Using Transformations, Computer Coding and 3D Printing to Investigate Solids

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This unit is designed to engage 10th grade geometry students with concepts relating to transformations, through the use of OpenSCAD computer software and a 3D Printer. The unit will help students to apply and expand their existing knowledge of transformations of figures on a coordinate plane to create and manipulate geometric solids in a 3D coordinate system.

Visualization of 3D objects is an area of geometry that often challenges students. This unit will offer an opportunity for the students to not only create, manipulate and print the actual 3D objects, it will also enable them to explore the cross sections of these objects throughout the 3D printing process.

Meeting the Common Core Standards: This unit will help to reinforce many of the Geometry Common Core Standards that are part of students' prior learning. In terms of content, they will need to recall and apply their knowledge of representing transformations in the plan using geometry software. They will need to recognize and determine which transformations are most appropriate to create the desired objects and position them correctly. Then, they will describe these transformations, using code, in order to create points, as well as entire figures, as output (G-CO.A.2, G-CO.A.4). In some cases, they may need to identify a sequence or combination of geometric figures (G-CO.A.5).

Through successful completion of this unit, students will gain new knowledge that is also aligned with the Common Core Standards. They will consider dissection arguments and Cavalieri's principle (G-GMD.A.1) and predict and identify the shapes of two-dimensional cross-sections of three-dimensional objects. They will also consider how two-dimensional objects can be rotated to generate three-dimensional shapes (G-GMD.B.4) Students will also use volume formulas for cylinders, pyramids, cones and spheres to solve problems and create their objects (G-GMD.A.3)

The following Mathematical Practice Standards will also be supported throughout this unit:

- Make Sense of Problems and Persevere in Solving Them
- Model with Mathematics
- Use appropriate tools strategically
- Attend to precision
- Look for and make use of structure

By nature, this unit offers opportunities for extensive differentiation as there is great capacity for self-directed learning and customizable outcomes. There are also many opportunities for extensions, adaptations and further applications for special courses at the high school or college level, and for interdisciplinary courses or projects.

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Unit Cover Page

Unit Title: Using Transformations, Computer Coding and 3D Printing to Investigate Solids

Grade Level: 10th grade

Subject/Topic Area: Geometry and computer science

Key Words/Related Topics: Transformations, translation, rotation, dilation, scale, 3D coordinates, solids, composite solids, cross sections, 3D printer, OpenSCAD.

Time allotted: 3 weeks

Brief Summary of Curriculum: Students will use the free software OpenSCAD to design and print objects on the 3D printer. They will explore the three-dimensional coordinate plane, develop their knowledge and ability to visualize geometric solids, and apply and further develop their knowledge of transformations. Students will learn some basic computer coding as they instruct the OpenSCAD software to create spheres, cylinders, cones, and rectangular prisms. They will also learn to construct solids by extruding and rotating two-dimensional objects. Lastly, students will print their creations using a 3D printer. This will allow them to explore the successive cross sections of their designs as they are printed layer by layer.

Geometry Unit Overview

Number of Lessons: 14

Time Period: Approximately 3 weeks/14 days of instruction

A. Common Core Standards

Prior skills used in this unit:

G-CO.A.2 Represent transformations in the plane using, e.g., transparencies and geometry software; describe transformations as functions that take points in the plane as inputs and give other points as outputs. Compare transformations that preserve distance and angle to those that do not (e.g., translation versus horizontal stretch).

G-CO.A.4 Develop definitions of rotations, reflections, and translations in terms of angles, circles, perpendicular lines, parallel lines, and line segments.

G-CO.A.5 Given a geometric figure and a rotation, reflection, or translation, draw the transformed figure using, e.g., graph paper, tracing paper, or geometry software. Specify a sequence of transformations that will carry a given figure onto another.

New skills obtained in this unit

G-GMD.A.1 Give an informal argument for the formulas for the circumference of a circle, area of a circle, volume of a cylinder, pyramid, and cone. Use dissection arguments, Cavalieri's principle, and informal limit arguments.

G-GMD.A.3 Use volume formulas for cylinders, pyramids, cones and spheres to solve problems.

G-GMD.B.4 Identify the shapes of two-dimensional cross-sections of three-dimensional objects, and identify three-dimensional objects generated by rotations of two-dimensional objects.

Mathematical Practice Standards supported by this unit

Make Sense of Problems and Persevere in Solving Them

Model with Mathematics

Use appropriate tools strategically

Attend to precision

Look for and make use of structure

B. Essential Questions

- How does the x-y-z-coordinate plane work?
- How can we create complex objects by transforming and combining basic solids?
- How can we use language from geometry to instruct a computer program to create a three-dimensional figure?
- How can we visualize three-dimensional objects described in words?
- How can we visualize the cross sections of three-dimensional objects?
- How does a 3D printer work?

C. Student Learning Objectives

What understandings are desired?

Students will understand how the three-dimensional coordinate space works like the two-dimensional coordinate space.

Students will understand that solid objects can be visualized as a stack of two dimensional objects and that the volume formulas can be derived from this process.

Students will understand that a 3D printer works by building objects one cross section at a time.

What key knowledge and skills will students acquire as a result of this unit?

Students will be able to use three-dimensional coordinates to define a point in space by observing "how much over, how much out, how much up".

Students will be able to translate, rotate, and scale solids in three-dimensional space.

Students will be able to use OpenSCAD to define basic solids and combine them into composite solids.

Students will be able to find the volumes of the objects they create.

Students will be able to define a rectangular prism as an extrusion of rectangles or a cylinder as an extrusion of circles.

Students will be able to visualize and draw the cross sections of various solids.

Students will be able to use the 3D printer software to export and print their designs.

D. Authentic Assessments

1. Cumulative Performance Tasks

Students will work in pairs to produce a project report and presentation including

- An composite solid they created using OpenSCAD
  - These objects can be defined based on the student level
  - Suggestions include:
    - Items from the Common Core Geometry curriculum, assessments and Regents exam
    - Molecules or atomic models from chemistry class
    - Animals made from combinations of simple solids
    - A print-out of their computer code annotated to show the effects of the transformations they used.
    - An explanation of how one of the solids in their design could be created by extruding a two-dimensional shape
    - An explanation of how the volume formula for this solid could be explained in terms of the extrusion.
    - The calculations they made to find the volume of their composite object and the final volume.
    - Several photographs or drawings of their object at different stages of the print process showing various cross sections
    - Their physical object as produced by the 3D printer

2. Formative Assessments

- The individual nature of this project work will require frequent teacher check-ins and observational assessments during class.
- Homework should include basic skills work on volume formulas and other basic skills as presented in the EngageNY Module 3.
- Students should submit drafts of their code at various stages of development. They should annotate the parts of their code that is working and write about what challenges they are having in creating their models.

High School Instructional Plan (calendar)

Monday	Tuesday	Wednesday	Thursday	Friday
1. Introduction to 3D printer hardware and software	2. Introduction to OpenSCAD. Basic solids and shapes: cube, sphere, cylinder (and cone).	3. The 3D coordinate plane and ordered triples.	4. Transformations: translate and scale.	5. Transformations: rotate.
6. Coding the difference and union functions	7. Student work day - creating composite solids	8. Linear_extrude, volume formulas, and Cavalieri's principle.	9. Rotate_extrude and solids of rotation.	10. Student work day - complete composite solids.
11. Finalize designs, export for printer, begin project write up	12. (printing day) Students work on reports or prepare for a test	13. (printing day) Students work on reports or prepare for a test	14. Project presentations	

High School Unit Plan Overview

1. Title and Brief Description of Unit

Title: Using Transformations and Computer Coding to Investigate Solids with the 3D Printer

Math: Students will use the free software OpenSCAD to design and print objects on the 3D printer. They will explore the three-dimensional coordinate plane, develop their knowledge and ability to visualize geometric solids, and call on their knowledge of transformations. Students will learn some basic computer coding as they instruct OpenSCAD to create spheres, cylinders, cones, and rectangular prisms. They will also learn to construct solids by extruding and rotating two-dimensional objects. Lastly, as their creations are printed on the 3D printer, students will explore the successive cross sections of their designs as they are printed layer by layer.

Technology: Students will learn basic computer coding as they program in OpenSCAD. They will also learn how the 3D printer works.

Science: This unit could be given a science connection having the students design composite solids that are models from science class. For example, students could use these skills to design and print models of molecules, incorporating electron radius, bond length, and bond angle and further deepening the math required for the project.

2. Common Core Standards

Common Core Math: Prior skills used in this unit

G-CO.A.2 Represent transformations in the plane using, e.g., transparencies and geometry software; describe transformations as functions that take points in the plane as inputs and give other points as outputs. Compare transformations that preserve distance and angle to those that do not (e.g., translation versus horizontal stretch).

G-CO.A.4 Develop definitions of rotations, reflections, and translations in terms of angles, circles, perpendicular lines, parallel lines, and line segments.

G-CO.A.5 Given a geometric figure and a rotation, reflection, or translation, draw the transformed figure using, e.g., graph paper, tracing paper, or geometry software. Specify a sequence of transformations that will carry a given figure onto another.

New skills obtained in this unit

G-GMD.A.1 Give an informal argument for the formulas for the circumference of a circle, area of a circle, volume of a cylinder, pyramid, and cone. Use dissection arguments, Cavalieri's principle, and informal limit arguments.

G-GMD.A.3 Use volume formulas for cylinders, pyramids, cones and spheres to solve problems.

G-GMD.B.4 Identify the shapes of two-dimensional cross-sections of three-dimensional objects, and identify three-dimensional objects generated by rotations of two-dimensional objects.

3. Content Overview

Background information about the content

Teacher should have knowledge of the required content from Module 1 and Module 2 of the EngageNY Common Core Geometry Curriculum. Teacher will likely not have knowledge of the theory and syntax of the OpenSCAD programming language. Please see the Appendix of this unit plan for a complete description of all of the OpenSCAD commands that are involved in this unit as well as further curriculum suggestions for each concept. Material from this Appendix is incorporated into the lesson plans.

4. Prior knowledge required by students:

Knowledge (facts): ● Knowledge of formulas for area of a rectangle and circle ● Familiarity with formulas for the volume of a sphere, rectangular prism, cylinder, and cone.
Skills: ● Understanding of two dimensional coordinates ● Understanding of transformations: translations, rotations, and dilations.

5. Possible Misconceptions

At first glance, the computer code to create these models seems very complicated, and the syntax may intimidate the students at first. If they start slow, work in groups, and are given time to experiment, they will be able to complete the tasks.

6. Materials & Resources Needed

All lessons	Computers - one per student or one per group of 2. 3D printer Suggested: Stands (possibly ring stands from the chemistry lab) hung with with stick models of the x-, y-, and z-axis to help student visualize the three-dimensional coordinate space.

7. Project Kickoff

Task Activating Prior Knowledge:

Have an object printing on the 3D printer. Have students observe the printer in action.

1. Describe what the printer is doing to create the object.
2. If you could print anything at all, what would you print?
3. How do you think you could design an object on the computer for the printer to print? What would the challenges be?

Tell the class that they will be working in pairs to design an object for the 3D printer.

As a class, create a Know-Need to Know chart about the project.

Depending on the general structure of the course, you can provide an entry document or project task sheet before or after the Know-Need to Know conversation. If you think it will encourage the students, show them an example of code from OpenSCAD and the object it produced.

High School Expanded Lessons

Lesson # 1 Title: Introduction to 3D printer hardware and software

Brief summary of the lesson

Students will see the 3D printer in action, be introduced to the project, and learn the basics of printing a design on the printer.

I. Introduction

Essential Questions:

1. How does a 3D printer work?
2. How is a 3D design different than a 2D design?
3. How can we download, prepare, and print a design on the 3D printer?

Have an object printing on the 3D printer. Have students observe the printer in action.

1. Describe what the printer is doing to create the object.
2. If you could print anything at all, what would you print?
3. How do you think you could design an object on the computer for the printer to print? What would the challenges be?

Tell the class that they will be working in pairs to design an object for the 3D printer.

As a class, create a Know-Need to Know chart about the project.

Depending on the general structure of the course, you can provide an entry document or project task sheet before or after the Know-Need to Know conversation. If you think it will encourage the students, show them an example of code from OpenSCAD and the object it produced.

II. Development

- Students explore Thingiverse and download at least one object they would like to print
- Students open the .stl file on the Makerbot software
- Teacher demonstrates how to adjust the View, the Position, the Rotation, and the Dimension of the object.
- Teacher asks for students to state the name of the transformation taking place with each change.
- Students take time to position, rotate, and scale their object.
- Teacher shows students how to export a print file, describing the Resolution, Rafts, and Supports. Teacher discusses the print time and shows students the Print Preview feature, further discussing rafts and supports.
- In print preview, Teacher shows the students how to adjust the slider on the left to show the cross sections that will appear as the printer works.

III. Closure

- Students export their print file and email the .makerbot file to the teacher. Each student should answer these questions in their email.
  - Why did you choose this object?
  - How long will it take your object to print? How could you reduce this print time?
  - Why does the printer sometimes need to build supports when printing an object?
  - Why is the concept of cross section so essential when working with the 3D printer?

IV Differentiation

- Some students may have trouble with basic computer skills. Pair these students with stronger students or sit them together at a table where they can receive teacher guidance together.
- Print a list of steps for each of the computer tasks for the students to follow.
- Some objects on Thingiverse come with OpenSCAD code attached. Find a few of these objects ahead of time. Have the stronger students download the object and the OpenSCAD code and begin to analyze the code.

Lesson # 2 Title: Introduction to OpenSCAD. Basic solids and shapes: sphere, rectangular prism, cylinder, and cone.

Brief summary of the lesson

Students will learn the basic syntax of the OpenSCAD language and use OpenSCAD to create spheres, rectangular prisms, cylinders, and cones.

I. Introduction

Essential Questions:

1. How can you give the computer instructions to print something specific on the 3D printer?
2. What are the important basic features of a computer language?
3. How can we compare and contrast a computer language and our language?

- Students open OpenSCAD on the computer. They have a blank programming window and a blank rendering window showing a x-y-z plane.
- Teacher instructs students to create a sphere with radius 2.
- Students struggle with this, and is unlikely that any student will complete the task. (Some may google how to do it, which is great!)

II. Development

- Teacher explains that you can tell a computer what to do, but you need to use a language the computer will understand -- a very precise language.
- Teacher models how to make a sphere: *sphere (r = 2);*
- Highlight the importance of the semicolon
- Discuss the differences between `sphere(r = 2);` and `sphere(d = 2);`.
- Observe the difference in resolution between `sphere(r = 2);`, `sphere(r = 20);`, and `sphere(r = 200);`.
- Discuss other arguments such as $fa (fragment angle in degrees), $fs (fragment size in mm), and $fn (resolution). Motivate how OpenSCAD renders curved surfaces such as spheres.
- Have the students practice.
- Show students that if they place the comment marker “//” in front of a line, it will not execute. This will allow them to remove their sphere from the rendering space but keep the code in the coding window.

- Introduce the `cube` command. Motivate the fact that this command can and will return a general rectangular solid.
- Discuss the differences between `cube(size = 1, center = false);` and `cube(size = [1,2,3], center = true);`.
- Have the students practice.

- Introduce the `cylinder` command. Motivate the fact that this command returns right cylinders with a certain orientation.
- Discuss the differences between providing one radius/diameter to the cylinder and two radii/diameters. Motivate that this is a frustum of a cone.
- Have the students practice.

- Show how to draw right cones using the `cylinder` command with the first or second radius input as 0.
- Have the students practice.

- Show students that they can place multiple objects in the space using the *translation* command. Have them input *translate ([10, 0, 0])* before the first object, *translate ([20, 0, 0])* before the second object, etc. You will explain this in lesson 4, but it is a good preview now, and advanced students will experiment more with this command.

III. Closure

- Students save their work with a meaningful name that includes their names.
- Teacher challenges students to come to the display computer and create objects that the teacher names: sphere with radius 12, prism with dimensions… Students volunteer and come to the front one at a time to try to create the object named.

IV. Differentiation

- A list of commands and examples of syntax is essential for this unit, though it does not need to be distributed to all students. Encourage students to google "How do I …. on OpenSCAD" when they are stuck.
- Stronger students should move more quickly to the translate command so they can experiment with placing their objects in different places in the space.
- Stronger students might be given a list of commands and syntax and allowed to experiment without following with the teacher demonstrations.

Lesson # 3 Title: The 3D coordinate plane and ordered triples.

Brief summary of the lesson

Students will learn about extending the two-dimensional x-y-coordinate plane into three dimensions.

Materials

- 2D coordinate plane represented on the floor of the classroom
- Physical representation of the 3D coordinate space for each group of students
- String or straight sticks or straws to represent lines in the space
- Construction paper to represent plane in the space
- Balls and boxes to represent spheres and prisms in the space

I. Introduction

Essential Questions:

1. How can we use coordinates to describe the position of a point in space?
2. What objects can we define in space that we cannot define on the plane?

- A square of space is cleared in the middle of the room with an x- and y-axis marked with tape (ideally your floor is tiled with squares).
- Teacher places a small object at a point on the floor and asks students to name the coordinates of the object. A student at the board draws a corresponding point on an x-y-plane. Repeat a couple times.
- Teacher then names a point and asks a student to place the object at the correct position. Repeat a couple times.
- Teacher then stand on the plane but holds the object up instead of placing it on the floor. Teacher asks for a volunteer to name the coordinates of the point where the object is now. Allow creative answers. Repeat a couple times. Students may ask to use a meter stick to measure the height. Ask the student at the board to draw this new position on the x-y plane. What is the problem? Highlight the difficulty in drawing a three-dimensional position or object on a two-dimensional plane.
- Teacher asks for a volunteer to place the object at position $(-2, 6, 4)$. Repeat a couple of times. What about $(5,\ 1, -6)$?

II. Development

- Students take their seats
- Teacher gives students definitions for their notes on the 3-dimensional coordinate system: x-y-z-coordinate system, coordinate triple (x, y, z), x-intercept, y-intercept, z-intercept.
- Teacher introduces students to the physical model of the 3D space: some type of stand -- maybe a ring stand from the chemistry lab, with an x-, y-, and z-axis hanging from it. Or, the stand itself can be the z-axis and the x- and y-axes can be affixed. Though it is arbitrary, maintain the vertical axis as the z-axis because this feature is fixed in OpenSCAD.
- Have students complete the following activities using the model of the x-y-z-space. They should describe their results in words and if possible, take a picture of each scenario and attach it later to a document with their work from class.
  1. Locate each point: (0, 0, 0) (1, 2, 3) (4, -3, 8) (-5, -5, -5)
  2. Represent the line that connects each pair of points
     a. (1, 4, -5) and (-2, -3, 5)
     b. (0, 0, 2) and (2, 0, 0)
  3. Represent the plane that is defined by each triple of points
     a. (1, 2, 3) and (4, -3, 8) and (-5, -5, -5)
     b. (1, 4, -5) and (-2, -3, 5) and (0, 0, 2)
  4. Locate the position (5, 8, 6)
     a. Make a string or stick approximately 4 units long
     b. Fix one end of the string or stick at the point (5, 8, 6)
     c. Move the other end of the string or stick freely.
     d. What is the name of the object formed by the collection of all points that are 4 units from a fixed point?
  5. Represent each of the following objects
     a. A sphere with center (-4, -3, 6) and radius 2.
     b. A sphere with center (5, -6, 7) and radius 4.
     c. For the two spheres you just made, is there any other way you could make the sphere? Is there only one sphere that fits these conditions?
     d. A rectangular prism with one vertex at (0, 0, 0) and another vertex at (3, 4, 5).
     e. For exercise d., is there any other way to make the rectangular prism? Is there only one rectangular prism that fits these conditions?
     f. Place a rectangular prism in the coordinate space and define the coordinate triple for each of the eight vertices.

III. Closure

- Have student pairs decide how they are going to package and present their information from class. They may have to share notes or upload or email photos.
- Bring the class together and take questions from the activities and questions. Discuss these issues as a class, reviewing the concepts of the lesson.

IV. Differentiation

- Students might work in larger groups so students with difficulty with the visualizations can have support from their peers. Be careful with grouping so that all students are involved in the hands-on activities.
- Students may also have trouble recording their work from this lesson, since the arrangements cannot be drawn on paper. Again, it might be helpful to group the students heterogeneously because some students might be better able to manage the verbal representations or photos.

Appendix: Curriculum guide for the OpenSCAD programming language

Unit: 10th grade - Geometry

Part I: Solids and Transformations

The first part of the unit is designed to introduce geometry students to some basic 3-dimensional shapes. We will also discuss various transformations and their effects on these shapes.

Section I.A: Review of basic solids

Recall that all 2-dimensional shapes have an interior and a boundary. The boundary gives the outline of the shape and consists of one or more straight or curved line segments (i.e., 1-dimensional shapes or edges). Similarly, a solid is a 3-dimensional shape and also possesses an interior and a boundary. In the case of a solid, the boundary consists of one or more faces. The faces are 2-dimensional shapes and they may be flat or curved.

Classroom exercises

- Present drawings of 2-dimensional shapes and identify the boundary and what sort of lines segments make up the boundary.
- Do the same for 3-dimensional shapes.
- Given a certain number of flat faces and curved faces, ask the students to think of a shape that conforms.

OpenSCAD exercises

- Make sure the students have OpenSCAD installed and give them a general orientation.

3-D printing exercises

- Make sure the students have the appropriate printer software installed (such as MakerBot Desktop) and give them a general orientation.

Subsection I.A.1: Rectangular solids/cube

A rectangular solid is a solid with six faces that are all flat rectangles. Furthermore, opposite sides are parallel and congruent. Because of this configuration, it possesses three parameters given by the lengths of the edges. These parameters are usually called length, width, and height. When the length, width, and height are all equal, the faces are congruent squares and the rectangular solid is known as a cube.

Classroom exercises

- Draw different rectangular solids of various shapes and sizes. Identify the differences in length, width, and height, as well as which is which.
- Draw rectangular solids in various perspectives.

OpenSCAD exercises

- Introduce the `cube` command. Motivate the fact that this command can and will return a general rectangular solid.
- Discuss the differences between `cube(size = 1, center = false);` and `cube(size = [1,2,3], center = true);`.
- Have the students practice.

3-D printing exercises

- Export rectangular solids from OpenSCAD and print. Possibly have some preprinted.

Subsection I.A.2: Spheres

A sphere is a solid with a single curved face for its boundary. Its boundary is given by the set of all points a fixed distance (called the radius) from a particular fixed point (called the center). In other words, it is essentially a 3-dimensional version of the 2-dimensional circle.

Classroom exercises

- Draw spheres of various sizes.
- Demonstrate some of the challenges of drawing a sphere.

OpenSCAD exercises

- Introduce the `sphere` command.
- Discuss the differences between `sphere(r = 2);` and `sphere(d = 2);`.
- Observe the difference in resolution between `sphere(r = 2);`, `sphere(r = 20);`, and `sphere(r = 200);`.
- Discuss other arguments such as $fa (fragment angle in degrees), $fs (fragment size in mm) and $fn (resolution). Motivate how OpenSCAD renders curved surfaces such as spheres.
- Have the students practice.

3-D printing exercises

- Export spheres from OpenSCAD and print. Possibly have some preprinted. This will probably get into the issue of support and possible rafts.

Subsection I.A.3: Cylinders

A cylinder is a solid with three faces, two of which are flat, and one of which is curved. The flat faces are called bases and consist of parallel congruent circles, with the third face connecting the two. The third face is the set of all points that are a fixed distance from the line segment called the axis that connects the centers of the two circular faces. The most common type of cylinder is the right circular cylinder, a cylinder whose axis is perpendicular to each of its circular faces.

Classroom exercises

- Draw cylinders of various shapes and sizes.
- Make sure to distinguish between a right cylinder and non-right cylinder.

OpenSCAD exercises

- Introduce the `cylinder` command. Motivate the fact that this command returns right cylinders with a certain orientation.
- Discuss the differences between providing one radius/diameter to the cylinder and two radii/diameters. Motivate that this is a frustum of a cone.
- Have the students practice.

3-D printing exercises

- Export cylinders from OpenSCAD and print. Possibly have some preprinted.

Subsection I.A.4: Prisms

A prism is very similar to a cylinder, except that all faces are flat. The faces on the ends are called bases. The bases are parallel congruent polygons and the other faces are parallelograms that connect the two end faces. If these parallelograms are rectangles, then the prism is called a right circular prism.

Classroom exercises

- Draw prisms of various shapes and sizes.
- Make sure to distinguish between a right prism and non-right prism.

OpenSCAD exercises

- Introduce the `polyhedron` command. Motivate the fact that this command is very general and must be told specifically how to generate a prism in OpenSCAD.
- Display various examples.
- Have the students practice.

3-D printing exercises

- Export prisms from OpenSCAD and print. Possibly have some preprinted. Non-right prisms could get into the issue of support.

Subsection I.A.5: Cones/pyramids

A cone is a solid with two faces, one flat and one curved. The flat face is circular and is called the base. The curved face can be thought of as the set of all lines that connect the edge of the base to a fixed point.

A pyramid is very similar to a cone, except that all faces are flat. The base is a polygon and the other faces are constructed by taking the set of all lines that connect the edges of the base to a fixed point.

Classroom exercises

- Draw cones and pyramids of various shapes and sizes.
- Make sure to distinguish between right cones/pyramids and non-right cones/pyramids.

OpenSCAD exercises

- Show how to draw right cones using the `cylinder` command with the first or second radius input as 0.
- Show how the `polyhedron` command can be used to generate a pyramid in OpenSCAD.
- Have the students practice.

3-D printing exercises

- Export cones and pyramids from OpenSCAD and print. Possibly have some preprinted. This could get into the issue of support. For cones, make sure the cones are pointed up.

Subsection I.A.6: Platonic solids

The platonic solids are the most regular solids that exist. To be a platonic solid, a solid's faces must consist entirely of congruent regular polygons and the same number of faces must meet at each vertex. These criteria provide enough restrictions such that there are only five platonic solids. Three platonic solids have faces that are all equilateral triangles. These are:

1. The tetrahedron, which has four faces.
2. The octahedron, which has eight faces.
3. The icosahedron, which has twenty faces.

The fourth platonic solid has faces that are all squares. This is:

4. The cube or hexahedron, which has six faces.

The fifth platonic solid has faces that are all regular pentagons. This is:

5. The dodecahedron, which has twelve faces.

Classroom exercises

- Draw the various platonic solids.
- Engage the class in a discussion of why there are no more than five of these. Perhaps the students can try to draw additional ones or one with hexagonal faces. One good way to motivate this is to consider the number of regular polygons that can meet at a vertex. There must be at least 3 and there must be less than the number that would tile the plane. This results in 3, 4, or 5 equilateral triangles, 3 squares, or 3 regular pentagons.

OpenSCAD exercises

- Show how to draw platonic solids using the polyhedron command.
- Have the students practice.

3-D printing exercises

- There are lots of online models of platonic solids, some of which print the faces and snap together. Having some of these preprinted would give the students some nice models to construct to get a feel for these.

Section I.B: Introduction to volume

Volume is a way of measuring the amount of space occupied by a solid. This is analogous to the concept of area, which measures the relative size of a 2-dimensional shape in the plane. One way of thinking of volume is to imagine a 3-dimensional grid tiled by cubes measuring each 1 unit by 1 unit by 1 unit. To obtain the volume, simply count the number of these cubes that the shape occupies. This turns out to be straightforward for some shapes but far more difficult for others.

Classroom exercises

- Display some technical drawings of shapes on a 3-dimensional grid and determine/estimate the volume using this method.

OpenSCAD exercises

- Show how the coordinate grid on OpenSCAD relates to these ideas.

Subsection I.B.1: Volume of rectangular solids

It is a relatively straightforward observation to determine that the volume of a rectangular solid is given by $V = lwh$, where l, w, and h denote length, width, and height respectively. In the case of a cube, the $l = w = h$, so the volume is given by l^3.

Classroom exercises

- Show how this stems from the approach of counting the number of cubes occupied on the grid that was outlined above.
- Compute the volumes of various rectangular solids.

3-D printing exercises

- Measure the dimensions of printed 3-dimensional models of rectangular solids and compute their volumes.

Subsection I.B.2: Volume of right cylinders and prisms

Note that the rectangular solid is actually a right prism with rectangular bases. This suggests that the volume of a general right prism (or right cylinder) is obtained by multiplying the area of the base by the height of the prism (or cylinder).

Classroom exercises

- Provide some justification by overlaying these objects on the grid.
- Compute the volumes of various right cylinders and right prisms. Note that this can be difficult for prisms depending on the degree of complication of the base shape.

3-D printing exercises

- Measure the dimensions of printed 3-dimensional models of right cylinders/simple right prisms and compute their volumes.

Section I.C: Transformations

Given a solid in space, there are a number of transformations that can be applied to move or change the shape of the solid. One way of expressing this information is to use a Cartesian coordinate system. In two dimensions, this is the standard coordinate system with two axes that specifies each point as an ordered pair (x, y) of coordinates. In three dimensions, a Cartesian coordinate system has three axes and each point is expressed as an ordered triple (x, y, z) of coordinates.

Using this notation, it is possible to express various types of transformations by designating what they do to each point (x, y, z). Sometimes this may depend on the values of x, y, and z, respectively.

Classroom exercises

- Draw a 3-dimensional Cartesian coordinate grid.
- Motivate the right-hand rule for labelling the x, y, and z axes.

OpenSCAD exercises

- Show how the students have been using this system all along when using OpenSCAD. Draw connections between how OpenSCAD displays solids and 3-dimensional Cartesian coordinates.

Subsection I.C.1: Translations

A translation takes every point in 3-dimensional space and moves it a fixed distance in a fixed direction. This is accomplished by taking each point (x, y, z) and adding/subtracting a fixed value from each coordinate. For example, taking the point (x, y, z) and adding 3 to x, subtracting 2 from y and adding 5 to z results in the point $(x + 3,\ y - 2,\ z + 5)$.
This can be expressed in a number of ways. One way is to show that the point (x, y, z) becomes the point $(x + 3,\ y - 2,\ z + 5)$ by writing $(x, y, z) \rightarrow (x + 3, y - 2, z + 5)$. Another is to the think of the translation as a function which we will call T. The translation takes an ordered triple as its input and returns another ordered triple as its output. In this notation, the aforementioned translation is written $T(x, y, z) = (x + 3,\ y - 2,\ z + 5)$. In fact, all transformations can be expressed in such a fashion. Note that a translation applied to a solid has no effect on its volume.

Classroom exercises

- Provide various examples of translations.

OpenSCAD exercises

- Demonstrate how the OpenSCAD `translate` command modifies another command to apply a translation. Show how `translate([3,-2,5])` performs the same role as the transformation T described above.

Subsection I.C.2: Rotations

A rotation takes every point in 3-dimensional space and rotates it a fixed angle around a fixed line. In general, this often requires some relatively complicated reasoning using trigonometry. However, certain rotations can be expressed more easily. For example, rotating the point (x, y, z) 90 degrees counterclockwise about the z-axis results in the point $(-y, x, z)$. In the notation detailed in the Subsection I.C.1, this can be expressed as $(x, y, z) \to (-y, x, z)$ or $R(x, y, z) = (-y, x, z)$. Note that a rotation applied to a solid has no effect on its volume.

Classroom exercises

- Provide various examples of rotations.
- Highlight which rotations are readily expressible and which would require more heavy-duty mathematics to perform.

OpenSCAD exercises

- Demonstrate how the OpenSCAD `rotate` command modifies another command to apply a rotation. Discuss how `rotate([0,90,0])` and `rotate(a=90, v=[0,1,0])` both rotate 90 degrees about the y-axis.
- To rotate about a more complicated axis, something like `rotate(a=45, v=[1,2,3])` rotates 45 degrees across the axis determined by the line through the origin (0,0,0) and (1,2,3).
- Provide the students some additional examples of the various ways of rotating objects in OpenSCAD. Give them a chance to explore the various syntaxes and see that the resulting rotations look like.

Subsection I.C.3: Reflections

A reflection takes every point in 3-dimensional space and reflects it across a fixed plane. Again, this may require complex geometric/trigonometric reasoning, although several reflections can be written down more easily. One example is reflection across the xy-plane (plane formed by the x and y axes). This results in the transformation $(x, y, z) \to (x, y, -z)$ or $F(x, y, z) = (x, y, -z)$. Note that a reflection applied to a solid has no effect on its volume.

Classroom exercises

- Provide various examples of reflections.
- Highlight which reflections are readily expressible and which would require more heavy-duty mathematics to perform.

OpenSCAD exercises

- Demonstrate how the OpenSCAD `mirror` command modifies another command to apply a reflection. Discuss how `mirror([1,2,3])` reflects an object across the plane perpendicular to the line through the origin (0,0,0) and (1,2,3).
- Provide the students some examples of common reflections in OpenSCAD. Give them a chance to explore what the resulting reflections look like.

Subsection I.C.4: Scalings

A scaling is a transformation that either stretches, shrinks, or simultaneously stretches and shrinks space. To keep things simple, we will consider scalings that keep the origin (0,0,0) fixed and scale along the x-, y-, and z-axes by a factor of a, b, and c, respectively, where a, b, and c are positive real numbers. In other words, a scaling is expressed as $(x, y, z) \rightarrow (ax, by, cz)$ or $S(x, y, z) = (ax, by, cz)$. If $a = b = c$, then the scaling is said to be uniform. Note that a scaling will have an effect on the volume of a solid. In particular, a uniform scaling will have the effect of multiplying the volume of a solid by a^3.

Classroom exercises

- Provide various examples of scalings.
- Highlight the distinction between uniform scalings and non-uniform scalings.
- Give demonstrations of scalings that shrink, scalings that stretch, and scalings that both shrink and stretch.

OpenSCAD exercises

- Demonstrate how the OpenSCAD scale command modifies another command to apply a scaling. Discuss how `scale([a,b,c])` performs the same role as the transformation S described above.
- Demonstrate how the OpenSCAD `resize` command modifies another command to apply a scaling. Discuss how `resize([a,b,c])` modifies an object so that it is sized to a units in the x direction, b units in the y direction, and c units in the z direction.
- Explain the differences between `scale` and `resize`.
- Have the students practice using `scale` and `resize`.

Subsection I.C.5: Combinations of transformations

It is worth noting that transformations can be combined in any order to produce new transformations. In particular, combinations of simple versions of some of the transformations detailed above can result in more complicated versions.

Classroom exercises

- Demonstrate how two parallel reflections can give a translation, as can two rotations.
- Give some other examples of composed transformations. Show that they are generally noncommutative, so order matters.

OpenSCAD exercises

- Have the students practice various examples and explorations of combining various transformations using OpenSCAD.

3-D printing exercises

- If the students are using the MakerBot Desktop software, the software itself support rotations, translations, and scalings of objects. Have the students explore these options.

- Discuss how these various transformations will affect the printing process. Note how certain orientations will work better for certain objects and the scaling can drastically impact the duration of the printing.

Section I.D: Effects of transformations on shapes

As one would expect, translations, rotations, and reflections will move, rotate, and reflect a shape. However, a scaling (perhaps in combination with the other translations) could potentially distort a shape and at the very least is expected to change its size.

A shape is said to exhibit symmetry if there is applying a transformation results in the exact same shape again. When this happens as a result of a rotation, we say that the shape has rotational symmetry. When it occurs following a reflection, we say that the shape has reflective symmetry.

Classroom exercises

- Discuss some examples of types of shapes that exhibit rotational and reflective symmetry.

OpenSCAD exercises

- Have the students explore what solids have what kind of symmetry.

Unit Part II: Composition/Decomposition of Complex Solids from/to Basic Solids

The second part of the instructional unit is to introduce the students to two parallel ideas. The first is a way of building up more complex solids from more common solids such as those listed above. Not only does this provide insight into how artists reproduce imagery on both a 2-dimensional and 3-dimensional level, but it is key concept behind modern notions of computer graphics, particularly the modeling and reproduction of 3-dimensional objects. The second idea is to reverse this notion and break a solid down into simpler building blocks. This is an important tool when it comes to finding volumes.

Section II.A: Combining [transformed] simple shapes to form more complex shapes

Imagine trying to build a 3-dimensional model of an animal, such as a cow. The shape of the cow is rather complex and is not well-represented by any of the simple solids list in section I. However, a simplified rendition of the cow could be created by combining four cylinders for the legs, a rectangular solid for the body, and a sphere for the head. Additional details, such as the tail, neck, and hooves could be built up from additional shapes, etc. As part of this process, each shape may need to go through a series of transformations. For example, the head may look better when the sphere is scaled to be oblong. A rotation may be necessary to obtain the angle of the head. A translation will put the head into the appropriate position. Combining more and more shapes in this fashion should eventually lead to a better and better model of a cow.

Classroom exercises

- Provide some drawings/examples that show how a complicated shape can be broken down into simpler shapes. Perhaps invite a guest art teacher into the classroom to discuss how simple shapes can be found within more complicated shapes.
- If possible, show the students a wooden posable mannequin to motivate some of these notions.
- Introduce the students to TinkerCAD and ask them to model a figure of their choice.

OpenSCAD exercises

- Have the students attempt to reproduce their TinkerCAD model in OpenSCAD.
- Discuss and highlight the differences between these pieces of software and motivate their advantages/disadvantages.

3-D printing exercises

- Have the students print their creations. If there is limited time/resources, select one or a few student creations for printing.

Section II.B: Decomposing a cube

Now imagine taking a simple solid. What other solids can be found within this solid? We begin by considering the cube.

Subsection II.B.1: Dividing the cube into smaller cubes

Suppose we are given a cube with edge length s. Recall that a uniform scaling of 1/2 applied to this cube will result in a cube of edge length $s/2$. The volume of the original cube is s^3, whereas the volume of the new cube is $(s/2)^3 = (s^3)/8$ or 1/8 the volume of the original cube. The suggests that a cube can be divided into eight small cubes with edge length one-half of the original cube.

Classroom exercises

- Discuss/motivate how a cube breaks down into the eight smaller cubes.

OpenSCAD exercises

- Have the students attempt to demonstrate this using OpenSCAD. One approach uses combinations of the `cube`, `translate`, and `scale` commands.
- Introduce the `color` modifier so that the subcubes can be highlighted in different colors.

3-D printing exercises

- Print or have already printed 8 uniform cubes and stack them into a larger cube.

Subsection II.B.2: Dividing the cube into pyramids

Again, suppose we are given a cube with edge length s. It turns out that any pyramid with a square bottom of edge length s and height s has a volume of $(s^3)/3$, which is 1/3 the volume of the cube. Again, this suggests that a cube can be divided into three pyramids with square bases and height s. The following exercises explore this idea further.

Classroom exercises

- Discuss/motivate how a cube breaks down into three pyramids.

OpenSCAD exercises

- Have the students attempt to demonstrate this using OpenSCAD. One approach uses combinations of the `polyhedron`, `translate`, and `rotate` commands.
- Use the `color` modifier so that the three pyramids can be highlighted in different colors.

3-D printing exercises

- Print or have already printed 3 such pyramids and stack them into a cube shape.

Section II.C: Decomposing/transforming a cylinder

We now consider a cylinder and ways of dividing it into smaller solids. We will also consider a way of transforming the cylinder into different shapes with the same volume.

Subsection II.C.1: Dividing the cylinder into discs

Imagine having a right cylinder with a base of radius r and height h. As mentioned above, the volume of this cylinder is $\pi r^2 h$. Now, imagine that it is sliced into ten identical smaller cylinders of base radius r and height h/10. The volume of each of these smaller cylinders is $\pi r^2 h$/10. If we do a similar procedure where we cut the cylinder into 100 identical smaller cylinders of base radius r and height h/100, the smaller cylinders or discs each have a volume of $\pi r^2 h$ /100.

Now imagine rearranging the 100 discs into another object. These discs can be thought of as the building blocks for a more complex shape than a cylinder. Any rearrangement of them will have the same volume of the original cylinder (namely $\pi r^2 h$).

In calculus, this principle is used to find the volumes of many different solids. The general idea is that if you can compute the volumes of very thin slices of a larger solid, you can add these volumes to obtain the volume of the larger solid.

Classroom exercises

- Show how a stack of coins is shaped like a cylinder.

OpenSCAD exercises

- Have the students attempt to demonstrate this using OpenSCAD. One approach uses combinations of the `cylinder`, `translate`, and `scale` commands.
- Use the `color` modifier so that the slices can be highlighted in different colors.

3-D printing exercises

- Print or have already printed several short uniform cylinders and stack them into a larger cylinder.

Subsection II.C.2: Cavalieri's principle

Imagine that you slice a solid in slices so thin that they are essentially 2-dimensional. Cavalieri's principle states that if two solids of the same height can both be cut into these thin slices such that each slice has equal area (possibly depending on the height), then the two solids will have equal volume. This has various consequences, one of which is that the formulae for volumes of spheres, cones and pyramids depends only on the area of the base and the height. It also tells us that performing what is known as a shear transformation on a solid will not change its volume.

Classroom exercises

- Show how a stack of coins can be adjusted to give different shapes of the same volume by staggering the coins/not putting them directly on top of each other.
- Do the same with a deck of cards.
- Have the students work out how to slice spheres, pyramids and cones so that the slices are different sizes of the same shapes. What estimates of volume does this provide?

OpenSCAD exercises

- Have the students attempt to demonstrate slicing spheres, pyramids, and cones using OpenSCAD.

3-D printing exercises

- Print or have already printed examples of spheres, cones, and pyramids (or rough approximations thereof) that are broken down into smaller slices.

Section II.D: Building solids from hollow cylinders

There is a class of solids called a solid of rotation. This is described as the set of points obtained by taking the points in a 2-dimensional region in the plane and looking at every point obtained by rotating these points around a fixed axis. The resulting object will have rotational symmetry about the fixed axis of rotation. If this were to be sliced very thinly as described above, the slices would either be circular or washer-shaped. However, in some cases, the solid can be broken down into very thin hollow cylinders. Now imagine reversing the process and building a solid out of thin hollow cylinders.

Classroom exercises

- Demonstrate how a solid can be decomposed into cylindrical shells. Provide an example of building a design from hollow cylinders.

OpenSCAD exercises

- Have the students build objects from cylinders/hollow cylinders using OpenSCAD. Have them freely explore what kind of shapes they can crank out.

3-D printing exercises

- Preprint an example of a solid broken into cylinders. Have students print their creations. If there is limited time/resources, select one or a few student creations for printing.

Section II.E: Cross sections of solids

When considering ways of slicing solids, it is important to be aware of what sort of cross sections you may encounter. Performing volume calculations will work best when all of the cross sections of a solid are 2-dimensional figures with an area that can be readily computed. For example, slicing a sphere with parallel slices that are perpendicular to a diameter of the sphere will give circular cross sections. However, slicing a sphere in a more haphazard fashion may give elliptical cross sections which have areas that are more difficult to compute.

What happens you slice a cube with parallel slices that are perpendicular to the axis connecting two opposite corners?

Imagine that you have two right cones that are placed point-to-point such that their axes both lie along the same line. What sort of figures can you obtain when you slice these cones? These figures are called conic sections.

Classroom exercises

- Cover/review the conic sections.

OpenSCAD exercises

- Have the students build a double cone in OpenSCAD. Using `difference` (applied perhaps to large rotated and translated rectangular solids), it should be possible to show the double cone sliced open. The students should try to realize the various conic sections (circles, ellipses, hyperbolas, parabolas, a point, two crossed lines, and a single line).
- Have the students build a cube in OpenSCAD. Using difference, show the cube sliced open at various angles. The students should try to obtain triangular and hexagonal cross sections, as well as squares, rectangles, and parallelograms (any other quadrilaterals or other polygons?).
- What other solids have interesting cross sections? What about the other platonic solids?

3-D printing exercises

- Print some interesting examples of these sliced-open objects. There should be resources online for printing nice models of conic sections, cubes, etc. in multiple pieces that demonstrate what is going on.

Unit Part III: Volumes

Section III.A: Estimating volumes of more complex objects

What is the volume of a sphere of radius r? What is the volume of a cone of radius r and height h? What is the volume of a pyramid? How can we estimate the volumes of more complicated objects? The previous unit gave us ways of thinking about how to build various solids up from smaller solids. We also considered ways of breaking down solids into smaller pieces with volumes that may be simpler to compute. How can we apply these methods to estimate the volumes of such things as spheres, cones, and pyramids?

Classroom exercises

- Based on all the previous classroom activities, try to estimate the formulae for various solids such as spheres, cones, and pyramids (see II.B.2 and II.C.2). How do these predictions match the actual formulae?

Section III.B: Displacement as a proxy for volume

Another method for measuring volume is to measure the amount of volume displaced by an object when it is immersed in water or sand. This method dates back to the time of Archimedes.

Classroom/3-D printing exercises

- Take examples of 3-D printed solids and immerse them in sand or water, being careful to measure the displacement (one issue with using water is that these models are very likely to float). How do these real-world estimates compare with the actual formulae, as well as the predictions?

References

Online OpenSCAD User Manual located at https://en.wikibooks.org/wiki/OpenSCAD_User_Manual