes_{\mathfrak{B}_0} (\mathfrak{B}_0 \otimes_{\mathfrak{A}_0} \mathfrak{A}) \\ &= (\mathfrak{B} \otimes_{\mathfrak{B}_0} \mathfrak{B}_0) \otimes_{\mathfrak{A}_0} \mathfrak{A} = \mathfrak{B} \otimes_{\mathfrak{A}_0} \mathfrak{A}\end{aligned}$$

The previous computations show that if we have a sequence of embeddings of *-algebras and we construct tensor products through the extensions of 'scalars', what we get at the end, it is always the tensor product built on the smallest ring.

4.2 *-Admissible topologies

Prior to investigating the topological structure of the tensor product normed (resp. Banach) quasi *-algebra, we give some notions about the *admissible* topologies on the algebraic tensor product quasi *-algebra.

Suppose now that $A[\tau_A]$, $B[\tau_B]$ are locally convex spaces and $A \otimes B$ their vector space tensor product. Denote with A^* and B^* the topological dual of A and B respectively.

Definition 4.2.1 A topology τ on $A \otimes B$ is called *compatible* [39] (with the tensor product vector space structure of $A \otimes B$) *topology* on $A \otimes B$ if the following conditions are satisfied

(1) The vector space $A \otimes B$ equipped with τ is a locally convex space, that will be denoted by $A \otimes^\tau B$;

(2) The tensor map $\Phi : A \times B \to A \otimes^\tau B : (x, y) \mapsto x \otimes y$ is separately continuous;

(3) For any equicontinuous subset M of A^* and N of B^*, the set $M \otimes N$ given by $\{x' \otimes y' : x' \in M, y' \in N\}$ is an equicontinuous subset of $\big(A \otimes^\tau B\big)^*$.

The completion of $A \otimes^\tau B$ is denoted by $A \widehat{\otimes}^\tau B$.

Let now $A[\| \cdot \|_A]$, $B[[\| \cdot \|_B]$ be Banach spaces. If a norm $\| \cdot \|$ on the tensor product space $A \otimes B$ satisfies the equality

$$\|x_1 \otimes x_2\| = \|x_1\|_A \|x_2\|_B, \quad \forall\, x_1 \in A,\ x_2 \in B, \tag{4.5}$$

is called a *cross-norm* on $A \otimes B$.

- **The injective cross-norm on** $A \otimes B$

Taking an arbitrary element $z = \sum_{i=1}^n x_i \otimes y_i$ in $A \otimes B$, we put

$$\|z\|_\lambda = \sup \left\{ \left| \sum_{i=1}^n f(x_i) g(y_i) \right| : f \in A^*, \|f\| \leq 1; g \in B^*, \|g\| \leq 1 \right\}. \tag{4.6}$$

The function $\| \cdot \|_\lambda$ is a well-defined cross-norm on $A \otimes B$, called the *injective cross-norm.* It fulfills Definition 4.2.1 and it is the least cross-norm on $A \otimes B$. The normed space induced by $A \otimes B[\| \cdot \|_\lambda]$, will be denoted as $A \otimes^\lambda B$; its respective completion, which is a Banach space, will be denoted by $A \widehat{\otimes}^\lambda B$.

When $\mathfrak{A}[\tau_\mathfrak{A}], \mathfrak{B}[\tau_\mathfrak{B}]$ are given locally convex *-algebras (in this case we shall always assume that involution is continuous and the multiplication is separately continuous), then Definition 4.2.1 can be modified as follows

Definition 4.2.2 [34] Let $\mathfrak{A}[\tau_\mathfrak{A}], \mathfrak{B}[\tau_\mathfrak{B}]$ be as before, with $\tau_\mathfrak{A}, \tau_\mathfrak{B}$ respectively defined by upwards directed families of seminorms $\{p\}$ and $\{q\}$. Let $\mathfrak{A} \otimes \mathfrak{B}$ be their corresponding tensor product *-algebra. A topology τ on $\mathfrak{A} \otimes \mathfrak{B}$ is called $*$-*admissible* (that is, compatible with the tensor product *-algebra structure of $\mathfrak{A} \otimes \mathfrak{B}$), if the following conditions are satisfied

(1) $\mathfrak{A} \otimes \mathfrak{B}$ endowed with τ is a locally convex $*$–algebra, denoted by $\mathfrak{A} \otimes^\tau \mathfrak{B}$;

(2) The tensor map $\Phi : \mathfrak{A} \times \mathfrak{B} \to \mathfrak{A} \otimes^\tau \mathfrak{B}$ is continuous, in the sense that if τ is determined by the family of $*$–seminorms $\{r\}$, then for every r there exist p, q, such that $r(x \otimes y) \leq p(x)q(y)$ for all $(x, y) \in \mathfrak{A} \times \mathfrak{B}$;

(3) For any equicontinuous subsets M of $\mathfrak{A}^*$ and N of $\mathfrak{B}^*$, the set $M \otimes N$ given by $\{x^* \otimes y^* : x^* \in M, y^* \in N\}$ is an equicontinuous subset of $\left(\mathfrak{A} \otimes^\tau \mathfrak{B}\right)^*$.

The completion of $\mathfrak{A} \otimes^\tau \mathfrak{B}$ is a complete locally convex *-algebra denoted by $\mathfrak{A} \widehat{\otimes}^\tau \mathfrak{B}$.

Let us now assume that $\mathfrak{A}[\|\cdot\|_{\mathfrak{A}}]$, $\mathfrak{B}[[\|\cdot\|_{\mathfrak{B}}]$ are normed *-algebras with isometric involution. We shall define on the tensor product *-algebra $\mathfrak{A} \otimes \mathfrak{B}$ the projective cross-norm (see, for instance, [63, p. 189] and [54]).

- **The projective cross-norm on $\mathfrak{A} \otimes \mathfrak{B}$**

Let $z = \sum_{i=1}^n x_i \otimes y_i$ be an arbitrary element in $\mathfrak{A} \otimes \mathfrak{B}$. Put

$$\|z\|_\gamma = \inf \left\{ \sum_{i=1}^n \|x_i\|_1 \|y_i\|_2 \right\} \tag{4.7}$$

where the infimum is taken over all representations $\sum_{i=1}^n x_i \otimes y_i$ of z. The function $\|\cdot\|_\gamma$ is a well-defined cross-norm that majorizes all other cross-norms on $\mathfrak{A} \otimes \mathfrak{B}$ and it is called the *projective cross-norm.* The normed *-algebra induced by $\mathfrak{A} \otimes \mathfrak{B}[\| \cdot \|_\gamma]$, will be denoted as $\mathfrak{A} \otimes^\gamma \mathfrak{B}$ and its respective completion, which is a Banach *-algebra, will be denoted by $\mathfrak{A} \widehat{\otimes}^\gamma \mathfrak{B}$. Note that the cross-norm $\| \cdot \|_\gamma$ satisfies Definition 4.2.2, therefore is a *-admissible cross-norm.

In particular, *any compatible cross-norm* $\|\cdot\|$ *on* $\mathfrak{A} \otimes \mathfrak{B}$ *lies between the injective and projective cross-norm*, i.e.,

$$\| \cdot \|_\lambda \;\leq\; \| \cdot \| \;\leq\; \| \cdot \|_\gamma. \tag{4.8}$$

Even more, *a cross-norm* $\| \cdot \|$ *on* $A \otimes B$ *is compatible, if and only if, the inequality* (4.8) *is valid.*

- **The maximal C*-cross-norm**

Let $\mathfrak{A}[\| \cdot \|_{\mathfrak{A}}]$, $\mathfrak{B}[\| \cdot \|_{\mathfrak{B}}]$ be two C*-algebras. In the usual way (as above), $\mathfrak{A} \otimes \mathfrak{B}$ becomes a *-algebra.

Given a C*-algebra $\mathfrak{A}$, denote by $\mathcal{R}(\mathfrak{A})$ the set of all *-representations of $\mathfrak{A}$ on a Hilbert space $\mathcal{H}$. Keep the same symbol for all *-representations of the *-algebra $\mathfrak{A} \otimes \mathfrak{B}$, on a Hilbert space $\mathcal{H}$ (see [63, Lemma 4.1]). For further details on the definitions that follow, see [63].

The *projective or maximal C*-cross-norm* on $\mathfrak{A} \otimes \mathfrak{B}$, denoted by $\| \cdot \|_{\max}$, is defined as

$$\|z\|_{\max} = \sup \left\{ \|\pi(z)\| : \pi \in \mathcal{R}(\mathfrak{A} \widehat{\otimes}^\gamma \mathfrak{B}) \right\}, \quad z \in \mathfrak{A} \otimes \mathfrak{B} \tag{4.9}$$

Thecompletion of $\mathfrak{A} \otimes \mathfrak{B}[\|\cdot\|_{\max}]$ is a C*-algebra denoted by $\mathfrak{A} \underset{\max}{\widehat{\otimes}} \mathfrak{B}$.

From (4.9) and automatic continuity of representations on a C*-algebra (see Theorem A.3.3), we obtain

$$\|z\|_{\max} \ \leq \ \|z\|_\gamma, \quad z \in \mathfrak{A} \otimes \mathfrak{B}. \tag{4.10}$$

- **The minimal C*-cross-norm**

If π_1, π_2 are *-representations of $\mathfrak{A}, \mathfrak{B}$ acting on the Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$ respectively, then there is a unique *-representation π of $\mathfrak{A} \otimes \mathfrak{B}$, acting on the Hilbert space tensor product of $\mathcal{H}_1, \mathcal{H}_2$, denoted by $\mathcal{H}_1 \widehat{\otimes} \mathcal{H}_2$ (see [60] and/or [34]). The *-representation π is defined by

$$\pi(z) = \sum_{i=1}^{n} \pi_1(x_i) \otimes \pi_2(y_i), \quad \forall\ z = \sum_{i=1}^{n} x_i \otimes y_i \in \mathfrak{A} \otimes \mathfrak{B}.$$

The *injective or minimal cross-norm* on $\mathfrak{A} \otimes \mathfrak{B}$ is denoted by $\|\cdot\|_{\min}$ and given as

$$\|z\|_{\min} = \sup\left\{\|(\pi_1 \otimes \pi_2)(z)\| : \pi_1 \in \mathcal{R}(\mathfrak{A}), \pi_2 \in \mathcal{R}(\mathfrak{B}), z \in \mathfrak{A} \otimes \mathfrak{B}\right\}. \tag{4.11}$$

The C*-algebra completion of $\mathfrak{A} \otimes \mathfrak{B}[\|\cdot\|_{\min}]$ is denoted by $\mathfrak{A} \underset{\min}{\widehat{\otimes}} \mathfrak{B}$.

By the very definitions (4.9), (4.10) and (4.11), we have that

$$\|z\|_{\min} \leq \|z\|_{\max} \leq |\|z\|_\gamma, \quad z \in \mathfrak{A} \otimes \mathfrak{B}.$$

Now, taking into account that $\|\cdot\|_\gamma$ is a cross-norm, i.e, it fulfills (4.5), from the definition of $\|\cdot\|_{\min}$ above and the standard C*-algebra theory, we conclude that both $\|\cdot\|_{\min}$ and $\|\cdot\|_{\max}$ satisfy (4.5), therefore they are cross-norms in the sense that

$$\|x_1 \otimes x_2\|_{\min} = \|x_1\|_{\mathfrak{A}}\|x_2\|_{\mathfrak{B}} = \|x_1 \otimes x_2\|_{\max}, \quad (x_1, x_2) \in \mathfrak{A} \times \mathfrak{B}.$$

On the other hand (see [63]), $\|z\|_\lambda \leq \|z\|_{\min}$, z in $\mathfrak{A} \otimes \mathfrak{B}$, therefore we finally obtain that

$$\|z\|_\lambda \leq \|z\|_{\min} \leq \|z\|_{\max} \leq \|z\|_\gamma, \quad z \in \mathfrak{A} \otimes \mathfrak{B}. \tag{4.12}$$

We observe that any C*-norm $\|\cdot\|$ on the *-algebra $\mathfrak{A} \otimes \mathfrak{B}$, is a cross-norm; this can be seen from the following relation (see [63, Theorem 4.19])

$$\|x_1\|_{\mathfrak{A}}\|x_2\|_{\mathfrak{B}} = \|x_1 \otimes x_2\|_{\min} \leq \|x_1 \otimes x_2\| \leq \|x_1 \otimes x_2\|_{\max} = \|x_1\|_{\mathfrak{A}}\|x_2\|_{\mathfrak{B}}, \tag{4.13}$$

for all $(x_1, x_2) \in \mathfrak{A} \times \mathfrak{B}$.

4.3 Topological tensor product

Given two normed (resp. Banach) quasi *-algebras $(\mathfrak{A}, \mathfrak{A}_0), (\mathfrak{B}, \mathfrak{B}_0)$, with an embedding $\mathfrak{A}_0 \hookrightarrow \mathfrak{B}_0$, we will construct their tensor product normed (resp. Banach) quasi *-algebra.

We have already seen from the discussion in Section 4.1 that $(\mathfrak{A} \otimes_{\mathfrak{A}_0} \mathfrak{B}, \mathfrak{A}_0 \otimes_{\mathfrak{A}_0} \mathfrak{B}_0)$ is a quasi *-algebra. Hence, according to Definition 1.3.1, we still have to show that $\mathfrak{A} \otimes_{\mathfrak{A}_0} \mathfrak{B}$ becomes a normed (resp. Banach) space, under a tensor norm that fulfills the conditions of Definition 1.3.1.

We start looking at the injective cross-norm (4.6) on $\mathfrak{A} \otimes_{\mathfrak{A}_0} \mathfrak{B}$. We denote the respective normed space by $\mathfrak{A} \overset{\lambda}{\otimes}_{\mathfrak{A}_0} \mathfrak{B} := \mathfrak{A} \otimes_{\mathfrak{A}_0} \mathfrak{B}[\| \cdot \|_\lambda]$ and its completion by $\mathfrak{A} \widehat{\otimes}^{\lambda}_{\mathfrak{A}_0} \mathfrak{B}$, which clearly is a Banach space. For distinction, we denote by $\|\cdot\|_{\mathfrak{A}}$, $\|\cdot\|_{\mathfrak{B}}$, the given norms on $\mathfrak{A}$, $\mathfrak{B}$, respectively.

We prove that $\mathfrak{A}_0 \otimes \mathfrak{B}_0$ is dense in $\mathfrak{A} \overset{\lambda}{\otimes}_{\mathfrak{A}_0} \mathfrak{B}$. It suffices to show the existence of an approximating sequence in $\mathfrak{A}_0 \otimes \mathfrak{B}_0$ for elementary tensors and then extend by linearity.

By Definition 1.3.1, $\mathfrak{A}_0$ in dense in $\mathfrak{A}[\| \cdot \|_{\mathfrak{A}}]$ and $\mathfrak{B}_0$ in $\mathfrak{B}[\| \cdot \|_{\mathfrak{B}}]$. Thus, if a is in $\mathfrak{A}$ and b in $\mathfrak{B}$ there exist sequences $\{x_n\}$ in $\mathfrak{A}_0$ and $\{y_n\}$ in $\mathfrak{B}_0$, such that

$$\|x_n - a\|_{\mathfrak{A}} \to 0 \quad \text{and} \quad \|y_n - b\|_{\mathfrak{B}} \to 0.$$

We prove now that the sequence $\{x_n \otimes y_n\}$ in $\mathfrak{A}_0 \otimes \mathfrak{B}_0$ is $\| \cdot \|_\lambda$-converging to $a \otimes b$. Indeed: from (4.6), we have

$$\|x_n \otimes y_n - x_m \otimes y_m\|_\lambda = \sup \big\{ |f(x_n)g(y_n) - f(x_m)g(y_m)| : \\ f \in \mathfrak{A}^*, \|f\| \leq 1, g \in \mathfrak{B}^*, \|g\| \leq 1 \big\},$$

where

$$\begin{aligned} &|f(x_n)g(y_n) - f(x_m)g(y_m)| \\ &\qquad = |f(x_n)g(y_n) - f(x_n)g(y_m) + f(x_n)g(y_m) - f(x_m)g(y_m)| \\ &\qquad \leq |f(x_n)|\,|g(y_n - y_m)| + |f(x_n - x_m)|\,|g(y_m)| \\ &\qquad \leq \|x_m\|_{\mathfrak{A}}\,\|y_n - y_m\|_{\mathfrak{B}} + \|x_n - x_m\|_{\mathfrak{A}}\|y_m\|_{\mathfrak{B}} \\ &\qquad \leq M_1\|y_n - y_m\|_{\mathfrak{B}} + M_2\|x_n - x_m\|_{\mathfrak{A}} \to 0 \end{aligned}$$

for certain positive constants M_1, M_2 determined by the boundedness of the sequences $\{\|x_n\|_{\mathfrak{A}}\}$ and $\{\|y_n\|_{\mathfrak{B}}\}$. This shows that $\mathfrak{A}_0 \otimes \mathfrak{B}_0$ is dense in $\mathfrak{A} \overset{\lambda}{\otimes}_{\mathfrak{A}_0} \mathfrak{B}$.

Furthermore, by Definition 1.3.1 (ii), the (extended) involution on $\mathfrak{A}$ and $\mathfrak{B}$ is isometric, therefore it defines a continuous involution on $\mathfrak{A} \overset{\lambda}{\otimes}_{\mathfrak{A}_0} \mathfrak{B}$ and (by continuous extension) on $\mathfrak{A} \widehat{\otimes}^{\lambda}_{\mathfrak{A}_0} \mathfrak{B}$, which is isometric for elementary tensors, since $\| \cdot \|_\lambda$ is a cross-norm. Namely,

$$\|(a \otimes b)^*\|_\lambda = \|a^* \otimes b^*\|_\lambda = \|a^*\|\,\|b^*\| = \|a\|\,\|b\| = \|a \otimes b\|_\lambda,$$

for all a in $\mathfrak{A}$ and b in $\mathfrak{B}$. With a similar argument, we get the same conclusion for finite sums of elementary tensors.

It remains to show that for every $z = \sum_{i\in F} x_i \otimes y_i$ in $\mathfrak{A}_0 \otimes \mathfrak{B}_0$, F a finite subset in $\mathbb{N}$, the (right) multiplication operator

$$R_z : \mathfrak{A}\otimes^{\lambda}_{\mathfrak{A}_0}\mathfrak{B} \to \mathfrak{A}\otimes^{\lambda}_{\mathfrak{A}_0}\mathfrak{B} \quad \text{defined as} \quad c \mapsto cz$$

is continuous.

First observe that R_z is well defined and that $R_{x\otimes y} = R_x \otimes R_y$, for all (x, y) in $\mathfrak{A}_0 \times \mathfrak{B}_0$. From the fact that R_x, R_y are continuous and $\|\cdot\|_\lambda$ is a cross-norm, we deduce that the operator $R_{x\otimes y}$ is continuous. Indeed, without loss of generality, consider a sequence $\{x_n \otimes y_n\}$ made of elementary tensors in $\mathfrak{A}_0 \otimes \mathfrak{B}_0$ that is λ-vanishing. Thus,

$$\begin{aligned}\|x_n x \otimes y_n y\|_\lambda &= \|x_n x\|_{\mathfrak{A}} \|y_n y\|_{\mathfrak{B}} \leq \|x\|_0 \|x_n\|_{\mathfrak{A}} \|y\|_0 \|y_n\|_{\mathfrak{B}} \\ &= \|x\|_0 \|y\|_0 \|x_n \otimes y_n\|_\lambda \to 0.\end{aligned}$$

By linearity and continuity we deduce that for any z in $\mathfrak{A}_0 \otimes \mathfrak{B}_0$, R_z is continuous on $\mathfrak{A}\otimes^{\lambda}_{\mathfrak{A}_0}\mathfrak{B}$ and thus it is uniquely extended to a continuous operator on $\mathfrak{A}\widehat{\otimes}^{\lambda}_{\mathfrak{A}_0}\mathfrak{B}$ too. We conclude that $(\mathfrak{A}\widehat{\otimes}^{\lambda}_{\mathfrak{A}_0}\mathfrak{B}, \mathfrak{A}_0 \otimes \mathfrak{B}_0)$ *is a Banach quasi *-algebra.*

Concerning, $\mathfrak{A}\widehat{\otimes}^{\lambda}_{\mathfrak{A}_0}\mathfrak{B}$ we have

$$\mathfrak{A}_0 \otimes_{\mathfrak{A}_0} \mathfrak{B}_0 \hookrightarrow \mathfrak{A} \otimes_{\mathfrak{A}_0} \mathfrak{B} \quad \Rightarrow \quad \overline{\mathfrak{A}_0 \otimes_{\mathfrak{A}_0} \mathfrak{B}_0}^{\|\cdot\|_\lambda} = \mathfrak{A}\widehat{\otimes}^{\lambda}_{\mathfrak{A}_0}\mathfrak{B}$$

where $\overline{\mathfrak{A}_0 \otimes \mathfrak{B}_0}^{\|\cdot\|_\lambda}$ means the closure of $\mathfrak{A}_0 \otimes \mathfrak{B}_0$ with respect to $\|\cdot\|_\lambda$, whereas the arrow $\hookrightarrow$ indicates a dense embedding.

We conclude that the pair $(\mathfrak{A}\widehat{\otimes}^{\lambda}_{\mathfrak{A}_0}\mathfrak{B}, \mathfrak{A}_0 \otimes \mathfrak{B}_0)$ *is a Banach quasi*-algebra.*

All the preceding arguments, as well as the fixed notation, can be equally well applied for the projective cross-norm $\|\cdot\|_\gamma$ (see (4.7)) and all the generic cross norms, to give that the pair $(\mathfrak{A}\otimes^{\gamma}_{\mathfrak{A}_0}\mathfrak{B}, \mathfrak{A}_0 \otimes \mathfrak{B}_0)$ is a *normed quasi *-algebra*, while $(\mathfrak{A}\widehat{\otimes}^{\gamma}_{\mathfrak{A}_0}\mathfrak{B}, \mathfrak{A}_0 \otimes \mathfrak{B}_0)$ is a *Banach quasi *-algebra.*

4.4 Representations of tensor products quasi *-algebras

If $(\mathfrak{A}, \mathfrak{A}_0)$ and $(\mathfrak{B}, \mathfrak{B}_0)$ are normed quasi *-algebras, with $\mathfrak{A}_0$ embedded in $\mathfrak{B}_0$, a compatible tensor norm $\overline{n}$ on $\mathfrak{A} \otimes \mathfrak{B}$ that respects the involutive structure of $\mathfrak{A} \otimes^{\overline{n}}_{\mathfrak{A}_0} \mathfrak{B}$, i.e., $\overline{n}$ makes $\mathfrak{A} \otimes^{\overline{n}}_{\mathfrak{A}_0} \mathfrak{B}$ into a normed *-space, is called **-compatible* (see Definitions 4.2.1 and 4.2.2). If moreover $\overline{n}$ is a cross-norm (see (4.5)), then we speak about a **-compatible cross-norm.* The projective and injective γ, λ respectively, tensor cross-norms, are *-compatible and any *-compatible tensor cross-norm $\overline{n}$ lies between λ and γ.

$\diamond$ From now on, we will assume that both $(\mathfrak{A}, \mathfrak{A}_0)$ and $(\mathfrak{B}, \mathfrak{B}_0)$ have unit $\mathbb{1}_{\mathfrak{A}}$ and $\mathbb{1}_{\mathfrak{B}}$ respectively.

Lemma 4.4.1 *Let* $(\mathfrak{A}, \mathfrak{A}_0)$ *and* $(\mathfrak{B}, \mathfrak{B}_0)$ *be Banach quasi *-algebras such that there is an embedding* $\mathfrak{A}_0 \hookrightarrow \mathfrak{B}_0$. *Let* $\overline{n}$ *be a *-compatible cross-norm on* $\mathfrak{A} \otimes_{\mathfrak{A}_0} \mathfrak{B}$. *Let* π *be a *-representation of the tensor product Banach quasi *-algebra* $\mathfrak{A}\widehat{\otimes}^{\overline{n}}_{\mathfrak{A}_0}\mathfrak{B}$ *into* $\mathcal{L}^\dagger(\mathcal{D}_\pi, \mathcal{H}_\pi)$. *Then there exist unique *-representations* π_1 *of* $(\mathfrak{A}, \mathfrak{A}_0)$ *and* π_2 *of* $(\mathfrak{B}, \mathfrak{B}_0)$*, such that*

$$\pi(a \otimes b) = \pi_1(a)\pi_2(b) = \pi_2(b)\pi_1(a), \quad \forall a \in \mathfrak{A}, b \in \mathfrak{B}; \tag{4.14}$$

*The *-representations* π_1, π_2 *are restrictions of the *-representation* π *to* $\mathfrak{A}$, $\mathfrak{B}$ *respectively.*

Proof. Before defining the *-representation π_1, we observe that there exists an isometric *-isomorphism between $\mathfrak{A}$ and $\mathfrak{A} \otimes \mathbb{1}_{\mathfrak{B}}$ given by the map $\mathfrak{A} \to \mathfrak{A} \otimes \mathbb{1}_{\mathfrak{B}}$ defined as $a \mapsto a \otimes \mathbb{1}_{\mathfrak{B}}$ and then extended by linearity.

This map is linear and bijective by definition, preserves the involution and the norm defined on $\mathfrak{A} \otimes \mathbb{1}_{\mathfrak{A}}$ is equivalent to that defined on $\mathfrak{A}$ by the cross-norm property: $\|a \otimes \mathbb{1}_{\mathfrak{A}}\| = \|a\|_{\mathfrak{A}} \|\mathbb{1}_{\mathfrak{B}}\|$.

In the same way, it is possible to show that $\mathfrak{B}$ is isometrically *-isomorphic to $\mathbb{1}_{\mathfrak{A}} \otimes \mathfrak{B}$.

Let $\pi : \mathfrak{A}\widehat{\otimes}^{\overline{n}}_{\mathfrak{A}_0}\mathfrak{B} \to \mathcal{L}^\dagger(\mathcal{D}_\pi, \mathcal{H}_\pi)$ be a *-representation of $\mathfrak{A}\widehat{\otimes}^{\overline{n}}_{\mathfrak{A}_0}\mathfrak{B}$. Then we define a map π_1 on $\mathfrak{A}$ in the following way

$$\pi_1(a)\xi := \pi(a \otimes \mathbb{1}_{\mathfrak{B}})\xi, \quad \forall a \in \mathfrak{A}, \xi \in \mathcal{D}_\pi.$$

The map π_1 is linear and it is a *-representation of $\mathfrak{A}$ such that $\mathcal{D}_{\pi_1} = \mathcal{D}_\pi$. With similar arguments, π_2 defined as

$$\pi_2(b)\xi := \pi(\mathbb{1}_{\mathfrak{A}} \otimes b)\xi, \quad \forall b \in \mathfrak{B}, \xi \in \mathcal{D}_\pi$$

is a *-representation of $\mathfrak{B}$ such that $\mathcal{D}_{\pi_2} = \mathcal{D}_\pi$.

Let us now show the equalities 4.14. Take $a \in \mathfrak{A}$ and $b \in \mathfrak{B}$, then

$$\pi(a \otimes b) = \pi\left[(a \otimes \mathbb{1}_{\mathfrak{B}})(\mathbb{1}_{\mathfrak{A}} \otimes b)\right] = \pi(a \otimes \mathbb{1}_{\mathfrak{B}})\pi(\mathbb{1}_{\mathfrak{A}} \otimes b) = \pi_1(a)\pi_2(b)$$

and also

$$\pi(a \otimes b) = \pi\left[(\mathbb{1}_{\mathfrak{A}} \otimes b)(a \otimes \mathbb{1}_{\mathfrak{B}})\right] = \pi(\mathbb{1}_{\mathfrak{A}} \otimes b)\pi(a \otimes \mathbb{1}_{\mathfrak{B}}) = \pi_2(b)\pi_1(a). \ \square$$

Proposition 4.4.2 *In the hypotheses of Lemma 4.4.1, there exist representable functionals* ω_1 *on* $(\mathfrak{A}, \mathfrak{A}_0)$ *and* ω_2 *on* $(\mathfrak{B}, \mathfrak{B}_0)$. *Moreover, if the *-representation of Lemma 4.4.1 is* ($\bar{n}$-τ_w)*-continuous, then* ω_1 *and* ω_2 *are continuous.*

Proof. Let π_1 be a *-representation of $(\mathfrak{A}, \mathfrak{A}_0)$ defined as in Lemma 4.4.1. Define now a linear functional $\omega_1 : \mathfrak{A} \to \mathbb{C}$ as

$$\omega_1(a) := \langle \pi(a \otimes \mathbb{1}_{\mathfrak{B}})\xi|\xi\rangle = \langle \pi_1(a)\xi|\xi\rangle, \quad \forall a \in \mathfrak{A}, \xi \in \mathcal{D}.$$

We want to show that ω_1 is representable. The conditions (R.1) and (R.2) are easily verified. To show (R.3), consider $a \in \mathfrak{A}$ and then we want to estimate $|\omega_1(a^*x)|$

$$\begin{aligned}|\omega_1(a^*x)| = |\langle \pi_1(a^*x)\xi|\xi\rangle| &= \left|\langle \pi_1(a)^\dagger \pi_1(x)\xi|\xi\rangle\right| = |\langle \pi_1(x)\xi|\pi_1(a)\xi\rangle| \\ &\leq \|\pi_1(a)\xi\|\|\pi_1(x)\xi\| \leq (\gamma_a + 1)\, |\langle \pi_1(x^*x)\xi|\xi\rangle|^{\frac{1}{2}} \\ &= (\gamma_a + 1)\omega(x^*x)^{\frac{1}{2}},\end{aligned}$$

where $\gamma_a = \|\pi_1(a)\xi\| \geq 0$.

With a similar argument, it is possible to show that $\omega_2 : \mathfrak{B} \to \mathbb{C}$ defined as

$$\omega_2(b) := \langle \pi(\mathbb{1}_{\mathfrak{A}} \otimes b)\xi|\xi\rangle = \langle \pi_2(b)\xi|\xi\rangle, \quad \forall b \in \mathfrak{B}, \xi \in \mathcal{D}$$

is a representable linear functional on $\mathfrak{B}$.

Suppose now $\pi : \mathfrak{A}\widehat{\otimes}^{\overline{n}}_{\mathfrak{A}_0}\mathfrak{B} \to \mathcal{L}^\dagger(\mathcal{D}_\pi, \mathcal{H}_\pi)$ is $(\overline{n}\text{-}\tau_w)$-continuous. Hence, also π_1 and π_2 are $(\overline{n}\text{-}\tau_w)$-continuous *-representations of $(\mathfrak{A}, \mathfrak{A}_0)$ and $(\mathfrak{B}, \mathfrak{B}_0)$ respectively. Indeed, let $\{a_n\}$ be a sequence of elements in $\mathfrak{A}$ such that $\|a_n - a\|_{\mathfrak{A}} \to 0$ for $n \to \infty$. By the isometric *-isomorphism, then $\|a_n \otimes \mathbb{1}_{\mathfrak{B}}\| \to 0$. Therefore, using the τ_w-continuity of π, we have

$$\langle \pi_1(a_n)\xi|\eta\rangle = \langle \pi(a_n \otimes \mathbb{1}_{\mathfrak{B}})\xi|\eta\rangle \to \langle \pi(a \otimes \mathbb{1}_{\mathfrak{B}})\xi|\eta\rangle = \langle \pi_1(a)\xi|\eta\rangle$$

for all $\xi, \eta \in \mathcal{D}_\pi$.

If π_1 is a $(\overline{n}\text{-}\tau_w)$-continuous *-representation of $(\mathfrak{A}, \mathfrak{A}_0)$, then ω_1 is continuous. Consider $a_n, a \in \mathfrak{A}$ for every $n \in \mathbb{N}$ such that $a_n \to a$ in norm $\|\cdot\|_{\mathfrak{A}}$. Hence,

$$\omega(a_n) = \langle \pi_1(a_n)\xi|\xi\rangle \to \langle \pi_1(a)\xi|\xi\rangle = \omega_1(a)$$

for every $\xi \in \mathcal{D}_\pi$.

Employing the same strategy, we show that π_2 is $(\overline{n}\text{-}\tau_w)$-continuous *-representation of $(\mathfrak{B}, \mathfrak{B}_0)$ if π is a $(\overline{n}\text{-}\tau_w)$-continuous of the tensor product. This allows to show that ω_2 is continuous representable functional on $(\mathfrak{B}, \mathfrak{B}_0)$. □

Proposition 4.4.3 *Let $(\mathfrak{A}, \mathfrak{A}_0)$ and $(\mathfrak{B}, \mathfrak{B}_0)$ be Banach quasi *-algebras and suppose $\mathfrak{A}_0$ is embedded in $\mathfrak{B}_0$. Let Ω be a representable and continuous linear functional on $\mathfrak{A}\widehat{\otimes}^{\overline{n}}_{\mathfrak{A}_0}\mathfrak{B}$ (where $\overline{n}$ is as in Lemma 4.4.1). Then there exist representable and continuous linear functionals ω_1 and ω_2 on $\mathfrak{A}$ and $\mathfrak{B}$ respectively, such that*

$$\Omega(a \otimes b) = \omega_1(a) \otimes \omega_2(b), \quad \forall\, a \otimes b \in \mathfrak{A} \otimes \mathfrak{B}. \tag{4.15}$$

Proof. For every $a \in \mathfrak{A}$, let us define $\omega_1 : \mathfrak{A} \to \mathbb{C}$ as

$$\omega_1(a) := \Omega(a \otimes \mathbb{1}_{\mathfrak{B}}), \quad \forall\, a \in \mathfrak{A}$$

and so extend ω_1 by linearity. Since $\mathfrak{A}[\|\cdot\|] \simeq \mathfrak{A} \otimes \{\mathbb{1}_{\mathfrak{B}}\}[\|\cdot\|_{\overline{n}}]$, ω_1 is representable and continuous on $\mathfrak{A}$.

Analogously, define $\omega_2(b) := \Omega(\mathbb{1}_{\mathfrak{A}} \otimes b)$ for every $b \in \mathfrak{B}$. By the above isometric isomorphism, we can interpret ω_1 as a functional $\omega_1 : \mathfrak{A} \otimes \mathbb{1}_{\mathfrak{B}} \to \mathbb{C}$ and ω_2 as a functional $\omega_2 : \mathbb{1}_{\mathfrak{A}} \otimes \mathfrak{B} \to \mathbb{C}$ have (4.15). Indeed,

$$\Omega(a \otimes b) = \Omega\left[(a \otimes \mathbb{1}_{\mathfrak{B}}) \otimes (\mathbb{1}_{\mathfrak{A}} \otimes b)\right] = \omega_1(a \otimes \mathbb{1}_{\mathfrak{B}})\omega_2(\mathbb{1}_{\mathfrak{A}} \otimes b) = \omega_1(a)\omega_2(b)$$

for every $a \in \mathfrak{A}$ and $b \in \mathfrak{B}$. □

Adapting [14, Theorem 7.3], we can show that

Proposition 4.4.4 *Let $(\mathfrak{A}, \mathfrak{A}_0)$ be a unital Banach quasi *-algebra. Then the following are equivalent:*

1. *there exists a faithful ($\|\cdot\|_{\mathfrak{A}}$-τ_{s^*})-continuous *-representation π of $\mathfrak{A}$;*

2. *$(\mathfrak{A}, \mathfrak{A}_0)$ is *-semisimple.*

Theorem 4.4.5 *Let $(\mathfrak{A}, \mathfrak{A}_0)$ and $(\mathfrak{B}, \mathfrak{B}_0)$ be Banach quasi *-algebras and suppose $\mathfrak{A}_0$ is embedded into $\mathfrak{B}_0$. Let $(\mathfrak{A}\widehat{\otimes}^{\overline{n}}_{\mathfrak{A}_0}\mathfrak{B}, \mathfrak{A}_0 \otimes \mathfrak{B}_0)$ be the tensor product Banach quasi *-algebra of $(\mathfrak{A}, \mathfrak{A}_0)$ and $(\mathfrak{B}, \mathfrak{B}_0)$, where $\overline{n}$ is a *-compatible cross norm on $\mathfrak{A} \otimes_{\mathfrak{A}_0} \mathfrak{B}$. Suppose $(\mathfrak{A}\widehat{\otimes}^{\overline{n}}_{\mathfrak{A}_0}\mathfrak{B}, \mathfrak{A}_0 \otimes \mathfrak{B}_0)$ is *-semisimple. Then both $(\mathfrak{A}, \mathfrak{A}_0)$, $(\mathfrak{B}, \mathfrak{B}_0)$ are *-semisimple Banach quasi *-algebras.*

Proof. By Proposition 4.4.4, there exists a faithful τ_{s^*}-continuous *-representation of $(\mathfrak{A}\widehat{\otimes}_{\mathfrak{A}_0}\mathfrak{B}, \mathfrak{A}_0 \otimes \mathfrak{B}_0)$. Hence, by Lemma 4.4.1 there exist τ_{s^*}-continuous *-representations π_1 and π_2 of $(\mathfrak{A}, \mathfrak{A}_0)$ and $(\mathfrak{B}, \mathfrak{B}_0)$ respectively. What remains to show is that π_1 and π_2 are faithful. Let $0 \neq a \in \mathfrak{A}$. By the isometric *-isomorphism between $\mathfrak{A}[\|\cdot\|]$ and $\mathfrak{A} \otimes \mathbb{1}_{\mathfrak{B}}[\|\cdot\|_\tau]$, we identify $a \equiv a \otimes \mathbb{1}_{\mathfrak{B}}$. Thus, $a \otimes \mathbb{1}_{\mathfrak{B}} \neq 0$.

By faithfulness of π, we have $\pi(a \otimes \mathbb{1}_{\mathfrak{B}}) \neq 0$. We conclude that

$$\pi_1(a) = \pi_1(a) Id_{\mathcal{H}_\pi} = \pi_1(a)\pi_2(\mathbb{1}_{\mathfrak{B}}) = \pi(a \otimes \mathbb{1}_{\mathfrak{B}}) \neq 0.$$

With the same argument, we get the conclusion for $(\mathfrak{B}, \mathfrak{B}_0)$. □

Theorem 4.4.6 *Let $(\mathfrak{A}, \mathfrak{A}_0)$ and $(\mathfrak{B}, \mathfrak{B}_0)$ be Banach quasi *-algebras and suppose $\mathfrak{A}_0$ is embedded into $\mathfrak{B}_0$. Let $(\mathfrak{A}\widehat{\otimes}^{\overline{n}}_{\mathfrak{A}_0}\mathfrak{B}, \mathfrak{A}_0 \otimes \mathfrak{B}_0)$ be the tensor product Banach quasi *-algebra, where $\overline{n}$ is a *-compatible cross norm on $\mathfrak{A} \otimes_{\mathfrak{A}_0} \mathfrak{B}$. Suppose that $\mathcal{R}_c(\mathfrak{A}\widehat{\otimes}^{\overline{n}}_{\mathfrak{A}_0}\mathfrak{B}, \mathfrak{A}_0 \otimes \mathfrak{B}_0)$ is sufficient. Then both $\mathcal{R}_c(\mathfrak{A}, \mathfrak{A}_0)$, $\mathcal{R}_c(\mathfrak{B}, \mathfrak{B}_0)$ are sufficient.*

Proof. Let $a \in \mathfrak{A}$ be a positive element. Hence $a = \lim_{n\to\infty} \sum_{finite} x_n^* x_n$ for $x_n \in \mathfrak{A}_0$ for every $n \in \mathbb{N}$. By the isometric *-isomorphism of $\mathfrak{A}[\|\cdot\|]$ and $\mathfrak{A} \otimes \mathbb{1}_{\mathfrak{B}}[\|\cdot\|_\tau]$, $a \otimes \mathbb{1}_{\mathfrak{B}}$ is again a positive element, i.e. $a \otimes \mathbb{1}_{\mathfrak{B}} = \lim_{n\to\infty} \sum_{finite} x_n^* x_n \otimes \mathbb{1}_{\mathfrak{B}}$.

$\mathcal{R}_c(\mathfrak{A}\widehat{\otimes}^{\overline{n}}_{\mathfrak{A}_0}\mathfrak{B}, \mathfrak{A}_0 \otimes \mathfrak{B}_0)$ is sufficient by assumption, therefore there exists a representable and continuous functional $\widehat{\Omega}$ on $\mathfrak{A}\widehat{\otimes}^{\overline{n}}_{\mathfrak{A}_0}\mathfrak{B}$ such that $\widehat{\Omega}(a \otimes \mathbb{1}_{\mathfrak{B}}) > 0$.

By Proposition 4.4.3, there exist $\widehat{\omega}_1$ and $\widehat{\omega}_2$ such that

$$\widehat{\Omega} = \widehat{\omega}_1 \otimes \widehat{\omega}_2, \quad \text{on } \mathfrak{A} \otimes \mathfrak{B}.$$

Hence $\widehat{\Omega}(a \otimes \mathbb{1}_{\mathfrak{B}}) = \widehat{\omega}_1(a)\widehat{\omega}_2(\mathbb{1}_{\mathfrak{B}}) > 0$. $\widehat{\Omega} \neq 0$, then $\widehat{\omega}_2(1) \neq 0$. We conclude that $\widehat{\omega}_1(a) > 0$ and $\mathcal{R}_c(\mathfrak{A}, \mathfrak{A}_0)$ is sufficient.

With the same strategy, it is possible to prove that $\mathcal{R}_c(\mathfrak{B}, \mathfrak{B}_0)$ is sufficient. □

Our aim would be to show that the following statements are equivalent

- $(\mathfrak{A}, \mathfrak{A}_0)$ and $(\mathfrak{B}, \mathfrak{B}_0)$ are fully representable;
- $(\mathfrak{A}\widehat{\otimes}^{\overline{n}}_{\mathfrak{A}_0}\mathfrak{B}, \mathfrak{A}_0 \otimes_{\mathfrak{A}_0} \mathfrak{B}_0)$ is fully representable.

This statement involves positive elements of $\mathfrak{A}\widehat{\otimes}^{\overline{n}}_{\mathfrak{A}_0}\mathfrak{B}$ that cannot be easily characterized in terms of positive elements of $\mathfrak{A}$ and $\mathfrak{B}$. Thus we want to study how *-semisimplicity passes from the tensor product to the factors and viceversa in the case of normed quasi *-algebras.

Proposition 4.4.7 *Let $(\mathfrak{A}, \mathfrak{A}_0)$ and $(\mathfrak{B}, \mathfrak{B}_0)$ be normed quasi *-algebras such that $\mathfrak{A}_0 \hookrightarrow \mathfrak{B}_0$. Then the following are equivalent*

1. *$(\mathfrak{A}, \mathfrak{A}_0)$ and $(\mathfrak{B}, \mathfrak{B}_0)$ are *-semisimple;*
2. *$(\mathfrak{A}\widehat{\otimes}^{\overline{n}}_{\mathfrak{A}_0}\mathfrak{B}, \mathfrak{A}_0 \otimes_{\mathfrak{A}_0} \mathfrak{B}_0)$ is *-semisimple.*

Proof. Before proving the equivalence, we show a bijective correspondence between sesquilinear forms in the tensor product and in the factors.

If $\Theta \in \mathcal{S}_{\mathfrak{A}_0 \otimes_{\mathfrak{A}_0} \mathfrak{B}_0}(\mathfrak{A} \otimes^{\bar{n}}_{\mathfrak{A}_0} \mathfrak{B})$, then we can define

$$\theta_1(a, a) := \Theta(a \otimes \mathbb{1}_{\mathfrak{B}}, a \otimes \mathbb{1}_{\mathfrak{B}}), \quad a \in \mathfrak{A}$$

$$\theta_2(b, b) := \Theta(\mathbb{1}_{\mathfrak{A}} \otimes b, \mathbb{1}_{\mathfrak{A}} \otimes b), \quad b \in \mathfrak{B}.$$

$\theta_1 \in \mathcal{S}_{\mathfrak{A}_0}(\mathfrak{A})$ and $\theta_2 \in \mathcal{S}_{\mathfrak{B}_0}(\mathfrak{B})$ as restrictions of $\Theta \in \mathcal{S}_{\mathfrak{A}_0 \otimes_{\mathfrak{A}_0} \mathfrak{B}_0}(\mathfrak{A} \otimes^{\bar{n}}_{\mathfrak{A}_0} \mathfrak{B})$.

On the contrary, if $\theta_1 \in \mathcal{S}_{\mathfrak{A}_0}(\mathfrak{A})$ and $\theta_2 \in \mathcal{S}_{\mathfrak{B}_0}(\mathfrak{B})$, then it is possible to define $\Theta \in \mathcal{S}_{\mathfrak{A}_0 \otimes_{\mathfrak{A}_0} \mathfrak{B}_0}(\mathfrak{A} \otimes^{\bar{n}}_{\mathfrak{A}_0} \mathfrak{B})$ in the following way

$$\Theta(a \otimes b, a \otimes b) := \theta_1(a, a)\theta_2(b, b), \quad a \in \mathfrak{A}, b \in \mathfrak{B}.$$

and thus extend Θ by linearity. By polarization identity we have the equality $\Theta(a \otimes b, c \otimes d) = \theta_1(a,c)\theta_2(b,d)$ for every $a \otimes b, c \otimes d \in \mathfrak{A} \otimes^{\bar{n}}_{\mathfrak{A}_0} \mathfrak{B}$. We show that the properties required are verified. Indeed, for $a \otimes b, c \otimes d \in \mathfrak{A} \otimes^{\bar{n}}_{\mathfrak{A}_0} \mathfrak{B}$ and $x \otimes y, t \otimes s \in \mathfrak{A}_0 \otimes_{\mathfrak{A}_0} \mathfrak{B}_0$ we have

$$\begin{aligned}
\Theta(a \otimes b, a \otimes b) &= \theta_1(a,a)\theta_2(b,b) \geq 0. \\
\Theta(ax \otimes by, t \otimes s) &= \theta_1(ax,t)\theta_2(by,s) = \theta_1(x,a^*t)\theta_2(y,b^*s) \\
&= \Theta(x \otimes y, a^*t \otimes b^*s). \\
|\Theta(a \otimes b, c \otimes d)| &= |\theta_1(a,c)|\,|\theta_2(b,d)| \leq \|a\|_1\|c\|_1\|b\|_2\|d\|_2 \\
&= \|a \otimes b\|\,\|c \otimes d\|.
\end{aligned}$$

Hence there exists a bijection between $\mathcal{S}_{\mathfrak{A}_0 \otimes_{\mathfrak{A}_0} \mathfrak{B}_0}(\mathfrak{A} \otimes^{\bar{n}}_{\mathfrak{A}_0} \mathfrak{B})$ and $\mathcal{S}_{\mathfrak{A}_0}(\mathfrak{A}) \otimes \mathcal{S}_{\mathfrak{B}_0}(\mathfrak{B})$ given by $\Theta \leftrightarrow (\theta_1, \theta_2)$ such that $\Theta = \theta_1 \otimes \theta_2$.

Suppose now 1. By hypothesis $(\mathfrak{A}, \mathfrak{A}_0)$ and $(\mathfrak{B}, \mathfrak{B}_0)$ are *-semisimple, then for $0 \neq a \in \mathfrak{A}$ and $0 \neq b \in \mathfrak{B}$ there exist $\widetilde{\theta}_1 \in \mathcal{S}_{\mathfrak{A}_0}(\mathfrak{A})$ and $\widetilde{\theta}_2 \in \mathcal{S}_{\mathfrak{B}_0}(\mathfrak{B})$ such that $\widetilde{\theta}_1(a,a) > 0$ and $\widetilde{\theta}_2(b,b) > 0$.

If $a \otimes b \neq 0$, then $a \neq 0$ and $b \neq 0$. Then $\Theta = \widetilde{\theta}_1 \otimes \widetilde{\theta}_2 \in \mathcal{S}_{\mathfrak{A}_0 \otimes_{\mathfrak{A}_0} \mathfrak{B}_0}(\mathfrak{A} \otimes^{\bar{n}}_{\mathfrak{A}_0} \mathfrak{B})$ is such that

$$\Theta(a \otimes b, a \otimes b) = \widetilde{\theta}_1(a,a)\widetilde{\theta}_2(b,b) > 0.$$

We conclude that $(\mathfrak{A} \otimes^{\bar{n}}_{\mathfrak{A}_0} \mathfrak{B}, \mathfrak{A}_0 \otimes_{\mathfrak{A}_0} \mathfrak{B}_0)$ is *-semisimple.

Suppose now 2. If $0 \neq a \otimes b \in \mathfrak{A} \otimes^{\bar{n}}_{\mathfrak{A}_0} \mathfrak{B}$, then there exists $\widetilde{\Theta} \in \mathcal{S}_{\mathfrak{A}_0 \otimes_{\mathfrak{A}_0} \mathfrak{B}_0}(\mathfrak{A} \otimes^{\bar{n}}_{\mathfrak{A}_0} \mathfrak{B})$ such that $\widetilde{\Theta}(a \otimes b, a \otimes b) > 0$.

On the other hand, $\widetilde{\Theta} \equiv (\widetilde{\theta}_1, \widetilde{\theta}_2)$. Hence for $a \neq 0$, $\widetilde{\theta}_1(a,a) > 0$ and as well for $b \neq 0$ we have $\widetilde{\theta}_2(b,b) > 0$. Therefore, $(\mathfrak{A}, \mathfrak{A}_0)$ and $(\mathfrak{B}, \mathfrak{B}_0)$ are *-semisimple. $\square$

*-semisimplicity has been characterized in Proposition 4.4.4 through the existence of faithful τ_w-continuous *-representations even in the normed case. Indeed, the proof of Proposition 4.4.4 is not depending on the completion of the Banach quasi *-algebra.

As a direct consequence of Proposition 4.4.7, we obtain

Proposition 4.4.8 *Let $(\mathfrak{A}, \mathfrak{A}_0)$ and $(\mathfrak{B}, \mathfrak{B}_0)$ be normed quasi *algebras such that $\mathfrak{A}_0 \hookrightarrow \mathfrak{B}_0$. Then the following are equivalent*

1. *There exist faithful $(\overline{n}$-$\tau_{s^*})$-continuous *-representations π_1 and π_2 of the normed quasi *-algebras $(\mathfrak{A}, \mathfrak{A}_0)$ and $(\mathfrak{B}, \mathfrak{B}_0)$ respectively*

2. *There exists a faithful $(\overline{n}$-$\tau_{s^*})$-continuous *-representation π of the tensor product normed quasi *-algebra $(\mathfrak{A} \otimes^{\bar{n}}_{\mathfrak{A}_0} \mathfrak{B}, \mathfrak{A}_0 \otimes_{\mathfrak{A}_0} \mathfrak{B}_0)$*

At this point, we want to investigate what happens for full representability. For reader's convenience, we recall the condition of positivity (P)

$$b \in \mathfrak{A} \;\text{ and }\; \omega(x^*bx) \geq 0 \;\;\forall \omega \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{A}_0);\; \forall x \in \mathfrak{A}_0 \;\;\Rightarrow\;\; b \in \mathfrak{A}^+.$$

For full representability we have to distinguish two cases, the equivalence is not perfectly holding in this case.

Proposition 4.4.9 *Let* $(\mathfrak{A}, \mathfrak{A}_0)$ *and* $(\mathfrak{B}, \mathfrak{B}_0)$ *be normed quasi *-algebras such that* $\mathfrak{A}_0 \hookrightarrow \mathfrak{B}_0$*. If* $(\mathfrak{A} \otimes^{\overline{n}}_{\mathfrak{A}_0} \mathfrak{B}, \mathfrak{A}_0 \otimes_{\mathfrak{A}_0} \mathfrak{B}_0)$ *is fully representable, then* $(\mathfrak{A}, \mathfrak{A}_0)$ *and* $(\mathfrak{B}, \mathfrak{B}_0)$ *are fully representable.*

Proof. With a similar argument employed in Proposition 4.4.7, it is possible to show that there exists a 1-1 correspondence between $\Omega \in \mathcal{R}_c(\mathfrak{A} \otimes^{\overline{n}}_{\mathfrak{A}_0} \mathfrak{B}, \mathfrak{A}_0 \otimes_{\mathfrak{A}_0} \mathfrak{B}_0)$ and $(\omega_1, \omega_2) \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{A}_0) \times \mathcal{R}_c(\mathfrak{B}, \mathfrak{B}_0)$. Indeed, if $\Omega \in \mathcal{R}_c(\mathfrak{A} \otimes^{\overline{n}}_{\mathfrak{A}_0} \mathfrak{B}, \mathfrak{A}_0 \otimes_{\mathfrak{A}_0} \mathfrak{B}_0)$, then $\omega_1(a) := \Omega(a \otimes \mathbb{1}_{\mathfrak{B}})$ for $a \in \mathfrak{A}$ and $\omega_2(b) := \Omega(\mathbb{1}_{\mathfrak{A}} \otimes b)$ for every $b \in \mathfrak{B}$ are representable and continuous functionals over $\mathfrak{A}$ and $\mathfrak{B}$ respectively.

If $\omega_1 \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{A}_0)$ and $\omega_2 \in \mathcal{R}_c(\mathfrak{B}, \mathfrak{B}_0)$, then $\Omega(a \otimes b) := \omega_1(a)\omega_2(b)$ is representable and continuous. Indeed, it is positive and invariant, so the first two properties are verified. Let us check also the third property. Let $a \otimes b \in \mathfrak{A} \otimes^{\overline{n}}_{\mathfrak{A}_0} \mathfrak{B}$ and $x \otimes y \in \mathfrak{A}_0 \otimes_{\mathfrak{A}_0} \mathfrak{B}_0$, then

$$\begin{aligned}
|\Omega(a^* x \otimes b^* y)| &= |\varphi_\Omega(x \otimes y, a \otimes b)| \\
&\leq \varphi_\Omega(x \otimes y, x \otimes y)^{\frac{1}{2}} \varphi_\Omega(a \otimes b, a \otimes b)^{\frac{1}{2}} \\
&\leq \gamma_{a \otimes b} \Omega(x^* x \otimes y^* y)^{\frac{1}{2}}
\end{aligned}$$

If $a \in \mathfrak{A}^+$ and $b \in \mathfrak{B}^+$, then $a \otimes b$ is positive. Indeed, there exist sequences $\{x_n\} \in \mathfrak{A}_0$ and $\{y_n\} \in \mathfrak{B}_0$ such that

$$a = \lim_n \sum_i x_i^* x_i \quad \text{and} \quad b = \lim_n \sum_j y_j^* y_j.$$

Hence, combining the two sequences and using the cross-norm property, we have

$$a \otimes b = \lim_n \sum_{i,j} x_i^* x_i \otimes y_j^* y_j \in (\mathfrak{A} \otimes^{\overline{n}}_{\mathfrak{A}_0} \mathfrak{B})^+.$$

Indeed,

$$\begin{aligned}
\left\| a \otimes b - \sum_{i,j} x_i^* x_i \otimes y_j^* y_j \right\| &= \left\| a \otimes b - \sum_i x_i^* x_i \otimes \sum_j y_j^* y_j \right\| \\
&\leq \left\| a - \sum_i x_i^* x_i \right\| \left\| b - \sum_j y_j^* y_j \right\| \to 0.
\end{aligned}$$

Therefore, by full representability of the tensor product $\mathfrak{A} \otimes^{\overline{n}}_{\mathfrak{A}_0} \mathfrak{B}$, there exists a functional $\widehat{\Omega} \in \mathcal{R}_c(\mathfrak{A} \otimes^{\overline{n}}_{\mathfrak{A}_0} \mathfrak{B}, \mathfrak{A}_0 \otimes_{\mathfrak{A}_0} \mathfrak{B}_0)$ such that $\widehat{\Omega}(a \otimes b) > 0$, then there exist $\widehat{\omega_1} \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{A}_0)$ and $\widehat{\omega_2} \in \mathcal{R}_c(\mathfrak{B}, \mathfrak{B}_0)$ such that

$$\widehat{\Omega}(a \otimes b) = \widehat{\omega_1}(a) \widehat{\omega_2}(b) > 0$$

We conclude that $\widehat{\omega_1}(a)$ and $\widehat{\omega_2}(b)$ are both positive real numbers or both negative. Hence the couple $(\widehat{\omega_1}, \widehat{\omega_2})$ is such that $\widehat{\omega_1}(a) > 0$ and $\widehat{\omega_2}(b) > 0$. If this is not the case, it is enough to consider the couple $(-\widehat{\omega_1}, -\widehat{\omega_2})$ for our purpose.

Now what remains to be shown is $\mathcal{D}(\overline{\varphi}_{\omega_1}) = \mathfrak{A}$ for every $\omega_1 \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{A}_0)$, as well for ω_2.

Let $a \in \mathfrak{A}$, then $a \otimes \mathbb{1}_{\mathfrak{B}} \in \mathfrak{A} \otimes \mathbb{1}_{\mathfrak{B}} \hookrightarrow \mathfrak{A} \otimes^{\bar{n}}_{\mathfrak{A}_0} \mathfrak{B}$. Then, by full representability of the tensor product, $a \otimes \mathbb{1}_{\mathfrak{B}} \in \mathcal{D}(\overline{\varphi}_\Omega)$ for every $\Omega \in \mathcal{R}_c(\mathfrak{A} \otimes^{\bar{n}}_{\mathfrak{A}_0} \mathfrak{B}, \mathfrak{A}_0 \otimes_{\mathfrak{A}_0} \mathfrak{B}_0)$. Hence, there exists $x_n \otimes \mathbb{1}_{\mathfrak{B}}$ in $\mathfrak{A}_0 \otimes \mathbb{1}_{\mathfrak{B}}$ such that $x_n \otimes \mathbb{1}_{\mathfrak{B}} \to a \otimes \mathbb{1}_{\mathfrak{B}}$ and

$$\begin{aligned}
&\varphi_\Omega(x_n \otimes \mathbb{1}_{\mathfrak{B}} - x_m \otimes \mathbb{1}_{\mathfrak{B}}, x_n \otimes \mathbb{1}_{\mathfrak{B}} - x_m \otimes \mathbb{1}_{\mathfrak{B}}) \\
&\quad = \Omega((x_n \otimes \mathbb{1}_{\mathfrak{B}} - x_m \otimes \mathbb{1}_{\mathfrak{B}})^*(x_n \otimes \mathbb{1}_{\mathfrak{B}} - x_m \otimes \mathbb{1}_{\mathfrak{B}})) \\
&\quad = \omega_1((x_n - x_m)^*(x_n - x_m)) \\
&\quad = \varphi_{\omega_1}(x_n - x_m, x_n - x_m) \to 0.
\end{aligned}$$

Hence we conclude that $a \in \mathfrak{A}$ belongs to every $\mathcal{D}(\overline{\varphi}_{\omega_1})$. The same argument holds for $\omega_2 \in \mathcal{R}_c(\mathfrak{B}, \mathfrak{B}_0)$, thus $(\mathfrak{A}, \mathfrak{A}_0)$ and $(\mathfrak{B}, \mathfrak{B}_0)$ are fully representable. □

For the other direction, we assume the condition (P) to be valid.

Proposition 4.4.10 *Let* $(\mathfrak{A}, \mathfrak{A}_0)$ *and* $(\mathfrak{B}, \mathfrak{B}_0)$ *be normed quasi *-algebras such that* $\mathfrak{A}_0 \hookrightarrow \mathfrak{B}_0$. *If* $(\mathfrak{A}, \mathfrak{A}_0)$ *and* $(\mathfrak{B}, \mathfrak{B}_0)$ *are fully representable and the condition (P) holds, then* $(\mathfrak{A} \otimes^{\bar{n}}_{\mathfrak{A}_0} \mathfrak{B}, \mathfrak{A}_0 \otimes_{\mathfrak{A}_0} \mathfrak{B}_0)$ *is fully representable.*

Proof. Following the same argument of Theorem 2.1.3, it is possible to show that full representability and positivity condition (P) implies *-semisimplicity. Then $(\mathfrak{A}, \mathfrak{A}_0)$ and $(\mathfrak{B}, \mathfrak{B}_0)$ are *-semisimple.

By Proposition 4.4.7, $(\mathfrak{A} \otimes^{\bar{n}}_{\mathfrak{A}_0} \mathfrak{B}, \mathfrak{A}_0 \otimes_{\mathfrak{A}_0} \mathfrak{B}_0)$ is *-semisimple. Hence the family $\mathcal{R}_c(\mathfrak{A} \otimes^{\bar{n}}_{\mathfrak{A}_0} \mathfrak{B}, \mathfrak{A}_0 \otimes_{\mathfrak{A}_0} \mathfrak{B}_0)$ is sufficient. We have to show that $\mathcal{D}(\overline{\varphi}_\Omega) = \mathfrak{A} \otimes^{\bar{n}}_{\mathfrak{A}_0} \mathfrak{B}$ for every $\Omega \in \mathcal{R}_c(\mathfrak{A} \otimes^{\bar{n}}_{\mathfrak{A}_0} \mathfrak{B}, \mathfrak{A}_0 \otimes_{\mathfrak{A}_0} \mathfrak{B}_0)$.

Let $a \otimes b \in \mathfrak{A} \otimes^{\bar{n}}_{\mathfrak{A}_0} \mathfrak{B}$. Then by full representability of the factors, there exist sequences

- $x_n \in \mathfrak{A}_0$ such that $\varphi_{\omega_1}(x_n - x_m, x_n, x_m) \to 0$ for every $\omega_1 \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{A}_0)$
- $y_n \in \mathfrak{B}_0$ such that $\varphi_{\omega_2}(y_n - y_m, y_n, y_m) \to 0$ for every $\omega_2 \in \mathcal{R}_c(\mathfrak{B}, \mathfrak{B}_0)$.

Hence $\{x_n \otimes y_n\}$ converges to $a \otimes b$ and

$$\varphi_\Omega = \varphi_{\omega_1} \otimes \varphi_{\omega_2}, \quad \forall \Omega \in \mathcal{R}_c(\mathfrak{A} \otimes^{\bar{n}}_{\mathfrak{A}_0} \mathfrak{B}, \mathfrak{A}_0 \otimes_{\mathfrak{A}_0} \mathfrak{B}_0).$$

We conclude that $a \otimes b \in \mathcal{D}(\overline{\varphi}_\Omega)$ for every $\Omega \in \mathcal{R}_c(\mathfrak{A} \otimes^{\bar{n}}_{\mathfrak{A}_0} \mathfrak{B}, \mathfrak{A}_0 \otimes_{\mathfrak{A}_0} \mathfrak{B}_0)$, thus $\mathcal{D}(\overline{\varphi}_\Omega) = \mathfrak{A} \otimes^{\bar{n}}_{\mathfrak{A}_0} \mathfrak{B}$. □

4.5 Examples of tensor product Banach quasi *-algebras

Example 4.5.1 Consider I to be the unit interval in the real line and λ the Lebesgue measure. Take the Banach quasi *-algebra $(L^p(I, d\lambda), \mathcal{C}(I))$, where $p > 2$ and $\mathcal{C}(I)$ is the C*-algebra of all $\mathbb{C}$-valued continuous functions on I.

Consider the tensor product of $(L^p(I, d\lambda), \mathcal{C}(I))$ with itself. Then we get the *Banach quasi* algebra* $(L^p(I, d\lambda)\widehat{\otimes}^{\gamma}_{\mathcal{C}(I)} L^p(I) L^p(I, d\lambda), \mathcal{C}(I))$.

The Banach quasi *-algebra $(L^p(I, d\lambda), \mathcal{C}(I))$ is both fully representable and *-semisimple. Also the positivity condition (P) holds. Hence, by Propositions 4.4.7 and 4.4.10, we know that $L^p(I, d\lambda) \otimes^{\gamma}_{\mathcal{C}(I)} L^p(I, d\lambda)$ is *-semisimple and fully representable. Unfortunately, we are not able to say anything about the tensor product Banach quasi *-algebra completion.

A non-commutative example is the following:

Example 4.5.2 Take two Hilbert quasi* algebras $(\mathcal{H}_1, \mathfrak{A}_0)$, $(\mathcal{H}_2, \mathfrak{B}_0)$, where $\mathfrak{A}_0$ is embedded in $\mathfrak{B}_0$ and both are *-algebras and pre-Hilbert spaces, with $\mathcal{H}_1$ the Hilbert space completion of $\mathfrak{A}_0$, with inner product $\langle\cdot|\cdot\rangle_1$ and $\mathcal{H}_2$ the Hilbert space completion of $\mathfrak{B}_0$ with inner product $\langle\cdot|\cdot\rangle_2$. Then, we get the tensor product Hilbert quasi *-algebra $(\mathcal{H}_1 \widehat{\otimes}^h_{\mathfrak{A}_0} \mathcal{H}_2, \mathfrak{A}_0 \otimes \mathfrak{B}_0)$, where $\mathcal{H}_1 \widehat{\otimes}^h_{\mathfrak{A}_0} \mathcal{H}_2$ is the Hilbert space completion of the pre-Hilbert space (and *-algebra) $\mathfrak{A}_0 \otimes \mathfrak{B}_0$, under the norm h induced by the inner product

$$\langle \xi, \xi' \rangle := \sum_{i=1}^{n} \sum_{j=1}^{m} \langle \xi_i, \xi'_j \rangle_1 \langle \eta_i, \eta'_j \rangle_2,$$

for any $\xi, \xi' \in \mathcal{H}_1 \otimes \mathcal{H}_2$, with $\xi = \sum_{i=1}^n \xi_i \otimes \eta_i$ and $\xi' = \sum_{j=1}^m \xi'_j \otimes \eta'_j$ (see [60, 34]).

In this case, the tensor product Hilbert quasi *-algebra $(\mathcal{H}_1 \widehat{\otimes}^h_{\mathfrak{A}_0} \mathcal{H}_2, \mathfrak{A}_0 \otimes \mathfrak{B}_0)$ is a *-semisimple and fully representable.

Conclusions

Locally convex quasi *-algebras have been widely studied in the last decades for the mathematical description of physical systems, especially those coming from Quantum Field Theory and Quantum Statistical Mechanics. They have been investigated also as abstract mathematical structures for their own interest.

In this thesis we mainly focused on the case of a Banach quasi *-algebra, i.e. the completion of a normed *-algebra for which the multiplication is just separately continuous. The main problem of the thesis concerns the continuity of representable functionals, i.e. those functionals that admit a GNS-triple made of a Hilbert space, a *-representation and a coset map.

Various ways have been employed to give a positive answer to this problem proving results about the continuity of representable functionals in the framework of Banach quasi *-algebras. These structures constitute a special family of locally convex quasi *-algebras. Hence, it would be interesting to consider the same problem in this more general background, considering that no examples of *discontinuous* representable functionals have been shown until now.

In this investigation a relevant role is played by sesquilinear forms, especially those associated to a representable (and continuous) functional. The first issue to face would be to find a different approach to prove analogous results in the locally convex case, because the results obtained here are often closely related to the structure properties of Banach spaces. In the locally convex case, it would be a good hint to prove a similar result about continuity of the aforementioned sesquilinear forms. Indeed, continuity of representable functionals is strictly linked to continuity of the *-representation in the GNS-triple and the continuity of sesquilinear forms associated to them.

As we have seen, a central role in the study of unbounded derivations is played again by sesquilinear forms occurring in the notion of *-semisimplicity. Indeed, *-semisimple Banach quasi *-algebras are those that possess a family of positive invariant continuous sesquilinear forms, shown to be related to the family of sesquilinear forms coming from representable and continuous functionals. The notion of *-semisimplicity is useful in extending classical results to the Banach quasi *-algebras case. Indeed, in the *-semisimple case it is possible to define a weaker notion of derivation and then give conditions on the weak *-derivation in order to get an infinitesimal generator of a certain weak *-automorphisms group.

In particular, if a weak *-derivation is the infinitesimal generator of a continuous one-parameter group of uniformly bounded weak *-automorphisms, then it is a closed map enjoying certain spectral properties. Conversely, if in addition the domain of the weak *-derivation consists of bounded elements, then the derivation generates such a weak *-automorphisms group. The condition of boundedness on the domain is however only *sufficient*, but not necessary, as shown in examples. Two main ingredients have been employed proving these results: the theory of one-parameter groups on Banach spaces and the properties of a convenient family of sesquilinear forms. In the literature, there are plenty of examples of locally convex quasi *-algebras for which the class of positive, invariant and *continuous* sesquilinear forms is trivial. Therefore, a direction for the future research concerns locally convex quasi *-algebras and the non *-semisimple case. Moreover, it would be interesting to investigate what happens if the continuity requirement on the weak *-automorphisms group is weakened, for instance studying continuous groups with respect to coarser topologies than the norm topology.

Constructing tensor product Banach quasi *-algebras could be a way to study representations of Banach quasi *-algebras. Indeed, it is interesting to study how related properties, like full representability and *-semisimplicity, pass from the tensor product to the tensor factors and viceversa. One direction, concerning properties passing from the tensor product to the factors, has been shown. About the other direction, it is possible to show that representations pass to the pre-completion, but nothing can be said about the completion. It is possible to show that the tensor product of two representable and continuous functionals is again representable and continuous functional on the pre-completion through sesquilinear forms. The main issue of dealing with the completion is that representation properties are not preserved through limits.

A possible approach to attack the problem could be constructing *CQ*-enveloping quasi *-algebra*, like in the Banach *-algebras case, but there are examples of CQ*-algebras with no representations. Hence, a more effective strategy would be constructing a *H*-enveloping quasi *-algebra*, since Hilbert quasi *-algebras are always fully representable.

We believe that this kind of construction could give a better glimpse on Banach quasi *-algebras and also on more concrete tensor products, for instance those of L^p-spaces, that remain mysterious at the moment.

Appendix A

Brief overview on Banach *-algebras

For the reader's convenience, we recall some notions about Banach *-algebras and C*-algebras and Operator Theory useful in the Chapters of this Thesis. For further reading, an introductory book on Operator Algebras is enough, for example see [63].

A.1 Banach *-algebras and C*-algebras

Definition A.1.1 Let $\mathfrak{A}_0$ be a Banach space over the complex numbers. If $\mathfrak{A}_0$ is an algebra over $\mathcal{C}$ in which the multiplication is such that

$$||xy|| \leq ||x||\,||y|| \quad \forall x, y \in \mathfrak{A}_0$$

then $\mathfrak{A}_0$ is said to be *Banach algebra*.

Definition A.1.2 If a Banach algebra $\mathfrak{A}_0$ is endowed with a linear map $* : \mathfrak{A}_0 \to \mathfrak{A}_0$ defined as $* : x \mapsto x^*$ with the following properties

1. $(x^*)^* = x$;
2. $(xy)^* = y^*x^*$;
3. $||x^*|| = ||x||$;

for every $x, y \in \mathfrak{A}_0$, then $\mathfrak{A}_0$ is called *Banach *-algebra*. The map * is called *involution*.

If the involution in $\mathfrak{A}_0$ satisfies the further C*-property

$$||x^*x|| = ||x||^2, \quad \forall x \in \mathfrak{A}_0, \tag{A.1}$$

then $\mathfrak{A}_0$ is said to be a *C*-algebra*.

Example A.1.3 Let $\mathcal{H}$ be a Hilbert space over $\mathbb{C}$ endowed with inner product denoted by $\langle \cdot | \cdot \rangle$, then the set $\mathcal{B}(\mathcal{H})$ of all the bounded operators on $\mathcal{H}$ is an algebra with the pointwise sum and scalar product. Moreover, it is a unital C*-algebra, if we endow it with involution given by $X \mapsto X^*$, where X^* is the adjoint operator of X, and norm given by

$$\|X\| = \sup_{0 \neq \xi \in \mathcal{H}} \|X\xi\|.$$

Example A.1.4 Let Ω be a compact space. The set $\mathcal{C}(\Omega)$ of all continuous functions defined on Ω is an algebra with the natural operation of sum and scalar product. Endowed with the involution given by

$$f^*(\omega) = \overline{f(\omega)}, \quad \forall \omega \in \Omega$$

and uniform convergence norm as follows

$$||f|| = \sup_{\omega \in \Omega} |f(\omega)|,$$

then $\mathcal{C}(\Omega)$ is a unital C$^*-$algebra.

Example A.1.5 Let $L^\infty[0,1]$ be the Banach space of essentially bounded functions in $I = [0,1]$. $L^\infty[0,1]$ endowed with the norm of essentially supremum

$$||f||_\infty = \operatorname{ess\,sup}_{x \in I} |f(x)|.$$

$L^\infty[0,1]$ is an algebra with the canonical pointwise sum and scalar product. With this norm, it becomes a unital C$^*-$algebra with involution $*$ given by the complex adjoint.

If the Banach $^*-$algebra $\mathfrak{A}_0$ is not unital, then $\mathfrak{A}_0$ can be embedded into a unital Banach *-algebra $\mathfrak{B}_0$ as its ideal.

$\mathcal{B}(\mathcal{H})$ in Example A.1.3 and $L^\infty[0,1]$ in Example A.1.5 constitute prototypes of a special class of C*-algebras, called *von Neumann algebras*.

If $\mathfrak{M}$ is a subset $\mathfrak{B}(\mathcal{H})$, then $\mathfrak{M}'$ denotes the set of all bounded operators on $\mathcal{H}$ commuting with every operator in $\mathfrak{M}$. $\mathfrak{M}'$ is said to be *commutant* of $\mathfrak{M}$ and it is always a Banach algebra containing the identity operator I.

Definition A.1.6 A *von Neumann algebra on* $\mathcal{H}$ is a *-subalgebra $\mathfrak{M}$ of $\mathcal{B}(\mathcal{H})$ containing the identity operator I such that

$$\mathfrak{M} = \mathfrak{M}'',$$

where $\mathfrak{M}''$ is the double commutant of $\mathfrak{M}$.

To von Neumann algebras, it is possible to associate some *unbounded* operators called *affiliated* operators. Although, these operators are densely defined and *closed*.

Definition A.1.7 Let $\mathcal{H}$ be a Hilbert space and $\mathcal{D}$ be a subspace of $\mathcal{H}$. An operator $T : \mathcal{D}(T) \to \mathcal{H}$ is said to be *closed* if for every sequence $\{x_n\}$ in $\mathcal{D}(T)$ such that $x_n \to x$ and Tx_n converging in $\mathcal{H}$, then $x \in \mathcal{D}(T)$ and $Tx = \lim_{n\to\infty} Tx_n$.

Definition A.1.8 Let $\mathfrak{M}$ be a von Neumann algebra. An operator T is *affiliated with a von Neumann algebra* and we will write $T\eta\mathfrak{M}$ if T is densely defined, closed and $TU \supseteq UT$ for every unitary operator U in $\mathfrak{M}'$.

A.2 Spectrum of a Banach *-algebra

Let $(\mathfrak{A}, \mathfrak{A}_0)$ be a unital Banach *-algebra. An important notion is given by the *spectrum* of an element $a \in \mathfrak{A}$. The definition is purely algebraic, but it is related to topological aspects of the Banach *-algebra

Definition A.2.1 Let $\mathfrak{A}_0$ be a Banach *-algebra with unit $\mathbb{1}$. The following set, denoted with $\sigma(x)$,

$$\sigma(x) := \{\lambda \in \mathbb{C} : \nexists(\lambda\mathbb{1} - x)^{-1} \in \mathfrak{A}_0\}$$

is said to be the *spectrum*. The *resolvent set*, indicated as $\rho(x)$, is given by $\rho(x) := \mathbb{C}\backslash\sigma(x)$.

$\sigma(x)$ is always nonempty and closed, for every $x \in \mathfrak{A}_0$. Hence, we can compute the *spectral radius* $r(x)$ defined below.

Theorem A.2.2 *Let $\mathfrak{A}_0$ be a C*-algebra. If $a \in \mathfrak{A}$, then*

$$r(x) := \sup\{|\lambda| : \lambda \in \sigma(x)\} = \inf_n \|a^n\|^{\frac{1}{n}} = \lim_{n\to\infty} \|a^n\|^{\frac{1}{n}}.$$

Moreover,

- *if $x \in \mathfrak{A}_0$ is normal, i.e. $xx^* = x^*x$, then $\|x\| = r(x)$;*
- *if $x = x^*$ then $\sigma(x) \subset \mathbb{R}$;*
- *$x \in \mathfrak{A}_0$ such that $x = x^*$ is positive if $\sigma(x) \subset [0, +\infty)$ and this is equivalent to the existence of $y \in \mathfrak{A}_0$ such that $x = y^*y$.*

A.3 *-Representations and positive functionals

According to Definition A.1.6, every von Neumann algebra is a *-subalgebra of $\mathcal{B}(\mathcal{H})$ for a certain Hilbert space $\mathcal{H}$. This is actually true also for C*-algebras, as proved in the celebrated Gelfand-Naimark Theorem.

We need to introduce first a class of functionals useful to construct *representations.

Definition A.3.1 Let $\mathfrak{A}_0$ be a C*-algebra. A **-representation* of $\mathfrak{A}_0$ is a *-homomorphism $\pi : \mathfrak{A}_0 \to \mathcal{B}(\mathcal{H})$, i.e. a linear map preserving sum, product, scalar product and involution.

Definition A.3.2 Let $\mathfrak{A}_0$ be a Banach *-algebra. A *-representation π is said to be

- *faithful* if $\pi(x) \neq 0$ implies $x \neq 0$;
- *continuous* if there exists $\gamma > 0$ such that $\|\pi(x)\| \leq \gamma\|x\|$ for all $x \in \mathfrak{A}_0$;
- *isometric* if $\|\pi(x)\| = \|x\|$ for every $x \in \mathfrak{A}$.

Theorem A.3.3 *Let $\mathfrak{A}_0$ be a C*-algebra and π a *-representation of $\mathfrak{A}_0$. Then π is automatically continuous. Moreover, if π is faithful, then π is automatically isometric.*

Definition A.3.4 Let $\mathfrak{A}_0$ be a C$^*-$algebra. A linear functional ω on $\mathfrak{A}_0$ is said to be *positive* if $\omega(x^*x) \geq 0$ for every $x \in \mathfrak{A}_0$.

Positive functionals own many interesting properties. In particular, for every C*-algebra, there exist as many functionals as many points in $\mathfrak{A}_0$ and all of them are automatically continuous.

Theorem A.3.5 *Let $\mathfrak{A}_0$ be a Banach *-algebra with unit $\mathbb{1}$ and let ω be a functional. Then*

- *ω is continuous and $\|\omega\| = \omega(\mathbb{1})$;*
- *$|\omega(y^*xy)| \leq \|x\|\omega(y^*y)$ for every $x, y \in \mathfrak{A}_0$;*

Theorem A.3.6 *Let $\mathfrak{A}_0$ be a C*-algebra with unit $\mathbb{1}$. If $x \in \mathfrak{A}_0$, then there exists a positive functional ω on $\mathfrak{A}_0$ such that $\omega(\mathbb{1}) = 1$ and $\omega(x^*x) = \|x\|_0^2$.*

From *-representations, we can define positive functionals, i.e. those functionals that are positive on positive elements.

Theorem A.3.7 *Let $\mathfrak{A}_0$ be a C*-algebra with unit $\mathbb{1}$. Let π be a *-representation of $\mathfrak{A}_0$ in a Hilbert space $\mathcal{H}$ and let ξ be a unit vector in $\mathcal{H}$. Then the map $\omega : \mathfrak{A}_0 \to \mathbb{C}$ defined as $\omega(x) = \langle \pi(x)\xi|\xi\rangle$ is a positive linear functional.*

Theorem A.3.8 *Let $\mathfrak{A}_0$ be a unital C^*-algebra. Given a positive functional ω, then there exists a *-representation π_ω on a Hilbert space $\mathcal{H}_\omega$ and a cyclic vector $\xi_\omega \in \mathcal{H}_\omega$, i.e.the subspace $\pi_\omega(\mathfrak{A}_0)\xi_\omega$ is dense in $\mathcal{H}_\omega$, such that*

$$\omega(x) = \langle \pi_\omega(x)\xi_\omega|\xi_\omega\rangle.$$

*The *-representation π_ω is unique up to unitary equivalence.*

Theorem A.3.9 *Every unital C^*-algebra is isometrically *-isomorphic to a C^*-subalgebra of $\mathcal{B}(\mathcal{H})$. This is equivalent to the existence of a faithful *-representation of $\mathfrak{A}_0$ on a Hilbert space $\mathcal{H}$*

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