

UNIVERSITY OF WARWICK



SECOND YEAR ESSAY

MA213

Knots and their Invariants

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Contents

| | | |
|----------|---------------------------------------|-----------|
| 1 | Knots: an introduction | 2 |
| 1.1 | Knots in time | 2 |
| 1.2 | Why knots? | 3 |
| 2 | Knots: a mathematical approach | 3 |
| 2.1 | Defining knots | 3 |
| 2.2 | The Reidemeister moves | 5 |
| 3 | Invariants | 7 |
| 3.1 | Crossing number | 7 |
| 3.2 | Unknotting number | 8 |
| 3.3 | Tricolourability | 9 |
| | Closing remarks | 12 |
| | References | 12 |

1 Knots: an introduction

1.1 Knots in time

From decorative art pieces (Figure 1) to information communication [2] (Figure 2), knots have been vital to our innovation as humans. Their role in mathematics however, is perhaps most key.



Figure 1: Example of a decorative, Celtic knot [1].

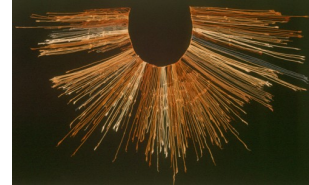


Figure 2: A method of transferring information called a Quipu. Often used by Incas [3].

Knot theory is a relatively new (first developed in the 18th century, with most advances coming much later [11]) subfield of the itself modern field of topology.

Mathematicians love to classify interesting, abstract objects: numbers, groups, rings, sets... But originally, our interest in the classification of knots was not simply motivated as to extend our mathematical knowledge. English physicist William Thomson (known as Lord Kelvin) believed knots could be the answer to our classification of all matter [17, § 1] - and hence the classification of knots began. But how do we classify knots?

If we first consider a simple knot as one may make using a piece of string - see Figure 3.



Figure 3: Our standard concept of a 'knot'.

It is clear that given freedom of movement we could once again unknot Figure 3. But now tie the ends to get a closed curve. See Figure 4.



Figure 4: A knot with the ends attached.

Can we now unknot this object? The simplest object we could make with a closed, continuous curve is quite clearly a circular knot and we call this the unknot (see Figure 5). So, is Figure 4 also the unknot with some extra twists and crossings? In our classification, we would surely want our knots in their simplest form.

Later on we will in fact prove Figure 4 is not the unknot and so a figure with three crossings exists that cannot be unknotted (this term refers to the ability - intuitively - to untangle our knot into the unknot). To see why it is not a 2 crossing knot, attempt to draw the possible 2 crossing knots.

And so, thanks to Lord Kelvin, we began to tabulate knots via their crossing number, making sure they could not be simplified further.

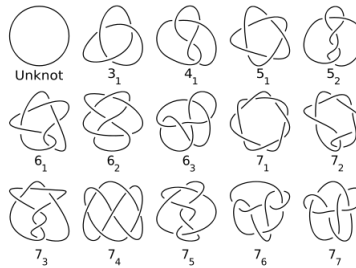


Figure 5: A knot table up to crossing number 7 [5].

1.2 Why knots?

Quite amazingly, knots can be seen to appear useful in various other areas of science. With the Sliding Knot Fish [17, pp. 50–51] and DNA analysis [8, pp. 181–195] in biology, the study of molecular synthesis and chirality [8, pp. 195–204] in chemistry and also that of quantum field theory [17, §. 8] in physics. It is clear therefore that a strict mathematical model will help provide key information to numerous other fields. However, the main questions that mathematicians have pondered are much more generalised (and as of yet lack complete solutions). Two such questions are:

1. (*The Classification Problem*) When are two knots equivalent?
2. (*The Unknotting Problem*) When is a knot the unknot?

Clearly problem 2 is a subset of problem 1, however, even this simplification of the problem has yet to be fully resolved [4].

The problem of classification is just one of many in knot theory, but in this essay it shall be explored how invariants have helped open up this fundamental question.

2 Knots: a mathematical approach

2.1 Defining knots

Definition 1. Given a homeomorphic function $f : \mathbb{S}^1 \rightarrow K \subset \mathbb{R}^3$. That is, a continuous, bijective function with a continuous inverse, with:

$$\mathbb{S}^1 = \{(x, y) : x^2 + y^2 = 1, x \text{ and } y \text{ real numbers}\}$$

Then let the image of f be equal to K and define K to be a **knot**.

Notice that this definition ensures a knot is a smooth, *closed* curve - as suggested to be desirable in section 1.1. In addition to this definition, the following subset of knots is further required in order to simplify our proofs and appeal to combinatorial topology.

Definition 2 ([9]). A polygonal knot is a knot whose image in \mathbb{R}^3 is the union of a finite set of line segments.

From now on it is assumed that all knots mentioned are polygonal without specifically stating so (even when represented as curves). This new class of knots then allows us to define strict, mathematical transformations that mimic the act of moving a knot in space.

Definition 3 (Elementary Knot Moves (EKMs) [13, p. 7]). For a knot K with arbitrary, adjacent vertices A and B :

(i) For any edge $AB \in K$ we may take a point $C \in AB$ and divide the edge AB into edges $AC \cup CB$.

(ii) Given $C \in \mathbb{R}^3$ such that $C \notin K$. If the triangle formed by joining A , B and C does not intersect K , we may replace edge AB with edges AC and CB .

We may also take the converse of these moves.

Definition 4. Two knots K_1 and K_2 are equivalent if we can obtain one from the other by applying EKMs. Denote this by $K_1 \sim K_2$.

A final fundamental concept to define before the exploration of knots begins is that of projections [12, pp. 6–7].

Definition 5. Given a knot K . Let:

$$\mathcal{P} : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \text{ with } \mathcal{P}(x, y, z) = (x, y, 0)$$

Then we call $\mathcal{P}(K)$ a **projection** of K .

This definition is then restricted further with the following, to ensure our projections can be used consistently and without confusion.

Definition 6. If the following conditions hold for \mathcal{P} :

(i) $\mathcal{P}(K)$ has finite points of intersection.

(ii) For any $p \in \mathcal{P}(K)$ such that p is a point of intersection, $|\mathcal{P}^{-1}(p) \cap K| = 2$.

(iii) For any v - a vertex of K , $|\mathcal{P}^{-1}(v) \cap K| = 1$.

Then \mathcal{P} is a **regular projection**. When drawing the diagram for a regular projection, at a point of intersection, the point from the knot with the greater z -value is drawn over (called an *overcrossing*) the secondary point (the *undercrossing*).

Example 2.1. It is now clear how the diagrams drawn in section 1 make mathematical sense (especially those of Figure 5). Consider the following example demonstrating our newfound mathematical object - the knot.

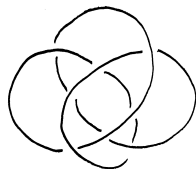


Figure 6: A diagram of the regular projection of a knot.

Using EKMs, it can now be shown that the knots in Figure 7 and 8 are in fact equivalent.

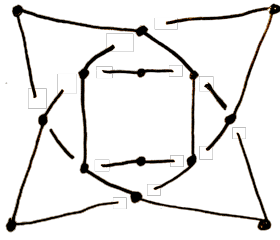


Figure 7: A polygonal representation of the knot in Figure 6.

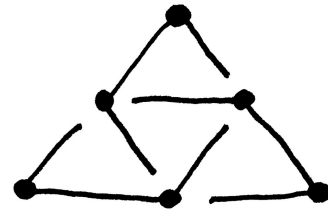


Figure 8: A polygonal representation of the 3_1 knot - the Trefoil knot.

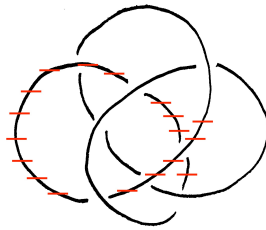


Figure 9:

The equivalence is easiest to see on our original diagram. Notice the shaded area in Figure 9 may be manipulated via EKMs and made into a straight line. Producing 3 crossings and hence we see the Trefoil.

Example 2.2. If there exists a homeomorphism of \mathbb{R}^3 onto itself, that maps a knot to a polygonal knot, this knot is called *tame* - otherwise it is *wild*. Our use of tame knots helps avoid problems with limit points and pathological behaviour associated to knots such as the example in Figure 10.

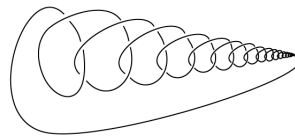
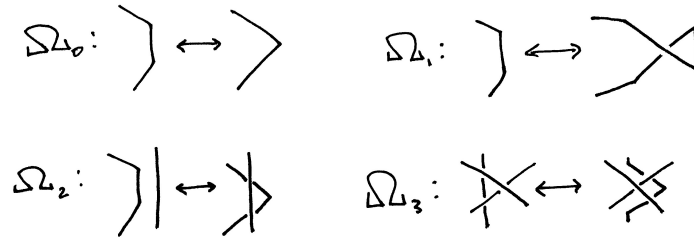


Figure 10: A wild knot. Notice the limit point as each iteration gets smaller [7].

2.2 The Reidemeister moves

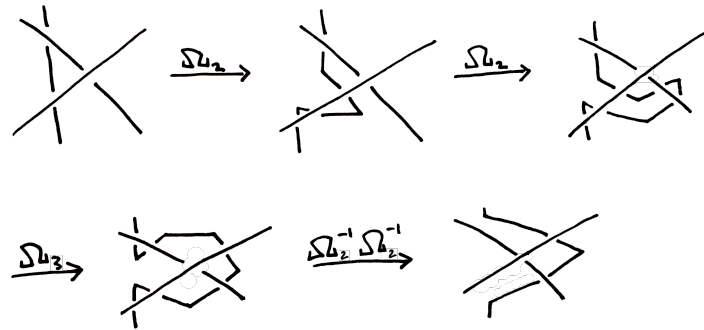
We may now apply EKMs to a knot K such that we transform its respective projection. The German mathematician Kurt Reidemeister however, discovered a set of moves we may apply to the diagram of $\mathcal{P}(K)$ - the regular projection of K , such that its corresponding knot may be transformed into any knot K^* such that $K \sim K^*$. We state this more formally in our first theorem. But before that we must state the moves themselves.

Definition 7 (The Reidemeister moves [14]). Pictured below are the moves, with their inverses given by traversing the arrows backwards. In each move it is clear how applying EKM (ii) allows us to apply the move along the forward arrow (similarly with the inverses).



The Reidemeister moves may at first seem quite arbitrary. They are very specifically presented and when applying them, the correct overcrossings and undercrossings must be seen. However, we accept the definitions to apply locally to a diagram and may form many other seemingly different moves from their composition.

Example 2.3. The following demonstrates how we may apply multiple Reidemeister moves to perform a seemingly different move to the original four (notice the alternate, central crossing).



We can clearly see that this equivalence will hold for the respective knots that the diagrams represent, and we will in fact see that the respective diagrams for any two equivalent knots may be navigated between using a set of Reidemeister moves.

One final definition is needed before the main theorem of this section.

Definition 8. Two regular knot diagrams D_1 and D_2 are equivalent if we can obtain one from the other by applying $\Omega_0, \Omega_1, \Omega_2, \Omega_3$ or their inverses. Denote this by $D_1 \sim D_2$.

And so, with all that we understand up to this point, Reidemeister then showed something incredible.

Reidemeister's Theorem ([13, p. 50][14]). *Given D and D^* are regular diagrams of two knots K and K^* , respectively. Then*

$$K \sim K^* \Leftrightarrow D \sim D^*$$

A standard proof of this is given by Burde and Zieschang [10], but I recommend looking to Murasugi [13] first, who follows a more combinatorial approach. The full proof is long and so here we shall omit it, but the reader is advised to look over it as given by Murasugi [13, pp. 52-56].

This theorem is astonishing. Given any property we can find that stays the same between Reidemeister moves, we now know that same property will stay the same between any equivalent knots. And so our exploration of invariants begins.

3 Invariants

An invariant, as the name suggests, is simply a mathematical object that does not vary as we alter our knot through EKMs. So we say the function $\mathcal{I} : \{K : K \text{ is a knot}\} \rightarrow X$, where X is any set, is an invariant if $\mathcal{I}(K) = \mathcal{I}(K^*)$ when $K \sim K^*$.

Remark. From this we also have that

$$\mathcal{I}(K) \neq \mathcal{I}(K^*) \Rightarrow K \not\sim K^*$$

And so the strength of invariants lies in their ability to tell knots apart.

3.1 Crossing number

By definition 6, the regular projection of a knot K has finite points of intersection, and so a corresponding diagram D of K has finite crossing points.

Example 3.1. The following shows the unknot and an equivalent diagram obtained by applying Ω_2 .



If we denote $c(D)$ to be the number of crossing points of a regular diagram D , then clearly by the above example, $c(D)$ is **not** an invariant. But all we need is something a little stronger, and so define:

$$\mathcal{D}_K = \{D : \mathcal{P}(K^*) = D \text{ any regular diagram of } K^* \text{ with } K^* \sim K\}$$

Theorem 3.1 ([13, p. 57]). *For a knot K let:*

$$c(K) = \min_{\mathcal{D}_K}(c(D))$$

Then $c(K)$ is a knot invariant.

Proof. Suppose $\arg \min_{\mathcal{D}_K}(c(D)) = D_0$. Now let K^* be any knot such that $K \sim K^*$, with corresponding, minimum, regular diagram D_0^* . By Reidemeister's theorem, D_0^* is also a regular diagram for K and so by definition $c(D_0) \leq c(D_0^*)$. Similarly, D_0 is also a regular diagram for K^* and so $c(D_0^*) \leq c(D_0)$. Therefore, $c(D_0) = c(D_0^*)$.

Hence, $c(D_0)$ is the minimum number of crossing points for all knots equivalent to K and $c(K)$ is our first invariant. \square

3.2 Unknotting number

We have seen that the crossing number is a fundamental property of knots, and so, does intuition lead us to anymore fundamental ways to assign a knot a specific value? Given the crossing number classifies the minimal form of a knot, is it then possible to unknot that knot by some step-by-step procedure?

Definition 9. In a given, regular diagram D of K , take a crossing point and switch the vertex with greater z value for the one with smaller z value. Call this an **unknotting move**.

Applying an unknotting move allows us to exchange an overcrossing with an undercrossing. It seems intuitive that multiple unknotting moves will unknot a given knot (the reader may wish to try this on some knots from Figure 5), but in order to utilise unknotting as an invariant we must show this rigorously.

Theorem 3.2. *Given any knot K there exists a sequence of unknotting moves that, along with the use of Reidemeister moves, will take its regular diagram D to the unknot.*

Proof. We induct on the crossing points $c(D)$. $c(D) = 0 \Rightarrow D = \mathbf{0}$ (where $\mathbf{0}$ is the unknot) and the base case is clear.

Now, assume true for any regular diagram D such that $c(D) < m \in \mathbb{N}$ and take a diagram D such that $c(D) = m$. Next choose an arbitrary point p on D such that $|\mathcal{P}^{-1}(p) \cup K| = 1$. Follow an arbitrary, fixed orientation around D . If ever an undercrossing is reached, perform an unknotting move, and as soon as we reach a crossing already visited we will have formed a loop with a single crossing. See Figure 11.



Figure 11

Everything below the loop may be moved clear of it, as it does not intersect it, and then the crossing itself may be removed via Ω_1 . Denote this new regular diagram by D^* . $c(D^*) = m - 1$ and by the induction hypothesis, D^* may now be taken to the unknot via more sequences of unknotting moves and Reidemeister moves. \square

Much like with crossing number, given we define $u(D)$ to be the minimum unknotting moves required to unknot D , this does not present an invariant of K (see Murasugi [13, p. 63]).

Theorem 3.3. *For a knot K let:*

$$u(K) = \min_{\mathcal{D}_K} (u(D))$$

Then $u(K)$ is a knot invariant.

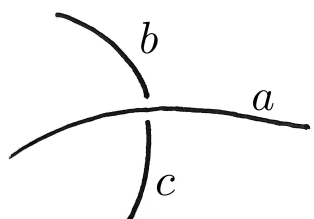
Proof. Identical to that of theorem 3.1. \square

3.3 Tricolourability

The two invariants we have met thus far are intuitive to our sense of what a knot is. However, in practice their calculation can be extremely difficult. Adams points out that $u(K)$ isn't necessarily realised in a regular projection of minimum crossings [8, pp. 67-69]. Furthermore, we have $c(\mathbf{0}) = 0$ and $c(3_1) \leq 3$, but how do we then show $\mathbf{0} \approx 3_1$? Our next step is to introduce colourability.

Definition 10. Given a knot K , call the section $s \in K$ between two undercrossings a **strand**. Further, let $S(D)$ be the set of all strands of a regular diagram D of K .

Definition 11 ([16, p. 13]). The function $x : S(D) \rightarrow \mathbb{Z}/n\mathbb{Z}$ is a colouring modulo n if it satisfies the following for each crossing of D :



$2x(a) \equiv x(b) + x(c) \pmod{n}$

Remark. If $x(s) \equiv \gamma$, some constant, for all $s \in S(D)$ then we note this as a trivial colouring since this gives

$$\begin{aligned} 2x(a) &\equiv x(b) + x(c) \pmod{n} \\ \Rightarrow 2\gamma &\equiv \gamma + \gamma \pmod{n} \\ \Rightarrow 2\gamma &\equiv 2\gamma \pmod{n} \end{aligned}$$

Which is true for all γ and n .

Definition 12. A knot K is n -colourable if there exists a colouring x (modulo n), that is non-trivial, of a regular diagram of K .

Example 3.2. If a knot is colourable for $n = 3$, we say it is **tricolourable**. The definition we have given is equivalent to colouring the strands of a knot with three colours, and at each crossing the strands are either all the same colour or all different colours. We demonstrate that the trefoil is tricolourable in Figure 12.

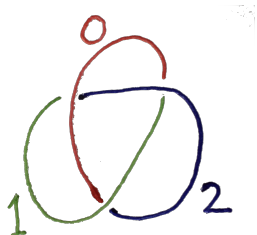
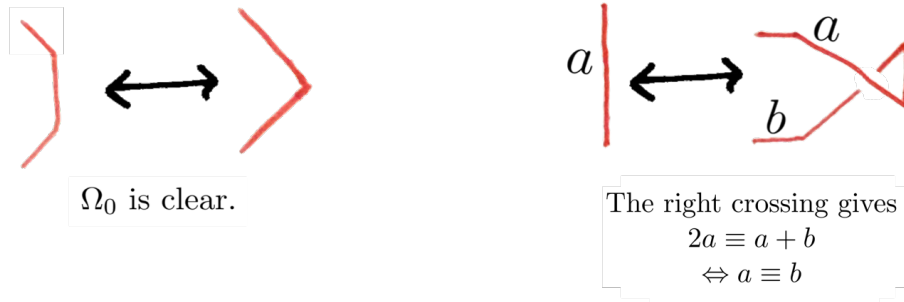


Figure 12: The 3_1 knot 3-coloured.

In its regular, minimal diagram, the unknot must clearly be trivially coloured (only having one strand). We have also seen that the trefoil may be tricoloured. Now, if colourability is preserved across equivalent diagrams, not only will we have found another invariant, but we will finally have that $\mathbf{0} \approx 3_1$.

Theorem 3.4 ([15, § 2.2]). *Given a knot K with corresponding, regular diagram D , if there exists a colouring $x(D)$ (modulo n), then for all D^* such that $D \sim D^*$, there exists a colouring $\tilde{x}(D^*)$ (modulo n).*

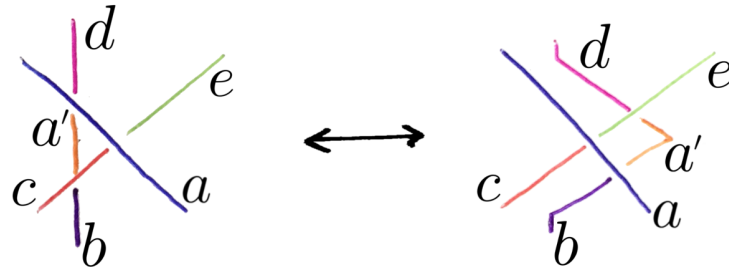
Proof. We show the Reidemeister moves preserve colourability between equivalent diagrams. From here onwards we work modulo n without specifically stating so.



The preservation of colour in Ω_0 is clear, whilst for Ω_1 it is shown above that if a loop is colourable then the strands are coloured the same. Hence, a single colour is needed for either direction.



For Ω_2 , if a and b are coloured the same, we simply take all strands on the right coloured the same. Alternatively, with $a \neq b$, from crossings on the right, we may colour a the same as a'' , and assign a third colour to a' . More simply, the crossing equations for the left diagram are satisfied if and only if the equations for the right are.



For Ω_3 , considering the top, middle and bottom crossings of the left hand side diagram gives us, respectively:

$$\begin{aligned} (c1) \quad & 2a \equiv e + c \\ (c2) \quad & 2a \equiv d + a' \\ (c3) \quad & 2c \equiv b + a' \end{aligned}$$

Where (c2) and (c3) give $2a - d \equiv 2c - b \Leftrightarrow 2a \equiv 2c - b + d$. Similarly with the right diagram:

$$\begin{aligned} (\tilde{c}1) \quad & 2e \equiv d + a' \\ (\tilde{c}2) \quad & 2a \equiv b + a' \\ (\tilde{c}3) \quad & 2a \equiv c + e \end{aligned}$$

($\tilde{c}1$) and ($\tilde{c}2$) then give:

$$\begin{aligned} & 2a - b \equiv 2e - d \\ \Leftrightarrow & -2a \equiv -2e - b + d \\ \Leftrightarrow & 2a \equiv (-2e - b + d) + 2(c + e) \quad [\text{using } (\tilde{c}3)] \\ \Leftrightarrow & 2a \equiv 2c - b + d \end{aligned}$$

And so, our congruence equations for the left diagram are satisfied if and only if those for the right are as well. Thus for $\Omega_0, \Omega_1, \Omega_2$ and Ω_3 , colouring of the left hand side diagram is achievable if and only if colouring of the right is. \square

Corollary 3.5. *The colourability of a knot K is a knot invariant.*

Proof. Immediate from Reidemeister's theorem. \square

Corollary 3.6. *The unknot and trefoil knot are not equivalent.*

Proof. For a knot K define

$$\mathcal{X}(K) = \begin{cases} 1 & \text{if } K \text{ is colourable} \\ 0 & \text{otherwise} \end{cases}$$

Then \mathcal{X} is a knot invariant and $\mathcal{X}(\mathbf{0}) \neq \mathcal{X}(3_1) \Rightarrow \mathbf{0} \not\approx 3_1$. \square

Corollary 3.7. $c(3_1) = 3$

Proof. Figure 13 depicts an example of a one crossing knot (with crossings left ambiguous) and Figure 14 that of two crossing knots. It is clear that each possibility is in fact equivalent to one or more unknots. Hence, $c(K) = 0, 1, 2 \Rightarrow K = \mathbf{0}$ since the unknot is the only knot with crossing number zero. Then, since we have presented a diagram of the trefoil with three crossings, the result is clear. \square



Figure 13



Figure 14

Closing remarks

Our closing results appear elementary in nature, seemingly intuitive before any theory was developed. However, the solid mathematical foundation we have laid paves the way to less intuitive knot theoretic results. The curious reader may wish to explore other, more involved invariants, such as the Jones and Alexander polynomial. The modules MA3F1 Introduction to Topology and, of course, MA3F2 Knot Theory, lead on very naturally from this essay.

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