Variational Principles for Nonpotential Operators

V. M. FILIPPOV

Volume 77

TRANSLATIONS OF
MATHEMATICAL MONOGRAPHS

American Mathematical Society

Variational Principles for Nonpotential Operators

This page intentionally left blank

10.1090/mmono/077

TRANSLATIONS OF MATHEMATICAL MONOGRAPHS

VOLUME 77

Variational Principles for Nonpotential Operators

V. M. FILIPPOV

American Mathematical Society · Providence · Rhode Island

В. М. ФИЛИППОВ

ВАРИАЦИОННЫЕ ПРИНЦИПЫ ДЛЯ НЕПОТЕНЦИАЛЬНЫХ ОПЕРАТОРОВ

«НАУКА», МОСКВА, 1985

Translated from the Russian by J. R. Schulenberger
Translation edited by Ben Silver

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 35A15, 47A50, 47H17; Secondary 46E35.

ABSTRACT. A variational method for solving linear equations with B-symmetric and B-positive operators and its generalization to nonlinear equations with nonpotential operators are developed. A constructive extension of the variational method to "nonvariational" equations (including parabolic equations) is carried out in classes of functionals which differ from the Euler-Lagrange functionals; in connection with this some new function spaces are considered.

The book is intended for mathematicians working in the areas of functional analysis and differential equations and also for specialists, graduate students, and students in advanced courses who use variational methods in solving linear and nonlinear boundary value problems in continuum mechanics and theoretical physics.

Bibliography: 356 titles.

Library of Congress Cataloging-in-Publication Data

Filippov, V. M.
[Variat͡sionnye print͡sipy dli͡a nepotent͡sial′nykh operatorov. English]
Variational principles for nonpotential operators/V.M. Filippov.
p. cm. – (Translations of mathematical monographs; v. 77)
Title on verso t.p.: Variat͡sionnye print͡sipy dli͡a nepotent͡sial′nykh operatorov.
Translation of: Variat͡sionnye print͡sipy dli͡a nepotent͡sial′nykh operatorov.
Bibliography: p.
ISBN 0-8218-4529-2 (alk. paper)
1. Nonlinear operators. 2. Variational principles. 3. Differential equations, Partial.
I. Title. II. Title: Variat͡sionnye print͡sipy dli͡a nepotent͡sial′nykh operatorov. III. Series.
QA329.8F5513 1989 89-6904
515′.7248–dc20 CIP

Printed in the United States of America

Information on Copying and Reprinting can be found at the back of this volume.
The paper used in this book is acid-free and falls within the guidelines
established to ensure permanence and durability. ♾
This publication was typeset using AMS-TEX,
the American Mathematical Society's TEX macro system.

Contents

Preface vii

Notation and Terminology xi

Introduction 1

CHAPTER I. Variational Problems for Linear Equations with B-Symmetric and B-Positive Operators 7

§1. Definitions and properties of operators symmetric and positive in a generalized sense 7

§2. Extensions of operators in the sense of Sobolev and Friedrichs 15

§3. The generalized space of Friedrichs 19

§4. The variational problem for equations with a B-symmetric and B-positive operator 22

§5. Dual variational principles for nonselfadjoint equations 31

Commentary to Chapter I 33

CHAPTER II. Classes of Functionals and Function Spaces 45

§6. The role of classes of functionals for the variational method 45

§7. On a class of functionals depending on linear operators 51

§8. On a quasiclassical solution of the inverse problem of the calculus of variations for linear differential equations 61

§9. The function spaces $S_{\bar{a}}^{-l}\overset{\circ}{W}_p(\Omega)$ 67

§10. Spaces of differentiable functions over $\tilde{S}_{\bar{a}}^{-l}W_p(\Omega)$ 80

Commentary to Chapter II 88

CHAPTER III. Construction and Investigation of Variational Principles for Linear Boundary Value Problems 91

§11. On various formulations of the inverse problem of the calculus of variations and some solutions of it 91

§12. A variational method for solving parabolic problems 107
§13. The principle of a minimum of a quadratic functional for the wave equation 124
§14. A variational principle for hypoelliptic PDE with constant coefficients 133
§15. Symmetrization of some nonclassical PDE 139
§16. A variational principle constructed on the basis of a priori estimates in negative spaces 146
Commentary to Chapter III 162

CHAPTER IV. Variational Principles for Nonlinear Equations 168
§17. On the inverse problem of the calculus of variations for nonlinear equations 168
§18. A variational method of solving nonlinear equations with nonpotential operators 182
Commentary to Chapter IV 198

Appendix 203

Bibliography 215

Preface

In the domestic and foreign literature there are remarkable books devoted to the variational method of investigating equations with selfadjoint operators (in the nonlinear case with potential operators). At a mathematical level these are primarily the monographs of S. L. Sobolev, *Applications of Functional Analysis in Mathematical Physics*, L. D. Kudryavtsev, *Direct and Inverse Imbedding Theorems. Applications to the Solution of Elliptic Equations by a Variational Method*, and M. M. Vaĭnberg, *Variational Methods of Investigating Nonlinear Equations*. Other books by S. G. Mikhlin, K. O. Friedrichs, and K. Rektorys were of major importance for the popularization of variational methods in applications.

The development of computational techniques and the possibility of automating variational methods aroused the current interest in questions of further improving them and constructive extension to new classes of equations. The books of W. Velte [337], G. I. Marchuk and V. I. Agoshkov [207], and P. Blanchard and E. Brüning [39] which have appeared in recent years are partially devoted to these problems. However, in the mathematical literature there is no single monograph specially devoted to variational principles for equations with nonsymmetric (nonpotential) operators, although the basic mathematical aspects of this direction were worked out by A. E. Martynyuk, W. V. Petryshyn, and V. M. Shalov back in 1957–1965. The present book is intended to fill this gap to some extent.

In applications most people know the practical value of the fact that for a particular boundary value problem (generally speaking, with an unbounded nonpotential operator) it is possible to construct a functional analogous in variational properties to the Dirichlet functional for the Laplace equation; but variational principles for nonsymmetric nonpotential operators have not been widely used either by mathematicians or in applications. One of the basic reasons for this is, apparently, the complexity of a constructive approach to the necessary "symmetrizing" operators. In Marchuk and Agoshkov's book [207] many results are presented for equations with B-symmetric and B-positive-definite operators (in the sense of W. V. Petryshyn), but the symmetrization of the neutron-transport equation presented as a substantial example (a result obtained by V. S. Vladimirov [340] in his doctoral dissertation) is one of the few cases of constructive execution of symmetrization of an operator of a rather complex boundary value problem.

The theoretical foundations of a variational method for investigating linear equations with, generally speaking, nonsymmetric and nonpositive operators are presented in the first part of the book (Chapters 1 and 2). Chapter 3 is devoted to a constructive approach to variational principles for various boundary value problems for partial differential equations. Generalizations of the variational method of investigating nonlinear equations with nonpotential operators proposed by M. Z. Nashed [235], A. D. Lyashko [202], and E. Tonti [324] are developed in Chapter 4.

The development of a variational method for investigating a differential equation $N(u) = f$ is closely connected with the inverse problem of the calculus of variations, and throughout the entire book a quasiclassical solution of this inverse problem is investigated in the sense of seeking functionals $F[u]$, bounded below in some Hilbert space, which contain derivatives of the unknown function u of lower order than in the equation $N(u) - f = 0$ and are such that the set of solutions of the equation coincides with the set of critical points of the functional.

Many of the results presented here were reported in 1961–1983 by V. M. Shalov and by the author to the Steklov Mathematical Institute of the Academy of Sciences of the USSR in the seminar of Academician S. M. Nikol′skiĭ and Corresponding Member of the Academy of Sciences of the USSR L. D. Kudryavtsev; the author is deeply grateful to all participants of the seminar for their attention. During one report of the author Professor Kudryavtsev pointed out the expediency of investigating nonclassical boundary value problems for partial differential equations (PDE) by the variational method developed here; in Chapter 3 we carry out constructive approaches to variational principles and their investigation, mainly for known equations of mathematical physics with nonclassical boundary conditions or for nonclassical PDE.

Also during one of the author's reports Academician Nikol′skiĭ noted that if a linear boundary value problem can be rigorously investigated by the Bubnov-Galerkin method, then it is possible to select a corresponding quadratic functional of the variational method, while in the nonlinear case with a nonnegative operator the selection is, of course, almost impossible; in Chapter 4 we prove the existence of solutions of the inverse problem of the calculus of variations for broad classes of nonlinear equations and develop a variational method for investigating nonlinear equations with nonpotential operators.

I am especially grateful to Corresponding Member of the Academy of Sciences of the USSR L. D. Kudryavtsev, whose ideas regarding the development and application of variational methods constantly aided me in the work.

A large part of the book was written during my stay in 1983–1984 at the Free University of Brussels, and I should thank Professors J. P. Gosset, I. R. Prigogine, and P. Glansdorf (Belgium) for the assistance and support they showed me. I am grateful also to Professor V. I. Burenkov, Professor A. N. Skorokhodov, and A. Yu. Rodionov, who examined individual sections and made a number of valuable remarks. Especially valuable were the remarks and discussion of the results obtained with one of the founders of the theory developed here—Professor W. V. Petryshyn (USA)—to whom I am deeply grateful.

The Author

This page intentionally left blank

Notation and Terminology

1. R_n is the n-dimensional Euclidean, arithmetic space of points $x = (x_1, \dots, x_n)$, $|x| = (\sum_1^n x_i^2)^{1/2}$, and $x\xi = \sum_1^n x_i\xi_i$.

2. Ω is a domain, an open connected set in R_n (which is bounded everywhere in this book), with boundary $\partial\Omega$ which is piecewise smooth unless higher smoothness is specified; and $\bar{\Omega}$ is the closure of Ω in R_n ($\bar{\Omega} = \Omega \cup \partial\Omega$; $\bar{\Omega}$ is compact).

3. $C^k(\Omega)$ ($C^k(\bar{\Omega})$) is the set of functions having uniformly continuous derivatives in Ω ($\bar{\Omega}$) through kth order, $k \geq 0$, and the norm is

$$\|u|C^k(\bar{\Omega})\| = \sum_{|\bar{\alpha}| \leq k} \max_{\Omega} |D^{\bar{\alpha}}u(x)|.$$

$\overset{\circ}{C}{}^l(\Omega) \equiv \overset{\circ}{C}{}^l(\bar{\Omega})$ is the set of functions in $C^l(\bar{\Omega})$ which vanish on $\partial\Omega$ together with all partial derivatives through order l.

$C_0^l(\Omega)$ is the set of functions in $C^l(\bar{\Omega})$ with compact support in Ω.

$\overset{\circ}{C}{}^\infty(\Omega)$ is the set of functions for which all partial derivatives of any order in Ω exist and vanish on $\partial\Omega$.

4. $D^{\bar{\alpha}} = \partial^{|\bar{\alpha}|}/\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}$, where $\bar{\alpha} = (\alpha_1, \dots, \alpha_n)$ and $|\bar{\alpha}| = \sum_1^n \alpha_j$; $\alpha_j \geq 0$, $j = 1, \dots, n$.

$\Delta = \sum_1^n \partial^2/\partial x_k^2$ is the Laplacian.

$\nabla u \equiv \operatorname{grad} u = (u_{x_1}, \dots, u_{x_n})$, where $u_{x_i} = \partial u(x)/\partial x_i$, $i = 1, \dots, n$,

Finally, $|\nabla u|^2 = \sum_1^n (u_{x_i})^2$.

5. $L_p(\Omega)$ is the (linear, normed, complete) real Banach space of measurable functions for which the norm

$$\|u|L_P(\Omega)\| = \left(\int_\Omega |u(x)|^p \, dx\right)^{1/p}, \qquad \infty > p \geq 1,$$

exists and is finite. An equivalent definition of $L_p(\Omega)$ is the completion in the norm $\| \cdot |L_p(\Omega)\|$ of the sets of functions $C(\Omega)$, $\overset{\circ}{C}{}^{\bar{\alpha}}(\Omega)$, $C_0^{\bar{\alpha}}(\Omega)$

($\bar{\alpha} = (\alpha_1, \dots, \alpha_n)$, $\alpha_j \geq 0$, $j = 1, \dots, n$), and $C_0^\infty(\Omega)$. The Sobolev spaces $W_p^m(\Omega)$ are defined as follows:

$$W_p^m(\Omega) = \{u(x)\colon D^{\bar{\Omega}} u \in L_p(\Omega),\ |\alpha| \leq m\}, \qquad \infty > m \geq 1,$$

$$\|u|W_p^m(\Omega)\| = \left[\int_\Omega \left\{ |u|^p + \sum_{|\bar{k}| \leq m} |D^{\bar{k}} u(x)|^p \right\} dx \right]^{1/p},$$

where the $D^{\bar{k}}$ are generalized partial derivatives in the Sobolev sense, $\bar{k} = (k_1, \dots, k_n)$; $\overset{\circ}{W}{}_p^m(\Omega)$ is the closure of $\overset{\circ}{C}{}^\infty(\Omega)$ in the norm of $W_p^m(\Omega)$; for the domains Ω considered here the functions $u(x) \in \overset{\circ}{W}{}_p^m(\Omega)$ vanish on $\partial\Omega$ in the sense of $L_p(\partial\Omega)$ together with all j generalized derivatives to order $m - 1$.

Note that we used both $\|\cdot|W\|$ and $\|\cdot\|_W$ as the norm for a function space W. Moreover, in the definition of the norm of a negative space

$$\|u|W_2^-\| = \sup_{v \neq 0} \frac{|(u, v)|}{\|v|W_2^+\|}$$

to shorten notation we shall not always indicate that the sup is taken over all functions v in W_2^+.

6. The symbol $\forall$ denotes "for every" or "for any"; $\exists \alpha \in M$ means "there exists an element α in the set M".

$\exists! \alpha \in M$ means "in the set M there exists a unique element α".

$M \Leftrightarrow N$ means that assertion M holds if and only if assertion N holds or, in other words, for the validity of M it is necessary and sufficient that N be satisfied.

$A := B$ means "A is set equal to B". $l = 1, \dots, n$ indicates that the quantity l takes integral values from 1 to n.

$C_0, C_1, C_2, \dots$ denote positive constants not depending on the functions explicitly written in the formula in question. The notation $C_n(g)$ for some n means that this coefficients $C_n > 0$ depends on the function g.

The enumeration of the constants $C_0, C_1, C_2, \dots$ is done independently in each section.

7. H and H_i everywhere denote Hilbert spaces; $\|u'|H_1\| \sim \|u'|H_2\|$ means the two norms are equivalent, i.e., there exist $C_1, C_2 > 0$ such that $\|u'|H_1\| \leq C_1 \|u'|H_2\| \leq C_2 \|u'|H_1\|$ for all $u' \in H_1$. A space H_1 is *equivalent* to a space H_2, $H_1 \sim H_2$, if $H_1 \subseteq H_2 \subseteq H_1$ and $\|u'|H_1\| \sim \|u'|H_2\|$.

$\bar{Q} = H$ means that the completion of the set Q in the norm of the space H is the entire space (set) H.

8. We shall sometimes write "a sequence $\{u_n\} \subset H$", having in mind that this sequence is formed from elements of the space H.

Strong convergence "$u_n \to u$ $(n \to \infty)$ in H" means convergence in the norm of this space: $\|u_n - u|H\| \to 0$ $(n \to \infty)$, u, $u_n \in H$ for all n.

Weak convergence "$u_n \rightharpoonup u$ $(n \to \infty)$ in H" means that $|(u_n - u, v)| \to 0$ $(n \to \infty)$ for all $v \in H$ and u_n, $u \in H$, where $(\cdot, \cdot)$ is the inner product in H.

9. $W_1 \to W_2$, i.e., "the space W_1 is imbedded in W_2", means that $W_1 \subseteq W_2$ and $\|u|W_2\| \leq C_n\|u'|W_1\|$ for all $u \in W_1$.

10. A function u is some function space $W_1(\Omega)$ has stable boundary values in the sense of a space $W_2(\Gamma)$ if the norm $\|u'|W_2(\Gamma)\|$ exists and the estimate

$$\|u|W_2(\Gamma)\| \leq C_n\|u|W_1(\Omega)\| \quad \forall u \in W_1(\Omega),\ \Gamma = \partial\Omega,$$

holds with a constant $C_n > 0$ not depending on u.

11. $D(A)$ is the domain and $R(A)$ the range of an operator A. Linearity of an operator A is understood only in the following sense:

$$A(\alpha_1 u_1 + \alpha_2 u_2) = \alpha_i A u_1 + \alpha_2 A u \quad \forall \alpha_1,\ \alpha_2 \in R_1,\ \forall u_1, u_2 \in D(A);^{(1)}$$
$$D(A, B) = \{v\colon v \in D(A) \cap D(B)\};$$
$$R_A(B) = \{Bv\colon v \in D(A)\}; (^1)$$

A^* is the adjoint operator, A^{-1} is the inverse operator, and I is the identity operator.

The notation $A \supseteq B$ for two operators A and B means that A is an extension of B, i.e., $D(A) \supseteq D(B)$ and $Au = Bu$ for all $u \in D(B)$.

12. The convolution of functions $(f * g)(x) = \int_{R_n} f(x - y)g(y)\,dy$.

13. Two equations are considered equivalent if any solution (in a particular sense) of one of them is a solution of the other as well.

14. Well-posedness of some problem $Au = f$ in a pair of spaces W, G is understood in the Hadamard sense: for any element $f \in G$ there exists a unique element u_0 in W—the solution of the equation—and this solution depends continuously on f: $\|u_0|W\| \leq C_0\|f|G\|$.

15. The book is broken into chapters, Chapters into sections, and some of the sections into subsections.

In a number $(m.n)$ of a formula the first number (m) denotes the number of the section, while the second (n) denotes the order number of this formula in §m. Similarly, "Theorem M.N" (or "Corollary M.N", "Lemma M.N", "Remark M.N", "Assertion M.N") means that this is the theorem (corollary, etc.) with order number N in §M.

---

(1)Operators possessing this property will sometimes be called *distributive* below.

This page intentionally left blank

This page intentionally left blank

APPENDIX

On the Nonexistence of Semibounded Solutions of Inverse Problems of the Calculus of Variations

V. M. FILIPPOV AND V. M. SAVCHIN

Suppose there is given a boundary value problem

$$\mathscr{L}u \equiv \sum_{i,j=1}^{n} p^{ij}u_{ij} + \sum_{i=1}^{n} q^{i}u_{i} + r \cdot u = f(x), \qquad x \in \Omega, \tag{1}$$

$$u(x) = 0, \qquad x \in \partial\Omega. \tag{2}$$

Here Ω is a bounded domain in R^n with piecewise smooth boundary $\partial\Omega$; the $p^{ij} = p^{ji}$ and q^i $(i, j = 1, \dots, n)$ are constant coefficients in $\overline{\Omega}$; $r(x) \in C(\overline{\Omega})$ and $f(x) \in C(\overline{\Omega})$ are given functions; $u = u(x) \in M = C^2(\Omega) \cap C^1(\overline{\Omega}) \cap \overset{\circ}{C}(\overline{\Omega})$, $x = (x_1, \dots, x_n)$; $u_i = \partial u/\partial x_i \equiv D_i u$, and $u_{ij} = D_j D_i u$ $(i, j = 1, \dots, n)$.

We introduce the class of Euler-Lagrange functionals

$$\mathscr{F}_{[u]} = \int_{\Omega} L(x, u, u_1, \dots, u_n)\, dx. \tag{3}$$

The following formulation of the inverse problem of the calculus of variations (IPCV) for equation (1) is known (see [54] and [18]).

FORMULATION 1. Find a function $\mu(x, u, u') \equiv \mu(x, u, u_1, \dots, u_n)$ which is continuously differentiable in R^{2n+1}, with $\mu(x, u, u') \neq 0$ $(x \in \overline{\Omega})$ for all $u \in M$, and in the class of functionals (3) find a functional $F_\mu[u]$ such that for all $u \in M$

$$\delta F_\mu[u] = \int_{\Omega} \mu(x, u, u') \cdot (Lu - f)\delta u\, dx. \tag{4}$$

If $\partial\mu/\partial u \not\equiv 0$ or $\partial\mu/\partial u_i \not\equiv 0$ for some $i = 1, \dots, n$, then because of the nonlinearity of the integrand in (4) it is obvious that the corresponding functional $F[u]$ is nonquadratic. Since for a linear equation (1) it is expedient to restrict attention to functionals (3) quadratic in u and $u_1, \dots, u_n$, we henceforth consider the following IPCV (see [54]).

FORMULATION 2. Find a function $\mu(x) \in C^1(\overline{\Omega})$, $\mu(x) \neq 0$ $(x \in \overline{\Omega})$, and a functional $F_\mu[u]$ in the class (3) quadratic in u and $u_1, \dots, u_n$ such that for all $u \in M$

$$\delta F_\mu[u] = \int_\Omega \mu(x) \cdot (\mathscr{L}u - f) \cdot \delta u \, dx. \tag{5}$$

The usual IPCV follows from this for $\mu(x) \equiv 1$.

Results of Copson [54] were presented in §11.

LEMMA 1. *For any equation* (1) *nonparabolic in* Ω *the general solution of the IPCV in Formulation* 2 *is given by*[47]

$$V[u] = \int_\Omega \exp\left(\sum_1^n k_i x_i\right) \cdot \left\{ \sum_{i,j=1}^n p^{ij} D_i u D_j u - \left[r(x) - \sum_{i=1}^n b^i k_i \right] \cdot u^2 + 2f \cdot u \right\} dx, \tag{6}$$

where

$$k_j = \det P_j / \det P, \qquad j = 1, \dots, n, \tag{7}$$

and the matrix P_j *is obtained from the matrix* $P = (p^{\alpha\beta})$ *by replacing the* j*th column by the column* $(q^1, \dots, q^n)$; *the constants* $b^1, \dots, b^n$ *do not depend on* x *or* u.

COROLLARY 1. *For any nonparabolic equation* (1) *with constant coefficients* p^{ij} $(i, j = 1, \dots, n)$ *and* $q^i = 0$ $(i = 1, \dots, n)$, *and* $r(x), f(x) \in C(\overline{\Omega})$, *solutions of the IPCV in Formulation* 2 *are given by*

$$\tilde{V}[u] = \int_\Omega \left[\sum_{i,j=1}^n p^{ij} D_i u D_j u - r(x) \cdot u^2 + 2f \cdot u \right] dx.$$

The functional $\tilde{V}[u]$ is obtained from (6), since for $q^i = 0$ $(i = 1, \dots, n)$ from (7) we have $k_i = 0$ $(i = 1, \dots, n)$.

---

(47) Sets of functions are henceforth given up to a factor and term not depending on x or u.

COROLLARY 2. *An element $u_0 \in M$ is a solution of the Dirichlet problem for the nonparabolic equation* (1) *if and only if u_0 is a critical point of the functional $V[u]$* (6), (7).

Thus, for any nonparabolic operator $\mathscr{L}$ of problem (1), (2) the IPCV in Formulation 2 always has a solution—this should be borne in mind with regard to assertions of "nonvariational" nonselfadjoint elliptic or hyperbolic equations.

We now consider in more detail the question of semiboundedness (that is, boundedness above or below) of the functionals (6) constructed for nonparabolic equations (1).

It is known ([57], vol. II, Chapter 3, §3) that any PDE (1) with constant coefficients (we further assume that $r(x) \equiv \text{const}$, $x \in \overline{\Omega}$) by a change of the independent variables

$$x_i = \sum_{j=1}^{n} t_{ij} y_j, \qquad i = 1, \dots, n \qquad (x = Ty) \tag{8}$$

can be transformed to the form

$$\mathscr{L}_1 u \equiv \sum_{i=1}^{n} a^i \frac{\partial^2 u}{\partial y_i^2} + \sum_{i=1}^{n} b_1^i \frac{\partial u}{\partial y_i} + r_1 \cdot u = g_1(y). \tag{9}$$

Here $\det(t_{ij})_{i,j=1}^n \neq 0$, the t_{ij} $(i, j = 1, \dots, n)$ are constant real coefficients in $\overline{\Omega}$, the constants a^i $(i = 1, \dots, n)$ assume the values $0, 1$, or -1, r_j is a constant in $\overline{\Omega}$, and $g_1(y) = f(Ty)$.

For a nonparabolic equation (1) in Ω it is possible ([57], vol. II) by a change of the unknown function

$$u(y) = v(y) \cdot \exp(-\tfrac{1}{2} \cdot z), \qquad z = \sum_{i=1}^{n} b_1^i y_i / a^i \tag{10}$$

to transform equation (9) into

$$\mathscr{L}_2 v \equiv \sum_{i=1}^{n} a^i \partial^2 v / \partial y_i^2 + \lambda \cdot v = \tilde{g}, \tag{11}$$

where

$$\lambda = r_1 - \frac{1}{4} \cdot \sum_{i=1}^{n} \frac{(b_1^i)^2}{a^i}, \qquad \tilde{g} = g_1 \cdot \exp(\tfrac{1}{2} z). \tag{12}$$

Thus, any equation (1) elliptic in Ω can be transformed into([48])

$$\Delta u + \lambda u = f(\xi), \qquad \lambda \equiv \text{const}, \ \xi \in \tilde{\Omega}, \tag{13}$$

([48])Appearance of the variables ξ_i $(i = 1, \dots, n)$ is connected with a possible renumbering of the variables in passing from (11) to (13) or (14).

while any ultrahyperbolic (including properly hyperbolic) equation (1) can be transformed into

$$\sum_{i=1}^{m} \frac{\partial^2 u}{\partial \xi_i^2} - \sum_{i=m+1}^{n} \frac{\partial^2 u}{\partial \xi_i^2} + \lambda \cdot u(\xi) = f(\xi), \qquad \xi \in \tilde{\Omega}, \tag{14}$$

where $1 \le m < n$, $n \ge 2$, and $\lambda = \text{const}$ in $\tilde{\Omega}$.

For equations (13) and (14) functionals are known which are solutions of the IPCV. A natural question arises: if for equation (13) or (14) a solution of the IPCV has been constructed, will the functional obtained by transformations inverse to (8) and (10) be a solution of the IPCV for the original equation (1) of the corresponding type? It is convenient to first consider the question of transformation of functionals of this type in a more general case.

Suppose we are given a functional

$$\mathscr{F}[u] = \int_\Omega F(x_k, u^j, D_k u^j)\, dx, \qquad k = 1, \dots, n;\ j = 1, \dots, m, \tag{15}$$

where

$$F(x_k, u^j, D_k u^j) \equiv F(x_1, \dots, x_n, u^1, \dots, u^m, D_1 u^1, \dots, D_n u^1, \dots, D_1 u^m \dots, D_n u^m)$$

is a function continuously differentiable in all its arguments in R^{n+m+mn}, and $u^j(x) \in M$ $(j = 1, \dots, m)$.

We introduce a nonsingular transformation $y = y(x)$,

$$\begin{cases} y_k = y_k(x_i), & i, k = 1, \dots, n,\ y_k \in C^1(\overline{\Omega}), \\ \det(\partial y_k / \partial x_i)_{i,k=1}^n \ne 0, & x \in \overline{\Omega}. \end{cases} \tag{16}$$

We denote the inverse transformation by $x = x(y)$.

The variational derivative $\delta F / \delta u^j$ for the functional $\mathscr{F}[u]$ is the left side of the jth Euler equation for the functional (15). We have

LEMMA 2. *For any $u^{j'} \in M$ $(j' = 1, \dots, m)$*

$$\left(\frac{\partial F}{\partial u^{j'}}\right)_{x=x(y)} = \det\left(\frac{\partial y_r}{\partial x_s}\right)_{r,s=1}^n \cdot \frac{\delta}{\delta \tilde{u}^{j'}} \left[\tilde{F}(y_k, \tilde{u}^j(y), D_k \tilde{u}^j(y)) \cdot \det\left(\frac{\partial x_r}{\partial y_s}\right)_{r,s=1}^n\right], \qquad j' = 1, \dots, m, \tag{17}$$

where

$$\tilde{F}(y_k, \tilde{u}^j(y), D_k\tilde{u}^j(y)) = F\left[x_i(y_k), u^j(x_i(y_k)), \sum_{k=1}^{n} \frac{\partial \tilde{u}^j}{\partial y_k} \cdot \frac{\partial y_k}{\partial x_i}\right],$$

$$u^j(x(y)) = \tilde{u}^j(y) \qquad (j = 1, \ldots, m).$$

PROOF. For any $u^j \in M$ $(j = 1, \ldots, m)$ we have

$$\frac{\partial u^j}{\partial x_i} = \sum_{k=1}^{n} \frac{\partial \tilde{u}^j}{\partial y_k} \cdot \frac{\partial y_k}{\partial x_i},$$

$$\int_\Omega (x_k, u^j(x), D_k u^j(x))\, dx = \int_{\tilde{\Omega}} \tilde{F}(y_k, \tilde{u}^j(y), D_k\tilde{u}^j(y)) / \det\left(\frac{\partial x_r}{\partial y_s}\right)_{r,s=1}^{n} / dy, \tag{18}$$

where $\tilde{\Omega}$ is the domain obtained from Ω by means of the mapping (16). From this we obtain

$$\begin{aligned}\int_\Omega \frac{\delta F}{\delta u^{j'}} \cdot \eta^{j'}\, dx &= \left[\frac{\partial}{\partial \alpha^{j'}} \int_\Omega F(x_k, u^j + \alpha^j \eta^j, D_k u^j + \alpha^j \cdot D_k \eta^j)\, dx\right]_{\alpha=0} \\ &= \left[\frac{\partial}{\partial \alpha^{j'}} \int_{\tilde{\Omega}} \tilde{F}(y_k, \tilde{u}^j(y) + \alpha^j \cdot \tilde{\eta}^j(y), D_k\tilde{u}^j(y) + \alpha^j \cdot D_k\tilde{\eta}^j(y)) \right. \\ &\qquad \left. \times \left|\det\left(\frac{\partial x_r}{\partial y_s}\right)_{r,s=1}^{n}\right| dy\right]_{\alpha=0} \\ &= \int_{\tilde{\Omega}} \frac{\delta}{\delta \tilde{u}^{j'}} \left[\tilde{F}(y_k, \tilde{u}^j(y), D_k\tilde{u}^j(y)) \cdot \left|\det\left(\frac{\partial x_r}{\partial y_s}\right)_{r,s=1}^{r}\right|\right] \\ &\qquad \cdot \eta^j(x(y)) \cdot dy, \qquad j' = 1, \ldots, m.\end{aligned} \tag{19}$$

Here $\alpha = (\alpha^1, \ldots, \alpha^m)$ is an arbitrary m-dimensional vector of real numbers, and $\eta^{j^1} \in M$ $(j' = 1, \ldots, m)$.

Using (19), we obtain

$$\begin{aligned}\int_\Omega \frac{\delta F}{\delta u^{j'}} \cdot \eta^{j'}\, dx &= \int_{\tilde{\Omega}} \left(\frac{\delta F}{\delta u^{j'}}\right)_{x=x(y)} \cdot \eta^{j'}(x(y)) \cdot \left|\det\left(\frac{\partial x_r}{\partial y_s}\right)_{r,s=1}^{n}\right| dy \\ &= \int_{\tilde{\Omega}} \frac{\delta}{\delta \tilde{u}^{j'}} \left[\tilde{F}(y_k, \tilde{u}^j, D_k\tilde{u}^j) \cdot \left|\det\left(\frac{\partial x_r}{\partial y_s}\right)_{r,s=1}^{n}\right|\right] \eta^{j'}(x(y))\, dy, \\ &\qquad j' = 1, \ldots, m.\end{aligned}$$

The required equality (17) follows from this, since $\eta^{j'} \in M$ $(j' = 1, \ldots, m)$ is arbitrary. ■

Suppose now that equation (1) is transformed into (9) by means of the transformation (8) (see (16)).

LEMMA 3. *A function* $\tilde{E}[v]$ *is a solution of the IPCV for the nonparabolic equation* (11), (12) *if and only if the functional*

$$\mathscr{I}[u(x)] = \tilde{E}[u \cdot \exp(\tfrac{1}{2} \cdot z)]|_{y=y(x)} \tag{20}$$

is a solution of the IPCV for equation (1).

PROOF. By Lemma 1 we find that the set of solutions of the IPCV for (11) and (12) is given by

$$\tilde{E}[v] = \int_{\tilde{\Omega}} \left[\sum_{i=1}^{n} a^i \left(\frac{\partial v}{\partial y_i} \right)^2 - \lambda \cdot v^2 + 2 \cdot \tilde{g} \cdot v \right] dy. \tag{21}$$

From (10) we have

$$v(y) = u(y) \cdot \exp(\tfrac{1}{2} \cdot z), \tag{22}$$

$$\frac{\partial v}{\partial y_i} = \left(\frac{\partial u}{\partial y_i} + \frac{1}{2} \frac{b_1^i}{a^i} u \right) \cdot \exp(\tfrac{1}{2} z), \qquad i = 1, \dots, n. \tag{23}$$

Substituting (22), (23) into (21), we obtain

$$\begin{aligned} \mathscr{I}_1[u] &= \tilde{E}[u \cdot \exp(\tfrac{1}{2} \cdot z)] \\ &= \int_{\tilde{\Omega}} \left[\sum_{i=1}^{n} a^i \cdot \left(\frac{\partial u}{\partial y_i} + \frac{1}{2} \cdot \frac{b_1^i}{a^i} u \right)^2 - \lambda \cdot u^2 + 2 g_1 \cdot u \right] \cdot \exp z \, dy. \end{aligned}$$

Hence

$$\begin{aligned} \delta \mathscr{I}_1[u] = 2 \cdot \int_{\tilde{\Omega}} \exp z \cdot & \left[\sum_{i=1}^{n} a^i \cdot \left(\frac{\partial u}{\partial y_i} + \frac{1}{2} \cdot \frac{b_1^i}{a^i} u \right) \right. \\ & \left. \times \left(\delta u_{y_i} + \frac{1}{2} \cdot \frac{b_1^i}{a^i} \cdot \delta u \right) - \lambda u \cdot \delta u + g_1 \cdot \delta u \right] dy \\ = 2 \cdot \int_{\tilde{\Omega}} \exp z \cdot & \left\{ \sum_{i=1}^{n} a^i \frac{\partial}{\partial y_i} \left[\left(\frac{\partial u}{\partial y_i} + \frac{1}{2} \cdot \frac{b_1^i}{a^i} u \right) \delta u \right] \right. \\ & - \sum_{i=1}^{n} a^i \left(\frac{\partial^2 u}{\partial y_i^2} + \frac{1}{2} \frac{b_1^i}{a^i} \cdot \frac{\partial u}{\partial y_i} \right) \delta u - \lambda \cdot u \cdot \delta u \\ & \left. + \frac{1}{2} \cdot \sum_{i=1}^{n} b_1^i \left(\frac{\partial u}{\partial y_i} + \frac{1}{2} \cdot \frac{b_1^i}{a^i} u \right) \cdot \delta u + g_1 \cdot \delta u \right\} dy. \end{aligned} \tag{24}$$

Noting that for all $u \in M$

$$\int_{\tilde\Omega} \exp z \cdot \sum_{i=1}^{n} a^i \frac{\partial}{\partial y_i}\left[\left(\frac{\partial u}{\partial y_i} + \frac{1}{2}\cdot\frac{b_1^i}{a^i}\cdot u\right)\delta u\right] dy$$
$$= \int_{\tilde\Omega}\left\{\sum_{i=1}^{n} a^i \frac{\partial}{\partial y_i}\left[\exp z\cdot\left(\frac{\partial u}{\partial y_i} + \frac{1}{2}\cdot\frac{b_1^i}{a^i}\cdot u\right)\delta u\right] - \sum_{i=1}^{n} b_1^i\left(\frac{\partial u}{\partial y_i} + \frac{1}{2}\cdot\frac{b_1^i}{a^i}u\right)\delta u\cdot\exp z\right\} dy,$$

from (24) we obtain, for all $u \in M$,

$$\delta\mathscr{I}_1[u] = -2\cdot\int_{\tilde\Omega}\exp z\cdot\left\{\sum_{i=1}^{n} b_1^i\cdot\left(\frac{\partial u}{\partial y_i} + \frac{1}{2}\frac{b_1^i}{a^i}\cdot u\right) + \right.$$
$$+\sum_{i=1}^{n} a^i\left(\frac{\partial^2 u}{\partial y_i^2} + \frac{1}{2}\cdot\frac{b_1^i}{a^i}\cdot\frac{\partial u}{\partial y_i}\right)$$
$$-\frac{1}{2}\cdot\sum_{i=1}^{n} b_1^i\cdot\left(\frac{\partial u}{\partial y_i} + \frac{1}{2}\cdot\frac{b_1^i}{a^i}u\right)$$
$$\left. + r_1\cdot u - \frac{1}{4}\cdot\sum_{i=1}^{n}\frac{(b_1^i)^2}{a^i}\cdot u - g_1\right\}\delta u\, dy$$
$$= -2\cdot\int_{\tilde\Omega}\exp z\cdot(\mathscr{L}_1 u - g_1)\cdot\delta u\cdot dy.$$

The lemma follows from this via (17) and (20). ■

From the lemmas proved above we obtain

COROLLARY 2. *Suppose problem* (11), (12), (2) *is obtained from* (1) *and* (2) *by means of transformations* (8) *and* (10). *Then the general solution of the IPCV in Formulation* 2 *for problem* (1), (2) *is the functional* $\mathscr{I}[u]$ *obtained from the functional* $\tilde E[v]$ *of* (21) *by the transformations* (22) *and* (16).

In theoretical investigations of differential equations by a variational method, and also in justifying direct methods of the calculus of variations, in applications it is important that the corresponding functional should have not a stationary point but an extremal point (a maximum or minimum). For the elliptic equation

$$-\Delta u + \lambda u = g(x), \qquad \lambda \equiv \text{const} < -\lambda_1^2,\ x \in \Omega \subset R^n, \tag{25}$$

or for the wave equation

$$u_{tt} - \Delta u = g(x,t), \qquad (x,t) \in Q \subset R^{n+1} \tag{26}$$

functionals which are solutions of the IPCV are well known: respectively,

$$V_1[u] = \int_\Omega [|\nabla u|^2 + \lambda \cdot u^2 - 2 \cdot g \cdot u]\, dx, \tag{27}$$

$$V_2[u] = \int_Q \{-u_t^2 + |\nabla u|^2 - 2 \cdot g \cdot u\}\, dx\, dt. \tag{28}$$

It can be shown, however (see below), that these functionals are not bounded on M above and below, and the following question arises: for equations (25) and (26) and for the more general nonparabolic equation (1) do there exist semibounded functionals which are solutions of the IPCV in Formulation 2?

THEOREM 1. *If*

$$\sum_{i,j=1}^{n} p^{ij}\xi_i\xi_j \geq \mu|\xi|^2 > 0,$$

then for an equation (1) *of elliptic type in* Ω *for*

$$r > r_0(\Omega) > 0 \tag{29}$$

in the class of Euler-Lagrange functionals (3) *there are no solutions of the IPCV which are semibounded on* M.

PROOF. *Step* 1. According to Corollary 1 and Lemma 3, the general solution of the IPCV in Formulation 2 for equation (9) is given by

$$\mathcal{J}_1[u] = \int_{\tilde\Omega} \left\{ \sum_{i=1}^{n} a_i \cdot \left(\frac{\partial u}{\partial y_i} + \frac{1}{2} \cdot \frac{b_1^i}{a^i} \cdot u \right)^2 - \left(r_1 - \frac{1}{4} \cdot \sum_{i=1}^{n} \frac{(b_1^i)^2}{a^i} \right) \cdot u^2 + 2 \cdot g_1 \cdot u \right\} \exp z\, dy. \tag{30}$$

Here (see (20))

$$\mathcal{J}_1[u] = \tilde{E}[u \cdot \exp(\tfrac{1}{2} \cdot z)], \tag{31}$$

where $\tilde{E}[v]$ is the functional (21) constructed for equation (11). Therefore, from (31) it is obvious by (10) that for unboundedness above and below on M of the functional $\mathcal{J}_1[u]$ of (31) it is necessary and sufficient that the functional $\tilde{E}[v]$ be unbounded above and below on M.

Step 2. We establish the unboundedness on M of the functional

$$E[v] = \int_{\tilde\Omega} \cdot \left\{ \sum_{i=1}^{n} \left(\frac{\partial v}{\partial y_i} \right)^2 - \lambda \cdot v^2 + 2 \cdot \tilde{g} \cdot v \right\} dy \tag{32}$$

of the form (21) (for an elliptic equation, as stipulated in (9), $a^i = 1$, $i = 1, \dots, n$). It is obvious that for all $v \in M$

$$E[v] - \int_{\tilde{\Omega}} \tilde{g}^2\, dy \leq \int_{\tilde{\Omega}} \left\{ \left(\frac{\partial v}{\partial y_i}\right)^2 + (1-\lambda)\cdot v^2 \right\} dy \equiv E_2[v], \quad (33)$$

$$E[v] + \int_{\tilde{\Omega}} \tilde{g}\, dy \geq \int_{\tilde{\Omega}} \left\{ \sum_{i=1}^{n} \left(\frac{\partial v}{\partial y_i}\right)^2 - (1+\lambda)\cdot v^2 \right\} dy \equiv E_1[v]. \quad (34)$$

Since with a piecewise smooth boundary $\partial\Omega$ the domain $\Omega \subset R^n$ contains some n-dimensional parallelepiped, because of the nondegeneracy of the transformation (16) taking Ω into $\tilde{\Omega}$, the domain $\tilde{\Omega}$ also contains some parallelepiped

$$Q = (a,b) \equiv (a_1, b_1) \times \cdots \times (a_n, b_n), \qquad |Q| = \prod_{i=1}^{n} (b_i - a_i) > 0.$$

We set

$$v_m(y) = \begin{cases} \prod_{i=1}^{n} \sin[m(l_i y_i + c_i)], & y \in Q, \\ 0, \quad y \in \tilde{\Omega} \backslash Q, \end{cases} \quad (35)$$

where $m = 1, 2, \dots$, and

$$l_i = 2\pi/(b_i - a_i), \quad c_i = -a_i \cdot l_i, \qquad i = 1, \dots, n. \quad (36)$$

It is obvious that $v_m(y) \in \overset{\circ}{W}{}_2^1(\tilde{\Omega})$ and

$$\|v_m | \overset{\circ}{W}{}_2^1(\tilde{\Omega})\| = \left\{ \int_{\tilde{\Omega}} \sum_{i=1}^{n} \left(\frac{\partial v_m}{\partial y_i}\right)^2 dy \right\}^{1/2} < \infty, \qquad m = 1, 2, 3, \dots,$$

and by direct computations we find

$$E_2[v_m] = \left(\frac{1}{2}\right)^n \cdot |Q| \cdot \left[1 - \lambda + m^2 \cdot \sum_{j=1}^{n} l_j^2 \right]. \quad (37)$$

Setting $m = 1$, from this we conclude that for any

$$\lambda > 1 + \sum_{j=1}^{n} l_j^2 \quad (38)$$

we have $E_2[v_1] < 0$. Then for any ρ, not depending on y or v, for the quadratic functional $E_2[v]$ of (33) we have $E_2[\rho v_1] = \rho^2 E_2[v_1] \to -\infty$ ($\rho \to \infty$). Therefore, the functional $E_2[v]$ and by (33) also $E[v]$ are unbounded below on $\overset{\circ}{W}{}_2^1(\tilde{\Omega})$.

Similarly, for the functional $E_1[v]$ of (34) by direct computations we find that

$$E_1[v_m] = \left(\frac{1}{2}\right)^n \cdot |Q| \cdot \left\{ -1 - \lambda + m^2 \sum_{j=1}^n l_j^2 \right\}.$$

Obviously for any $\lambda \in R^1$ we can find $N = N(\lambda)$ such that $E_1[v_N] > 0$. Therefore, $E_1[\rho v_N] = \rho^2 E_1[v_N] \to +\infty$ as $\rho \to \infty$. From this and (34) we conclude also that the functional $E[v]$ is not bounded above on $\overset{\circ}{W}{}_2^1(\Omega)$.

It is now not hard to obtain the unboundedness above and below on the set M of the functional $E[v]$ of (32) and (38) from the fact that the functional $E[v]$ is continuous on $\overset{\circ}{W}{}_1^2(\Omega)$, the set M is dense in $\overset{\circ}{W}{}_2^1(\Omega)$, and $E[v]$ of (32) and (38) is unbounded on $\overset{\circ}{W}{}_2^1(\Omega)$.

Since the functional $E[v]$ of (32) by Corollary 1 is the general solution of the IPCV for the equation

$$\Delta v + \lambda \cdot v = \tilde{g}(y), \quad y \in \tilde{\Omega}, \qquad v|_{\partial\tilde{\Omega}} = 0, \tag{39}$$

it is useful to distinguish the result of this second step separately.

COROLLARY 4. *For problem* (39), (38) *in the class of Euler-Lagrange functionals there are no solutions of the IPCV which are semibounded on* M.

Step 3. Since the functional $\mathscr{I}_1[u]$ of (30) is the general solution of the IPCV for equation (9), from the results of Step 2 and the arguments of Step 1 we also obtain

COROLLARY 5. *For the elliptic equation* (9) ($a^i = 1$, $i = 1, \dots, n$), *where*

$$\lambda \equiv r_1 - \frac{1}{4} \cdot \sum_{i=1}^n (b_1^i)^2 > 1 + \sum_{j=1}^n l_j^2. \tag{40}$$

in the class of Euler-Lagrange functionals there do not exist solutions of the IPCV which are semibounded on M.

Thus, the functional $\mathscr{I}_1[u]$ of (30) is unbounded on M above and below under condition (40). The functional $V[u]$, which is the general solution of the IPCV for equation (1), is obtained, according to Lemma 2, by a nondegenerate change of the independent variables $y = y(x)$ in the functional $\mathscr{I}_1[u(y)]$, i.e., a change of the independent variable occurs under the integral which does not change the value of the integral (see (18)). Therefore, the functional $V[u(x)] = \mathscr{I}_1[u(y)|_{y=y(x)}]$, which is the general solution of the IPCV for equation (1), is also unbounded on M above and below for $r > r_0(\Omega)$. The theorem is proved. ■

Theorem 2. *For any ultrahyperbolic (including properly hyperbolic) equation* (1) *in the class of Euler-Lagrange functionals there are no semibounded solutions of the IPCV.*

From the proof of Theorem 1 it is not hard to see that we need only establish unboundedness on M of the functional

$$G[v] = \int_{\tilde{\Omega}} \left\{ \sum_{i=1}^{m} v_{y_i}^2 - \sum_{k=m+1}^{n} v_{y_k}^2 - \lambda \cdot v^2 + 2 \cdot \tilde{g} \cdot v \right\} dy \tag{41}$$

which is the general solution (see Corollary 1) of the IPCV for the ultrahyperbolic equation (14).

For all $v \in M$ we have

$$G[v] - \int_{\tilde{\Omega}} \tilde{g}^2 \, dy \le \int_{\tilde{\Omega}} \left\{ \sum_{i=1}^{m} v_{y_i}^2 - \sum_{k=m+1}^{n} v_{y_k}^2 + (1-\lambda) \cdot v^2 \right\} dy \equiv G_1[v]; \tag{42}$$

$$G[v] + \int_{\tilde{\Omega}} \tilde{g}^2 \, dy \ge \int_{\tilde{\Omega}} \left\{ \sum_{i=1}^{m} v_{y_i}^2 - \sum_{k=m+1}^{n} v_{y_k}^2 - (1+\lambda) \cdot v^2 \right\} dy \equiv G_2[v]. \tag{43}$$

Since the domain Ω contains some parallelepiped and the transformation (16) mapping Ω to $\tilde{\Omega}$ is nondegenerate, the domain $\tilde{\Omega}$ also contains some parallelepiped

$$Q_1 = (a_1, b_1) \times \cdots \times (a_m, b_m) \times (\bar{a}_{m+1}, \bar{b}_{m+1}) \times \cdots \times (\bar{a}_n, \bar{b}_n),$$
$$|Q_1| = \prod_{i=1}^{m} (b_i - a_i) \prod_{i=m+1}^{n} (\bar{b}_i - \bar{a}_i) > 0.$$

For $t, s = 1, 2, \dots$ we set

$$v_{ts}(y) = \begin{cases} \prod_{i=1}^{m} \sin[t(l_i y_i + c_i)] \prod_{i=m+1}^{n} \sin[s(l_i y_i + c_i)], & y \in Q_1, \\ 0, \quad y \in \tilde{\Omega} \setminus Q_1. \end{cases} \tag{44}$$

Here the l_i and c_i for $i = 1, \dots, m$ are defined in (36), while

$$l_i = \frac{2\pi}{\bar{b}_i - \bar{a}_i}, \quad c_i = -\bar{a}_i \cdot l_i, \qquad i = m+1, \dots, n. \tag{45}$$

Obviously $v_{ts}(y) \in \overset{\circ}{W}{}^1_2(\tilde{\Omega})$ for all $t, s = 1, 2, \ldots$, and by direct computations we find that

$$G_1[v_{ts}] = \left(\frac{1}{2}\right)^n \cdot |Q_1| \cdot \left\{ (t^2 - s^2) \cdot \sum_{j=1}^n l_j + 1 - \lambda \right\}, \tag{46}$$

$$G_2[v_{ts}] = \left(\frac{1}{2}\right)^n \cdot |Q_1| \cdot \left\{ (t^2 - s^2) \cdot \sum_{j=1}^n l_j^2 - 1 - \lambda \right\}. \tag{47}$$

From this it is easy to see that the functional $G_1[v_{ts}]$, $(G_2[v_{ts}])$ for any $\lambda \in R^1$ is unbounded below (respectively, above), and from (42) (from (43)) we find that the functional $G[v]$, which is the general solution of the IPCV for equation (14), is unbounded below (respectively, above) on $\overset{\circ}{W}{}^1_2(\Omega)$.

Further, repeating the arguments of Steps 1 and 3 of the proof of Theorem 1, we can show that the functional $\tilde{G}[u]$ obtained from $G[v]$ of (41) by the changes inverse to (10) and (8) (see Lemmas 2 and 3), which is the general solution of the IPCV for the ultrahyperbolic equation (1), is unbounded on M both above and below for any constant coefficients p^{ij}, b_1^i $(i, j = 1, \ldots, n)$, and r. ■

For a parabolic equation there does not exist a solution of the IPCV in the class of Euler-Lagrange functionals (3) [54]. For any nonparabolic equation (1) in Ω such solutions exist (Lemma 1). However, for any ultrahyperbolic (including hyperbolic) equation (1) and also for an elliptic equation (1) for $r \geq r_0 > 0$ (assuming that $\sum_{i,j=1}^r p^{ij}\xi_i\xi_j > \mu|\xi|^2 > 0$) in the class of Euler-Lagrange functionals there are no functionals bounded above or below which are solutions of the IPCV in Formulation 2. It is therefore necessary to invoke other classes of functionals distinct from the class of Euler-Lagrange functionals (3) to develop a direct variational method of investigating even these basic equations of mathematical physics and to apply minimization methods to the functionals of the corresponding variational problem.

Bibliography

1. György Adler, *Sulla caratterizzabilità dell'equazione del calore dal punto di vista del calcolo delle variazioni*, Magyar Tud. Akad. Mat. Kutató Int. Közl. **2** (1957), 153–157.

2. L. Ainola, *Variational principles and general formulas for a mixed problem for the wave equation*, Eesti NSV Teod. Akad. Toimetised Füüs.-Mat. **18** (1969), 48–56. (Russian)

3. Yu. R. Akopyan and L. A. Oganesyan, *A variational-difference method for the solution of two-dimensional linear parabolic equations*, Zh. Vychisl. Mat. i Mat. Fiz. **17** (1977), 109–118; English transl. in USSR Comput. Math. and Math. Phys. **17** (1977).

4. S. J. Aldersley, *Higher Euler operators and some of their applications*, J. Mathematical Phys. **20** (1979), 522–531.

5. M. A. Aleksidze et al., *On the automation of variational methods for solving boundary value problems*, Trudy Vychisl. Tsentra Akad. Nauk Gruzin. SSR **10** (1971), no. 4, 16–25. (Russian)

6. T. I. Amanov, *Representation and imbedding theorems for the function spaces* $S^{(r)}_{p,\theta}B(R^n)$ *and* $S^{(r)}_{p^*,\theta}B(0 \leq x_j \leq 2\pi;\ j = 1, \ldots, n)$, Trudy Mat. Inst. Steklov. **77** (1965), 3–34; English transl. in Proc. Steklov Inst. Math. **77** (1975).

7. I. N. Anan'ev, Yu. I. Nyashin, and A. N. Skorokhodov, *A variational method for calculating a time-dependent temperature field*, Inzh.-Fiz. Zh. **26** (1974), 470–476; English transl. in J. Engrg. Phys. **26** (1974).

8. I. M. Anderson and T. Duchamp, *On the existence of global variational Principles*, Amer. J. Math. **102** (1980), 781–868.

9. R. S. Anderssen, *Variational methods and parabolic differential equations*, Ph. D. thesis, University of Adelaide, Adelaide, 1967.

10. T. A. Andreeva and L. D. Gordinskiĭ, *On a boundary value problem of filtration theory*, Boundary Value Problems of the Theory of Heat

Conduction (Yu. A. Mitropol'skiĭ and A. A. Berezovskiĭ, editors), Izdanie Inst. Mat. Akad. Nauk Ukrain. SSR, Kiev, 1975, pp. 5–16. (Russian)

11. R. I. Andrushkiev, *On the approximate solution of K-positive eigenvalue problems* $Tu - \lambda Su = 0$, J. Math. Anal. Appl. **50** (1975), 511–529.

12. A. M. Arthurs, *Dual extremum principles and error bounds for a class of boundary value problems*, J. Math. Anal. Appl. **41** (1973), 781–795.

13. ____, *Complementary variational principles for linear equations*, J. Math. Anal. Appl. **49** (1975), 237–239.

14. ____, *Complementary variational principles*, 2nd ed., Clarendon Press, Oxford, 1980.

15. T. M. Atanacković and Dj. S. Djukić, *An extremum variational principle for a class of boundary value problems*, J. Math. Anal. Appl. **93** (1983), 344–362.

16. R. W. Atherton and G. M. Homsy, *On the existence and formulation of variational principles for nonlinear differential equations*, Studies in Appl. Math. **54** (1975), 31–60.

17. Marc Authier, *Espaces du type Sobolev associés à une seule dérivée partielle, théorème de traces, applicatoin à certains problèmes aux limites*, C. R. Acad. Sci. Paris Sér. A-B **262** (1966), A1158–A1161.

18. F. Belatoni, *Über die Charakterisierbarkeit partieller Differentialgleichungen zweiter Ordnung mit Hilfe der Variationsrechnung*, Magyar Tud. Akad. Mat. Kutató Int. Közl. **5** (1960), 229–233.

19. M. F. Barnsley, *Padé approximant bounds for the difference of two series of Stieltjes*, J. Mathematical Phys. **17** (1976), 559–565.

20. M. F. Barnsley and George A. Baker, Jr., *Bivariational bounds in a complex Hilbert space, and correction terms for Padé approximants*, J. Mathematical Phys. **17** (1976), 1019–1027.

21. M. F. Barnsley and P. D. Robinson, *Dual variational principles and Padé-type approximants*, J. Inst. Math. Appl. **14** (1974), 229–249.

22. ____, *Bivariational bounds*, Proc. Roy. Soc. (London) Ser. A **338** (1974), 527–533.

23. Martin Becker, *The principles and applications of variational methods*, MIT Press, Cambridge, Mass., 1964.

24. Gian Cesare Belli, *Variational formulation for scalar wave and diffusion equations*, Ist. Lombardo Accad. Sci. Lett. Rend. A **106** (1972), 158–166.

25. G. Benthien and M. E. Gurtin, *A principle of minimum transformed energy in linear elastodynamics*, Trans. ASME Ser. E: J. Appl. Mech. **37** (1970), 1147–1149.

26. V. I. Berdichevskiĭ, *A variational equation of continuum mechanics*, Problems of the Mechanics of a Solid Deformable Body (V. V. Novozhilov Sixtieth Birthday Vol.; L. I. Sedov and Yu. N. Robotnov, editors), "Sudostroenie", Leningrad, 1970, pp. 55–66. (Russian)

27. ——, *On a variational principle*, Dokl. Akad. Nauk SSSR **215** (1974), 1329–1332; English transl. in Soviet Phys. Dokl. **19** (1974/75).

28. ——, *Variational principles of continuum mechanics*, "Nauka", Moscow, 1983. (Russian)

29. Yu. M. Berezanskiĭ, *On boundary value problems for general partial differential operators*, Dokl. Akad. Nauk SSSR **122** (1958), 959–962. (Russian)

30. ——, *Expansion in eigenfunctions on selfadjoint operators*, "Naukova Dumka", Kiev, 1965; English transl., Amer. Math. Soc., Providence, R. I., 1968.

31. P. W. Berg, *Calculus of variations*, Handbook of Engineering Mechanics (W. Flügge, editor), McGraw-Hill, 1962, Chapter 16.

32. O. V. Besov and A. D. Dzhabrailov, *Interpolation theorems for some spaces of differentiable functions*, Trudy Mat. Inst. Steklov. **105** (1969), 15–20; English transl. in Proc. Steklov Inst. Math. **105** (1969).

33. O. V. Besov, V. P. Il'in, and S. M. Nikol'skiĭ, *Integral representations of functions and imbedding theorems*, "Nauka", Moscow, 1975; English transl., Vols. 1, 2, Wiley, 1979.

34. Maurice A. Biot, *Variational principles in heat transfer*, Clarendon Press, Oxford, 1970.

35. M. Sh. Birman, *On minimal functionals for elliptic differential equations of second order*, Dokl. Akad. Nauk SSSR **93** (1953), 953–956. (Russian)

36. ——, *On Trefftz's variational method for the equation* $\Delta^2 u = f$, Dokl. Akad. Nauk SSSR **101** (1955), 201–204. (Russian)

37. ——, *Variational methods analogous to Trefftz's for solving boundary value problems*, Vestnik Leningrad. Univ. **1956**, no. 13 (Ser. Mat. Mekh. Astr. vyp. 3), 69–89. (Russian)

38. L. Bittner, *Abschätzungen bei Variationsmethoden mit Hilfe von Dualitätssätzen.* I, Numer. Math. **11** (1968), 129–143.

39. Philippe Blanchard and Erwin Brüning, *Direkte Methoden der Variationsrechnung. Ein Lehrbuch*, Springer-Verlag, 1982.

40. Oskar Bolza, *Lectures in the calculus of variations*, Univ. of Chicago Press, Chicago, Ill., 1904; reprints, Chelsea and Dover, New York, 1960; German transl., Teubner, Leipzig, 1909.

41. V. A. Borovikov, *Fundamental solutions of linear partial differential equations with constant coefficients*, Trudy Moskov. Mat. Obshch. **8** (1959), 199–257; English transl. in Amer. Math. Soc. Transl. (2) **25** (1963).

42. J. S. Bradley and B. D. Sleeman, *K-positive ordinary differential operators of the third order*, J. Math. Anal. Appl. **70** (1979), 249–257.

43. H. Brézis, *Quelques propriétés des opérateurs monotones et des semi-groupes non linéaires*, Nonlinear Operators and the Calculus of Variations (Summer School, Brussels, 1975), Lecture Notes in Math., vol. 543, Springer-Verlag, 1976, pp. 56–82.

44. Haïm Brézis and Ivar Ekeland, *Un principe variationnel associé à certaines équations paraboliques. Le cas dépendant du temps*, C. R. Acad. Sci. Paris Sér. A-B **282** (1976), A1197–A1198.

45. Felix E. Browder, *Remarks on nonlinear functional equations*. I, Proc. Nat. Acad. Sci. U.S.A. **51** (1964), 985–989.

46. ——, *Remarks on nonlinear functional equations*. II, III, Illinois J. Math. **9** (1965), 608–616, 617–622.

47. ——, *Existence and uniqueness theorems for solutions of nonlinear boundary value problems*, Applications of Nonlinear Partial Differential Equations in Mathematical Physics, Proc. Sympos. Appl. Math., vol. 17, Amer. Math. Soc., Providence, R. I., 1965, pp. 24–49.

48. ——, *Nonlinear monotone and accretive operators in Banach spaces*, Proc. Nat. Acad. Sci. U.S.A. **61** (1968), 388–393.

49. V. I. Burenkov, *The density of infinitely differentiable functions in spaces of functions specified on an arbitrary open set*, Theory of Cubature Formulas and Applications of Functional Analysis to Problems of Mathematical Physics (Materials, School-Conf., Tashkent, 1974; S. L. Sobolev, editor), Inst. Mat., Sibirsk. Otdel Akad. Nauk SSSR, Novosibirsk, 1975, pp. 9–22. (Russian)

50. Alfred Carasso, *A least squares procedure for the wave equation*, Math. Comp. **28** (1974), 757–767.

51. ——, *Error bounds in the final value problem for the heat equation*, SIAM J. Math. Anal. **7** (1976), 195–199.

52. E. Conjura and W. V. Petryshyn, *Extension of nonlinear densely defined operators, rates of convergence for the error and the residual, and application to differential equations*, J. Math. Anal. Appl. **64** (1978), 651–694.

53. Philip Cooperman, *An extension of the method of Trefftz for finding local bounds on the solutions of boundary value problems, and on their derivatives*, Quart. Appl. Math. **10** (1953), 359–373.

54. E. T. Copson, *Partial differential equations and the calculus of variations*, Proc. Roy. Soc. Edinburgh **46** (1925/26), 126–135.

55. R. W. Cottle et al. (editors), *Variational inequalities and complementarity problems* (Proc. Internat. School, Erice, 1978), Wiley, 1980.

56. Richard Courant, *Remarks about the Rayleigh-Ritz method*, Boundary Problems in Differential Equations (Proc. Sympos., Madison, Wisc., 1959; R. E. Langer, editor), Univ. of Wisconsin Press, Madison, Wisc., 1960, pp. 273–277.

57. R. Courant and D. Hilbert, *Methoden der mathematischen Physik*. Vols. I (2nd ed.), II, Springer-Verlag, 1931, 1937; English transl. of Vol. I, Interscience, 1953; of Vol. II (drastically revised), Interscience, 1962.

58. James W. Daniel, *Applications and methods for the minimization of functionals*, Nonlinear Functional Analysis and Applications (Proc. Sem., Madison, Wisc., 1970; L. B. Rall, editor), Academic Press, 1971, pp. 399–424.

59. ——, *The approximate minimization of functionals*, Prentice-Hall, Englewood Cliffs, N. J., 1971.

60. David R. Davis, *The inverse problem of the calculus of variations in higher space*, Trans. Amer. Math. Soc. **30** (1928), 710–736.

61. Paul Dedecker, *Sur un problème inverse du calcul des variations*, Acad. Roy. Belgique Bull. Cl. Sci. (5) **36** (1950), 63–70.

62. V. F. Dem′yanov and A. M. Rubinov, *Approximate methods in optimization problems*, Izdat. Leningrad. Univ., Leningrad, 1968; English transl., Amer. Elsevier, New York, 1970.

63. A. A. Dezin, *Existence and uniqueness theorems for boundary value problems for partial differential equations in function spaces*, Uspekhi Mat. Nauk **14** (1959), no. 3(87), 21–73; English transl. in Amer. Mat. Soc. Transl. (2) **42** (1964).

64. ——, *General questions in the theory of boundary value problems*, "Nauka", Moscow, 1980. (Russian)

65. A. A. Dezin and V. P. Mikhaĭlov, *On boundary value problems for linear differential operators*, Proc. Fourth All-Union Math. Congr. (Leningrad, 1961), vol. 2, "Nauka", Leningrad, 1964, pp. 499–501. (Russian)

66. J. B. Diaz, *Upper and lower bounds for quadratic functionals*, Proc. Sympos. Spectral Theory and Differential Problems, Dept. of Math., Oklahoma Agric. and Mech. Coll., Stillwater, Okla., 1951, pp. 279–289.

67. ——, *Upper and lower bounds for quadratic functionals*, Collectanea Math. **4** (1951), 3–49.

68. ——, *Upper and lower bounds for quadratic integrals, and at a point, for solutions of linear boundary value problems*, Boundary Problems in

Differential Equations (Proc. Sympos., Madison, Wisc., 1959; R. E. Langer, editor), Univ. of Wisconsin Press, Madison, Wisc., 1960, pp. 47–83.

69. J. B. Diaz and H. J. Greenberg, *Upper and lower bounds for the solution of the first boundary value problem of elasticity*, Quart. Appl. Math. **6** (1948), 326–331.

70. ——, *Upper and lower bounds for the solution of the first biharmonic boundary value problem*, J. Math. and Phys. **27** (1948), 193–201.

71. V. P. Didenko, *Generalized solvability of boundary value problems for systems of differential equations of mixed type*, Differentsial'nye Uravneniya **8** (1972), 24–29; English transl. in Differential Equations **8** (1972).

72. ——, *Generalized solvability of the Tricomi problem*, Ukrain. Mat. Zh. **25** (1973), 14–24; English transl. in Ukrainian Math. J. **25** (1973).

73. ——, *Finding strong solutions by variational methods*, Proc. Sci. Conf. Computational Mathematics in Modern Scientific and Technological Progress (Kanev, 1974), Inst. Kibernet. Akad. Nauk Ukrain. SSR and Kiev. Gos. Univ., Kiev, 1974, pp. 154–159. (Russian)

74. ——, *A variational problem for equations of mixed type*, Differentsial'nye Uravneniya **13** (1977), 44–49; English transl. in Differential Equations **13** (1977).

75. ——, *A variational method of solving boundary value problems whose operator is not symmetric*, Dokl. Akad. Nauk SSSR **240** (1978), 1277–1280; English transl. in Soviet Math. Dokl. **19** (1978).

76. V. P. Didenko and A. A. Popova, *Inequalities with a negative norm and variational problems for asymmetric differential operators*, Preprint 81–29, Inst. Kibernet. Akad. Nauk Ukrain. SSR, Kiev, 1981. (Russian) MR **83j**:35009.

77. Dj. S. Djukić and T. M. Alancković, *Error bounds via a new extremum variational principle, mean square residual and weighted mean square residual*, J. Math. Anal. Appl. **75** (1980), 203–218.

78. V. V. Dodonov, V. I. Man'ko, and V. D. Skarzhinsky [Skarzhinskiĭ], *The inverse problem of the variational calculus and the nonuniqueness of the quantization of classical systems*, Hadronic J. **4** (1980/81), 1734–1804.

79. Jesse Douglas, *Solution of the inverse problem of the calculus of variations*, Trans. Amer. Math. Soc. **50** (1941), 71–128.

80. A. D. Dzhabrailov, *On some function spaces. Direct and inverse imbedding theorems*, Dokl. Akad. Nauk SSSR **159** (1964), 254–257; English transl. in Soviet Math. Dokl. **5** (1964).

81. ——, *On the theory of "imbedding theorems"*, Trudy Mat. Inst. Steklov. **89** (1967), 80–118; English transl. in Proc. Steklov Inst. Math. **89** (1967).

82. A. V. Dzhishkariani, *The Bubnov-Galerkin method*, Zh. Vychisl. Mat. i Mat. Fiz. **7** (1967), 1398–1402; English transl. in USSR Comput. Math. and Math. Phys. **7** (1967).

83. ——, *On the question of the stability of approximate methods of variational type*, Zh. Vychisl. Mat. i Mat. Fiz. **11** (1971), 569–579; English transl. in USSR Comput. Math. and Math. Phys. **11** (1971).

84. T. D. Dzhuraev, *Boundary value problems for equations of mixed and mixed-composite types*, "Fan", Tashkent, 1979. (Russian)

85. Arne Engvist, *On fundamental solutions supported by a convex cone*, Ark. Mat. **12** (1974), 1–40.

86. V. T. Erofeenko, *The connection between the fundamental solutions in cylindrical and spherical coordinates (with the same origin) for certain equations of mathematical physics*, Differentsial′nye Uravneniya **9** (1973), 1310–1317; English transl. in Differential Equations **9** (1973).

87. M. A. Evgrafov, *Estimation of the fundamental solution of an elliptic equation with constant coefficients*, Preprint No. 11, Keldysh Inst. Appl. Math. Acad. Sci. USSR, Moscow, 1980. (Russian) MR **81d**:35015.

88. I. I. Fedik, V. N. Mikhaĭlov, and V. I. Kozhuklovskiĭ, *Formalization of a variational method for solving boundary value problems in complicated two-dimensional domains*, Dokl. Akad. Nauk Ukrain. SSR Ser. A **1982**, no. 10, 51–54. (Russian)

89. M. Filar, *Construction of the fundamental solution of the equation* $\Delta^p u(x) + ku(x) = 0$, Prace Mat. **10** (1966), 131–140.

90. A. F. Filippov, *Uniqueness of generalized solutions*, Sibirsk Mat. Zh. **10** (1969), 217–222; English transl. in Siberian Math. J. **10** (1969).

91. V. M. Filippov, *A variational method for solving boundary value problems of mathematical physics, and function spaces*, Differentsial′nye Uravneniya **15** (1979), 2056–2065; English transl. in Differential Equations **15** (1979).

92. ——, *Strongly B-positive operators*, Differential Equations and Inverse Problems of Dynamics (R. G. Muklarlyamov, editor), Univ. Druzhby Narodov, Moscow, 1983, pp. 95–99. (Russian)

93. ——, *The relation of the least squares method to symmetrization of differential operators*, Differential Equations and Functional Analysis (V. N. Maslennikova, editor), Univ. Druzhby Narodov, Moscow, 1984, pp. 148–153. (Russian)

94. ——, *A variational method for solving boundary value problems for the wave equation*, Differentsial′nye Uravneniya **20** (1984), 1961–1968; English transl. in Differential Equations **20** (1984).

95. ——, *On some consequences of the isometry of the spaces* $W_{2(x,y)}^{(+\alpha,-\beta)}(\Omega)$ *to the spaces* $W_2^k(\Omega)$, Differential Equations and Functional Analysis (V. N. Maslennikova, editor), Univ. Druzhby Narodov, Moscow, 1984, pp. 141–147. (Russian)

96. ——, *A variational method for solving a hyperbolic equation with boundary conditions on the entire boundary of the domain*, Differential Equations and Functional Analysis (V. N. Maslennikova, editor), Univ. Druzhby Narodov, Moscow, 1983, pp. 114–119. (Russian)

97. ——, *On a direct variational method for solving an elliptic equation with a nonsymmetric and nonpositive operator*, Numerical Methods in Problems of Mathematical Physics, Univ. Druzhby Narodov, Moscow, 1985, pp. 61–66. (Russian)

98. ——, *On classes of operators in the variational method for solving nonlinear equations*, Numerical Methods in Problems of Mathematical Physics, Univ. Druzhby Narodov, Moscow, 1985, pp. 46–55. (Russian)

99. ——, *A direct variational method for studying boundary value problems for an integrodifferential hyperbolic equation*, Preprint, Univ. Druzhby Narodov, Moscow, 1985=Manuscript No. 5444-85, deposited at VINITI, 1985. (Russian) R.Zh.Mat. **1985**, 11Б793.

100. V. M. Filippov and A. N. Skorokhodov, *A quadratic functional for the heat equation*, Differentsial′nye Uravneniya **13** (1977), 1113–1123; English transl. in Differential Equations **13** (1977).

101. ——, *The principle of a minimum of a quadratic functional for a boundary value problem of heat conduction*, Differentsial′nye Uravneniya **13** (1977), 1434–1445; English transl. in Differential Equations **13** (1977).

102. Bruce A. Finlayson, *The method of weighted residuals and variational principles*, Academic Press, 1972.

103. ——, *Variational principles for heat transfer*, Numerical Properties and Methodologies in Heat Transfer (Proc. Second Nat. Sympos., College Park, Md., 1981; T. M. Shih, editor), Hemisphere, Washington, D. C., 1983, pp. 17–31.

104. G. B. Folland, *A fundamental solution for a subelliptic operator*, Bull. Amer. Math. Soc. **79** (1973), 373–376.

105. Kurt O. Friedrichs, *Ein Verfahren der Variationsrechnung das Minimum eines Integrals als das Maximum eines anderen Ausdruckes darzustellen*, Nachr. Ges. Wiss. Göttingen Math.-Phys. Kl. **1929**, 13–20.

106. ——, *Spektraltheorie halbbeschränkter Operatoren und Anwendung auf die Spektralzerlegung von Differentialoperatoren.* I, II, Math. Ann. **109** (1934), 465–487, 685–713.

107. ——, *The identity of weak and strong extensions of differential operators*, Trans. Amer. Math. Soc. **55** (1944), 132–151.

108. ——, *Symmetric positive linear differential equations*, Comm. Pure Appl. Math. **11** (1958), 333–418.

109. Bent Fuglede and Laurent Schwartz, *Un nouveau théorème sur les distributions*, C. R. Acad. Sci. Paris Sér. A-B. **263** (1966), A899–A901.

110. Tsutomu Fujino, *Variational principles of linear differential equations*, Mitsubishi Tech. Bull. No. 77 (1972), 1–25.

111. Hiroshi Fújita, *Contribution to the theory of upper and lower bounds in boundary value problems*, J. Phys. Soc. Japan **10** (1955), 1–8.

112. Paul Funk, *Variationsrechnung und ihre Anwendung in Physik und Technik*, Springer-Verlag, 1962.

113. D. H. Gage et al., *The non-existence of a general thermokinetic variational principle*, Non-equilibrium Thermodynamics, Variational Techniques and Stability (Proc. Sympos., Chicago, Ill., 1965; R. J. Donnelly et al., editors), Univ. of Chicago Press, Chicago, Ill., 1966, pp. 283–286.

114. Herbert Gajewski and Arno Langenbach, *Zur Konstruktion von Minimalfolgen für das Funktional des ebenen elastisch-plastischen Spannungszustandes*, Math. Nachr. **30** (1965), 165–180.

115. V. S. Gamidov, *Variational principles for B-positive-definite operators*, Seventh School on Theory of Operators in Function Spaces, Abstracts of Reports, Minsk, 1982, pp. 40–41. (Russian)

116. V. S. Gamidov and A. A. Popova, *A variational problem for B-positive-definite operators*, Manuscript No. 1276-80, deposited at VINITI, 1980. (Russian) R.Zh.Mat. **1980**, 7Б259.

117. I. M. Gel′fand and L. A. Dikiĭ, *Asymptotic behavior of the resolvent of Sturm-Liouville equations and the algebra of the Korteweg-deVries equations*, Uspekhi Mat. Nauk **30** (1975), no. 5(185), 67–100; English transl. in Russian Math. Surveys **30** (1975).

118. I. M. Gel′fand and I. Ya. Dorfman, *Hamiltonian operators and algebraic structures associated with them*, Funktsional. Anal. i Prilozhen. **13** (1979), no. 4, 13–30; English transl. in Functional Anal. Appl. **13** (1979).

119. I. M. Gel′fand and S. V. Fomin, *Calculus of variations*, Fizmatgiz, Moscow, 1961; English transl., Prentice-Hall, Englewood Cliffs, N. J., 1963.

120. I. M. Gel′fand and G. E. Shilov, *Generalized functions*. Vol. 2: *Spaces of test functions*, Fizmatgiz, Moscow, 1958; English transls., Academic Press, 1968; Gordon and Breach, 1968.

121. P. Glansdorff, *Sur une forme norwelle en cascade du terme aux limites de la variation d'une intégrale multiple*, Acad. Roy. Belgique Bull. Cl. Sci. (5) **38** (1952), 136–153.

122. R. Glowinski, J.-L. Lions, and R. Trémolières, *Analyse numérique des inéquations variationnelles.* Vols. 1, 2, Dunod, Paris, 1976.

123. E. A. Gorin, *Partially hypoelliptic partial differential equations with constant coefficients*, Sibirsk. Mat. Zh. **3** (1962), 500–526. (Russian)

124. L. M. Graves (editor), *Calculus of variations and its applications*, Proc. Sympos. Appl. Math., vol. 8, Amer. Mat. Soc., Providence, R. I., 1958.

125. H. J. Greenberg, *The determination of upper and lower bounds for the solution of the Dirichlet problem*, J. Math. and Phys. **27** (1948), 161–182.

126. V. V. Grushin, *Relation between the local and global properties of solutions of hypoelliptic equations with constant coefficients*, Mat. Sb. **66**(**108**) (1965), 525–550; English transl. in Amer. Math. Soc. Transl. (2) **67** (1968).

127. K. A. Gubaĭdullin, *Some boundary value problems for an equation of composite type*, Volzh. Mat. Sb. Vyp. 5 (1966), 104–113. (Russian)

128. F. Guil Guerrero and L. Martinez Alonzo, *Generalized variational derivatives in field theory*, J. Phys. A **13** (1980), 689–700.

129. M. E. Gurtin, *Variational principles for linear initial-value problems*, Quart. Appl. Math. **22** (1964/65), 252–256.

130. A. F. Guseĭnov, *On boundary value problems for quasilinear equations*, Trudy Inst. Fiz. i Mat. Akad. Nauk Azerbaĭdzhan. SSR Ser. Mat. **7** (1955), 129–162. (Russian)

131. István Gyarmati, *Non-equilibrium thermodynamics; field theory and variational principles*, Springer-Verlag, 1970.

132. P. Havas, *The range of application of the Lagrange formalism*, Nuovo Cimento (10) **5** (1957), Suppl., 363–388.

133. H. von Helmholtz, *Über die physikalische Bedeutung des Princips der kleinsten Wirkung*, J. Reine Angew. Math. **100** (1887), 133–166, 213–222.

134. Ismail Herrera and Jacobo Bielak, *A simplified version of Gurtin's variational principles*, Arch. Rational Mech. Anal. **53** (1974), 131–149.

135. Joseph Hersch, *Une transformation variationnelle apparentée à celle de Friedrichs, conduisant à la méthode des problèmes auxiliares unidimensionnels*, L'Enseignement Math. (2) **11** (1965), 159–169.

136. Heinrich Kertz, *Die Prinzipien der Meckanik in neuem Zusammenhangedargestellt*, Barth, Leipzig, 1894; English transl., Macmillan, 1899; reprint, Dover, New York, 1956.

137. David Hilbert, *Über das Dirichlet'sche Princip*, Jber. Deutsch. Math.-Verein. **8** (1900), 184–188.

138. Arthur Hirsch, *Über eine charakteristische Eigenschaft der Differentialgleichungen der Variationsrechnung*, Math. Ann. **49** (1897), 49–72.

139. Ivan Hlaváček, *Variational principles for parabolic equations*, Apl. Mat. **14** (1969), 278–297.

140. Lars Hörmander, *On the theory of general partial differential operators*, Acta Math. **94** (1955), 161–248.

141. Gregory Walter Horndeski, *Differential operators associated with the Euler-Lagrange operator*, Tensor (N.S.) **28** (1974), 303–318.

142. V. A. Il'in, *On the solvability of mixed problems for hyperbolic and parabolic equations*, Uspekhi Mat. Nauk **15** (1960), no. 2(92), 97–154; English transl. in Russian Math. Surveys **15** (1960).

143. V. P. Il'in, *Some inequalities in function spaces and their application to the study of the convergence of variational processes*, Trudy Mat. Inst. Steklov. **53** (1959), 64–127; English transl. in Amer. Math. Soc. Transl. (2) **81** (1969).

144. B. Frank Jones, Jr., *A fundamental solutions for the heat equation which is supported in a strip*, J. Math. Anal. Appl. **60** (1977), 314–324.

145. P. B. Kagan, *On the solvability of a mixed boundary value problem for a parabolic equation*, Trudy Novokuznetsk. Gos. Ped. Inst. **4** (1962), 52–55. (Russian)

146. L. I. Kamynin and V. N. Maslennikova, *Solution of the first boundary value problem for a quasilinear parabolic equation in a noncylindrical domain*, Mat. Sb. **57**(**99**) (1962), 241–264. (Russian)

147. L. V. Kantorovich, *Functional analysis and applied mathematics*, Uspekhi Mat. Nauk **3** (1948), no. 6(28), 89–185; English transl., Report No. 1509, Nat. Bur. Standards, Washington, D. C., 1952.

148. S. Kaplan, *An analogy between the variational principles of reactor theory and those of classical mechanics*, Nuclear Sci. and Engrg. **23** (1965), 234–237.

149. ——, *Canonical and involutory transformations of variational problems involving higher derivatives*, J. Math. Anal. Appl. **22** (1968), 45–53.

150. S. Kaplan and James A. Davis, *Canonical involutory transformations of the variational problems of transport theory*, Nuclear Sci. and Engrg. **28** (1967), 166–176.

151. Tosio Kato, *On some approximate methods concerning the operators T^*T*, Math. Ann. **126** (1953), 253–262.

152. ——, *Demicontinuity, hemicontinuity and monotonicity*, Bull. Amer. Math. Soc. **70** (1964), 548–550.

153. ——, *Perturbation theory for linear operators*, Springer-Verlag, 1966.

154. G. G. Kazaryan, *The first boundary value problem for a general nonregular equation with constant coefficients*, Dokl. Akad. Nauk SSSR **251** (1980), 22–24; English transl. in Soviet Math. Dokl. **21** (1980).

155. ——, *A variational problem for a nonregular equation and uniqueness of the classical solution*, Differential'nye Uravneniya **18** (1982), 1907–1917; English transl. in Differential Equations **18** (1982).

156. P. Keast, *On the solutions of the wave equation in one space dimension under derivative boundary conditions*, SIAM J. Appl. Math. **17** (1969), 223–230.

157. Wilhelm Kecs, *On the fundamental solution of the Cauchy problem for a class of partial differential equations with constant coefficients*, An. Univ. Bucureşti Ser. Şti. Nat. Mat.-Mec. **17** (1968), no. 2, 59–68. (Romanian)

158. A. A. Kerefov, *Nonlocal boundary value problems for parabolic equations*, Differentsial'nye Uravneniya **15** (1979), 74–78; English transl. in Differential Equations **15** (1979).

159. Michael Kerner, *Die Differentiale in der allgemeinen Analysis*, Ann. of Math. (2) **34** (1933), 546–572.

160. D. F. Kharazov, *Symmetrizable linear operators which depend meromorphically on a parameter, and their applications*. I, II, Izv. Vyssh. Uchebn. Zaved. Mat. **1967**, no. 8(63), 82–93; no. 11(66), 90–97. (Russian)

161. Yu. S. Kolesov, *Periodic solutions of quasilinear parabolic equations of the second order*, Trudy Moskov. Mat. Obshch. **21** (1970), 103–134; English transl. in Trans. Moscow. Math. Soc. **21** (1970).

162. A. N. Kolmogorov and S. V. Fomin, *Elements of the theory of functions and of functional analysis*, 4th ed., "Nauka", Moscow, 1976; English transl. of 1st ed., Vols. I, II, Graylock Press, Albany, N. Y., 1957, 1961.

163. Vadim Komkov, *Application of Rall's theorem to classical elastodynamics*, J. Math. Anal. Appl. **14** (1966), 511–521.

164. ——, *Another look at the dual variational principles*, J. Math. Anal. Appl. **63** (1978), 319–323.

165. V. I. Kondrashov, *On the theory of boundary value problems and eigenvalue problems in domains with a degenerate contour for variational and differential equations*, Doctoral Dissertation, Steklov Inst. Math. Acad. Sci. USSR, Moscow, 1948. (Russian)

166. Yu. I. Kovach, *The Goursat problem for an n-wave equation*, Dokl. i Soobshch. Uzhgorod. Gos. Univ. Ser. Fiz.-Mat. Nauk No. 5 (1962), 98–101. (Russian)

167. L. A. Kozdoba, *Methods of solving nonlinear problems of heat conduction*, "Nauka", Moscow, 1975. (Russian)

168. M. Krawtchouk [M. Kravchuk], *Sur la résolution des équations linéaires différentielles et intégrales parla méthode des moments*. Vol. I, Vseukraïn. Akad. Nauk Prirod.-Tekhn. Vīddīl, Kiev, 1932. (Ukrainian; 25-page French summary).

169. M. G. Kreĭn, *On the theory of weighted integral equations*, Bul. Akad. Shtiintse RSS Moldoven, **1965**, no. 7, 40–46. (Russian)

170. S. G. Kreĭn et al., *Functional analysis*, "Nauka", Moscow, 1964; English transl., Noordhoff, 1972.

171. Nicolas Kryloff [N. V. Krylov], *Sur une méthode d'intégration approchée conterant comme cas particulier la méthode de W. Ritz, ainsi que celle des moindres carrés*, C. R. Acad. Sci. Paris **182** (1926), 676–678.

172. L. D. Kudryavtsev, *Direct and inverse imbedding theorems*, Trudy Mat. Inst. Steklov. **55** (1959), English transl., Amer. Math. Soc., Providence, R. I., 1974.

173. ——, *On the solution by the variational method of elliptic equations which degenerate on the boundary of the region*, Dokl. Akad. Nauk SSSR **108** (1956), 16–19. (Russian)

174. ——, *Onan integral inequality*, Nauchn. Dokl. Vyssh. Shkoly Fiz.-Mat. Nauki **1959**, no. 3, 25–32. (Russian)

175. ——, *A course in mathematical analysis*. Vols. I, II, "Vysshaya Shkola", Moscow, 1981. (Russian)

176. W. Kundt and E. T. Newman, *Hyperbolic differential equations in two dimensions*, J. Mathematical Phys. **9** (1968), 2193–2210.

177. B. A. Kupershmidt, *Lagrangian formalism in the calculus of variations*, Funktsional. Anal. i Prilozhen. **10** (1976), no. 2, 77–78; English transl. in Functional Anal. Appl. **10** (1976).

178. V. D. Kupradze, *Approximate solution of problems of mathematical physics*, Uspekhi Mat. Nauk **22** (1967), no. 2(134), 59–107; English transl. in Russian Math. Surveys **22** (1967).

179. L. P. Kuptsov, *The fundamental solution of a class of elliptic-parabolic equations of second order*, Differentsial'nye Uravneniya **8** (1972), 1649–1660; English transl. in Differential Equations **8** (1972).

180. O. A. Ladyzhenskaya and N. N. Ural'tseva, *Linear and quasilinear elliptic equations*, "Nauka", Moscow, 1964; English transl., Academic Press, 1968.

181. Arno Langenbach, *Variationsmethoden in der nichtlinearen Elastizitäts- und Plastizitätstheorie*, Wiss. Z. Humboldt-Univ. Berlin Math.-Nat. Reihe **9** (1959/60), 146–164.

182. ——, *Die Regularisierung nichtlinearer Gleichungen*, Math. Nachr. **24** (1962), 33–51.

183. ____, *Über Gleichungen mit Potentialoperatoren und Minimalfolgen nichtquadratischer Funktionale*, Math. Nachr. **32** (1966), 9–24.

184. ____, *On the application of a variational principle to some nonlinear differential equations*, Dokl. Akad. Nauk SSSR **121** (1958), 214–217. (Russian)

185. ____, *On an application of the method of least squares to nonlinear equations*, Dokl. Akad. Nauk SSSR **143** (1962), 31–34; English transl. in Soviet Math. Dokl. **3** (1962).

186. M. M. Lavrent′ev and B. Imomnazarov, *Strongly positive operators*, Dokl. Akad. Nauk SSSR **238** (1979), 23–25; English transl. in Soviet Math. Dokl. **19** (1978).

187. Peter D. Lax, *Symmetrizable linear transformations*, Comm. Pure Appl. Math. **7** (1954), 633–647.

188. ____, *On Cauchy's problem for hyperbolic equations and the differentiability of solutions of elliptic equations*, Comm. Pure Appl. Math. **8** (1955), 615–633.

189. Henri Lebesgue, *Sur le problème de Dirichlet*, Rend. Circ. Mat. Palermo **24** (1907), 371–402.

190. Jean Leray, *Hyperbolic differential equations*, Inst. Adv. Study, Princeton, N. J., 1953.

191. Beppo Levi, *Sul principo di Dirichlet*, Rend. Circ. Mat. Palermo **22** (1906), 293–360.

192. J. L. Lions, *Sur quelques problèmes aux limites relatifs à des opérateurs différentiels elliptiques*, Bull. Soc. Math. France **83** (1955), 225–250.

193. ____, *Problèmes mixtes abstraits*, Proc. Internat. Congr. Math. (Edinburgh, 1958), Cambridge Univ. Press, 1960, pp. 389–397.

194. J. L. Lions and E. Magenes, *Problèmes aux limites non homogènes et applications*. Vol. 1, Dunod, Paris, 1968; English transl., Springer-Verlag, 1972.

195. P. I. Lizorkin and S. M. Nikol′skiĭ, *A classification of differentiable functions in some fundamental spaces with dominant mixed derivative*, Trudy Mat. Inst. Steklov. **77** (1965), 143–167; English transl. in Proc. Steklov Inst. Math. **77** (1965).

196. A. D. Lyashko, *On the Galerkin-Petrov method*, Kazan. Gos. Univ. Uchen. Zap. **117** (1957), kn. 2, 42–44. (Russian)

197. ____, *Convergence of Galerkin type methods*, Dokl. Akad. Nauk SSSR **120** (1958), 242–244. (Russian)

198. ____, *A generalization of Galerkin's method*, Izv. Vyssh. Uchebn. Zaved. Mat. **1958**, no. 4(5), 153–160; correction, **1959**, no. 2(9), 275. (Russian)

199. ___, *Convergence of methods analogous to Galerkin's*, Izv. Vyssh. Uchebn. Zaved. Mat. **1958**, no. 6(7), 176–179. (Russian)

200. ___, *Some versions of the Galerkin-Krylov method*, Dokl. Akad. Nauk SSSR **128** (1959), 468–470. (Russian)

201. ___, *An approximate solution of one-dimensional boundary value problems*, Izv. Vyssh. Uchebn. Zaved. Mat. **1962**, no. 2(27), 95–99. (Russian)

202. ___, *A variational method for nonlinear operator equations*, Kazan. Gos. Univ. Uchen. Zap. **125** (1965), kn. 2, 95–101. (Russian)

203. A. D. Lyashko and M. M. Karchevskiĭ, *A study of the method of straight lines for nonlinear elliptic equations*, Zh. Vychils. Mat. i Mat. Fiz. **7** (1967), 677–680; English transl. in USSR Comput. Math. and Math. Phys. **7** (1967).

204. ___, *Study of difference schemes for nonlinear equations by a variational method*, Izv. Vyssh. Uchebn. Zaved. Mat. **1967**, no. 3(58), 59–65; English transl. in Amer. Math. Soc. Transl. (2) **111** (1978).

205. F. Magri, *Variational formulation for every linear equation*, Internat. J. Engrg. Sci. **12** (1974), 537–549.

206. Franco Mandras, *Su una classe di equazioni ellittiche degeneri di tipo non variazionale in due variabili*, Boll. Un. Mat. Ital. B (5) **18** (1981), 605–618.

207. G. I. Marchuk and V. I. Agoshkov, *Introduction to projection-difference methods*, "Nauka", Moscow, 1981. (Russian)

208. Joseph Marty, *Valeurs singulièrs d'une équation de Fredholm*, C. R. Acad. Sci. Paris **150** (1910), 1499–1502.

209. A. E. Martynyuk, *On a generalization of a variational method*, Dokl. Akad. Nauk SSSR **117** (1957), 374–377. (Russian)

210. ___, *On Galerkin's method*, Kazan. Gos. Univ. Uchen. Zap. **117** (1957), kn. 2, 70–74. (Russian)

211. ___, *Variational methods in boundary value problems for weakly elliptic equations*, Dokl. Akad. Nauk SSSR **122** (1959), 1222–1225. (Russian)

212. ___, *Some new applications of Galerkin type methods*, Mat. Sb. **49(91)** (1959), 85–108. (Russian)

213. ___, *Applications of Galerkin's method and Galerkin moments to a certain partial differential equation of Vekua type*, Izv. Vyssh. Uchebn. Zaved. Mat. **1960**, no. 3(16), 188–204. (Russian)

214. ___, *Solution of a fundamental boundary value problem for certain linear partial differential equations of even order*, Dokl. Akad. Nauk SSSR **147** (1962), 1288–1291; English transl. in Soviet Math. Dokl. **3** (1962).

215. ——, *Some new criteria for convergence of the method of successive approximations*, Kazan. Gos. Univ. Uchen. Zap. **124** (1964), kn. 6, 183–188. (Russian)

216. ——, *Some approximate methods for solving nonlinear equations with unbounded operators*, Izv. Vyssh. Uchebn. Zaved. Mat. **1966**, no. 6(55), 85–94; addendum, ibid. **1967**, no. 8(63), 111. (Russian)

217. ——, *Some approximate methods of Galerkin type and combined type*, Izv. Vyssh. Uchebn. Zaved. Mat. **1967**, no. 10(65), 62–75. (Russian)

218. V. N. Maslennikova, *On mixed problems for a system of equations of mathematical physics*, Dokl. Akad. Nauk SSSR **102** (1955), 885–888. (Russian)

219. T. Matsuzawa, *Construction of fundamental solutions of hypoelliptic equations by the use of a probabilistic method*, Nagoya Math. J. **72** (1978), 103–126.

220. Krzyztof Maurin, *Methods of Hilbert spaces*, PWN, Warsaw, 1959; English transl., 1967.

221. V. P. Mikhaĭlov, *The Dirichlet problem and the first mixed problem for a parabolic equation*, Dokl. Akad. Nauk SSSR **140** (1961), 303–306; English transl. in Soviet Math. Dokl. **2** (1961).

222. ——, *The Dirichlet problem for a parabolic equation*. I, Mat. Sb. **61**(**103**) (1963), 40–64. (Russian)

223. S. G. Mikhlin, *The problem of the minimum of a quadratic functional*, GITTL, Moscow, 1952; English transl., Holden-Day, San Francisco, Calif., 1965.

224. ——, *Two theorems on regularizers*, Dokl. Akad. Nauk SSSR **125** (1959), 737–739. (Russian)

225. ——, *Variational methods in mathematical physics*, 2nd rev. aug. ed., "Nauka", Moscow, 1970; English transl. of 1st ed., Pergamon Press, Oxford, and Macmillan, New York, 1964.

226. Clark B. Millikan, *On the study of the motions of viscous, incompressible fluids with particular reference to a variation principle*, Philos. Mag. and J. Sci. (7) **7** (1929), 641–662.

227. George J. Minty, *Integrability conditions for vector fields in Banach spaces*, Ph. D. Thesis, Univ. of Michigan, Ann Arbor, Mich., 1958.

228. ——, *Monotone* (*nonlinear*) *operators in Hilbert space*, Duke Math. J. **29** (1962), 341–346.

229. A. R. Mitchell and R. Wait, *The finite element method in partial differential equations*, Wiley, 1977.

230. B. L. Moiseiwitsch, *Variational principles*, Interscience, 1966.

231. Cathleen S. Morawetz, *A weak solution for a system of equations of elliptic-hyperbolic type*, Comm. Pure Appl. Math. **11** (1958), 315–331.

232. P. P. Mosolov, *A boundary value problem for hypoelliptic operators*, Mat. Sb. **55(97)** (1961), 307–328. (Russian)

233. ——, *Variational methods in nonstationary problems*, Izv. Akad. Nauk SSSR Ser. Mat. **34** (1970), 425–457; English transl. in Math. USSR Izv. **4** (1970).

234. M. Nagumo, *Lectures on the contemporary theory of partial differential equations*, Kyōritsu Shuppan, Tokyo, 1957 (Japanese); Russian transl., "Mir", Moscow, 1967.

235. M. Z. Nashed, *The convergence of the method of steepest descents for nonlinear equations with variational or quasi-variational operators*, J. Math. and Mech. **13** (1964), 765–794.

236. ——, *Differentiability and related properties of nonlinear operators: some aspects of the role of differentials in nonlinear functional analysis*, Nonlinear Functional Analysis and Applications (Proc. Sem., Madison, Wisc., 1970; L. B. Rall, editor), Academic Press, 1971, pp. 103–309.

237. J. von Neumann, *Über adjungierte Funktionaloperatoren*, Ann. of Math. (2) **33** (1932), 294–310.

238. S. M. Nikol′skiĭ, *On the question of solving the polyharmonic equation by a variational method*, Dokl. Akad. Nauk SSSR **88** (1953), 409–411. (Russian)

239. ——, *Properties of differentiable functions of several variables at the boundary of their domain of definition*, Dokl. Akad. Nauk SSSR **146** (1962), 542–545; English transl. in Soviet Math. Dokl. **3** (1962).

240. ——, *On stable boundary values of differentiable functions of several variables*, Mat. Sb. **61(103)** (1963), 224–252. (Russian)

241. ——, *A variational problem*, Mat. Sb. **62(104)** (1963), 53–75; English transl. in Amer. Math. Soc. Transl. (2) **51** (1966).

242. ——, *Approximation of functions of several variables and embedding theorems*, "Nauka", Moscow, 1969; English transl., Springer-Verlag, 1975.

243. B. Noble, *Complementary variational principles for boundary value problems*. I, V. S. Army Math. Res. Center Report No. 473, Univ. of Wisconsin, Madison, Wisc., 1964.

244. B. Noble and M. J. Sewell, *On dual extremum principles in applied mathematics*, J. Inst. Math. Appl. **9** (1972), 123–193.

245. Yu. I. Nyashin, *Variational formulation of the time-dependent problem of heat conduction*, Perm. Politekhn. Inst. Sb. Nauchn. Trudov Vyp. 152 (1974), 3–9. (Russian)

246. Yu. I. Nyashin, A. N. Skorokhodov, and I. N. Aran′ev, *A variational method for calculating a time-dependent temperature field*, Inzh.-Fiz. Zh. **26** (1974), 770–776; English transl. in J. Engrg. Phys. **26** (1974).

247. M. R. Osborne, *The numerical solution of a periodic parabolic problem subject to a nonlinear boundary condition*. II, Numer. Math. **12** (1968), 280–287.

248. V. P. Palamodov, *On the singularity of fundamental solutions of hypoelliptic equations*, Sibirsk. Mat. Zh. **4** (1963), 1365–1375; English transl. in Amer. Math. Soc. Transl. (2) **54** (1966).

249. B. P. Paneyakh, *On the existence of well-posed boundary value problems for systems of differential equations*, Uspekhi Mat. Nauk **18** (1963), no. 2(110), 175–176. (Russian)

250. W. V. Petryshyn, *Direct and iterative methods for the solution of linear operator equations in Hilbert space*, Trans. Amer. Math. Soc. **105** (1962), 136–175.

251. ——, *The generalized overrelaxation method for the approximate solution of operator equations in Hilbert space*, J. Soc. Indust. Appl. Math. **10** (1962), 675–690.

252. ——, *On a general iterative method for the approximate solution of linear operator equations*. Math. Comp. **17** (1963), 1–10.

253. ——, *On the eigenvalue problem $Tu - \lambda Su = 0$ with unbounded and nonsymmetric operators T and S*, Philos. Trans. Roy. Soc. London Ser. A **262** (1967/68), 413–458.

254. ——, *On the extrapolated Jacobi or simultaneous displacements method in the solution of matrix and operator equations*, Math. Comp. **19** (1965), 37–55.

255. ——, *On two variants of a method for the solution of linear equations with unbounded operators and their applications*, J. Math. and Phys. **44** (1965), 297–312.

256. ——, *On generalized inverses and on the uniform convergence of $(I - \beta K)^n$ with application to iterative methods*, J. Math. Anal. Appl. **18** (1967), 417–439.

257. ——, *On a class of K-p.d. and non-K-p.d. operators and operator equations*, J. Math. Anal. Appl. **10** (1965), 1–24.

258. ——, *On the extension and the solution of nonlinear operator equations*, Illinois J. Math. **10** (1966), 255–274.

259. ——, *On the inversion of matrices and linear operators*, Proc. Amer. Math. Soc. **16** (1965), 893–901.

260. ——, *On the iteration, projection and projection-iteration methods in the solution of nonlinear functional equations*, J. Math. Anal. Appl. **21** (1968), 575–607.

261. ——, *Remarks on the generalized overrelaxation and the extrapolated Jacobi methods for operator equations in Hilbert space*, J. Math. Anal. Appl. **29** (1970), 558–568.

262. ——, *On the approximation-solvability of equations involving A-proper and pseudo-A-proper mappings*, Bull. Amer. Math. Soc. **81** (1975), 223–312.

263. ——, *Variational solvability of quasilinear elliptic boundary value problems at resonance*, Nonlinear Anal. **5** (1981), 1095–1108.

264. T. H. H. Pian, *Finite element methods by variational principles with relaxed continuity requirement*, Variational Methods in Engineering (Proc. Internat. Sympos., Southhampton, 1973; C. Brebbia and H. Tottenham, editors), Vol. I, Southampton Univ. Press, Southhampton, 1973, pp. 3/1–3/25.

265. H. Poincaré, *Les idées de Hertz sur la mécanique*, Rev. Gén. Sci. Pures Appl. **8** (1897), 734–743.

266. G. C. Pomraning, *Complementary variational principles and their application to neutron transport problems*, J. Mathematical Phys. **8** (1967), 2096–2108.

267. ——, *Reciprocal and canonical forms of variational problems involving linear operators*, J. Math. and Phys. **47** (1968), 155–169.

268. A. A. Popova, *Application of the method of steepest descent to the solution of asymmetric operator equations*, Vychisl. i Prikl. Mat. (Kiev) Vyp. 42 (1980), 17–24. (Russian)

269. ——, *Variational methods for the approximate solution of boundary value problems in mathematical physics with a nonsymmetric operator*, Author's Summary of Candidate's Dissertation, Kiev State Univ., Kiev, 1981. (Russian)

270. I. Prigogine and P. Glansdorff, *Variational properties and fluctuation theory*, Physica **3** (1965), 1242–1256.

271. L. B. Rall, *Variational methods for nonlinear integral equations*, Nonlinear Integral Equations (Proc. Adv. Sem., Madison, Wisc., 1963; P. M. Anselone, editor), Univ. of Wisconsin Press, Madison, Wisc., 1964, pp. 155–189.

272. ——, *On complementary variational principles*, J. Math. Anal. Appl. **14** (1966), 174–184.

273. I. M. Rapoport, *The inverse problem of the calculus of variations*, Zh. Inst. Mat. Akad. Nauk Ukrain. SSR **1938**, no. 4, 105–122. (Russian)

274. ———, *Le problème inverse du calcul des variations*, C. R. (Dokl.) Acad. Sci. URSS **18** (1938), 131–135.

275. William T. Reid, *Symmetrizable completely continuous linear transformations in Hilbert space*, Duke Math. J. **18** (1951), 41–56.

276. Karel Pektorys, *On application of direct variational methods to the solution of parabolic boundary value problems of arbitrary order in the space variables*, Czechoslovak Math. J. **21**(**96**) (1971), 318–339.

277. ———, *Variational methods in mathematics, science and engineering*, Reidel, 1977.

278. Peter D. Robinson, *Complementary variational principles*, Nonlinear Functional Analysis and Applications (Proc. Sem., Madison, Wisc., 1970; L. B. Rall, editor), Academic Press, 1971, pp. 507–576.

279. Marcel N. Roşculet, *Contributions to the study of partial differential equations of parabolic type*. IV, Acad. R. P. Romîne Stud. Cerc. Mat. **13** (1962), 419–437. (Romanian; French summary)

280. Hanno Rund, *The reduction of certain boundary value problems to variational problems by means of transversality conditions*, Numer. Math. **15** (1970), 49–56.

281. Kyuichi Sakuma, *On the fundamental solutions of ultrahyperbolic operators*, Sûgaku **12** (1960/61), 107–111. (Japanese)

282. David A. Sánchez, *Calculus of variations for integrals depending on a convolution product*, Ann. Scuola Norm. Sup. Pisa (3) **18** (1964), 233–254.

283. ———, *On extremals for integrals depending on a convolution product*, J. Math. Anal. Appl. **11** (1965), 213–216.

284. Ranbir S. Sandhu and Karl S. Pister, *Variational principles for boundary value and initial-boundary value problems in continuum mechanics*, Internat. J. Solids and Structures **7** (1971), 639–654.

285. ———, *Variational methods in continuum mechanics*, Variational Methods in Engineering (Proc. Internat. Sympos., Southhampton, 1973; C. Brebbia and H. Tottenham, editors), Vol. I, Southampton Univ. Press, Southhampton, 1973, pp. 1/13–1/25.

286. Ruggero Maria Santilli, *Foundations of theoretical mechanics*. Vol. I, Springer-Verlag, 1978.

287. Leonard Sarason, *Differentiable solutions of symmetrizable and singular symmetric first order systems*, Arch. Rational Mech. Anal. **26** (1967), 357–384.

288. R. S. Schechter, *The variational method in engineering*, McGraw-Hill, 1967.

289. Laurent Schwartz, *Les travaux de L. Gårding sur les équations aux dérivés partielles elliptiques*, Sém. Bourbaki 1951/52, Exposé 67, Secretariat Math., Paris, 1952.

290. L. I. Sedov, *Application of a basic variational equation for the construction of models of continuous media*, Selected Questions of Modern Mechanics (S. S. Grigoryan Fiftieth Birthday Vol.), Izdat. Moskov. Gos. Univ., Moscow, 1981, pp. 11–64. (Russian) cf. also *Applying the basic variational equation for building models of matter and fields*, Proc. IUTAM-ISIMM Sympos.Modern Developments in Analytical Mech.(Torino, 1982). Vol. II, Atti Accad. Sci. Torino Cl. Sci. Fis. Mat. Nat. **117** (1983), Suppl. 2, 765–816.

291. M. J. Sewell, *On dual approximation principles and optimization in continuum mechanics*, Philos. Trans. Roy. Soc. London Ser. A **265** (1969/70), 319–351.

292. V. M. Shalov, *A generalization of Friedrichs space*, Dokl. Akad. Nauk SSSR **151** (1963), 292–294; English transl. in Soviet Math. Dokl. **4** (1963).

293. ——, *Solution of nonselfadjoint equations by the variational method*, Dokl. Akad. Nauk SSSR **151** (1963), 511–512; English transl. in Soviet Math. Dokl. **4** (1963).

294. ——, *The principle of a minimum of a quadratic functional for a hyperbolic equation*, Differentsial'nye Uravneniya **1** (1965), 1338–1365; English transl. in Differential Equations **1** (1965).

295. ——, *Equations of continuum mechanics*, Differentsial'nye Uravneniya **9** (1973), 912–921; English transl. in Differential Equations **9**(1973).

296. F. A. Shelkovnikov, *A generalized Cauchy formula*, Uspekhi Mat. Nauk **6** (1951), no. 3(43), 157–159. (Russian)

297. G. E. Shilov, *Local properties of solutions of partial differential equations with constant coefficients*, Uspekhi Mat. Nauk **14** (1959), no. 5(89), 3–44; English transl. in Amer. Math. Soc. Transl. (2) **42** (1964).

298. I. I. Shmulev, *Periodic solutions of the first boundary value problem for parabolic equations*, Mat. Sb. **66**(**108**) (1965), 398–410; English transl. in Amer. Math. Soc. Transl. (2) **79** (1969).

299. N. N. Shopolov, *A mixed problem for the heat equation with a nonlocal initial condition*, C. R. Acad. Bulgare Sci. **34** (1981), 935–936. (Russian)

300. J. P. O. Silberstein, *Symmetrisable operators*, II, III, J. Austral. Math. Soc. **4** (1964), 15–30, 31–48.

301. L. N. Slobodetskiĭ, *Generalized Sobolev spaces and their application to boundary value problems for partial differential equations*, Leningrad,

Gos. Ped. Inst. Uchen. Zap. **197** (1958), 54–112; English transl. in Amer. Math. Soc. Transl. (2) **57** (1966).

302. M. G. Slobodyanskiĭ, *Estimate of the error of the quantity sought for in the solution of linear problems by a variational method*, Dokl. Akad. Nauk SSSR **86** (1952), 243–246. (Russian)

303. ——, *Estimates of errors in approximate solutions of linear problems*, Prikl. Mat. Mekh. **17** (1953), 229–244. (Russian)

304. ——, *Approximate solution of some boundary value problems for elliptic differential equations, and error estimates*, Dokl. Akad. Nauk SSSR **89** (1953), 221–224. (Russian)

305. ——, *On the construction of an approximate solution in linear problems*, Prikl. Mat. Mekh. **19** (1955), 571–588. (Russian)

306. ——, *On transformation of the problem of the minimum of a functional to the problem of the maximum*, Dokl. Akad. Nauk SSSR **91** (1953), 733–736. (Russian)

307. M. M. Smirnov, *The first boundary value problem for a hyperbolic equation of the fourth order*, Vestnik Leningrad. Univ. **1958**, no. 19 (Ser. Mat. Mekh. Astr. vyp. 4), 55–57. (Russian)

308. W. Smit, *Complementary variational principles for the solution of a large class of operator equations*, Variational Methods in Engineering (Proc. Internat. Sympos., Southhampton, 1973; C. Brebbia and H. Tottenham, editors), Vol. I, Southampton Univ. Press, Southhampton, 1973, pp. 1/29–1/42.

309. Donald R. Smith and C. V. Smith, Jr., *When is Hamilton's principle an extremum principle?*, AIAA J. **12** (1974), 1573–1576.

310. P. Smith [Peter Smith], *Approximate operators and dual extremum principles for linear problems*, J. Inst. Math. Appl. **22** (1978), 457–465.

311. S. L. Sobolev, *Applications of functional analysis in mathematical physics*, Izdat. Leningrad. Gos. Univ., Leningrad, 1950; English transl., Amer. Math. Soc., Providence, R. I., 1963.

312. S. L. Sobolev and M. I. Vishik, *Some functional methods in the theory of partial differential equations*, Proc. Third All-Union Math. Congr. (Moscow, 1956), Vol. 3, Izdat. Akad. Nauk SSSR, Moscow, 1956, pp. 152–162. (Russian)

313. P. E. Sobolevskiĭ, *On equations with operators forming an acute angle*, Dokl. Akad. Nauk SSSR **116** (1957), 754–757. (Russian)

314. Ivar Stakgold, *Green's functions and boundary value problems*, Wiley, 1979.

315. J. L. Synge, *The method of the hypercircle in function-space for boundary-value problems*, Proc. Roy. Soc. London Ser. A 191 (1947), 447–467.

316. ——, *The method of the hypercircle in elasticity when body forces are present*, Quart. Appl. Math. **6** (1947), 15–19.

317. ——, *The hypercircle in mathematical physics: a method for the approximate solution of boundary value problems*, Cambridge Univ. Press, 1957.

318. Zofia Szmydt, *Fourier transformation and linear differential equations*, PWN, Warsaw, and Peidel, Dordrecht, 1977.

319. A. T. Taldykin, *Variational methods for solving equations*, Differentsial'nye Uravneniya **10** (1974), 1714–1720; English transl. in Differential Equations **10** (1974).

320. Enzo Tonti, *Condizioni iniziali neiprincipi variazionali*, Ist. Lombardo Accad. Sci. Lett. Rend. A **100** (1966), 982–988.

321. ——, *Variational formulation of nonlinear differential equations*. I, II, Acad. Roy. Belgique Bull. Cl. Sci. (5) **55** (1969), 137–165, 262–278.

322. ——, *On the variational formulation for linear initial value problems*, Ann. Mat. Pura Appl. (4) **95** (1973), 331–359.

323. ——, *A systematic approach to the search for variational principles*, Variational Methods in Engineering (Proc. Internat. Sympos., Southhampton, 1973; C. Brebbia and H. Tottenham, editors), Vol. I, Southampton Univ. Press, Southhampton, 1973, pp. 1/1–1/12.

324. ——, *A general solution of the inverse problem of the calculus of variations*, Proc. First Internat. Conf. Nonpotential Interactions and Their Lie-Admissible Treatment (Orléans, 1982), Part C, Hadronic J. **5**(1981/82), 1404–1450.

325. E. Trefftz, *Ein Gegenstück zum Ritzschen Verfahren*, Proc. Second Internat. Congr. Appl. Mech. (Zürich, 1926), Orell Füssli Verlag, Zürich, 1927, pp. 131–138.

326. ——, *Konvergenz und Fehlerabschätzung beim Ritzschen Verfahren*, Math. Ann. **100** (1928), 503–521.

327. J. F. Treves [François Trèves], *Lectures on linear partial differential equations with constant coefficients*, Notas de Mat., no. 27, Inst. Mat. Pura Apl. Conselho Nac. Pesquisas, Rio de Janeiro, 1961.

328. Hans Triebel, *Interpolation theory, function spaces, differential operators*, VEB Deutscher Verlag Wiss., Berlin, 1977, and North-Holland, Amsterdam, 1978.

329. Tu Guizhang, *On formal variational calculus of higher dimensions*, J. Math. Anal. Appl. **94** (1983), 348–365.

330. M. M. Vaĭnberg, *Some questions of the differential calculus in linear spaces*, Uspekhi Mat. Nauk **7** (1952), no. 4(50), 55–102. (Russian)

331. ———, *Variational methods for the study of nonlinear operators*, GITTL, Moscow, 1956; English transl., Holden-Day, San Francisco, Calif., 1964.

332. ———, *On some new principles in the theory of nonlinear equations*, Uspekhi Mat. Nauk **15** (1960), no. 1(19), 243–244. (Russian)

333. ———, *Convergence of the method of steepest descent for nonlinear equations*, Sibirsk. Mat. Zh. **2** (1961), 201–220. (Russian)

334. M. M. Vaĭnberg and R. I. Kachurovskiĭ, *On the variational theory of nonlinear operators and equations*, Dokl. Akad. Nauk SSSR **129** (1959), 1199–1202. (Russian)

335. A. L. Vanderbauwhede, *Potential operators and the inverse problem of classical mechanics*, Hadronic J. **1** (1978), 1177–1197.

336. R. S. Varga, *Functional analysis and approximation theory in numerical analysis*, Conf. Board Math. Sci. Regional Conf. Ser. Appl. Math., vol. 3, SIAM, Philadelphia, Pa., 1971.

337. Waldemar Velte, *Direkte Methoden der Variationsrechnung*, Teubner, Stuttgart, 1976.

338. ———, *Komplementäre Extremalprobleme*, Methoden und Verfahren der Mathematischen Physik, vol. 15 (B. Brosowski and E. Martensen, editors), Bibliograpisches Inst., Mannheim, 1976, pp. 1–44.

339. M. I. Vishik, *Mixed boundary value problems for equations having a first derivative with respect to time, and an approximate method for solving them*, Dokl. Akad. Nauk SSSR **99** (1954), 189–192. (Russian)

340. V. S. Vladimirov, *Mathematical problems in the uniform-speed theory of transport*, Trudy Mat. Inst. Steklov. **61** (1961). (Russian)

341. L. R. Volevich and B. P. Paneyakh, *Some spaces of generalized functions and imbedding theorems*, Uspekhi Mat. Nauk **20** (1965), no. 1(121), 3–74; English transl. in Russian Math. Surveys **20** (1965).

342. Vito Volterra, *Leçons sur les fonctions des lignes*, Gauthier-Villars, Paris, 1913.

343. Eugene L. Wachspress, *A rational finite element basis*, Academic Press, 1975.

344. K. Washizu, *Bounds for solutions of boundary value problems in elasticity*, J. Math. and Phys. **32** (1953), 117–128.

345. H. F. Weinberger, *Upper and lower bounds for torsional rigidity*, J. Math. and Phys. **32** (1953), 54–62.

346. Jürgen Weyer, *Über maximale Monotonie von Operatoren des Typs* $L^*\Phi \circ L$, Report 78-24, Math. Inst., Univ. Köln, 1978; published as Manuscripta Math. **28** (1979), 305–316.

347. S. Ya. Yakubov, *Hilbert-Schmidt theory for integral and integro-differential equations with nonsymmetric kernels*, Izv. Akad. Nauk Azerbaidzhan. SSR Ser. Fiz.-Mat. i Tekhn. Nauk **1962**, no. 1, 35–45. (Russian)

348. A. C. Zaanen, *Über vollstetige symmetrische und symmetrisierbare Operatoren*, Nieuw Arch. Wiskunde (2) **22** (1943), 57–80.

349. ——, *Normalisable transformations in Hilbert space and systems of linear integral equations*, Acta Math. **83** (1950), 197–248.

350. B. Zainea, *The generalized energy method*, An. Univ. Bucureşti Mat.-Mec. **21** (1972), 119–128. (Romanian)

351. V. I. Zaplstnyĭ [V. Ī. Zaplatniĭ], *Construction of the density of the Lagrangian function from a given fourth-order partial differential equation*, Dokl. Akad. Nauk Ukrain. SSR Ser. A **1980**, no. 1, 33–35.

352. Eduardo H. Zarantonello, *The closure of the numerical range contains the spectrum*, Bull. Amer. Math. Soc. **70** (1964), 781–787.

353. Stanislas [Stanisław] Zaremba, *Sur le principe de minimum*, Bull. Internat. Acad. Sci. Cracovie Cl. Sci. Math. Nat. **1909** (deuxième semestre), no. 7, 199–264.

354. ——, *Sur un problème toujours possible comprenant à titre de cas particuliers, le problème de Dirichlet et celui de Neumann*, J. Math. Pures Appl. (9) **6** (1927), 127–163.

355. O. C. Zienkiewicz, *The finite element method in engineering science*, 2nd rev. ed., McGraw-Hill, 1971; German transl., Carl Hanser Verlag, Munich and Vienna, 1975.

356. George Dincă and Daniel Mateescu, *On the structure of linear and K-positive definite operators*, Rev. Roumaine Math. Pures Appl. **27** (1982), 677–687.

ABCDEFGHIJ – 89

ISBN 978-0-8218-4529-5

9 780821 845295

MMONO/77