Using Congruence Theorems

6

6.1

Time to Get Right

Right Triangle Congruence Theorems

LEARNING GOALS

In this lesson, you will:

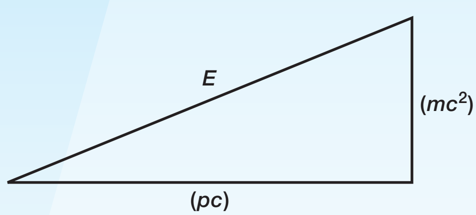
- Prove the Hypotenuse-Leg Congruence Theorem using a two-column proof and construction.
- Prove the Leg-Leg, Hypotenuse-Angle, and Leg-Angle Congruence Theorems by relating them to general triangle congruence theorems.
- Apply right triangle congruence theorems.

KEY TERMS

- Hypotenuse-Leg (HL) Congruence Theorem
- Leg-Leg (LL) Congruence Theorem
- Hypotenuse-Angle (HA) Congruence Theorem
- Leg-Angle (LA) Congruence Theorem

You know the famous equation $E = mc^2$. But this equation is actually incomplete. The full equation is $E^2 = (m^2)^2 + (pc)^2$, where E represents energy, m represents mass, p represents momentum, and c represents the speed of light.

You can represent this equation on a right triangle.



So, when an object's momentum is equal to 0, you get the equation $E = mc^2$.

But what about a particle of light, which has no mass? What equation would describe its energy?

PROBLEM 1 Hypotenuse-Leg (HL) Congruence Theorem

1. List all of the triangle congruence theorems you explored previously.

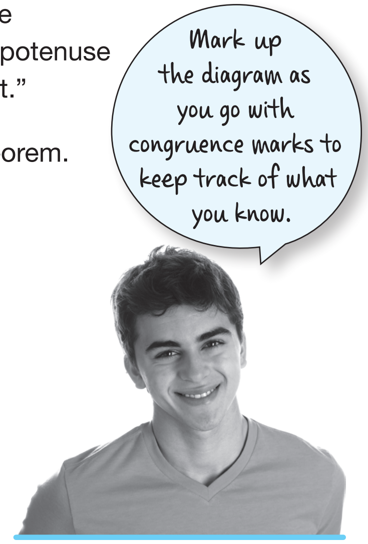
The congruence theorems apply to all triangles. There are also theorems that only apply to right triangles. Methods for proving that two right triangles are congruent are somewhat shorter. You can prove that two right triangles are congruent using only two measurements.

2. Explain why only two pairs of corresponding parts are needed to prove that two right triangles are congruent.

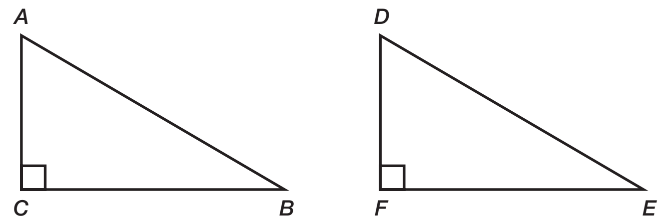
3. Are all right angles congruent? Explain your reasoning.

6

The **Hypotenuse-Leg (HL) Congruence Theorem** states: "If the hypotenuse and leg of one right triangle are congruent to the hypotenuse and leg of another right triangle, then the triangles are congruent."



4. Complete the two-column proof of the HL Congruence Theorem.



Given: $\angle C$ and $\angle F$ are right angles

$\overline{AC} \cong \overline{DF}$

$\overline{AB} \cong \overline{DE}$

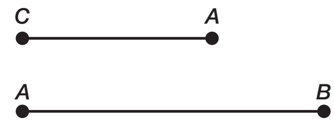
Prove: $\triangle ABC \cong \triangle DEF$

Statements	Reasons
1. $\angle C$ and $\angle F$ are right angles	
2. $\angle C \cong \angle F$	
3. $\overline{AC} \cong \overline{DF}$	
4. $\overline{AB} \cong \overline{DE}$	
5. $AC = DF$	
6. $AB = DE$	
7. $AC^2 + CB^2 = AB^2$	
8. $DF^2 + FE^2 = DE^2$	
9. $AC^2 + CB^2 = DF^2 + FE^2$	
10. $CB^2 = FE^2$	
11. $CB = FE$	
12. $\overline{CB} \cong \overline{FE}$	
13. $\triangle ABC \cong \triangle DEF$	

6

You can also use construction to demonstrate the Hypotenuse-Leg Theorem.

5. Construct right triangle *ABC* with right angle *C*, given leg $\overline{CA}$ and hypotenuse $\overline{AB}$. Then, write the steps you performed to construct the triangle.



a. How does the length of side $\overline{CB}$ compare to the lengths of your classmates' sides $\overline{CB}$?

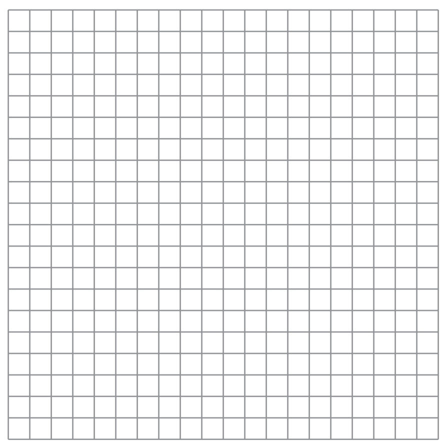
b. Use a protractor to measure ∠*A* and ∠*B* in triangle *ABC*. How do the measures of these angles compare to the measures of your classmates' angles *A* and *B*?

c. Is your triangle congruent to your classmates' triangles? Why or why not?

Through your two-column proof and your construction proof, you have proven that Hypotenuse-Leg is a valid method of proof for any right triangle. Now let's prove the Hypotenuse-Leg Theorem on the coordinate plane using algebra.

6. Consider right triangle *ABC* with right angle *C* and points *A* (0, 6), *B* (8, 0), and *C* (0, 0).

a. Graph right triangle *ABC*.



b. Calculate the length of each line segment forming the sides of triangle *ABC* and record the measurements in the table.

Sides of Triangle *ABC*	Lengths of Sides of Triangle *ABC* (units)
$\overline{AB}$	
$\overline{BC}$	
$\overline{AC}$	

c. Rotate side AB, side AC, and $\angle C$ 180° counterclockwise about the origin. Then, connect points B' and C' to form triangle $A'B'C'$. Use the table to record the coordinates of triangle $A'B'C'$.

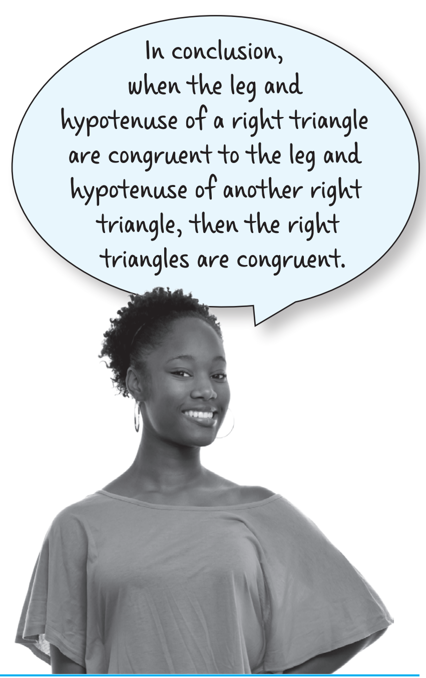
Coordinates of Triangle *ABC*	Coordinates of Triangle *A′B′C′*
$A(0,6)$	
$B(8,0)$	
$C(0,0)$	

d. Calculate the length of each line segment forming the sides of triangle $A'B'C'$, and record the measurements in the table.

Sides of Triangle *A′B′C′*	Lengths of Sides of Triangle *A′B′C′* (units)
$\overline{A'B'}$	
$\overline{B'C'}$	
$\overline{A'C'}$	

e. What do you notice about the side lengths of the image and pre-image?

f. Use a protractor to measure $\angle A$, $\angle A'$, $\angle B$, and $\angle B'$. What can you conclude about the corresponding angles of triangle ABC and triangle $A'B'C'$?



You have shown that the corresponding sides and corresponding angles of the pre-image and image are congruent. Therefore, the triangles are congruent.

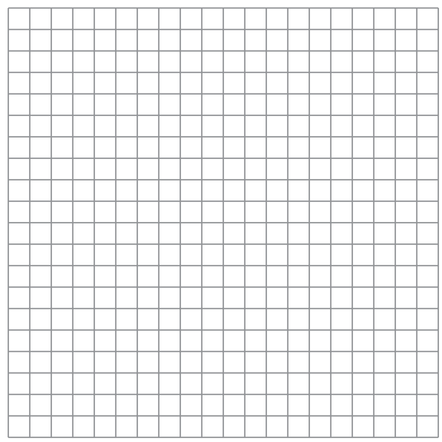
PROBLEM 2 Proving Three More Right Triangle Theorems

You used a two-column proof, a construction, and rigid motion to prove the Hypotenuse-Leg Congruent Theorem. There are three more right triangle congruence theorems that we are going to explore. You can prove each of them using the same methods but you'll focus on rigid motion in this lesson.

The **Leg-Leg (LL) Congruence Theorem** states: "If two legs of one right triangle are congruent to two legs of another right triangle, then the triangles are congruent."

1. Consider right triangle *ABC* with right angle *C* and points *A* (0, 5), *B* (12, 0), and *C* (0, 0).

a. Graph right triangle *ABC*.



b. Calculate the length of each line segment forming the sides of triangle *ABC,* and record the measurements in the table.

Sides of Triangle *ABC*	**Lengths of Sides of Triangle *ABC* (units)**
$\overline{AB}$	
$\overline{BC}$	
$\overline{AC}$	

c. Translate side *AC*, and side *BC,* to the left 3 units, and down 5 units. Then, connect points *A′*, *B′* and *C′* to form triangle *A′B′C′*. Use the table to record the image coordinates.

Coordinates of Triangle *ABC*	Coordinates of Triangle *A′B′C′*
A(0, 5)	
B(12, 0)	
C(0, 0)	

d. Calculate the length of each line segment forming the sides of triangle *A′B′C′*, and record the measurements in the table.

Sides of Triangle *A′B′C′*	Lengths of Sides of Triangle *A′B′C′* (units)
$\overline{A'B'}$	
$\overline{B'C'}$	
$\overline{A'C'}$	

e. What do you notice about the side lengths of the image and pre-image?

f. Use a protractor to measure ∠*A*, ∠*A′*, ∠*B*, and ∠*B′*. What can you conclude about the corresponding angles of triangle *ABC* and triangle *A′B′C′*?

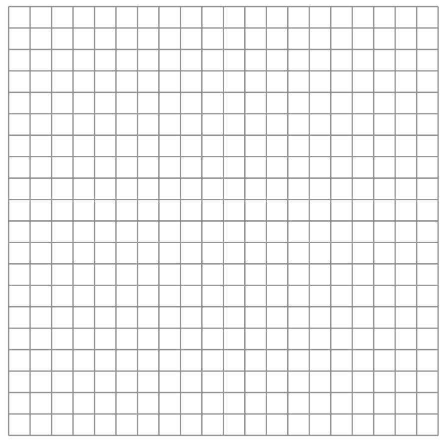
6

You have shown that the corresponding sides and corresponding angles of the pre-image and image are congruent. Therefore, the triangles are congruent.

The **Hypotenuse-Angle (HA) Congruence Theorem** states: "If the hypotenuse and an acute angle of one right triangle are congruent to the hypotenuse and acute angle of another right triangle, then the triangles are congruent."

2. Consider right triangle *ABC* with right angle *C* and points *A* (0, 9), *B* (12, 0), and *C* (0, 0).

a. Graph right triangle *ABC* with right $\angle C$, by plotting the points *A* (0, 9), *B* (12, 0), and *C* (0, 0).



b. Calculate the length of each line segment forming the sides of triangle *ABC,* and record the measurements in the table.

Sides of Triangle *ABC*	Lengths of Sides of Triangle *ABC* (units)
$\overline{AB}$	
$\overline{BC}$	
$\overline{AC}$	

c. Translate side *AB*, and ∠*A,* to the left 4 units, and down 8 units. Then, connect points *A*′, *B*′ and *C*′ to form triangle *A*′*B*′*C*′. Use the table to record the image coordinates.

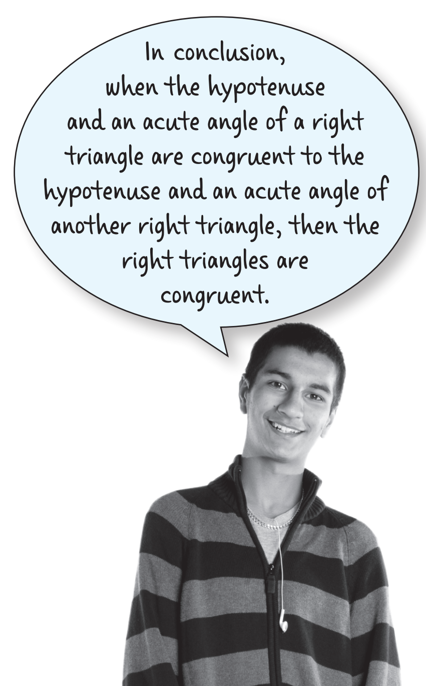
Coordinates of Triangle *ABC*	Coordinates of Triangle *A*′*B*′*C*′
A(0, 9)	
B(12, 0)	
C(0, 0)	

d. Calculate the length of each line segment forming the sides of triangle *A*′*B*′*C*′, and record the measurements in the table.

Sides of Triangle *A*′*B*′*C*′	Lengths of Sides of Triangle *A*′*B*′*C*′ (units)
$\overline{A'B'}$	
$\overline{B'C'}$	
$\overline{A'C'}$	

e. What do you notice about the side lengths of the image and pre-image?

f. Use a protractor to measure ∠*A*, ∠*A*′, ∠*B*, and ∠*B*′. What can you conclude about the corresponding angles of triangle *ABC* and triangle *A*′*B*′*C*′?



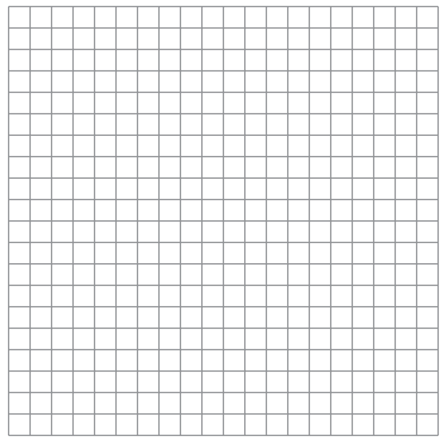
You have shown that the corresponding sides and corresponding angles of the pre-image and image are congruent. Therefore, the triangles are congruent.

6

The **Leg-Angle (LA) Congruence Theorem** states: "If a leg and an acute angle of one right triangle are congruent to a leg and an acute angle of another right triangle, then the triangles are congruent."

3. Consider right triangle *ABC* with right angle *C* and points *A* (0, 7), *B* (24, 0), and *C* (0, 0).

a. Graph right triangle *ABC* with right $\angle C$, by plotting the points *A* (0, 7), *B* (24, 0), and *C* (0, 0).



b. Calculate the length of each line segment forming the sides of triangle *ABC,* and record the measurements in the table.

Sides of Triangle *ABC*	**Lengths of Sides of Triangle *ABC* (units)**
$\overline{AB}$	
$\overline{BC}$	
$\overline{AC}$	

6

c. Reflect side AC, and $\angle B$ over the x-axis. Then, connect points A', B' and C' to form triangle $A'B'C'$. Use the table to record the image coordinates.

Coordinates of Triangle *ABC*	**Coordinates of Triangle *A′B′C′***
$A(0, 7)$	
$B(24, 0)$	
$C(0, 0)$	

d. Calculate the length of each line segment forming the sides of triangle $A'B'C'$, and record the measurements in the table.

Sides of Triangle *A′B′C′*	**Lengths of Sides of Triangle *A′B′C′* (units)**
$\overline{A'B'}$	
$\overline{B'C'}$	
$\overline{A'C'}$	

e. What do you notice about the side lengths of the image and pre-image?

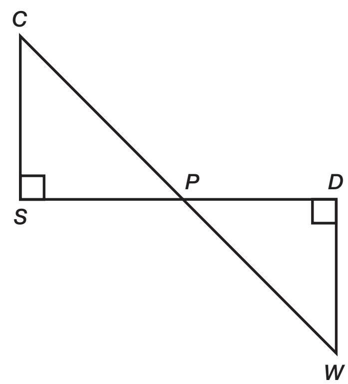
f. Use a protractor to measure $\angle A$, $\angle A'$, $\angle B$, and $\angle B'$. What can you conclude about the corresponding angles of triangle ABC and triangle $A'B'C'$?

You have shown that the corresponding sides and corresponding angles of the pre-image and image are congruent. Therefore, the triangles are congruent.

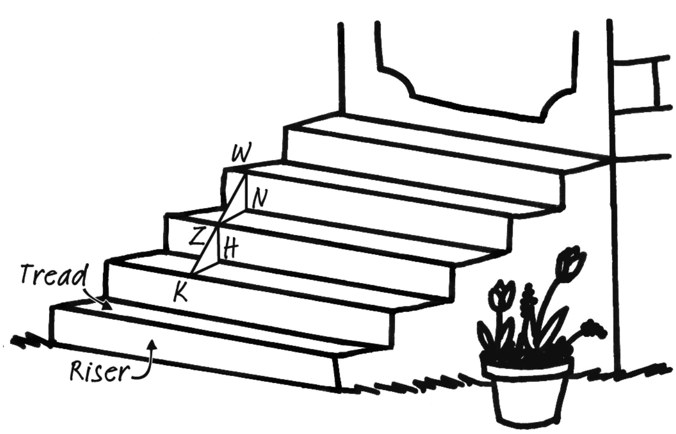
PROBLEM 3 Applying Right Triangle Congruence Theorems

Determine if there is enough information to prove that the two triangles are congruent. If so, name the congruence theorem used.

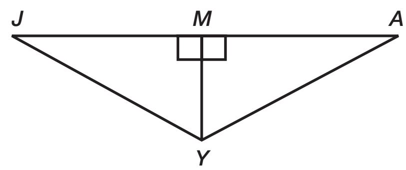
1. If $\overline{CS} \perp \overline{SD}$, $\overline{WD} \perp \overline{SD}$, and P is the midpoint of $\overline{CW}$, is $\triangle CSP \cong \triangle WDP$?



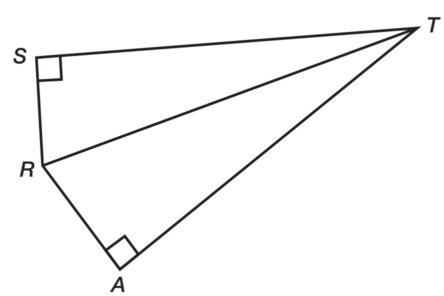
2. Pat always trips on the third step and she thinks that step may be a different size. The contractor told her that all the treads and risers are perpendicular to each other. Is that enough information to state that the steps are the same size? In other words, if $\overline{WN} \perp \overline{NZ}$ and $\overline{ZH} \perp \overline{HK}$, is $\triangle WNZ \cong \triangle ZHK$?



3. If $\overline{JA} \perp \overline{MY}$ and $\overline{JY} \cong \overline{AY}$, is $\triangle JYM \cong \triangle AYM$?



4. If $\overline{ST} \perp \overline{SR}$, $\overline{AT} \perp \overline{AR}$, and $\angle STR \cong \angle ATR$, is $\triangle STR \cong \triangle ATR$?



6

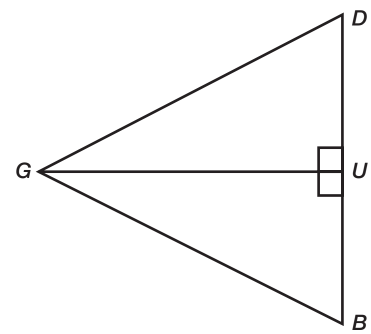
It is necessary to make a statement about the presence of right triangles when you use the Right Triangle Congruence Theorems. If you have previously identified the right angles, the reason is the definition of right triangles.

5. Create a proof of the following.

Given: $\overline{GU} \perp \overline{DB}$

$\overline{GB} \cong \overline{GD}$

Prove: $\triangle GUD \cong \triangle GUB$

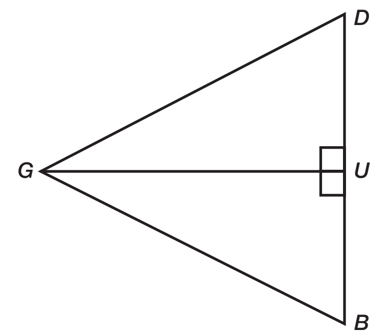


6

6. Create a proof of the following.

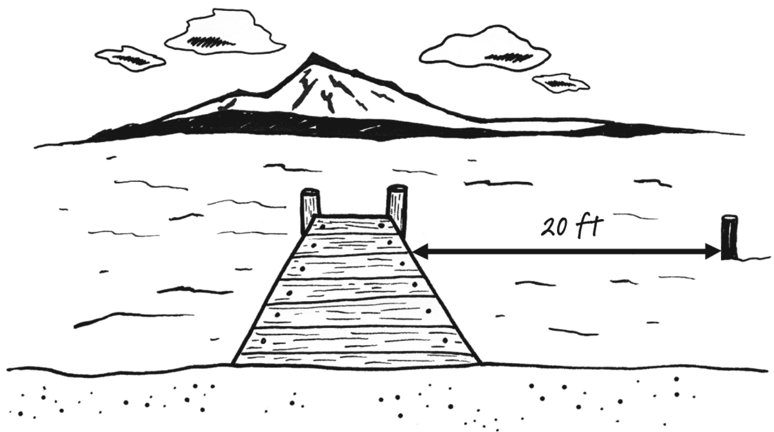
Given: $\overline{GU}$ is the $\perp$ bisector of $\overline{DB}$

Prove: $\triangle GUD \cong \triangle GUB$



6

7. A friend wants to place a post in a lake 20 feet to the right of the dock. What is the minimum information you need to make sure the angle formed by the edge of the dock and the post is a right angle?



6

Talk the Talk

1. Which triangle congruence theorem is most closely related to the LL Congruence Theorem? Explain your reasoning.

2. Which triangle congruence theorem is most closely related to the HA Congruence Theorem? Explain your reasoning.

3. Which triangle congruence theorem is most closely related to the LA Congruence Theorem? Explain your reasoning.

4. Which triangle congruence theorem is most closely related to the HL Congruence Theorem? Explain your reasoning.

6

Be prepared to share your solutions and methods.

CPCTC

Corresponding Parts of Congruent Triangles are Congruent

LEARNING GOALS

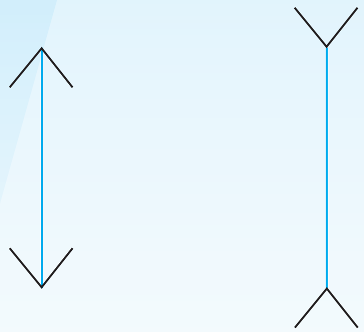
In this lesson, you will:

- Identify corresponding parts of congruent triangles.
- Use corresponding parts of congruent triangles are congruent to prove angles and segments are congruent.
- Use corresponding parts of congruent triangles are congruent to prove the Isosceles Triangle Base Angle Theorem.
- Use corresponding parts of congruent triangles are congruent to prove the Isosceles Triangle Base Angle Converse Theorem.
- Apply corresponding parts of congruent triangles.

KEY TERMS

- corresponding parts of congruent triangles are congruent (CPCTC)
- Isosceles Triangle Base Angle Theorem
- Isosceles Triangle Base Angle Converse Theorem

Which of the blue lines shown is longer? Most people will answer that the line on the right appears to be longer.



But in fact, both blue lines are the exact same length! This famous optical illusion is known as the Mueller-Lyer illusion. You can measure the lines to see for yourself. You can also draw some of your own to see how it almost always works!

PROBLEM 1 CPCTC

If two triangles are congruent, then each part of one triangle is congruent to the corresponding part of the other triangle. "**Corresponding parts of congruent triangles are congruent**," abbreviated as **CPCTC**, is often used as a reason in proofs. CPCTC states that corresponding angles or sides in two congruent triangles are congruent. This reason can only be used after you have proven that the triangles are congruent.

To use CPCTC in a proof, follow these steps:

Step 1: Identify two triangles in which segments or angles are corresponding parts.

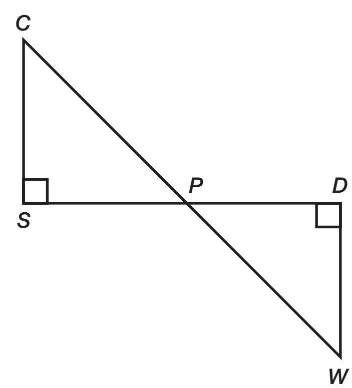
Step 2: Prove the triangles congruent.

Step 3: State the two parts are congruent using CPCTC as the reason.

1. Create a proof of the following.

Given: $\overline{CW}$ and $\overline{SD}$ bisect each other

Prove: $\overline{CS} \cong \overline{WD}$

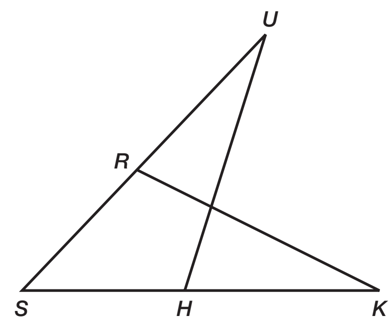


6

2. Create a proof of the following.

Given: $\overline{SU} \cong \overline{SK}$, $\overline{SR} \cong \overline{SH}$

Prove: $\angle U \cong \angle K$



6

PROBLEM 2 Isosceles Triangle Base Angle Theorem and Its Converse

CPCTC makes it possible to prove other theorems.

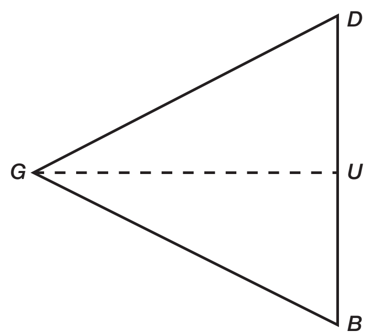
The **Isosceles Triangle Base Angle Theorem** states: “If two sides of a triangle are congruent, then the angles opposite these sides are congruent.”

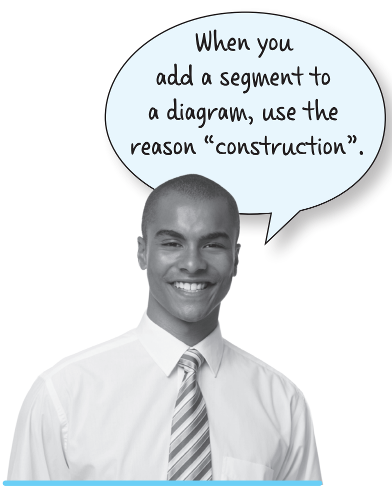
To prove the Isosceles Triangle Base Angle Theorem, you need to add a line to an isosceles triangle that bisects the vertex angle as shown.

1. Create a proof of the following.

Given: $\overline{GB} \cong \overline{GD}$

Prove: $\angle B \cong \angle D$





6

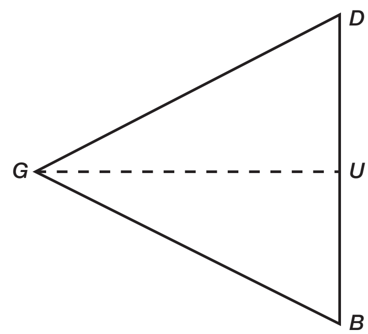
The **Isosceles Triangle Base Angle Converse Theorem** states: "If two angles of a triangle are congruent, then the sides opposite these angles are congruent."

To prove the Isosceles Triangle Base Angle Converse Theorem, you need to again add a line to an isosceles triangle that bisects the vertex angle as shown.

2. Create a proof of the following.

Given: $\angle B \cong \angle D$

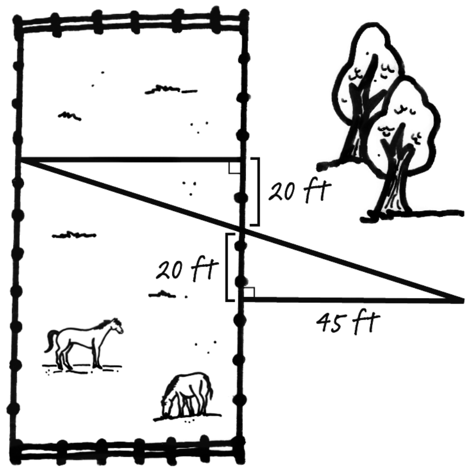
Prove: $\overline{GB} \cong \overline{GD}$



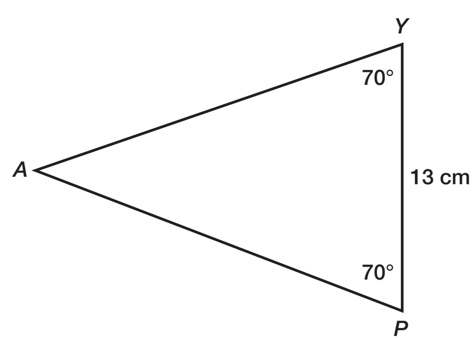
6

PROBLEM 3 Applications of CPCTC

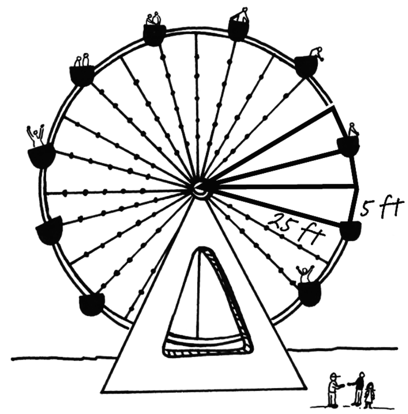
1. How wide is the horse's pasture?



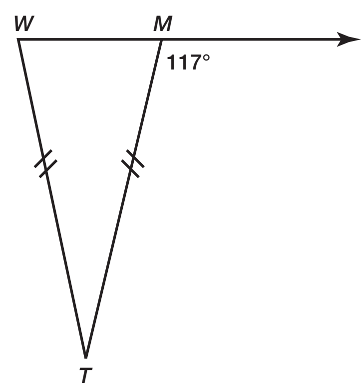
2. Calculate AP if the perimeter of $\triangle AYP$ is 43 cm.



3. Lighting booms on a Ferris wheel consist of four steel beams that have cabling with light bulbs attached. These beams, along with three shorter beams, form the edges of three congruent isosceles triangles, as shown. Maintenance crews are installing new lighting along the four beams. Calculate the total length of lighting needed.

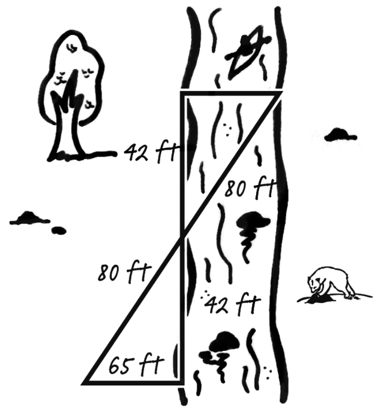


4. Calculate $m\angle T$.



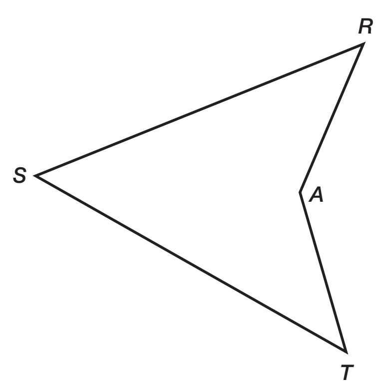
6

5. What is the width of the river?



6. Given: $\overline{ST} \cong \overline{SR}$, $\overline{TA} \cong \overline{RA}$

Explain why $\angle T \cong \angle R$.



6

Be prepared to share your solutions and methods.

6.3

Congruence Theorems in Action

Isosceles Triangle Theorems

LEARNING GOALS

In this lesson, you will:

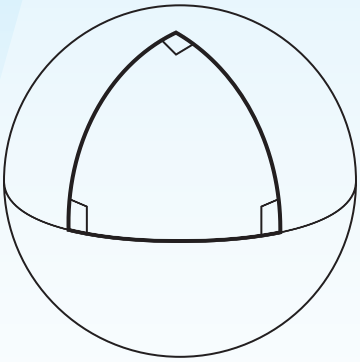
- Prove the Isosceles Triangle Base Theorem.
- Prove the Isosceles Triangle Vertex Angle Theorem.
- Prove the Isosceles Triangle Perpendicular Bisector Theorem.
- Prove the Isosceles Triangle Altitude to Congruent Sides Theorem.
- Prove the Isosceles Triangle Angle Bisector to Congruent Sides Theorem.

KEY TERMS

- vertex angle of an isosceles triangle
- Isosceles Triangle Base Theorem
- Isosceles Triangle Vertex Angle Theorem
- Isosceles Triangle Perpendicular Bisector Theorem
- Isosceles Triangle Altitude to Congruent Sides Theorem
- Isosceles Triangle Angle Bisector to Congruent Sides Theorem

You know that the measures of the three angles in a triangle equal 180°, and that no triangle can have more than one right angle or obtuse angle.

Unless, however, you're talking about a spherical triangle. A spherical triangle is a triangle formed on the surface of a sphere. The sum of the measures of the angles of this kind of triangle is always greater than 180°. Spherical triangles can have two or even three obtuse angles or right angles.



The properties of spherical triangles are important to a certain branch of science. Can you guess which one?

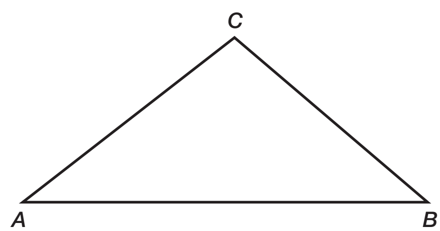
PROBLEM 1 Isosceles Triangle Theorems

You will prove theorems related to isosceles triangles. These proofs involve altitudes, perpendicular bisectors, angle bisectors, and *vertex angles*. A **vertex angle of an isosceles triangle** is the angle formed by the two congruent legs in an isosceles triangle.

The **Isosceles Triangle Base Theorem** states: "The altitude to the base of an isosceles triangle bisects the base."

1. Given: Isosceles $\triangle ABC$ with $\overline{CA} \cong \overline{CB}$.

   a. Construct altitude $\overline{CD}$ from the vertex angle to the base.



2. Prove the Isosceles Triangle Base Theorem.

The **Isosceles Triangle Vertex Angle Theorem** states: “The altitude to the base of an isosceles triangle bisects the vertex angle.”

3. Draw and label a diagram you can use to help you prove the Isosceles Triangle Vertex Angle Theorem. State the “Given” and “Prove” statements.

4. Prove the Isosceles Triangle Vertex Angle Theorem.

6

The **Isosceles Triangle Perpendicular Bisector Theorem** states: “The altitude from the vertex angle of an isosceles triangle is the perpendicular bisector of the base.”

5. Draw and label a diagram you can use to help you prove the Isosceles Triangle Perpendicular Bisector Theorem. State the “Given” and “Prove” statements.

6. Prove the Isosceles Triangle Perpendicular Bisector Theorem.

6

PROBLEM 2 More Isosceles Triangle Theorems

The **Isosceles Triangle Altitude to Congruent Sides Theorem** states: "In an isosceles triangle, the altitudes to the congruent sides are congruent."

1. Draw and label a diagram you can use to help you prove this theorem. State the "Given" and "Prove" statements.

2. Prove the Isosceles Triangle Altitude to Congruent Sides Theorem.

6

The **Isosceles Triangle Angle Bisector to Congruent Sides Theorem** states:
"In an isosceles triangle, the angle bisectors to the congruent sides are congruent."

3. Draw and label a diagram you can use to help you prove this theorem. State the "Given" and "Prove" statements.

4. Prove the Isosceles Triangle Angle Bisector to Congruent Sides Theorem.

6

Talk the Talk

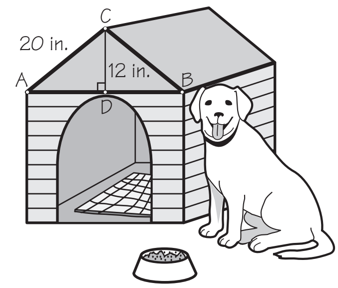
1. Solve for the width of the dog house.

   $\overline{CD} \perp \overline{AB}$

   $\overline{AC} \cong \overline{BC}$

   $CD = 12''$

   $AC = 20''$



Use the theorems you have just proven to answer each question about isosceles triangles.

2. What can you conclude about an altitude drawn from the vertex angle to the base?

3. What can you conclude about the altitudes to the congruent sides?

4. What can you conclude about the angle bisectors to the congruent sides?

Be prepared to share your solutions and methods.

Making Some Assumptions

6.4

Inverse, Contrapositive, Direct Proof, and Indirect Proof

LEARNING GOALS

In this lesson, you will:

- Write the inverse and contrapositive of a conditional statement.
- Differentiate between direct and indirect proof.
- Use indirect proof.

KEY TERMS

- inverse
- contrapositive
- direct proof
- indirect proof or proof by contradiction
- Hinge Theorem
- Hinge Converse Theorem

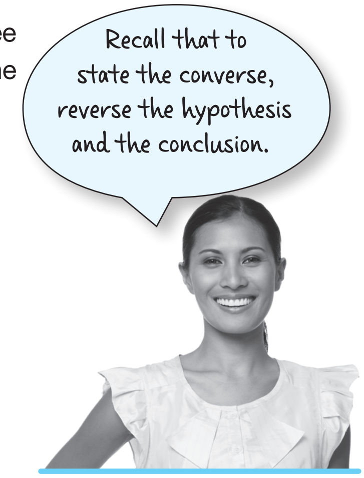
The Greek philosopher Aristotle greatly influenced our understanding of physics, linguistics, politics, and science. He also had a great influence on our understanding of logic. In fact, he is often credited with the earliest study of formal logic, and he wrote six works on logic which were compiled into a collection known as the *Organon*. These works were used for many years after his death. There were a number of philosophers who believed that these works of Aristotle were so complete that there was nothing else to discuss regarding logic. These beliefs lasted until the 19th century when philosophers and mathematicians began thinking of logic in more mathematical terms.

Aristotle also wrote another book, *Metaphysics*, in which he makes the following statement: "To say of what is that it is not, or of what is not that it is, is falsehood, while to say of what is that it is, and of what is not that it is not, is truth."

What is Aristotle trying to say here, and do you agree? Can you prove or disprove this statement?

PROBLEM 1 The Inverse and Contrapositive

Every conditional statement written in the form "If *p*, then *q*" has three additional conditional statements associated with it: the converse, the *contrapositive*, and the *inverse*. To state the **inverse**, negate the hypothesis and the conclusion. To state the **contrapositive**, negate the hypothesis and conclusion, and reverse them.



Conditional Statement	If *p*, then *q*.
Converse	If *q*, then *p*.
Inverse	If not *p*, then not *q*.
Contrapositive	If not *q*, then not *p*.

1. If a quadrilateral is a square, then the quadrilateral is a rectangle.

   a. Hypothesis *p*:

   b. Conclusion *q*:

   c. Is the conditional statement true? Explain your reasoning.

   d. Not *p*:

   e. Not *q*:

   f. Inverse:

   g. Is the inverse true? Explain your reasoning.

   h. Contrapositive:

   i. Is the contrapositive true? Explain your reasoning.

2. If an integer is even, then the integer is divisible by two.

a. Hypothesis *p*:

b. Conclusion *q*:

c. Is the conditional statement true? Explain your reasoning.

d. Not *p*:

e. Not *q*:

f. Inverse:

g. Is the inverse true? Explain your reasoning.

h. Contrapositive:

i. Is the contrapositive true? Explain your reasoning.

3. If a polygon has six sides, then the polygon is a pentagon.

a. Hypothesis *p*:

b. Conclusion *q*:

c. Is the conditional statement true? Explain your reasoning.

d. Not *p*:

e. Not *q*:

f. Inverse:

g. Is the inverse true? Explain your reasoning.

h. Contrapositive:

i. Is the contrapositive true? Explain your reasoning.

4. If two lines intersect, then the lines are perpendicular.
   a. Hypothesis *p*:
   b. Conclusion *q*:
   c. Is the conditional statement true? Explain your reasoning.
   d. Not *p*:
   e. Not *q*:
   f. Inverse:
   g. Is the inverse true? Explain your reasoning.
   h. Contrapositive:
   i. Is the contrapositive true? Explain your reasoning.

5. What do you notice about the truth value of a conditional statement and the truth value of its inverse?

6. What do you notice about the truth value of a conditional statement and the truth value of its contrapositive?

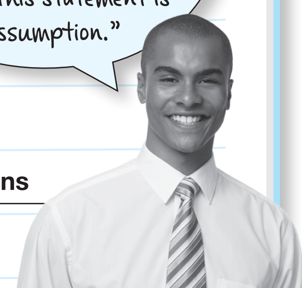
PROBLEM 2 Proof by Contradiction

All of the proofs up to this point were *direct proofs*. A **direct proof** begins with the given information and works to the desired conclusion directly through the use of givens, definitions, properties, postulates, and theorems.

An **indirect proof**, or **proof by contradiction**, uses the contrapositive. If you prove the contrapositive true, then the original conditional statement is true. Begin by assuming the conclusion is false, and use this assumption to show one of the given statements is false, thereby creating a contradiction.

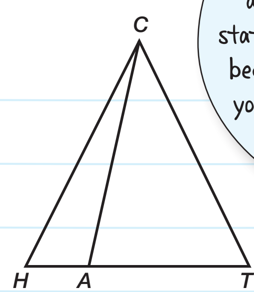
Let's look at an example of an indirect proof.

Notice, you are trying to prove $\triangle CHA \ncong \triangle CTA$. You assume the negation of this statement, $\triangle CHA \cong \triangle CTA$. This becomes the first statement in your proof, and the reason for making this statement is "assumption."



Given: In $\triangle CHT$, $\overline{CH} \cong \overline{CT}$,
$\overline{CA}$ does not bisect $\overline{HT}$

Prove: $\triangle CHA \ncong \triangle CTA$



Statements	Reasons
1. $\triangle CHA \cong \triangle CTA$	**1.** Assumption
2. $\overline{CA}$ does not bisect $\overline{HT}$	**2.** Given
3. $\overline{HA} \cong \overline{TA}$	**3.** CPCTC
4. $\overline{CA}$ bisects $\overline{HT}$	**4.** Definition of bisect
5. $\triangle CHA \cong \triangle CTA$ is false	**5.** This is a contradiction. Step 4 contradicts step 2; the assumption is false
6. $\triangle CHA \ncong \triangle CTA$ is true	**6.** Proof by contradiction

In step 5, the "assumption" is stated as "false." The reason for making this statement is "contradiction."

In an indirect proof:

- State the assumption; use the negation of the conclusion, or "Prove" statement.
- Write the givens.
- Write the negation of the conclusion.
- Use the assumption, in conjunction with definitions, properties, postulates, and theorems, to prove a given statement is false, thus creating a contradiction.

Hence, your assumption leads to a contradiction; therefore, the assumption must be false. This proves the contrapositive.

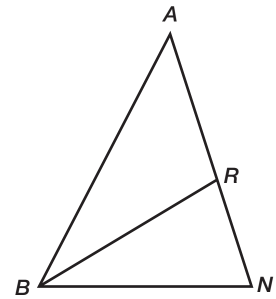
6

1. Create an indirect proof of the following.

Given: $\overline{BR}$ bisects $\angle ABN$,

$\angle BRA \not\cong \angle BRN$

Prove: $\overline{AB} \not\cong \overline{NB}$

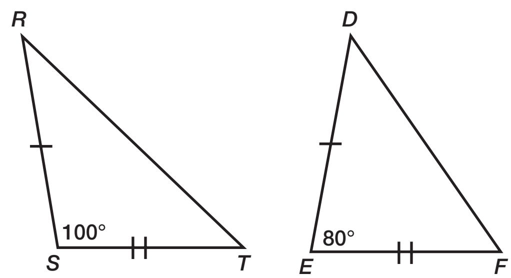


2. Create an indirect proof to show that a triangle cannot have more than one right angle.

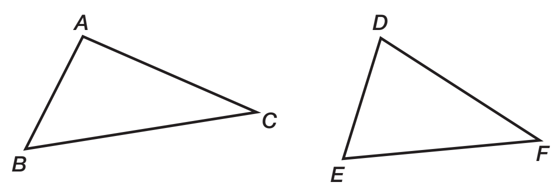
PROBLEM 3 Hinge Theorem and Its Converse

The **Hinge Theorem** states: "If two sides of one triangle are congruent to two sides of another triangle, and the included angle of the first pair is larger than the included angle of the second pair, then the third side of the first triangle is longer than the third side of the second triangle."

In the two triangles shown, notice that $RS = DE$, $ST = EF$, and $\angle S > \angle E$. The Hinge Theorem says that $RT > DF$.



1. Use an indirect proof to prove the Hinge Theorem.



Given: $AB = DE$

$AC = DF$

$m\angle A > m\angle D$

Prove: $BC > EF$

Negating the conclusion, $BC > EF$, means that either BC is equal to EF, or BC is less than EF. Therefore, this theorem must be proven for both cases.

Case 1: $BC = EF$

Case 2: $BC < EF$

6

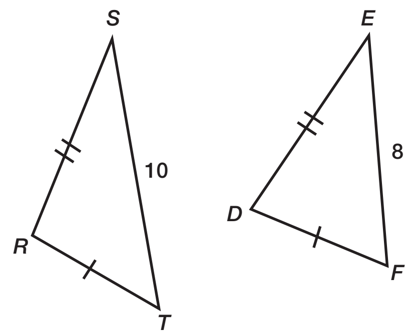
a. Write the indirect proof for Case 1, $BC = EF$.

b. Write the indirect proof for Case 2, $BC < EF$.

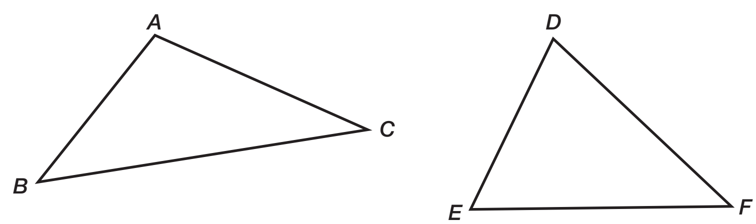
6

The **Hinge Converse Theorem** states: "If two sides of one triangle are congruent to two sides of another triangle and the third side of the first triangle is longer than the third side of the second triangle, then the included angle of the first pair of sides is larger than the included angle of the second pair of sides."

In the two triangles shown, notice that $RT = DF$, $RS = DE$, and $ST > EF$. The Hinge Converse Theorem guarantees that $m\angle R > m\angle D$.



2. Create an indirect proof to prove the Hinge Converse Theorem.



Given: $AB = DE$

$AC = DF$

$BC > EF$

Prove: $m\angle A > m\angle D$

This theorem must be proven for two cases.

Case 1: $m\angle A = m\angle D$

Case 2: $m\angle A < m\angle D$

6

a. Create an indirect proof for Case 1, $m\angle A = m\angle D$.

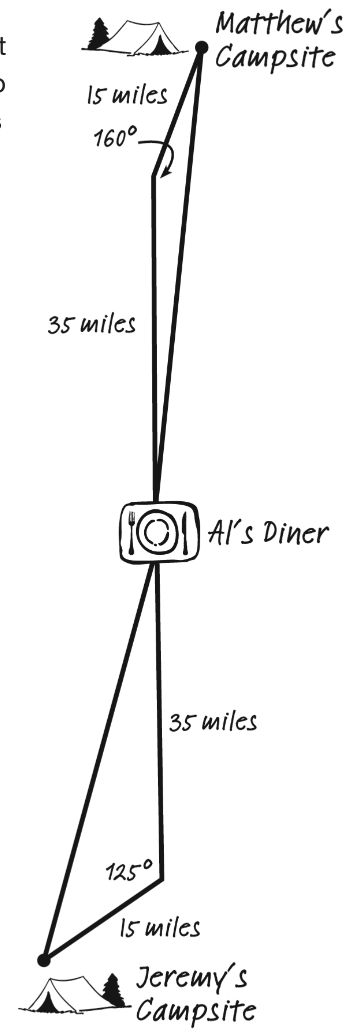
b. Create an indirect proof for Case 2, $m\angle A < m\angle D$.

6

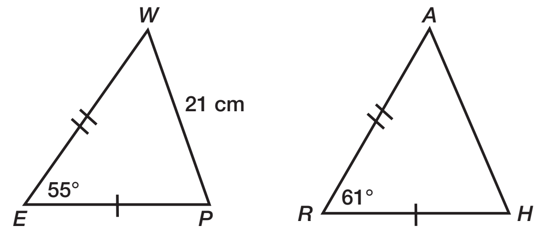
PROBLEM 4 It All Hinges on Reason

1. Matthew and Jeremy's families are going camping for the weekend. Before heading out of town, they decide to meet at Al's Diner for breakfast. During breakfast, the boys try to decide which family will be further away from the diner "as the crow flies." "As the crow flies" is an expression based on the fact that crows, generally fly straight to the nearest food supply.

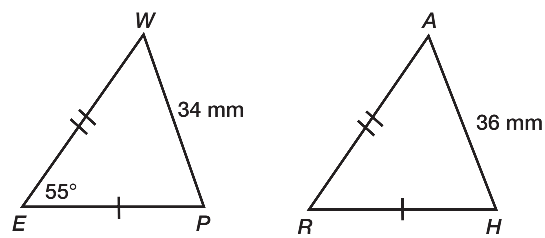
   Matthew's family is driving 35 miles due north and taking an exit to travel an additional 15 miles northeast. Jeremy's family is driving 35 miles due south and taking an exit to travel an additional 15 miles southwest. Use the diagram shown to determine which family is further from the diner. Explain your reasoning.



2. Which of the following is a possible length for *AH*: 20 cm, 21 cm, or 24 cm? Explain your choice.



3. Which of the following is a possible angle measure for ∠*ARH:* 54°, 55° or 56°? Explain your choice.



6

Be prepared to share your solutions and methods.

6

Chapter 6 Summary

KEY TERMS

- corresponding parts of congruent triangles are congruent (CPCTC) (6.2)
- vertex angle of an isosceles triangle (6.3)
- inverse (6.4)
- contrapositive (6.4)
- direct proof (6.4)
- indirect proof or proof by contradiction (6.4)

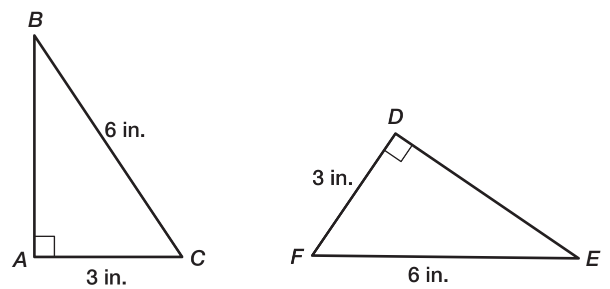
THEOREMS

- Hypotenuse-Leg (HL) Congruence Theorem (6.1)
- Leg-Leg (LL) Congruence Theorem (6.1)
- Hypotenuse-Angle (HA) Congruence Theorem (6.1)
- Leg-Angle (LA) Congruence Theorem (6.1)
- Isosceles Triangle Base Angle Theorem (6.2)
- Isosceles Triangle Base Angle Converse Theorem (6.2)
- Isosceles Triangle Base Theorem (6.3)
- Isosceles Triangle Vertex Angle Theorem (6.3)
- Isosceles Triangle Perpendicular Bisector Theorem (6.3)
- Isosceles Triangle Altitude to Congruent Sides Theorem (6.3)
- Isosceles Triangle Angle Bisector to Congruent Sides Theorem (6.3)
- Hinge Theorem (6.4)
- Hinge Converse Theorem (6.4)

6.1 Using the Hypotenuse-Leg (HL) Congruence Theorem

The Hypotenuse-Leg (HL) Congruence Theorem states: "If the hypotenuse and leg of one right triangle are congruent to the hypotenuse and leg of another right triangle, then the triangles are congruent."

Example



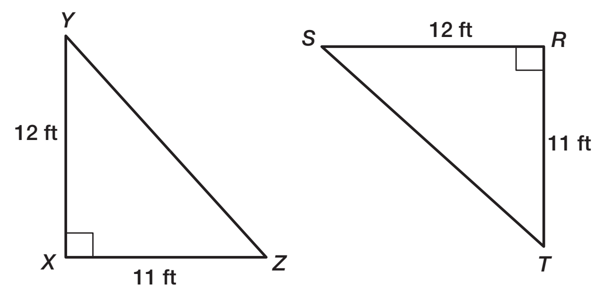
$\overline{BC} \cong \overline{EF}$, $\overline{AC} \cong \overline{DF}$, and angles A and D are right angles, so $\triangle ABC \cong \triangle DEF$.

6

6.1 Using the Leg-Leg (LL) Congruence Theorem

The Leg-Leg (LL) Congruence Theorem states: "If two legs of one right triangle are congruent to two legs of another right triangle, then the triangles are congruent."

Example

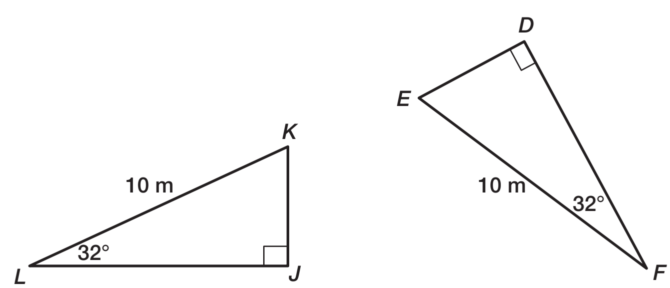


$\overline{XY} \cong \overline{RS}$, $\overline{XZ} \cong \overline{RT}$, and angles X and R are right angles, so $\triangle XYZ \cong \triangle RST$.

6.1 Using the Hypotenuse-Angle (HA) Congruence Theorem

The Hypotenuse-Angle (HA) Congruence Theorem states: "If the hypotenuse and an acute angle of one right triangle are congruent to the hypotenuse and acute angle of another right triangle, then the triangles are congruent."

Example



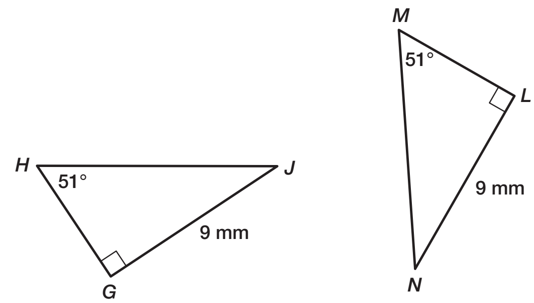
$\overline{KL} \cong \overline{EF}$, $\angle L \cong \angle F$, and angles J and D are right angles, so $\triangle JKL \cong \triangle DEF$.

6

Using the Leg-Angle (LA) Congruence Theorem

The Leg-Angle (LA) Congruence Theorem states: "If a leg and an acute angle of one right triangle are congruent to the leg and an acute angle of another right triangle, then the triangles are congruent."

Example



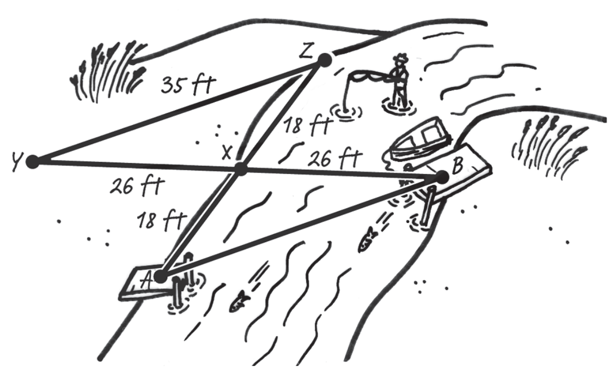
$\overline{GN} \cong \overline{LN}$, $\angle H \cong \angle M$, and angles G and L are right angles, so $\triangle GHJ \cong \triangle LMN$.

Using CPCTC to Solve a Problem

If two triangles are congruent, then each part of one triangle is congruent to the corresponding part of the other triangle. In other words, "corresponding parts of congruent triangles are congruent," which is abbreviated CPCTC. To use CPCTC, first prove that two triangles are congruent.

Example

You want to determine the distance between two docks along a river. The docks are represented as points *A* and *B* in the diagram below. You place a marker at point *X*, because you know that the distance between points *X* and *B* is 26 feet. Then, you walk horizontally from point *X* and place a marker at point *Y*, which is 26 feet from point *X*. You measure the distance between points *X* and *A* to be 18 feet, and so you walk along the river bank 18 feet and place a marker at point *Z*. Finally, you measure the distance between *Y* and *Z* to be 35 feet.

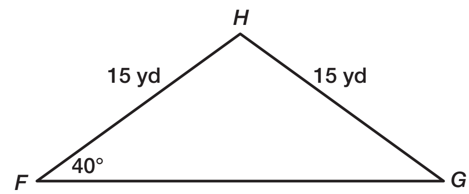


From the diagram, segments *XY* and *XB* are congruent and segments *XA* and *XZ* are congruent. Also, angles *YXZ* and *BXA* are congruent by the Vertical Angles Congruence Theorem. So, by the Side-Angle-Side (SAS) Congruence Postulate, $\triangle YXZ \cong \triangle BXA$. Because corresponding parts of congruent triangles are congruent (CPCTC), segment *YZ* must be congruent to segment *BA*. The length of segment *YZ* is 35 feet. So, the length of segment *BA*, or the distance between the docks, is 35 feet.

6.2 Using the Isosceles Triangle Base Angle Theorem

The Isosceles Triangle Base Angle Theorem states: "If two sides of a triangle are congruent, then the angles opposite these sides are congruent."

Example

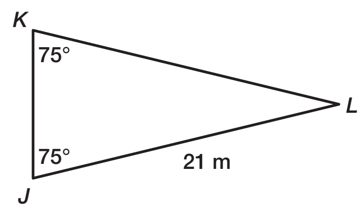


$\overline{FH} \cong \overline{GH}$, so $\angle F \cong \angle G$, and the measure of angle G is 40°.

6.2 Using the Isosceles Triangle Base Angle Converse Theorem

The Isosceles Triangle Base Angle Converse Theorem states: "If two angles of a triangle are congruent, then the sides opposite these angles are congruent."

Example

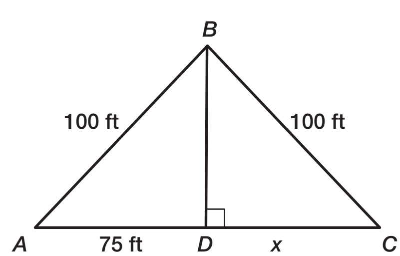


$\angle J \cong \angle K$, $\overline{JL} \cong \overline{KL}$, and the length of side KL is 21 meters.

6.3 Using the Isosceles Triangle Base Theorem

The Isosceles Triangle Base Theorem states: "The altitude to the base of an isosceles triangle bisects the base."

Example



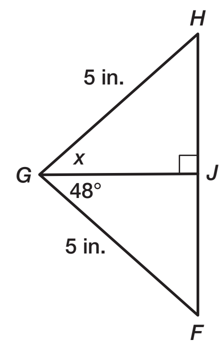
$CD = AD$, so $x = 75$ feet.

6

6.3 Using the Isosceles Triangle Vertex Angle Theorem

The Isosceles Triangle Base Theorem states: "The altitude to the base of an isosceles triangle bisects the vertex angle."

Example

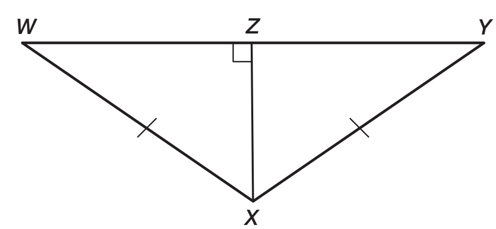


$m\angle FGJ = m\angle HGJ$, so $x = 48°$.

6.3 Using the Isosceles Triangle Perpendicular Bisector Theorem

The Isosceles Triangle Perpendicular Bisector Theorem states: "The altitude from the vertex angle of an isosceles triangle is the perpendicular bisector of the base."

Example



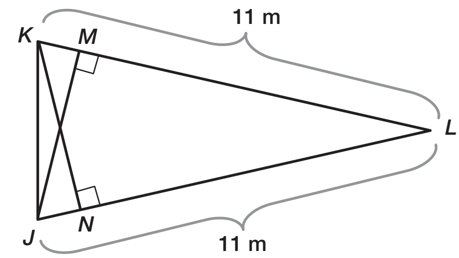
$\overline{WY} \perp \overline{XZ}$ and $WZ = YZ$

6

6.3 Using the Isosceles Triangle Altitude to Congruent Sides Theorem

The Isosceles Triangle Perpendicular Bisector Theorem states: "In an isosceles triangle, the altitudes to the congruent sides are congruent."

Example

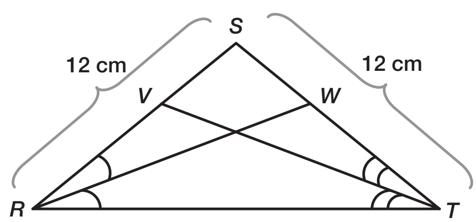


$\overline{KN} \cong \overline{JM}$

6.3 Using the Isosceles Triangle Bisector to Congruent Sides Theorem

The Isosceles Triangle Perpendicular Bisector Theorem states: "In an isosceles triangle, the angle bisectors to the congruent sides are congruent."

Example



$\overline{RW} \cong \overline{TV}$

6.4 Stating the Inverse and Contrapositive of Conditional Statements

To state the inverse of a conditional statement, negate both the hypothesis and the conclusion. To state the contrapositive of a conditional statement, negate both the hypothesis and the conclusion and then reverse them.

Conditional Statement: If *p*, then *q*.

Inverse: If not *p*, then not *q*.

Contrapositive: If not *q*, then not *p*.

Example

Conditional Statement: If a triangle is equilateral, then it is isosceles.

Inverse: If a triangle is not equilateral, then it is not isosceles.

Contrapositive: If a triangle is not isosceles, then it is not equilateral.

6

6.4 Writing an Indirect Proof

In an indirect proof, or proof by contradiction, first write the givens. Then, write the negation of the conclusion. Then, use that assumption to prove a given statement is false, thus creating a contradiction. Hence, the assumption leads to a contradiction, therefore showing that the assumption is false. This proves the contrapositive.

Example

Given: Triangle *DEF*

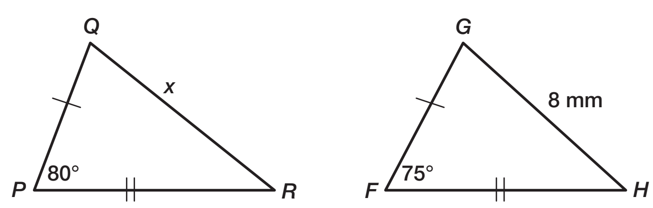
Prove: A triangle cannot have more than one obtuse angle.

Given $\triangle DEF$, assume that $\triangle DEF$ has two obtuse angles. So, assume $m\angle D = 91°$ and $m\angle E = 91°$. By the Triangle Sum Theorem, $m\angle D + m\angle E + m\angle F = 180°$. By substitution, $91° + 91° + m\angle F = 180°$, and by subtraction, $m\angle F = -2°$. But it is not possible for a triangle to have a negative angle, so this is a contradiction. This proves that a triangle cannot have more than one obtuse angle.

6.4 Using the Hinge Theorem

The Hinge Theorem states: "If two sides of one triangle are congruent to two sides of another triangle and the included angle of the first pair is larger than the included angle of the second pair, then the third side of the first triangle is longer than the third side of the second triangle."

Example

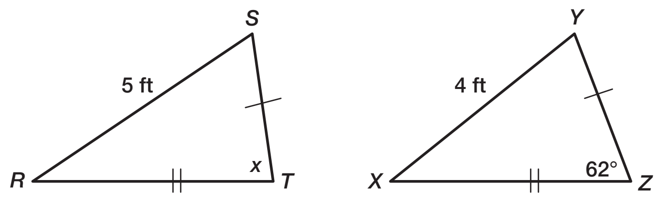


$QR > GH$, so $x > 8$ millimeters.

6.4 Using the Hinge Converse Theorem

The Hinge Converse Theorem states: "If two sides of one triangle are congruent to two sides of another triangle and the third side of the first triangle is longer than the third side of the second triangle, then the included angle of the first pair of sides is larger than the included angle of the second pair of sides."

Example



$m\angle T > m\angle Z$, so $x > 62°$.

6