Using Mathcad to Evaluate View Factors and Orbital Heats for General Surfaces of Revolution

by

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Abstract

The general integral for calculating view factors is presented along with its evaluation using Mathcad. The method is applicable to any surface that has an analytical definition for the surface normal vector and is not self viewing. Techniques are discussed on adapting the integral for general use. Examples are presented and limitations discussed. The method is extended to the calculation of spacecraft orbital heats.

1.0 Introduction

Many closed form solutions exist for the calculation of the view factor [1]. These solutions are limited to cases where the surface area integration can be evaluated and the surfaces have geometric relationships that are conducive for change of variables techniques. Specialized computer programs have been developed to calculate view factors either by numerical evaluation of the integral or by Monte Carlo ray casting techniques. These programs remain specialized, requiring specific training, are not necessarily intuitive in approach and are generally costly. Furthermore these programs have a limited variety of surface types they can handle.

A view factor integral is presented for use in Mathcad [2], a generally available engineering mathematics program. The integral is easy to read, understand and alter, requiring only general Mathcad skills. The integral can be applied to general surfaces of revolution. Shadowing presents limitations to the method. Techniques are presented to overcome this limitation in some situations.

The method is extended for use in evaluating spacecraft orbital heats. Any conical section orbit can be analyzed with a spacecraft surface at any orientation in either the spacecraft coordinate system or the celestial coordinate system. All orbit parameters are readily modified. Solar heating calculations can account for orbit intersections with the umbral cone. Planetary heat calculations can accommodate transitions through the terminator as well as the assumption of solar inclination anlge dependence of the heats. This integral formulation can be used for quick orbit evaluations and spot checking Monte Carlo results. The method is limited to direct heating calculations.

2.0 The View Factor Integral

The general integral to calculate view factor for surface i to surface j is given by Eq.(1) [1].

$$F_{ij} = \frac{-1}{A_i \cdot \pi} \cdot \int \int \frac{\left| n_i \cdot r \right| \cdot \left| n_j \cdot r \right|}{(r \cdot r)^2} \, dA_i \, dA_j$$

Eq.(1)

Where $\mathbf{n_i}$, $\mathbf{n_j}$ are surface unit normal vectors, **r** is a vector from a point on surface i to a point on surface j. The integration is taken over areas of the surfaces that are viewable from the other surface. The negative sign accounts for the reversal of **r** when viewing dA_j from dA_i.

The task of evaluating Eq.(1) becomes one of formulating the differential areas, limits of integration and the spatial dependence of the vectors. This formulation is possible, not only for the limited set of surfaces types available in commercial Monte Carlo codes, but to the much larger set of all revolved surfaces and in fact any surface that an analytical expression for surface normal can be formulated. The formulation is most easily accomplished for surfaces that do not have view factors to themselves and situations where shading is not involved. But in some cases, as shown in examples below, can be handled with proper limits of integration or with a binary conditional statements within the integral.

The integral formulation is easily adapted to situations where one surface moves or is rotated with respect to the other. Spatial variations are accomplished through use of rotational matrices, variable limits on integration, and linear transformations. Some of these techniques are demonstrated in the examples.

2.0 View Factor Examples

In the examples below Monte Carlo results are occasionally included for comparison. These were calculated with Thermal Desktop [4]. Unless otherwise stated 5000 rays per surface was used in the calculations.

2.1 Parallel Plates of Equal Areas with Aligned Centers with Varying Separation Distance

This is most easily accomplished by defining **r** as a function of separation distance d.

$$r\left(x_i, x_j, z_i, z_j\right) := \begin{pmatrix} x_j - x_i \\ d \\ z_j - z_i \end{pmatrix} \quad \text{Eq.(3)}$$

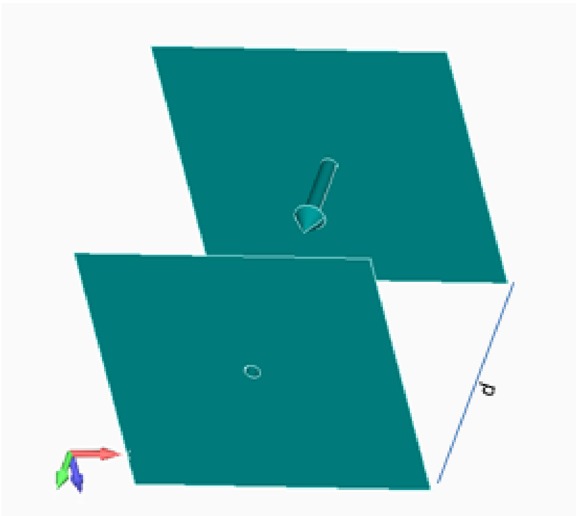

Figure 1 Parallel plates with varying separation distance d.

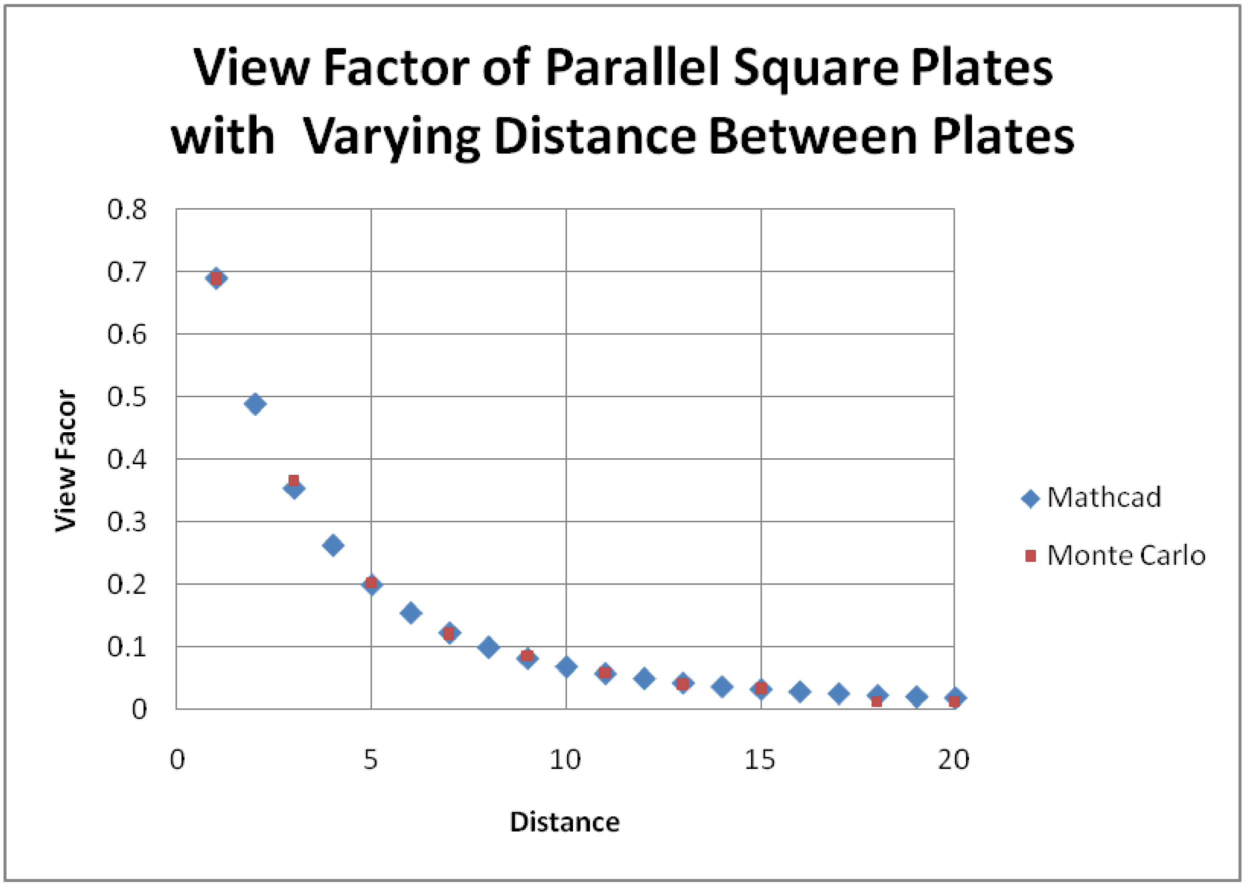

Figure 2 View Factor of Parallel Square Plates with Varying Distance Between Plates. Plate edges are 5 units long.

2.2 Parallel Plates of Equal Areas with Initially Aligned Centers One Plate Moving Along an Edge

In this example a variable was used in the limits of integration.

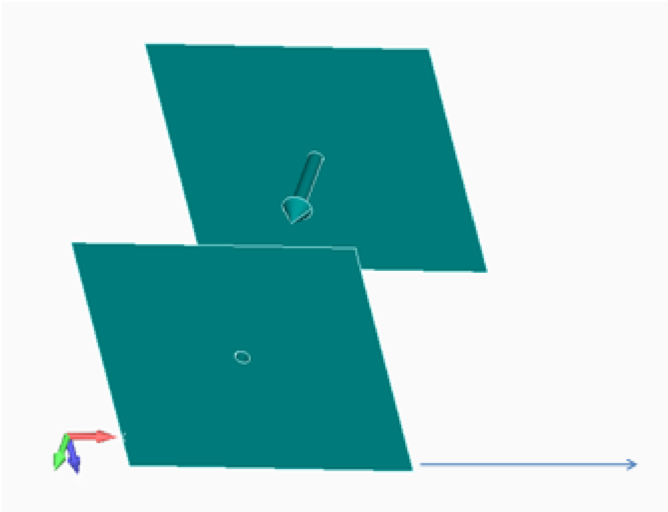

Figure 3 Parallel plates with one plate moving in direction parallel to edge.

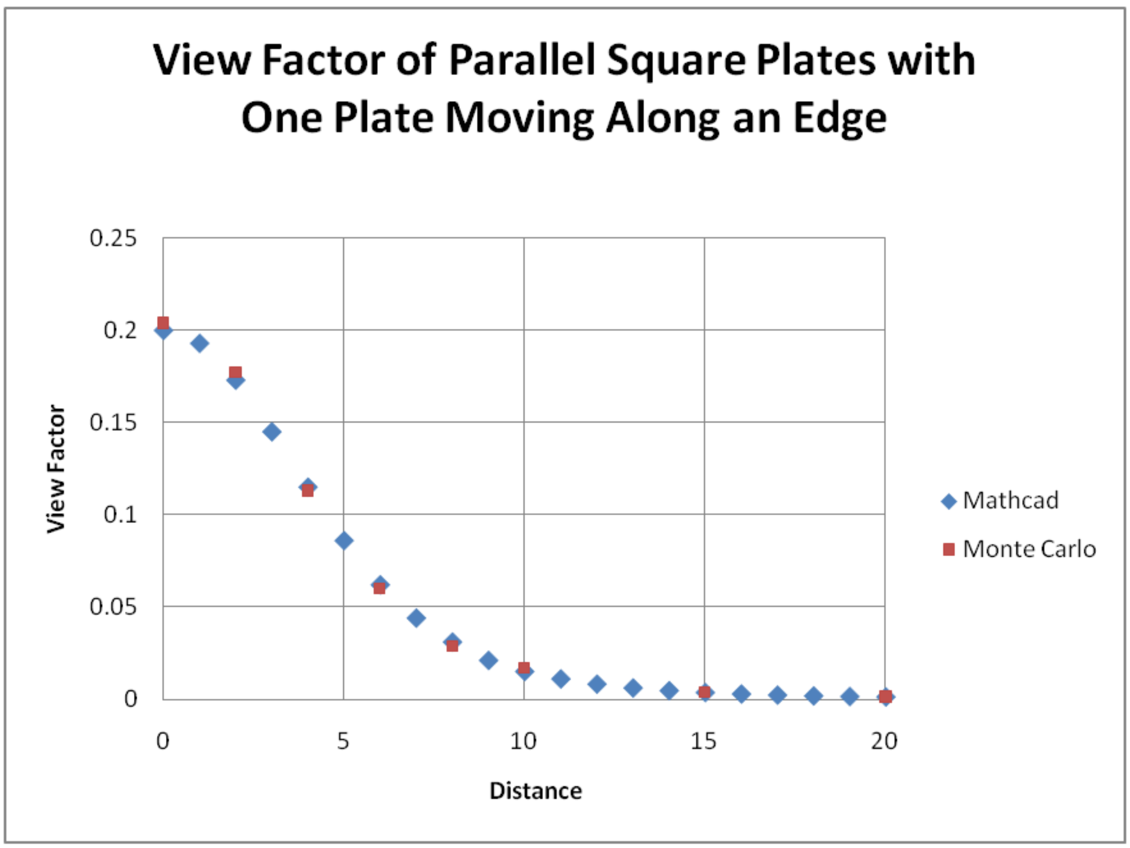

Figure 4 View Factor of Parallel Square Plates with One Plate Moving Along an Edge. Edge length equals 5. Calculated using a variable in integration limits.

2.3 Parallel Plates of Equal Areas with Aligned Centers With One Plate Rotated About Axis Through Centers.

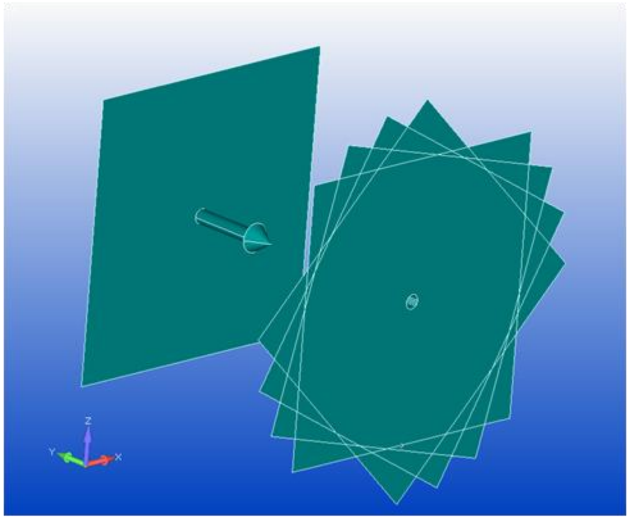

Figure 5 Parallel Plates of Equal Areas With One Plate Rotated About Central Axis.

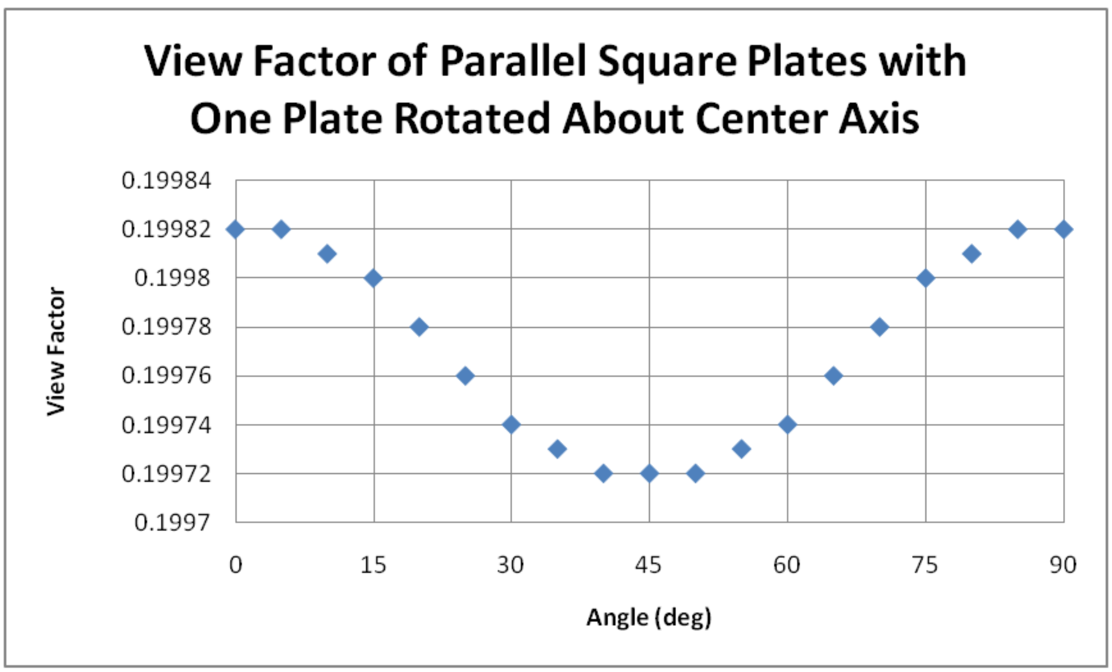

Figure 6 View Factor of Parallel Square Plates with One Plate Rotated About Center Axis. This was accomplished by using a rotation matrix to alter the i surface variables in the r vector.

2.4 Parallel Plates of Equal Areas with Initially Aligned Centers One Plate Rotated About Axis Perpendicular to Plates and Located at a Corner

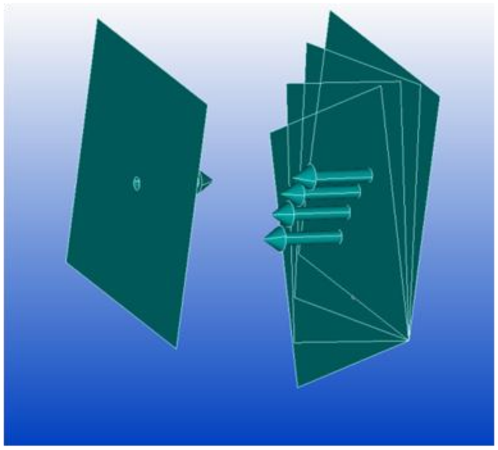

Figure 7 Aligned Parallel Plates With One Plate Rotated About Axis Through Opposing Corners.

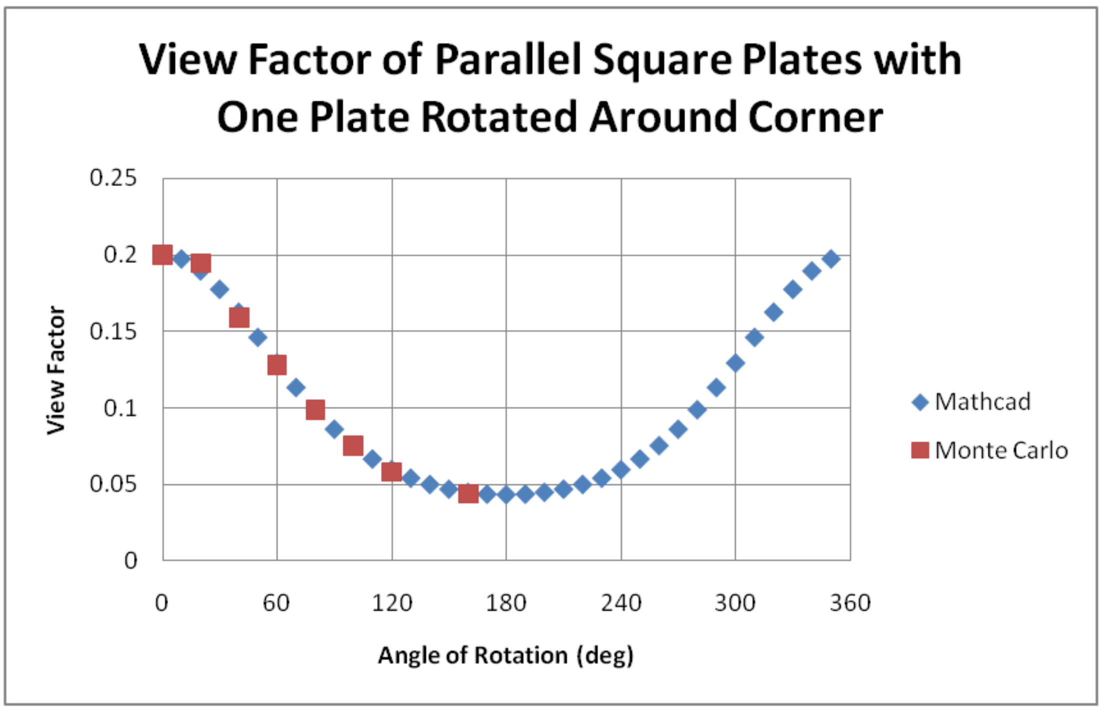

Figure 8 View Factor of Parallel Square Plates with One Plate Rotated Around Corner.

2.5 Plates of Equal Areas with Initially Aligned Centers With One Plate Rotated About an Edge

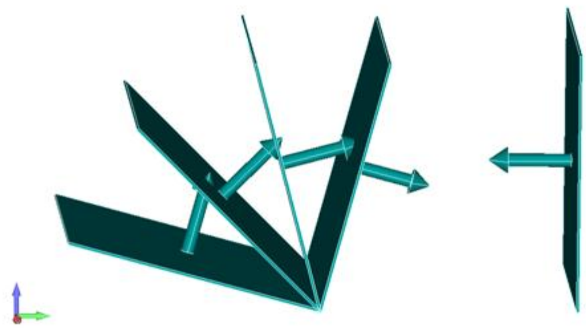

Figure 9 Parallel Plates with One Plate Rotated About An Edge.

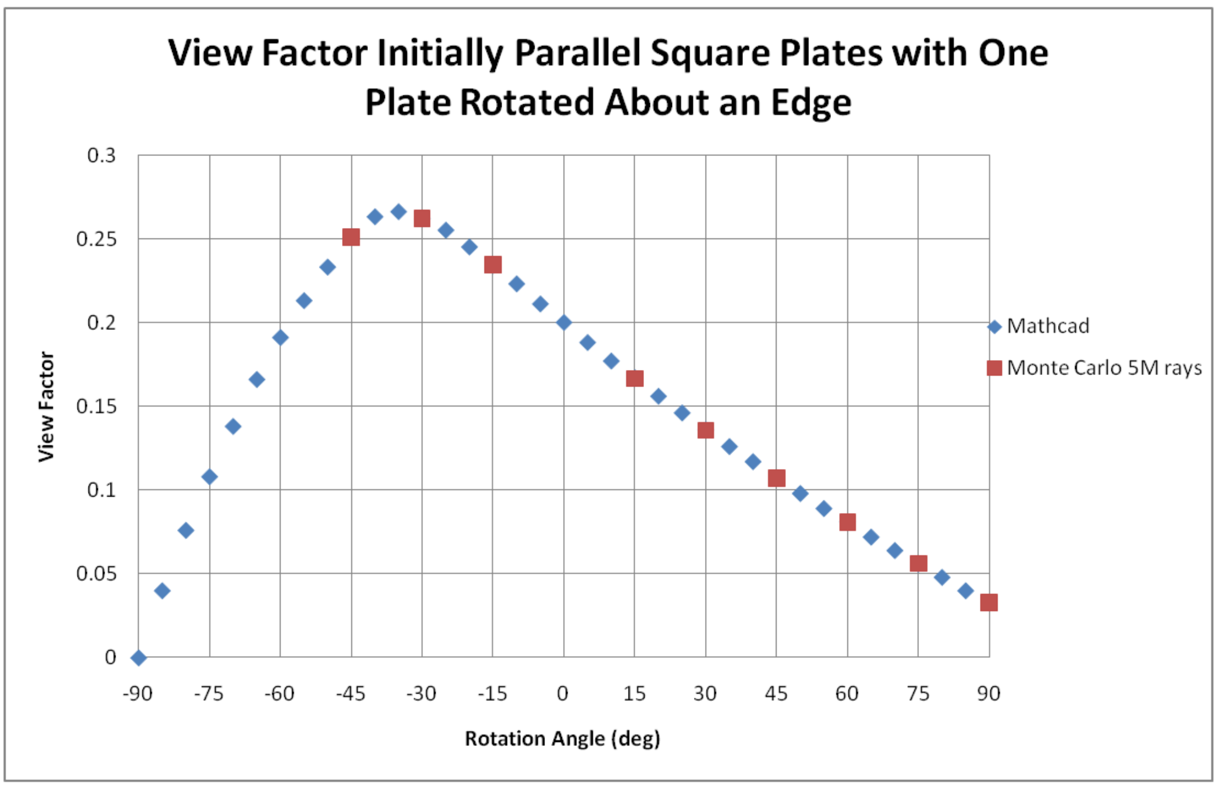

Figure 10 View Factor Initially Parallel Square Plates with One Plate Rotated About an Edge. This was achieved by using rotation matrix on ith surface coordinates in r vector definition.

2.6 Parallel Cylinder and Plate of Equal Heights with Plate Moving Parallel (along an) to Edge Perpendicular to Cylinder Axis

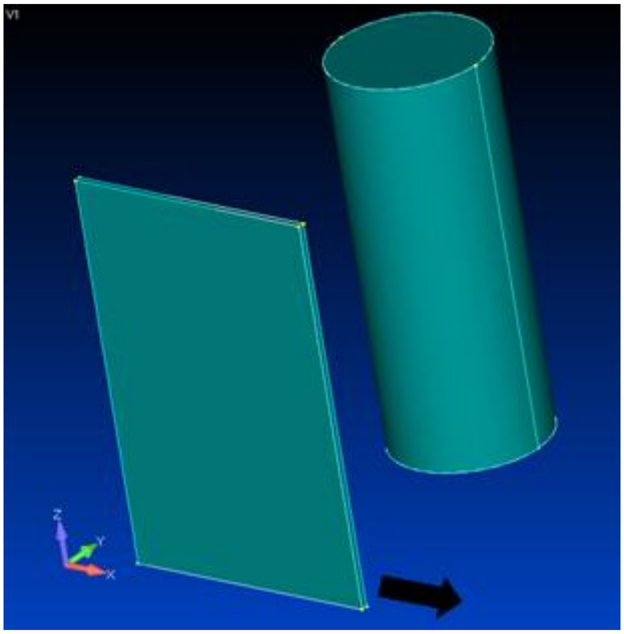

Figure 11. Plate and Parallel Cylinder.

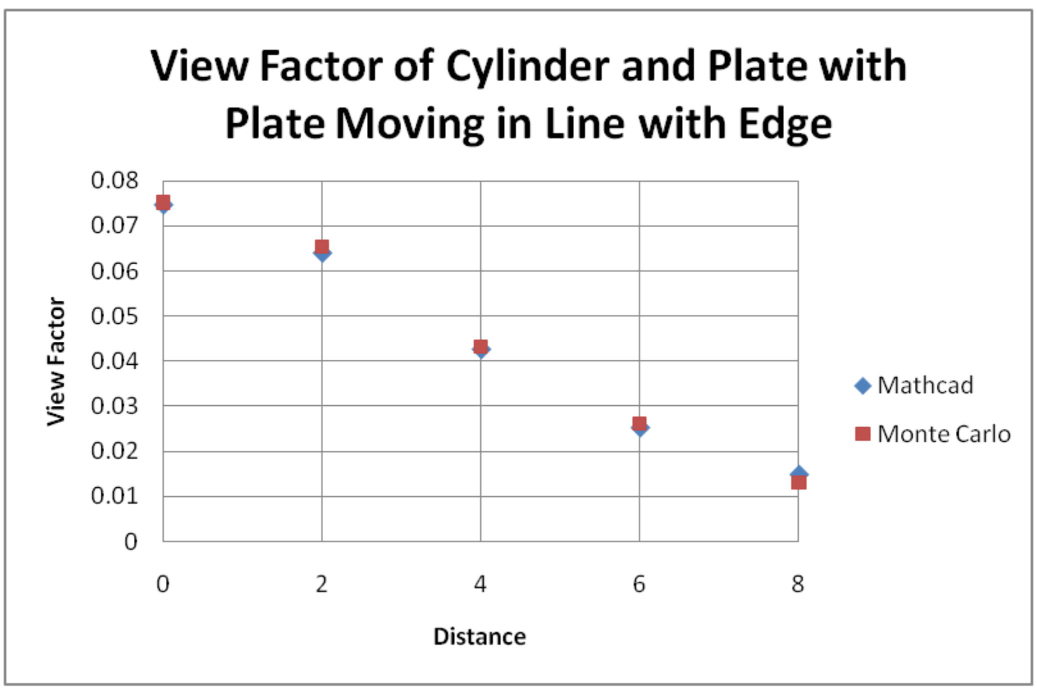

Figure 12 View Factor of Cylinder and Plate with Plate Moving in Line with Edge.

The normal vector, $\mathbf{n_c}$, of the cylinder was defined in cylindrical coordinates. It was not necessary to formulate the integral with complex limits of integration. The positions on the cylinder not visible to the plate did not contribute to the integral because a conditional statement within the integral Eq.(4).

$$\mathrm{if}\left(n_c\left(\theta_i\right)\cdot r\left(\theta_i, y_j, z_i, z_j\right) > 0, 1, 0\right) \qquad \text{Eq.(4)}$$

The if conditional stipulates that if the dot product, $\mathbf{n_c * r}$, is greater than 0 then 1 is returned, else 0.

2.7 Partially Closed Cylinder to Itself

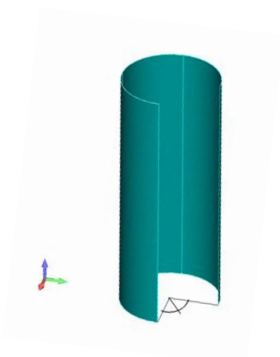

Figure 13 View Factor of Partial Cylinder to Itself as Opening Angle Varies.

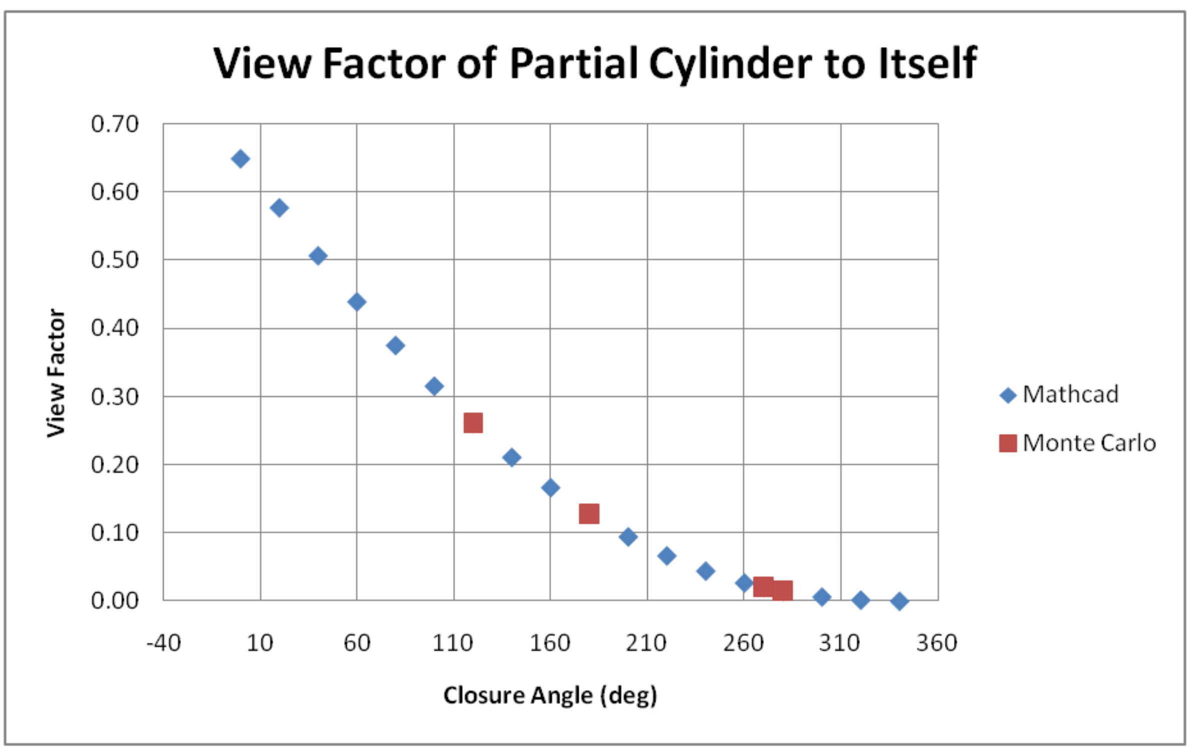

Figure 14 View Factor of Partial Cylinder to Itself.

2.8 Identical Planar Cylinders Located on a Hub

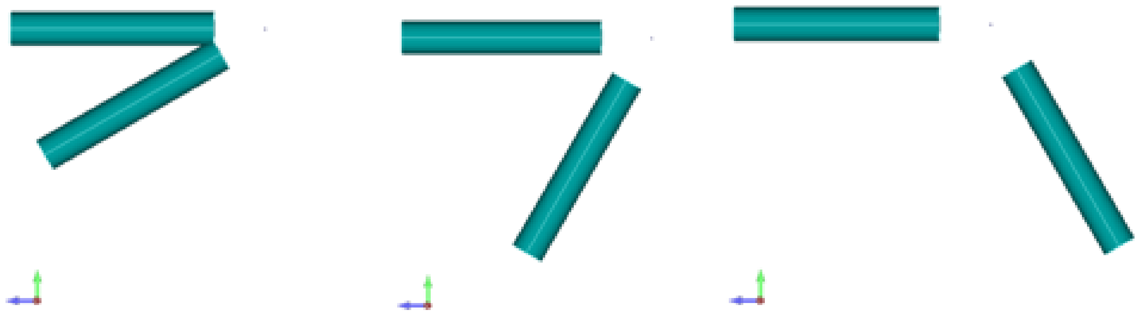

Figure 15 Cylinders Located on Hub at Varying Angles.

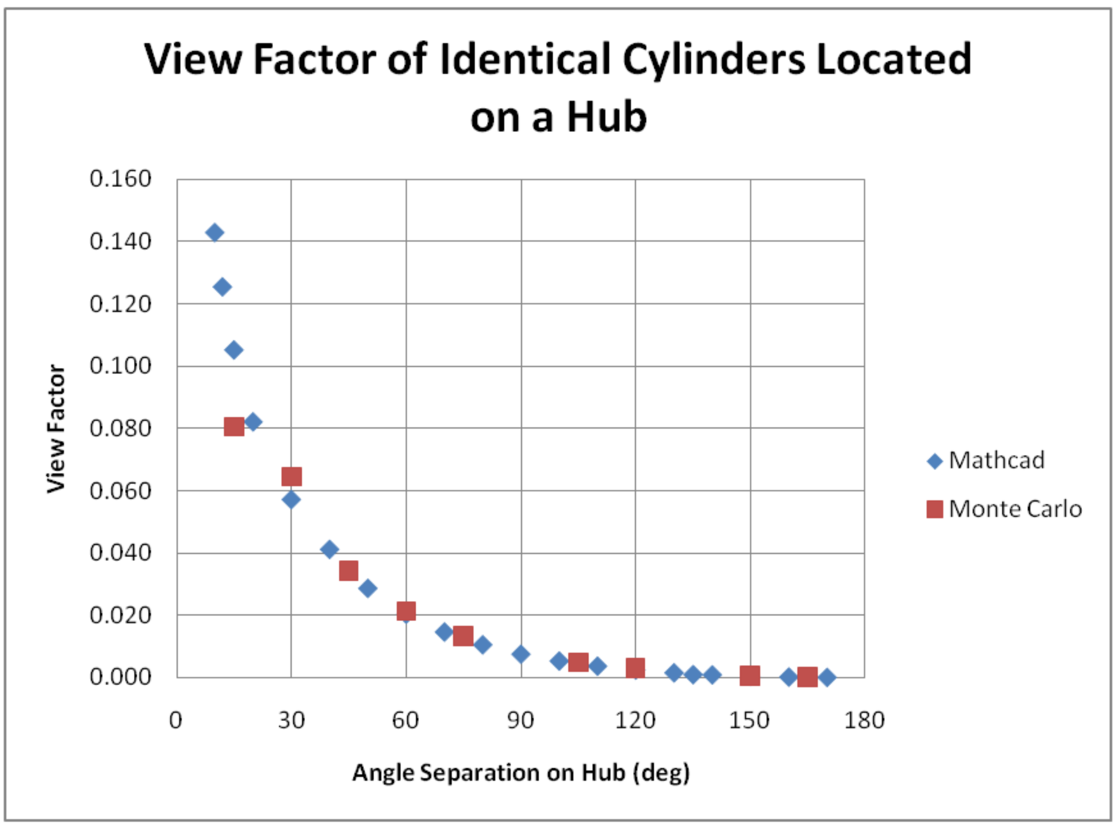

Figure 16 View Factor of Identical Cylinders Located on a Hub.

3.0 Keplerian Orbital Mechanics and Calculation of Orbital Heats

One of the surfaces in the view factor integral can be a planet surface and the other surface a spacecraft surface in orbit about the planet. The integration takes place over the viewable area of the planet, the view cone. The basic formulation is stated in Eq.(5). Planet radius is assumed equal to 1.

$$q_{ir} = \frac{-1}{A_i \cdot \pi} \cdot \int_0^{R_v} \int_0^{2\pi} \frac{n_i \cdot r_a \cdot r \cdot r_a}{\left| r_a \cdot r_a \right|^2} \cdot \frac{R}{\sqrt{1 - R^2}} \, d\tau \, dR$$

Eq.(5)

where A_i is the surface area, $\mathbf{n_i}$ is the surface normal vector, $\mathbf{r}$ is a the vector from the planet center to point on its surface, $\mathbf{r_a}$ is the vector from the planet surface to the spacecraft. R is the radius of the intermediate view cone and R_v is the view cone radius. τ is azimuthal angle of integration.

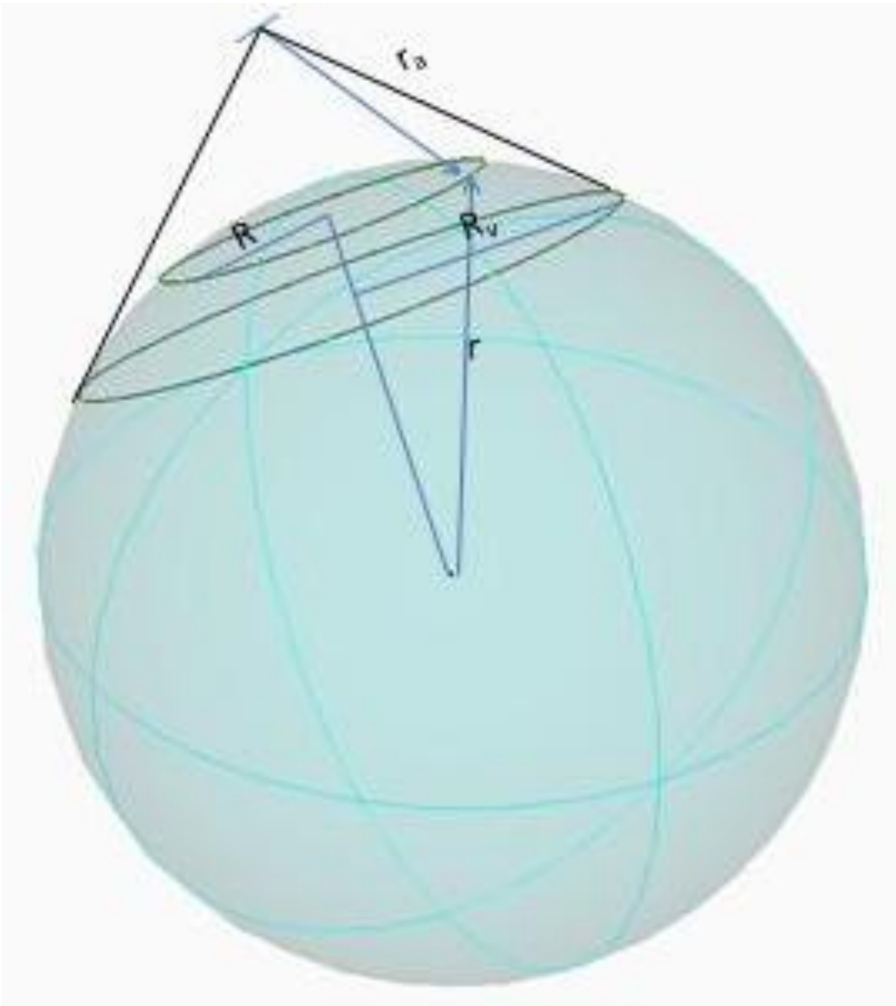

Figure 17 View Cone and Variables of Integration Used to Calculate Planetary Heats.

The integral can be amended to account for surface areas that do not have full view of the view cone and for view cones that intersect the terminator and for the inclination angle dependence of planetary heats. Integration over the spacecraft surface is also included. Eq.(6) shows the resulting integral.

$$q_{ir}(\theta) = \frac{-S}{A_i \cdot \pi} \cdot \int_0^{R_v(\theta)} \int_0^{2\cdot\pi} \int_0^{h_{tr}} \int_0^{2\cdot\pi} \frac{n_i\left(\theta_i\right) \cdot r_a(\theta,R,\tau) \cdot r(\theta,R,\tau) \cdot r_a(\theta,R,\tau) \cdot (1-\rho) \cdot z \cdot r(\theta,R,\tau) \cdot \mathrm{if}\left(r(\theta,R,\tau)_3 > 0, 1, 0\right) \cdot \mathrm{if}\left(-n_i\left(\theta_i\right) \cdot r_a(\theta,R,\tau) > 0, 1, 0\right)}{\left(\left|r_a(\theta,R,\tau) \cdot r_a(\theta,R,\tau)\right|\right)^2} \cdot dA_i \cdot \frac{R}{\sqrt{1-R^2}}\, d\theta_i\, dz_i\, d\tau\, dR$$

Eq.(6)

3.1 General Form of Orbits in Cylindrical Coordinates

The general form of the orbit equation is given by

$$r_r(\theta) := \frac{d \cdot e}{1 + e \cdot \cos(\theta)} \qquad \text{Eq.(7)}$$

Where e is eccentricity and d is distance to directrix. For elliptical orbits 0 < e <1.

3.2 Elliptical Orbits

The example elliptical orbit discussed in the sections below has e = 0.4 and d = 5. The planet radius is 1. Orbit period is 1. Rotation matrices are employed to orient the orbit in the celestial coordinate system. These transformations can be described in traditional orbital mechanic terms of β angle and line of nodes.

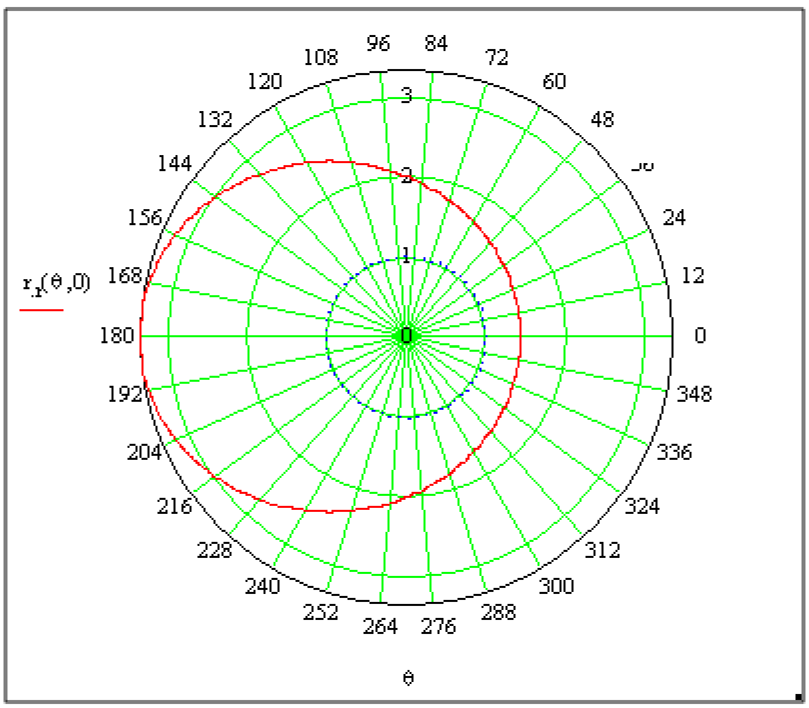

Figure 18 Polar Plot of an Example Orbit. The periapsis is at the sub-solar point. A planet of radius 1 is located at the focus.

3.2.1 Equal Areas in Equal Time Calculations

By the conservation of angular momentum an object in orbit sweeps outs equal area per unit of time. Dividing the orbit into equal swept areas allows the mapping orbit position dependent functions to be defined in terms of time. The root function in Mathcad is applied to the function of Eq.(8) to find orbit points of equal areas.

$$f(\theta_2) = \int_{\theta_1}^{\theta_2} \frac{1}{2}\cdot\left(\frac{d\cdot e}{1 + e\cdot\cos(\theta)}\right)^2 d\theta - \Delta A$$

Eq.(8)

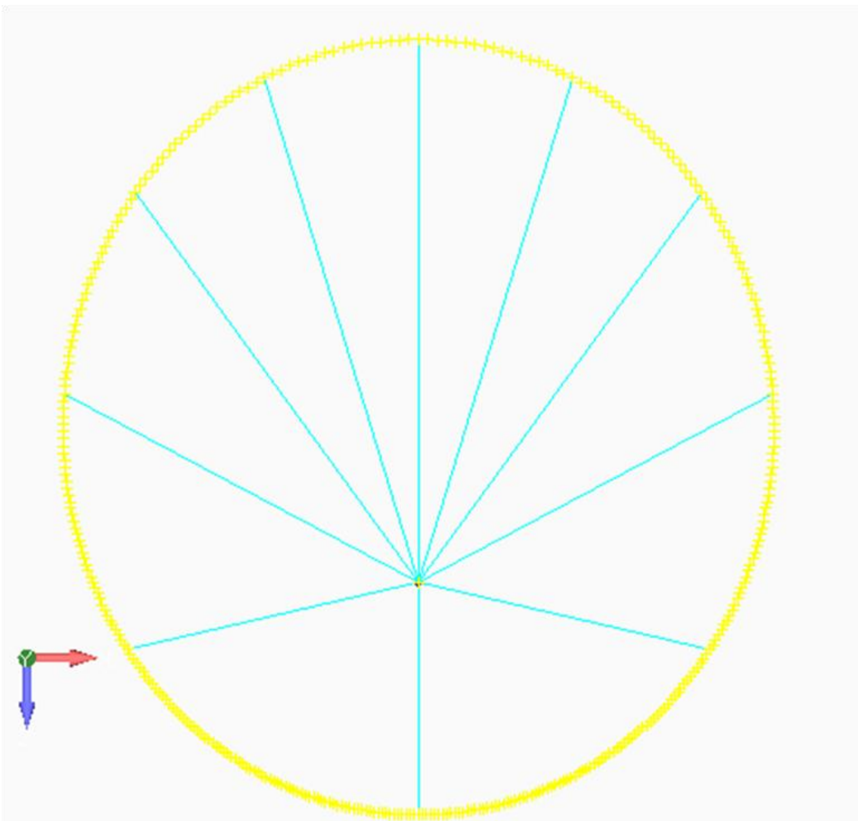

Figure 19 Example Elliptical Orbit with Equal Sweep Areas Indicated.

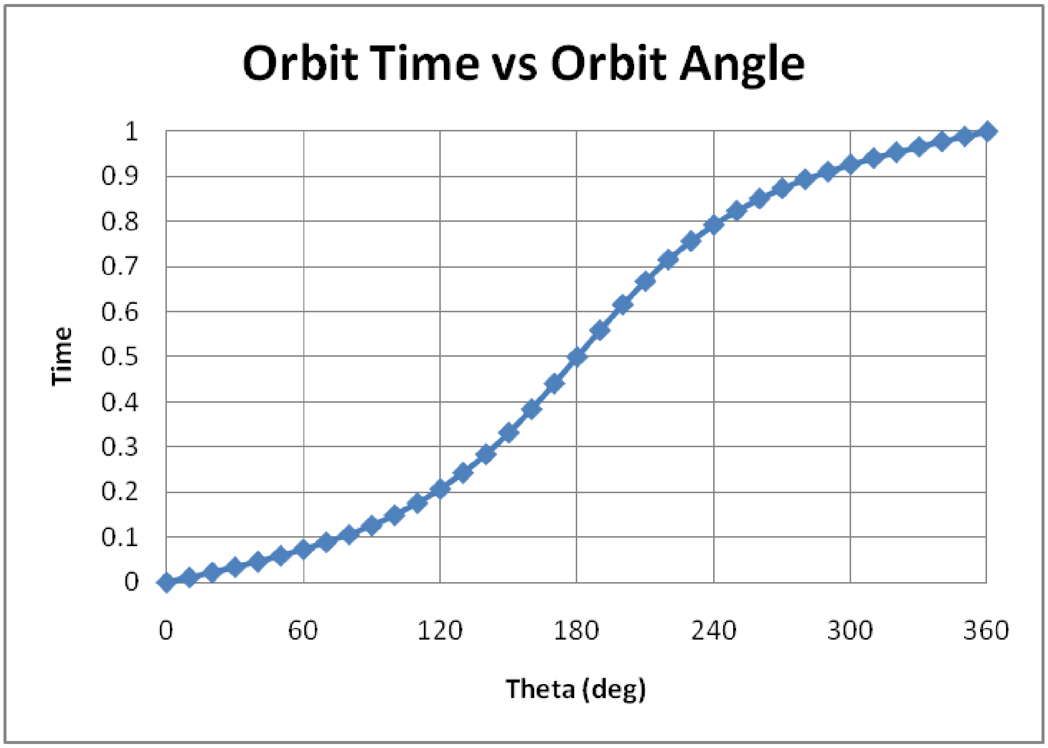

Figure 20 Orbit Time vs Orbit Angle. Angle is measured from periapsis.

3.3 Solar Heating Examples

3.3.1 Orbit in Elliptic and Periapsis at Sub-Solar Point

The Mathcad root function is used to find the entry and exit points of the umbral cone.

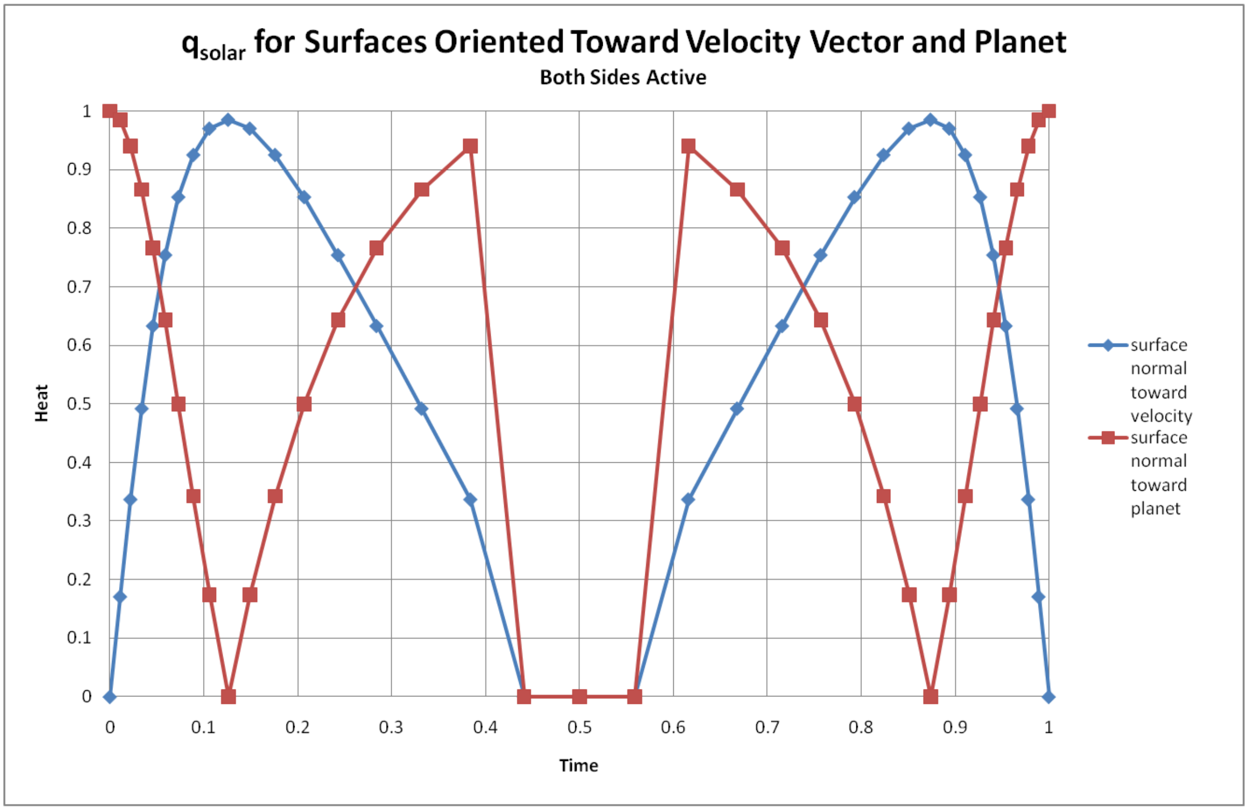

Figure 21 q_{solar} For Surfaces with Normals Pointed Toward Velocity Vector and Planet. Both sides are active.

5.2.1.1 Orbit in Elliptic and Periapsis at 60° from Sub-Solar Point

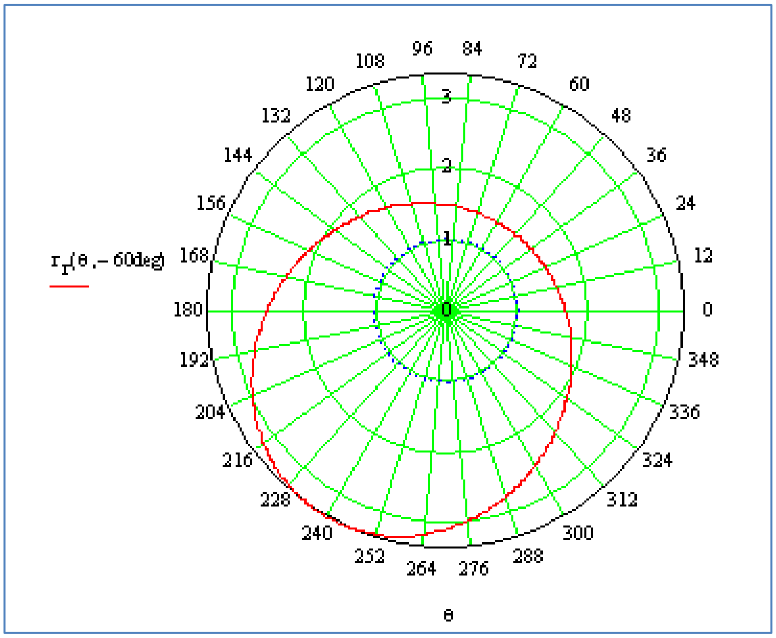

Figure 22 Orbit in Elliptic With Periapsis Rotated 60° from Sub-Solar Point.

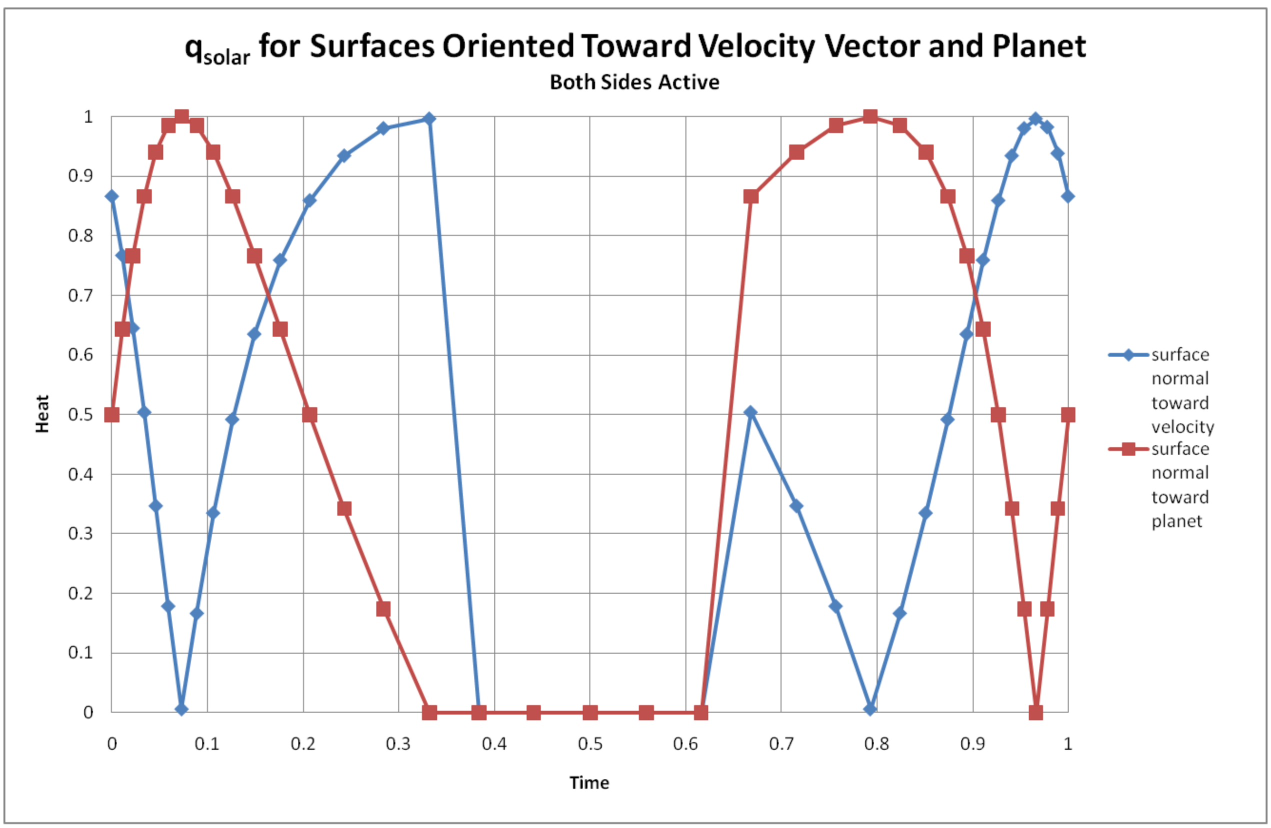

Figure 23 Same as Figure 20 but with orbit rotated 60° from solar vector.

Rotation of the orbit is accomplished by matrix multiplication on the solar vector.

3.3.2 Planetary Heating Examples

Under adiabatic regolith and diffuse reflection assumptions the infra-red and albedo heat fluxes are given by

$$IR_{planet} = S\cdot(1-\rho)\cdot\cos(\gamma) \qquad \text{Eq.(9)}$$

$$Albedo_{planet} = S\cdot\rho\cdot\cos(\gamma) \qquad \text{Eq.(10)}$$

where γ is the angle off the sub-solar point. In the planetary heat integral the $\cos(\gamma)$ term is accounted for in the dot product **S*r** between the solar vector and the planet surface vector.

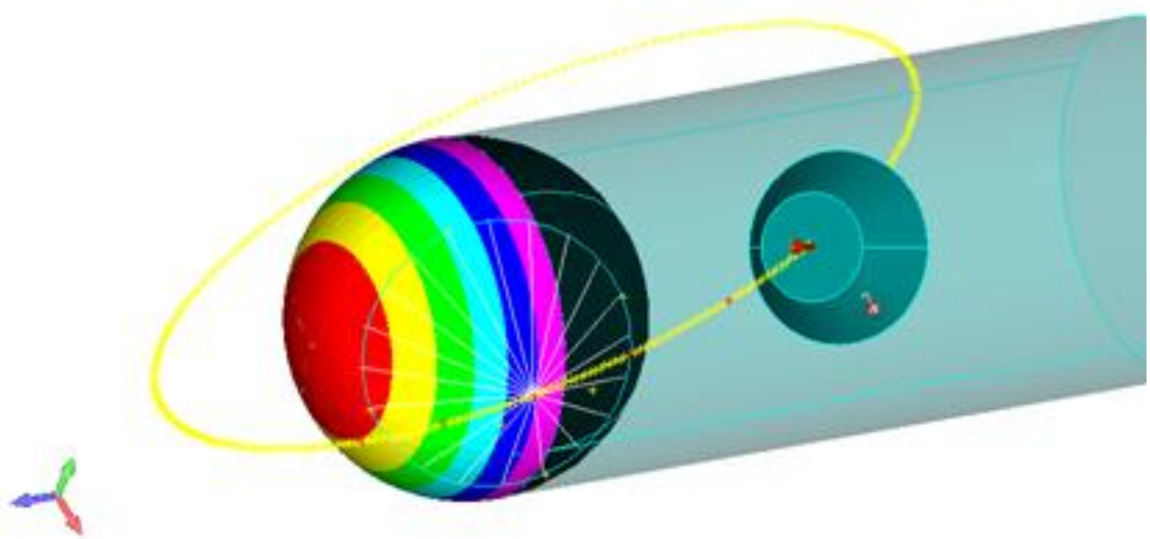

Figure 24 Spacecraft Surface and Orbit. The cosine dependence of infra-red or albedo planetary heat is indicated by the color contours. Also shown is the view cone of the planet at one orbit position. The Mathcad integral incorporates a conditional statement to account for that portion of the view cone in the planet shadow.

Planetary heats are calculated by integrating over the view cone at each orbit position. Heats are calculated by adding heats from two integrals that separately calculate the heat contributions from the illuminated and shaded sides of the planet. Binary conditional statements are used in the integrals to account for portions of the spacecraft surface that are in, or not, in view of the planet and portions of the view cones that are illuminated or shaded.

5.2.3.1 Spacecraft Surface Fixed in Celestial Coordinate System

A right-handed celestial coordinate system is defined for reference, with the z axis pointed at the vernal equinox and the y axis in the direction of the elliptic normal. The example orbit is considered with the following rotations:

$R_z(40°)R_x(20°)R_y(30°)$ Eq.(11)

this results in a $\beta = 33.3°$ and a solar-periaspsis angle of 35.5°. The analysis was run for a plate surface fixed in the celestial coordinate system with a normal vector:

$$n_i = \begin{pmatrix} -0.224 \\ 0.375 \\ -0.9 \end{pmatrix}$$ Eq.(12)

The results are shown in Figure 26.

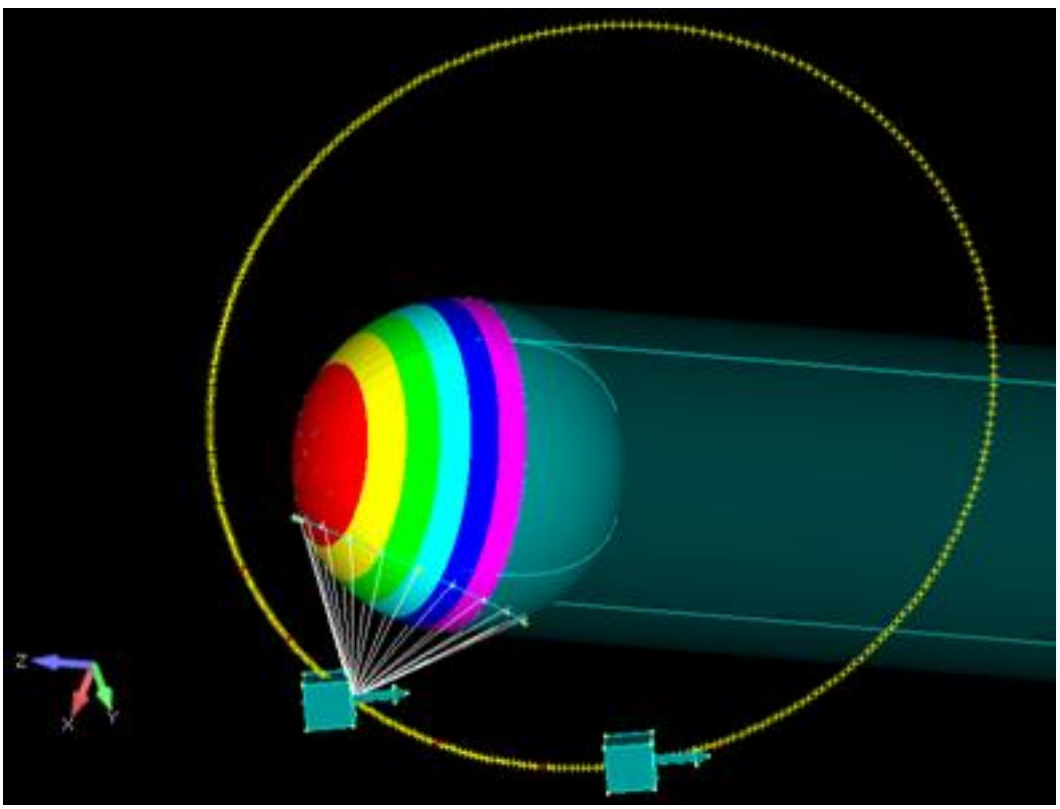

Figure 25 View of Orbit and Spacecraft Surface From Orbit Normal. The orbit does not intercept the umbral cone.

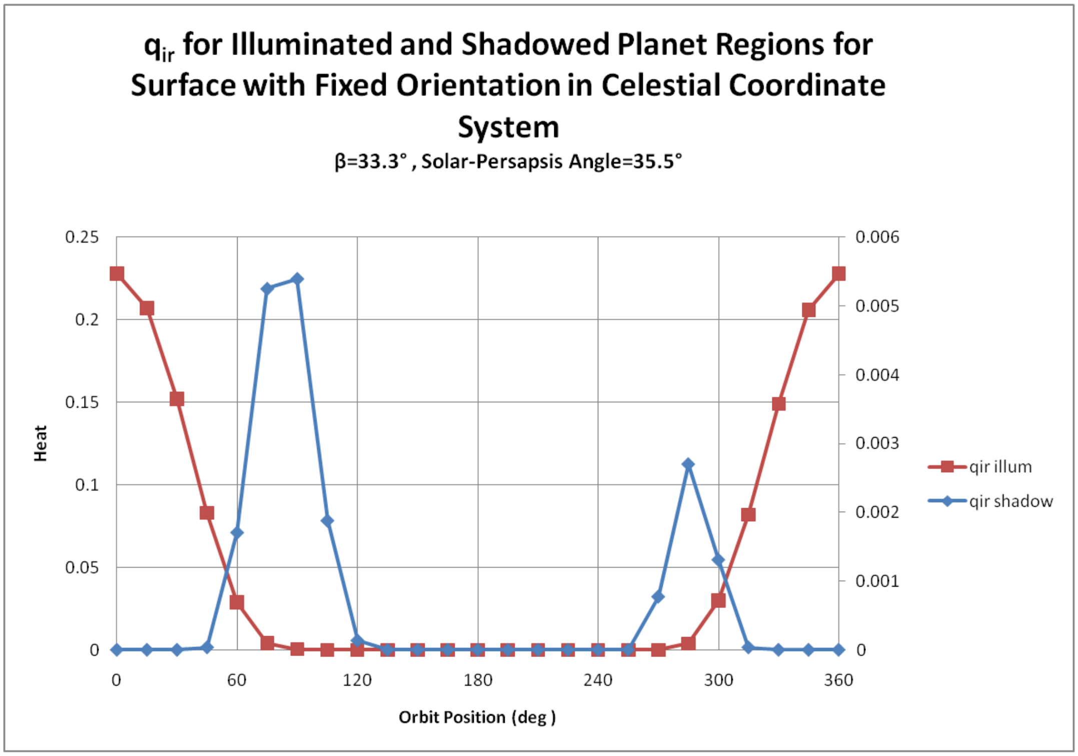

Figure 26 Planetary Infra-red for Illuminated and Shadowed Planet Regions for Fixed in Celestial Coordinate System. The transition between the illuminated and shadowed planet regions is captured. The infra-red constant is one tenth of the solar constant in this example.

4.0 Conclusions

Mathcad has proven to be a useful tool for calculating view factors and spacecraft orbital heats. View factors calculations are straight forward for any surface that has an analytical formulation and is not self viewable. This includes most revolved surfaces, including revolutions of all conic sections. In some cases shadowing can be accounted for by a simple conditional statement placed in the integral. The view factor integral is easily adaptable to calculate view factors as functions of rotations and translations of surfaces with respect to one another.

The view factor integral combined with orbit functions can be used to calculate solar and planetary orbital heats. Surfaces can be oriented in the spacecraft or celestial coordinate systems, orientation can be easily adapted to be time dependent. All common spacecraft surface types can be accounted for. Shadowing by other surfaces would require special techniques. All conic section orbits are readily handled.

The orbital heat calculations capture transitions through the planet terminator and the solar inclination angle dependence of albedo and infra-red heats. Rotation of matrices can be used to vary orbit inclination. E.g the solar vector can be rotated to analyze the variations in orbital heat averages throughout the year or orbital maximum heats can be calculated as a function of β angle. The method is useful for rapid parametric analysis of orbits and as screening tool for detailed Monte Carlo models [5].

Using the methods presented here gives the student an intuitive feel for view factors and is an excellent tool to acquaint oneself with orbital mechanics and the phenomenon of orbital heating.

5.0 Nomenclature

T	orbit period
θ	orbit angle
a	semi-major axis
b	semi-minor axis
d	distance, distance to directrix
e	eccentricity
t	time
$\mathbf{r}$	sight vector between surfaces, planet center vector
$\mathbf{r_a}$	vector from planet surface to spacecraft
R	radius of differential view cone
R_v	radius of view cone
$\mathbf{n_{i,j}}$	surface normal vectors
$\mathbf{r_r}$	orbit distance
$\mathbf{R_x}, \mathbf{R_y}, \mathbf{R_z}$	rotation matrices
τ	azimuthal angle of view cone
$\mathbf{q_{solar}}$	solar heat
$\mathbf{q_{ir}}$	infra-red heat
$\theta_{i,j}$	surface azimuthal angle
$A_{i,j}$	surface area
$\mathbf{S}$	solar vector

6.0 References

[1] Siegal, Robert, Howell, John R., “Thermal Radiation Heat Transfer”; Taylor & Francis, 3rd edition, 1992

[2] MathCad 14.0 MO3O, Parametric Technology Corporation, 140 Kendrick Street, Needham, MA 02494 USA

[3] Incropera, F.P, DeWitt, D.P., “Fundamentals of Heat Transfer and Mass Transfer”; John Wiley & Son, 4th edition, 1996

[4] Thermal Desktop 5.2 Patch 3

[5] Rickman, S.L., “A Simplified Closed-Form Method for Screening Spacecraft Orbital Heating Variations”; Proceedings Thermal Analysis and Fluid Workshop 2002