VANISHING CYCLES

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Vanishing cycle sheaves and their corresponding D-modules form the basis for Saito's constructions described later.

1. Vanishing cycles

We will start with the classical picture. Suppose that $f : X \to \mathbb{C}$ is a morphism from a nonsingular variety. The fiber $X_0 = f^{-1}(0)$ may be singular, but the nearby fibers X_t, $0 < |t| < \epsilon \ll 1$ are not. The premiage of the ϵ-disk $f^{-1}\Delta_\epsilon$ retracts onto X_0, and $f^{-1}(\Delta_\epsilon - \{0\}) \to \Delta_\epsilon - \{0\}$ is a fiber bundle. Thus we have a monodromy action by the (counterclockwise) generator $T \in \pi_1(\mathbb{C}^*, t)$ on $H^i(X_t)$. (From now on, we will tend to treat algebraic varieties as an analytic spaces, and will no longer be conscientious about making a distinction.) The image of the restriction map

$$H^i(X_0) = H^i(f^{-1}\Delta_\epsilon) \to H^i(X_t),$$

lies in the kernel of $T - 1$. The restriction is dual to the map in homology which is induced by the (nonholomorphic) collapsing map of X_t onto X_0; the cycles which die in the process are the vanishing cycles.

Let us reformulate things in a more abstract way following [SGA7]. The *nearby cycle* functor applied to $F \in D^b(X)$ is

$$\mathbb{R}\Psi F = i^*\mathbb{R}p_*p^*F,$$

where $\tilde{\mathbb{C}}^*$ is the universal cover of $\mathbb{C}^* = \mathbb{C} - \{0\}$, and $p : \tilde{\mathbb{C}}^* \times_{\mathbb{C}} X \to X$, $i : X_0 = f^{-1}(0) \to X$ are the natural maps. The *vanishing cycle* functor $\mathbb{R}\Phi F$ is the mapping cone of the adjunction morphism $i^*F \to \mathbb{R}\Psi F$, and hence it fits into a distinguished triangle

$$i^*F \to \mathbb{R}\Psi F \xrightarrow{can} \mathbb{R}\Phi F \to i^*F[1]$$

Both $\mathbb{R}\Psi F$ and $\mathbb{R}\Phi F$ are somewhat loosely refered to as sheaves of vanishing cycles. These objects possess natural monodromy actions by T. If we give i^*F the trivial T action, then the diagram with solid arrows commutes.

$$\begin{array}{ccccccc} i^*F & \longrightarrow & \mathbb{R}\psi_*F & \xrightarrow{can} & \mathbb{R}\phi_*F & \longrightarrow & i^*F[1] \\ \downarrow & & \downarrow{\scriptstyle T-1} & & \downarrow{\scriptstyle var} & & \downarrow \\ 0 & \longrightarrow & \mathbb{R}\psi_*F & \xrightarrow{=} & \mathbb{R}\psi_*F & \longrightarrow & 0 \end{array}$$

Thus we can deduce a morphism var, which completes this to a morphism of triangles. In particular, $T - 1 = var \circ can$. One can also show that $can \circ var = T - 1$.

Given $p \in X_0$, let B_ϵ be an ϵ-ball in X centered at p. Then $f^{-1}(t) \cap B_\epsilon$ is the so called Milnor fiber. The stalks

$$\mathcal{H}^i(\mathbb{R}\Psi\mathbb{Q})_p = H^i(f^{-1}(t) \cap B_\epsilon, \mathbb{Q})$$

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$$\mathcal{H}^i(\mathbb{R}\Phi\mathbb{Q})_p = \tilde{H}^i(f^{-1}(t) \cap B_\epsilon, \mathbb{Q})$$

give the (reduced) cohomology of the Milnor fiber. And

$$H^i(X_0, \mathbb{R}\Psi\mathbb{Q}) = H^i(f^{-1}(t), \mathbb{Q})$$

is, as the terminology suggests, the cohomology of the nearby fiber. We have a long exact sequence

$$\ldots H^i(X_0, \mathbb{Q}) \to H^i(X_t, \mathbb{Q}) \xrightarrow{can} H^i(X_0, \mathbb{R}\Phi\mathbb{Q}) \to \ldots$$

There is an étale version of this as well, where Δ is replaced by the spectrum of a Henselian DVR.

2. Vanishing cycles and perverse sheaves

The following is a key ingredient in the whole story [BBD]:

Theorem 2.1 (Gabber)**.** *If L is perverse, then so are $\mathbb{R}\Psi L[-1]$ and $\mathbb{R}\Phi L[-1]$.*

We concentrate on the first statement which can be reduced to several separate assertions.

Theorem 2.2. *$\mathbb{R}\Psi$ preserves constructibility.*

Given a triangulated functor $F : D_1 \to D_2$ between triangulated categories equipped with t-structures, F is called left t-exact (respectively right t-exact, or t-exact) if F preserves $D_i^{\geq 0}$ (respectively $D_i^{\leq 0}$, or both). For example, if j (respectively i) is the inclusion of a Zariski open (respectively closed) set, then j^* is t-exact (respectively right t-exact) for the perverse t-structure more or less by definition.

Theorem 2.3. *Given an open immersion j, $\mathbb{R}j_*$ is left t-exact. It is t-exact if j is also affine, e.g. the inclusion of the complement of a divisor.*

Proof. The first statement is completely formal. If $\mathcal{F} \in {}^pD^{\geq 0}$, we have to show that $\mathcal{G} = {}^p\tau_{<0}\mathbb{R}j_*\mathcal{F} = 0$ Since $j^*\mathcal{G} \in {}^pD^{\leq 0}$ by the previous discussion, we have

$$Hom(\mathcal{G}, \mathcal{G}) = Hom(\mathcal{G}, \mathbb{R}j_*\mathcal{F}) = Hom(j^*\mathcal{G}, \mathcal{F}) = 0$$

This yields the vanishing of $\mathcal{G}$.

The second statement is deeper and follows from Artin's vanishing theorem for affine schemes. See [BBD, 4.1.3]. □

It is convenient to set ${}^p\psi_f L = {}^p\psi L = \mathbb{R}\Psi L[-1]$ and ${}^p\phi_f L = {}^p\phi L = \mathbb{R}\Phi L[-1]$.

Theorem 2.4. *${}^p\psi$ is right t-exact with respect to the perverse t-structure.*

Proof. Let $\mathcal{F} \in {}^pD^{\leq 0}$, we have to prove that ${}^p\psi\mathcal{F} \in {}^pD^{\leq 0}$, or equivalently that $\mathbb{R}\psi\mathcal{F} \in {}^pD^{\leq -1}$. We give a proof, based on [R], under the special case that the monodromy acts *quasi-unipotently* on $\mathcal{F}$. This assumption will hold in all the examples that we care about. After taking a ramified covering of Δ, we can assume that the monodromy is in fact unipotent. Let us suppose that $f : X \to \Delta$ is given by the restriction of an algebraic family $X^* \to C$ over a curve. Let $j : X^* - X_0 \to X'$ denote the inclusion. Then there is a distinguished triangle

$$i^*\mathbb{R}j_*j^*\mathcal{F} \to \mathbb{R}\psi\mathcal{F} \xrightarrow{T-1} \mathbb{R}\psi\mathcal{F} \to \tag{1}$$

[R, (4)]. By the results discussed above, $\mathbb{R}j_*$ and j^* are t-exact and i^* right t-exact. Therefore from (1) we obtain

$${}^pH^i(\mathbb{R}\psi\mathcal{F}) \xrightarrow{T-1} {}^pH^i(\mathbb{R}\psi\mathcal{F}) \to {}^pH^i(i^*\mathbb{R}j_*j^*\mathcal{F}) = 0$$

when $i \geq 0$. Since $T-1$ is surjective and nilpotent, by assumption, ${}^pH^i(\mathbb{R}\psi\mathcal{F})$ must vanish. Thus $\mathbb{R}\psi\mathcal{F} \in {}^pD^{\leq -1}$.

□

Proposition 2.5. *For any* $\mathcal{F} \in D_c^b(X)$, ${}^p\psi(\mathbb{D}\mathcal{F}) \cong \mathbb{D}\,{}^p\psi\mathcal{F}$

Proof of theorem 2.1. If $\mathcal{F}$ is perverse, then so is $\mathbb{D}\mathcal{F}$. Therefore ${}^p\psi\mathcal{F}$ and $\mathbb{D}\,{}^p\psi\mathcal{F}$ both lie in ${}^pD^{\leq 0}$.

□

2.1. **Perverse Sheaves on a polydisk.** Let Δ be a disk with the standard coordinate function t, and inclusion $j : \Delta - \{0\} = \Delta^* \to \Delta$. For simplicity assume $1 \in \Delta^*$. Consider a perverse sheaf F on Δ which is locally constant on Δ^*. Then we can form the diagram

$$ {}^p\psi_t F \underset{var}{\overset{can}{\rightleftarrows}} {}^p\phi_t F $$

Note that the objects in the diagram are perverse sheaves on $\{0\}$ i.e. vector spaces. This leads to the following elementary description of the category due to Deligne and Verdier (c.f. [V, sect 4]).

Proposition 2.6. *The category of perverse sheaves (with quasi-unipotent monodromy) on the disk Δ which are locally constant on Δ^* is equivalent to the category of quivers (diagrams of vector spaces) of the form*

$$ \psi \underset{v}{\overset{c}{\rightleftarrows}} \phi $$

where $I + c \circ v$ and $I + v \circ c$ are invertible (with eigenvalues which are roots of unity).

It is instructive to consider some basic examples. We see immediately that

$$ 0 \rightleftarrows V $$

corresponds to the sky scraper sheaf V_0.

Let L be a local system L on Δ^* with monodromy given by $T : L_1 \to L_1$. Then the perverse sheaf $j_*L[1]$ corresponds to

$$ L_1 \underset{v}{\overset{c}{\rightleftarrows}} \frac{L_1}{\ker(T-I)} $$

where c is the projection, v is induced by $T - I$. Thus a quiver

$$ \psi \underset{v}{\overset{c}{\rightleftarrows}} \phi $$

with c surjective arises from $j_*L[1]$, where $L_1 = \psi$ with $T = I + v \circ c$.

The above description can be extended to polydisks Δ^n [GGM]. For simplicity, we spell this out only for $n = 2$. Let t_i denote the coordinates. Then we can attach to any perverse sheaf F, four vector spaces $V_{11} = {}^p\psi_{t_1}{}^p\psi_{t_2}F$, $V_{12} = {}^p\psi_{t_1}{}^p\phi_{t_2}F$... along with maps induced by *can* and *var*.

Theorem 2.7. *The category of perverse sheaves on the polydisk Δ^2 which are constructible for the stratification $\Delta^2 \supset \Delta \times \{0\} \cup \{0\} \times \Delta \supset \{(0,0)\}$ is equivalent* *to the category of quivers of the form*

$$\begin{array}{ccc} V_{11} & \rightleftarrows & V_{12} \\ \uparrow\downarrow & & \uparrow\downarrow \\ V_{21} & \rightleftarrows & V_{22} \end{array}$$

It will be useful to characterize the subset of intersection cohomology complexes among all the perverse ones. In the one dimensional case these are just sky scraper sheaves V_0 in which case $\phi = ker(v)$, or sheaves of the form $j_*L[1]$ for which $\phi = image(c)$. In general, we have:

Lemma 2.8. *A quiver corresponds to a direct sum of intersection cohomology complexes if and only if*

$$\phi = image(c) \oplus ker(v)$$

holds for every subdiagram of the form

$$\psi \underset{v}{\overset{c}{\rightleftarrows}} \phi$$

3. Kashiwara-Malgrange filtration

By the Riemann-Hilbert correspondence, the previous picture for perverse sheaves should be translatable into D-module language. Suppose that M is a regular holonomic D-module on the disk Δ which is a connection on Δ^*. We will try to build the quiver associated to $DR(M)$ directly from M. The nearby cycles ψ can be identified with the solutions of the space of multivalued solutions $Hom(M, \mathcal{O}(\tilde{\Delta}^*))$. We claim that the vanishing cycles ϕ would be $Hom(M, \mathcal{O}(\tilde{\Delta}^*)/\mathcal{O}(\Delta))$, which gives rise to $c : \psi \to \phi$. Since monodromy T acts trivially on $\mathcal{O}(\Delta)$, we get an induced map $v : \phi \to \psi$, with $T - I = v \circ c$. To get a better sense of these constructions, and to check they are correct, let us calculate these when M is simple. There are 3 cases:

(1) $M = \mathcal{O}_\Delta$, we see obtain

$$\psi = \mathbb{C} \underset{0}{\overset{0}{\rightleftarrows}} \phi = 0$$

(2) Let $a \in \mathbb{C}^*$, $r = \frac{1}{2\pi i} \log a$, and $M = \mathcal{O}_\Delta[z^{-1}]$ with $\partial \cdot 1 = \frac{r}{z}$. Then we have

$$\psi = \mathbb{C} \underset{a-1}{\overset{1}{\rightleftarrows}} \phi = \mathbb{C}$$

(3) $M = DR(\mathcal{O}[z^{-1}]/\mathcal{O}) = \mathbb{C}_0$. Then we have

$$\psi = 0 \underset{0}{\overset{0}{\rightleftarrows}} \phi = \mathbb{C}$$

From now on assume in addition that $DR(M)$ *has quasi unipotent monodromy.* Then M is built from simple modules as above with $r \in \mathbb{Q}$. The quiver associated to $DR(M)$ can be decomposed as a direct sum

$$\bigoplus_\lambda \psi_\lambda \rightleftarrows \phi_\lambda$$

where $\psi_\lambda, \phi_\lambda$ are the generalized eigenspaces of T. On the D-module side, we consider the generalized eigenspaces

$$M^\alpha = \bigcup_k \ker(z\partial - \alpha)^k,\ \alpha \in \mathbb{C}$$

We will see in a moment that these zero unless α is rational. Then we define the Kashiwara-Malgrange filtration by

$$V^\alpha M = \bigoplus_{\beta \geq \alpha} M^\beta,\ \alpha, \beta \in \mathbb{Q}$$

$$V^{>\alpha} M = \bigoplus_{\beta > \alpha} M^\beta,\ \alpha, \beta \in \mathbb{Q}$$

so that

$$M^\alpha = Gr_V^\alpha M := V^\alpha M / V^{>\alpha} M$$

Proposition 3.1. *For a D-module satisfying the previous assumptions*

1. $zM^\alpha \subseteq M^{\alpha+1}$ *and* $\partial M^\alpha \subseteq M^{\alpha-1}$
2. $\alpha \neq -1$ *the previous inclusions are equalities.*
3. M^α *is finite dimensional, and is nonzero only when* $\alpha \in \mathbb{Q}$.
4. $V^\alpha M$ *is a decreasing filtration which is exhaustive, i.e.* $\cup V^\alpha M = M$.
5. $\phi(DRM)_1 = M^{-1}$, $\psi(DRM)_1 = M^0$.
6. *If* $\lambda = \exp(2\pi i\alpha)$ *with* $\alpha \in (-1, 0)$*, then* $\psi(DRM)_\lambda \cong \phi(DRM)_\lambda = M^\alpha$.

Proof. Suppose for simplicity that $m \in M^\alpha$ is an eigenvalue for $z\partial$. Then $z\partial(zm) = zm + z(z\partial m) = (\alpha+1)zm$. The other cases of (1) are similar.

For the remaining statements, we can reduce to the case of M simple. Then we can check case by case:

1. $M = \mathcal{O}_\Delta$, $M^\alpha = \mathbb{C}z^\alpha$ if $\alpha \in \mathbb{N}$, and the others are zero. We have $\psi = \mathbb{C}$, $\phi = 0$.
2. $M = \mathcal{O}_\Delta[z^{-1}]$ with $\partial \cdot 1 = \frac{r}{z}$, $M^{\alpha+r} = \mathbb{C}z^\alpha$ if $\alpha \in \mathbb{Z}$, and the others are zero. We have $\psi = \mathbb{C}$, $\phi = \mathbb{C}$.
3. $M = DR(\mathcal{O}[z^{-1}]/\mathcal{O})$, $M^\alpha = \mathbb{C}z^{-1}$ if $\alpha = -1$, and the others are zero. We have $\psi = 0$, $\phi = \mathbb{C}$

□

In view of the proposition, we define

$$\psi M = \bigoplus_{0 \leq \alpha < 1} M^\alpha \cong \bigoplus_{-1 < \alpha \leq 0} M^\alpha$$

$$\phi M = \bigoplus_{-1 \leq \alpha < 0} M^\alpha$$

These are isomorphic to the corresponding vector spaces associated to $DR(M)$. The map

$$can : \psi M \to \phi M$$

is given ∂. The variation map is a bit more complicated to express directly. It is simpler to modify the variation map to a map $Var : \phi \to \psi$ such that $Var \circ can$ and $can \circ Var$ are given by $1/2\pi i$ times the logarithm of the unipotent part of T. It is clear that var can be written as a function of can and Var. The modified map

$$Var : \phi M \to \psi M$$

is simply given by multiplication by z.

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